Poiseuille Flow of Liquid Crystals of the Nematic Type

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Contents

1. Introduction

It has been known since 1888 that many chemicals, upon heating, exhibit a phase which is ordered on the molecular level yet possesses the mechanical properties of a fluid. This so-called liquid crystal phase was classified by FRIEDEL into three types; smectic, nematic and cholesteric. All types consist of large, relatively rigid elongated molecules, the elongation giving rise to a preferred direction. The smectic type is thought to have a stratified structure, the molecules lying in layers with their long axes roughly normal to the plane of the layers. In the nematic and cholesteric type, however, the long rod like molecules appear to be free to move randomly, except that they align themselves approximately parallel to their neighbours.

The theoretical description of these mesophases is still being developed. The OSEEN-FRANK static theory for liquid crystals of nematic and cholesteric type, proposed by OSEEN [1] in 1929 and later modified by FRANK [2], has been successful in describing observed orientation patterns for these crystals* and, at present, is regarded as a reasonable static theory. In an attempt to describe the dynamical behaviour of these crystals, ERICKSEN proposed a simple continuum theory of anisotropic fluids. A review of his theory, together with some of its predictions, can be found in the treatise by TRUESDELL & NOLL [4, pp. 523 – 537]. As ERICKSEN [5] pointed out, this theory, as well as that proposed by LESLIE [6], in which thermodynamic restrictions are considered, does not reduce to the OSEEN-FRANK static theory or a likely alternative. More general theories are required to remedy this defect and LESLIE [7, 8] has proposed such theories for liquid crystals of both nematic and cholesteric type. In this paper, we are concerned with the former.

The viscometry of liquid crystals seems to be one area in which theory and experiment may be compared. LESLIE [7, pp. 279 – 281] has already considered

* See, for example, ERICKSEN [3].

an exact solution for shear flow between two parallel flat plates, a fixed distance h apart, one of which is at rest and the other moving with constant speed V along a straight line in its plane. In this solution, the orientation of the preferred direction varies, in general, with the distance from the plates, and, for a given material, the nature of this variation depends upon the product Vh. Defining an apparent viscosity η as the ratio of the shear stress producing the motion and the nominal shear rate V/h, LESLIE found that η is a function of the product Vh for a given liquid crystal. Also, he showed that, when the argument Vh tends to zero, η tends to a limiting value, and when Vh tends to infinity, η tends to a second limiting value, which is in general different from the first. ERICKSEN [9] has shown that the scaling found by LESLIE holds more generally.

Since most of the experiments are conducted in either capillary tubes or Couette viscometers,* it is desirable to examine the predictions of LESLIE's theory for these flows. ATKIN & LESLIE [12] have considered an exact solution for the flow between two rotating coaxial circular cylinders. The results are similar to those found for shear flow. In general, the orientation of the preferred direction varies throughout the liquid crystal. The manner in which the orientation, as well as a suitably defined apparent viscosity, varies, depends upon the torque applied per unit length of either cylinder about the common axis and upon the ratio of the radii of the cylinders. Couette flow has also been considered by CURRIE [13].

In this paper, we consider the flow due to a pressure gradient in both a stationary capillary tube and two coaxial stationary tubes, all cross sections being circular. We also mention briefly the case when the outer tube moves with a constant velocity whilst the inner tube remains stationary.

After outlining LESLIE's theory in Section 2, we apply it, in Section 3, to a possible exact solution of the governing equations for the flows under consideration. For the two coaxial tubes, we find that it is possible, under assumptions specified in Section 3, to establish the existence and uniqueness of this particular solution. At this stage we restrict our attention to orientations of the preferred direction below a critical angle θ_0.** For the flow in a capillary tube, the type of solution considered here has been examined previously by ERICKSEN [14] and LESLIE [6] using their earlier theories. ERICKSEN found that a "plug" flow, in which the orientation of the preferred direction along the axis of the tube was parallel to the streamlines, was necessary for the stresses to be finite throughout the region. LESLIE eliminated this effect, which so far has not been observed experimentally, by thermodynamic considerations. However, he found that the orientation was constant throughout the region, thus making it impossible for the solid boundary to induce a different orientation to that of the mainstream. In the present analysis, we find that no "plug" flow is necessary, but, for finite stresses throughout the region, the orientation along the axis has to be parallel to the streamlines. We also show that solutions for the orientation having the required property at $r=0$ exist for finite r. However, we are unable to establish the existence of solutions satisfying, in addition, a specified boundary orientation on the tube. In Section 4, we consider a more general flow and adapt ERICKSEN's scaling analysis to relate the measurable

* See, for example, PETER & PETERS [10] and PORTER & JOHNSON [11].

** For definition of θ_0 see equation (3.13).

quantities in the two viscometers. For a capillary tube, the relation is found to agree with recent data obtained by FISHER & FREDRICKSON [15] when the boundary orientation of the preferred direction is perpendicular to the tube. In the case when this orientation is parallel to the tube, there is still a discrepancy between the analysis presented here and the experimental data available.

2. Basic Theory

In this section, we outline the continuum theory proposed recently by LESLIE [7] for liquid crystals of the nematic type.

As is customary in continuum theories, LESLIE represents the preferred direction of these mesophases by assigning to each particle, occupying a place $\boldsymbol{x}$* at time t, a vector (or director) $\boldsymbol{d}(\boldsymbol{x}, t)$. Throughout this paper, we assume that

$$\boldsymbol{d}\cdot\boldsymbol{d}=1. \tag{2.1}$$

Further, we restrict our attention to isothermal flows, in which the liquid is incompressible and, for simplicity, we suppose that all *external* body forces, which influence orientation and motion, are absent.** The kinematic quantities required for the theory are the director gradient tensor, the director velocity vector $\boldsymbol{w}(\boldsymbol{x}, t)$, the fluid velocity vector $\boldsymbol{v}(\boldsymbol{x}, t)$, the rate of deformation tensor $\boldsymbol{A}(\boldsymbol{x}, t)$ and the vorticity tensor $\boldsymbol{\theta}(\boldsymbol{x}, t)$. In the absence of external body forces, the mechanical quantities, which are required to describe the behaviour of the crystals, are an intrinsic director body force vector $\boldsymbol{g}(\boldsymbol{x}, t)$, a stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ and a director stress tensor $\boldsymbol{\pi}(\boldsymbol{x}, t)$. The latter tensor represents the surface forces which do work in changing the orientation and, theoretically, it is through this quantity that solid boundaries may act to influence the orientation. Although no experimental data for such a force is available, it is known that, statically, the orientation can be influenced by surface actions. For example, when a liquid crystal is in contact with a glass wall the surface orientation can be influenced by rubbing the glass***. Finally we have the specific Helmholtz free energy $F(\boldsymbol{x}, t)$.

The basic field equations of the theory were first proposed by ERICKSEN [17]. If we refer the motion of the continuum to a fixed system of rectangular Cartesian co-ordinates x_1, x_2, x_3 (the x-co-ordinate system) and use the conventional notation for the x components of the various vectors and tensors introduced, under our assumptions these equations take the form

$$\begin{gathered} v_{p,p}=0, \\ \rho(\dot{v}_i+v_p v_{i,p})=\sigma_{pi,p}, \\ \rho_1(\dot{w}_i+v_p w_{i,p})=g_i+\pi_{pi,p}, \end{gathered} \tag{2.2}$$

* See TRUESDELL & NOLL [4, p. 37].

** For the analysis given in Section 3, ordinary body forces, which LESLIE [7, equation (3.1)] denotes by $\boldsymbol{F}(\boldsymbol{x}, t)$, may be included, provided we assume that they are conservative. Equations (3.7), (3.9) and (3.10) are then modified in the obvious way. If these forces are included in Section 4, they have to be scaled appropriately (see, for example, ATKIN & LESLIE [12]).

*** See BROWN & SHAW [16], Section X G.

where
$$w_i = \dot{d}_i + v_p d_{i,p}. \tag{2.3}$$

The superposed dot notation denotes partial derivatives with respect to the time t and the comma notation partial derivatives with respect to the co-ordinates x_i. Tensor indices take the values 1, 2, 3 and are subject to the summation convention. The symbols ρ and ρ_1 denote positive constants. Further, the relation

$$\sigma_{ji} - \pi_{pj} d_{i,p} + g_j d_i = \sigma_{ij} - \pi_{pi} d_{j,p} + g_i d_j \tag{2.4}$$

must hold. LESLIE also postulates a generalized form of the entropy-production inequality due to MÜLLER [18].

The theory is completed by postulating constitutive assumptions for F, $\boldsymbol{g}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$. Since the liquid is incompressible and the director is of unit magnitude, the stresses and the intrinsic director body force are indeterminate to a certain extent. Referring these quantities to the x-co-ordinate system, we therefore write

$$\begin{aligned} \sigma_{ji} &= -p\,\delta_{ij} + \hat{\sigma}_{ji}, \\ \pi_{ji} &= \beta_j d_i + \hat{\pi}_{ji}, \\ g_i &= \gamma\, d_i - (\beta_p d_i)_{,p} + \hat{g}_i, \end{aligned} \tag{2.5}$$

where $\hat{\boldsymbol{\sigma}}$, $\hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{g}}$ are referred to as the extra stress tensor, extra director stress tensor and extra intrinsic director body force vector respectively. The arbitrary scalars $p(\boldsymbol{x}, t)$, $\gamma(\boldsymbol{x}, t)$* are called the pressure and the director tension respectively and are determined from the equations of motion.

The arbitrary vector $\boldsymbol{\beta}(\boldsymbol{x}, t)$ is indeterminate. However, for problems in which either the couple stress** or the orientation of the preferred direction is prescribed on the boundary, its value does not affect the predictions of the theory, since it appears neither in the differential equations (2.2) governing the behaviour within the liquid crystal nor in the boundary conditions. For the problems which follow, we may therefore set $\boldsymbol{\beta} = \boldsymbol{0}$ without loss of generality.

LESLIE simplifies his theory by assuming that the non-equilibrium parts of the extra stress tensor and the extra intrinsic director body force vector are independent of the director gradients and depend linearly upon the kinematic variables. This assumption, together with motivation from the entropy-production inequality, leads him to propose the following constitutive equations,

$$\begin{aligned} \hat{\sigma}_{ji} &= -\rho \frac{\partial F}{\partial d_{p,j}} d_{p,i} + \mu_1 d_p d_q A_{pq} d_i d_j + \mu_2 d_j N_i + \mu_3 d_i N_j + \mu_4 A_{ij} \\ &\quad + \mu_5 d_j d_p A_{pi} + \mu_6 d_i d_p A_{pj}, \\ \hat{\pi}_{ji} &= \rho \frac{\partial F}{\partial d_{i,j}}, \\ \hat{g}_i &= -\rho \frac{\partial F}{\partial d_i} + \lambda_1 N_i + \lambda_2 d_p A_{pi}, \end{aligned} \tag{2.6}$$

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* The quantity denoted here by γ differs from the corresponding quantity in [7] by a term div $\boldsymbol{\beta}$.

** For definition see ERICKSEN [19].

where the x-components of the tensors A and θ and the vector N are given by

$$2A_{ij}=v_{i,j}+v_{j,i}, \quad 2\theta_{ij}=v_{i,j}-v_{j,i}, \quad N_i=w_i+\theta_{pi}d_p. \tag{2.7}$$

LESLIE assumes that the specific Helmholtz free energy is given by

$$\begin{aligned}2\rho F=&k_{22}d_{p,q}d_{p,q}+(k_{11}-k_{22}-k_{24})d_{p,p}d_{q,q}\\&+(k_{33}-k_{22})d_p d_q d_{r,p}d_{r,q}+k_{24}d_{p,q}d_{q,p}.\end{aligned} \tag{2.8}$$

Here we retain FRANK's [2] notation for the coefficients.

Under our assumptions, the coefficients λ_1, λ_2; μ_j ($j=1, 2, \ldots, 6$) and k_{11}, k_{22}, k_{33} and k_{24} are constants. λ_1, λ_2; μ_j ($j=1, 2, \ldots, 6$) have the dimensions of viscosity and

$$\lambda_1=\mu_2-\mu_3, \quad \lambda_2=\mu_5-\mu_6. \tag{2.9}$$

They are also restricted by the entropy-production inequality, which takes the form

$$\left(\hat{\sigma}_{qp}+\rho\frac{\partial F}{\partial d_{r,q}}d_{r,p}\right)A_{pq}-\left(\hat{g}_p+\rho\frac{\partial F}{\partial d_p}\right)N_p\geqq 0. \tag{2.10}$$

LESLIE [6] has shown that this is satisfied if

$$\begin{aligned}&\mu_4\geqq 0, \quad 2\mu_1+3\mu_4+2(\mu_5+\mu_6)\geqq 0, \quad 2\mu_4+\mu_5+\mu_6\geqq 0\\&\lambda_1\leqq 0, \quad -4\lambda_1(2\mu_4+\mu_5+\mu_6)\geqq(\mu_2+\mu_3-\lambda_2)^2.\end{aligned} \tag{2.11}$$

For the specific Helmholtz free energy to be positive definite, ERICKSEN [20] has shown that k_{11}, k_{22}, k_{33} and k_{24} must satisfy certain inequalities. Of these we note

$$k_{11}\geqq 0, \quad k_{33}\geqq 0. \tag{2.12}$$

ERICKSEN [21, 22] has suggested that two possible techniques for measuring some of the λ's and μ's are the observation of normal stress effects and the application of a magnetic field. This latter technique has been employed by MIESOWICZ [23] for the compound p-azoxyanisole. For the same compound, SAUPE [24] has given estimates of k_{11}, k_{22}, k_{33} by measuring static deformations in the presence of a magnetic field and he found that k_{11} and k_{33} are of the order of 10^{-6} dynes. Since k_{24} does not appear in the governing differential equation, a new technique is necessary for its measurement.* From ERICKSEN's [26] analysis of the so-called "twist" wave, which produces little or no motion of the fluid, ρ_1 emerges in the expression for the attenuation coefficient of this wave. A technique for measuring this quantity could therefore yield estimates of ρ_1. However, SAUPE [27] has suggested that the possibility of observing such waves may be remote.

3. An Exact Solution

One of the simplest viscometric flows is that in which steady laminar flow is caused by a pressure gradient along one, or more, tubes, the tube being stationary and of infinite length so that end effects can be neglected. In this section, we first

* See ERICKSEN [25].

consider such a flow, often called Poiseuille flow, through two coaxial circular tubes. Later we consider the case when the radius of the inner tube is zero. We also refer briefly to the case when the outer tube moves with a constant velocity V whilst the inner tube remains stationary. Henceforth, we refer the motion to a cylindrical polar co-ordinate system (r, ϕ, z), the z-axis being along the axis of the tubes and, when necessary, use physical components of the vectors and tensors involved.

One possible solution, which we now consider, is

$$\begin{aligned} &v_r=0, && v_\phi=0, && v_z=v(r), \\ &d_r=\sin\theta(r), && d_\phi=0, && d_z=\cos\theta(r). \end{aligned} \tag{3.1}$$

Here we assume that the orientation for the steady flow remains in the (r, z) plane and makes an angle $\theta(r)$ with the z-axis. This form for the components of $\boldsymbol{d}$ is motivated by the earlier work of ERICKSEN [14], where it was shown, using the simplified theory, that, for a steady solution, the flow tended to turn the directors into the flow plane. It is not clear whether a similar result is implied by the present theory. The solution (3.1) automatically satisfies the continuity equation (2.2_1).

For this solution, $\hat{\sigma}_{zz}$, $\hat{\pi}_{zr}$ and $\hat{\pi}_{zz}$ are functions of r only and the other non-vanishing constitutive equations (2.6) may be written in the form*

$$\begin{aligned} \hat{\sigma}_{rr} &= -f(\theta)\{\theta'(r)\}^2-(2r)^{-1}(k_{11}-k_{22}-k_{24})\sin 2\theta\,\theta'(r)+h(\theta)v'(r) \\ \hat{\sigma}_{rz} &= g(\theta)v'(r) \\ \hat{\sigma}_{\phi\phi} &= -(2r)^{-1}(k_{11}-k_{22}-k_{24})\sin 2\theta\,\theta'(r)-r^{-2}k_{11}\sin^2\theta \\ 2\hat{\sigma}_{zr} &= -f'(\theta)\{\theta'(r)\}^2+2r^{-1}(k_{11}-k_{22}-k_{24})\sin^2\theta\,\theta'(r) \\ &\quad +\{2g(\theta)+\lambda_1+\lambda_2\cos 2\theta\}v'(r), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \hat{\pi}_{rr} &= \{k_{11}+(k_{33}-k_{22})\sin^2\theta\}\cos\theta\,\theta'(r)+r^{-1}(k_{11}-k_{22}-k_{24})\sin\theta \\ \hat{\pi}_{rz} &= -\{k_{22}\cos^2\theta+k_{33}\sin^2\theta\}\sin\theta\,\theta'(r) \\ \hat{\pi}_{\phi\phi} &= (k_{11}-k_{22}-k_{24})\cos\theta\,\theta'(r)+r^{-1}k_{11}\sin\theta, \end{aligned} \tag{3.3}$$

$$\begin{aligned} g_r &= \gamma\sin\theta-(k_{33}-k_{22})\sin\theta\{\theta'(r)\}^2+\tfrac{1}{2}(\lambda_1+\lambda_2)\cos\theta\,v'(r) \\ g_z &= \gamma\cos\theta-\tfrac{1}{2}(\lambda_1-\lambda_2)\sin\theta\,v'(r), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} f(\theta) &= k_{11}\cos^2\theta+k_{33}\sin^2\theta \\ 2g(\theta) &= 2\mu_1\sin^2\theta\cos^2\theta+(\mu_5-\mu_2)\sin^2\theta+(\mu_6+\mu_3)\cos^2\theta+\mu_4 \\ 4h(\theta) &= (2\mu_1\sin^2\theta+\mu_2+\mu_3+\mu_5+\mu_6)\sin 2\theta. \end{aligned} \tag{3.5}$$

It follows from the inequalities (2.12) and the entropy-production inequality (2.10), which for the particular solution (3.1) reduces to

$$g(\theta)\{v'(r)\}^2 \geqq 0, \tag{3.6}$$

* Throughout, the notation $c'(\eta)$, $c''(\eta)$ denotes $dc/d\eta$ and $d^2c/d\eta^2$, respectively. Also we have placed $\boldsymbol{\beta}=\mathbf{0}$.

that f and g are non-negative functions of θ. We restrict our attention to the case when both functions are strictly positive.

Finally, for the particular solution (3.1) the field equations $(2.2_{2,3})$ reduce to

$$\begin{aligned} \frac{d\hat{\sigma}_{rr}}{dr}+\frac{1}{r}(\hat{\sigma}_{rr}-\hat{\sigma}_{\phi\phi})-\frac{\partial p}{\partial r}&=0\\ \frac{\partial p}{\partial \phi}&=0\\ \frac{d}{dr}(r\hat{\sigma}_{rz})-r\frac{\partial p}{\partial z}&=0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{d\hat{\pi}_{rr}}{dr}+\frac{1}{r}(\hat{\pi}_{rr}-\hat{\pi}_{\phi\phi})+g_r&=0\\ \frac{d}{dr}(r\hat{\pi}_{rz})+r g_z&=0. \end{aligned} \tag{3.8}$$

It follows from equations (3.7) that

$$p=-az+k(r), \tag{3.9}$$

where a is an arbitrary constant, usually referred to as the specific driving force of the flow, and $k(r)$ is a function determined from (3.7_1). Solving for $k(r)$, with the aid of equations $(3.2_{1,3})$, we see that p is given by

$$p=p_0-az+\hat{\sigma}_{rr}+\int\left\{\frac{h[\theta(s)]\,v'(s)-f[\theta(s)]\,\{\theta'(s)\}^2}{r}+\frac{k_{11}\sin^2\theta(s)}{r^3}\right\}ds, \tag{3.10}$$

p_0 being an arbitrary constant. Integrating equation (3.7_3) and using (3.2_2), (3.9) we have

$$g(\theta)\,v'(r)=-\frac{1}{2}ar+\frac{b}{r}, \tag{3.11}$$

where b is an arbitrary constant. Unless otherwise stated we take a and b to be non-zero. Two further differential equations are obtained on substituting (3.3) and (3.4) into (3.8). Eliminating γ from these equations, we obtain

$$\begin{aligned} 2f(\theta)\,\theta''(r)+f'(\theta)\,\{\theta'(r)\}^2+2r^{-1}f(\theta)\,\theta'(r)-r^{-2}k_{11}\sin 2\theta&\\ +\lambda_2(\cos 2\theta-\cos 2\theta_0)\,v'(r)=0&, \end{aligned} \tag{3.12}$$

where

$$\cos 2\theta_0=-\frac{\lambda_1}{\lambda_2}. \tag{3.13}$$

Henceforth we assume that

$$|\lambda_1|\leqq|\lambda_2|\neq 0 \tag{3.14}$$

so that the angle θ_0 is real. This assumption is equivalent to that used by ERICKSEN [14, 21] and by LESLIE [6, 7]. The angle θ_0 has arisen in previous analyses of "Poiseuille" flow in a capillary tube using the simpler theories of ERICKSEN [14] and LESLIE [6]. In particular, LESLIE [6] found that, for a solution of the type (3.1), his simpler theory predicts that the angle, which the orientation of the preferred direction makes with the z-axis, is θ_0 throughout the tube. LESLIE [7, p. 279] has also shown that, for the particular solution he takes for simple shear, it is the orientation predicted by this theory for crystals flowing at large distances from a solid boundary.

(i) *Existence and uniqueness of solutions of* (3.11), (3.12) *for two coaxial cylinders of radii* r_1, r_2 $(0<r_1<r_2)$. Eliminating $v'(r)$ from equations (3.11) and (3.12), we obtain the differential equation

$$2f(\theta)\,\theta''(r)+f'(\theta)\{\theta'(r)\}^2+2r^{-1}f(\theta)\,\theta'(r)-r^{-2}k_{11}\sin 2\theta + \lambda_2\frac{(\cos 2\theta-\cos 2\theta_0)}{g(\theta)}\left(-\frac{1}{2}ar+\frac{b}{r}\right)=0. \tag{3.15}$$

We first investigate the existence of solutions of this equation subject to the boundary conditions

$$\theta(r_1)=\theta_1, \qquad \theta(r_2)=\theta_2. \tag{3.16}$$

Throughout this subsection, we restrict our attention to orientations which satisfy the inequalities

$$|\theta(r)|\leqq\theta_0 \qquad 0<\theta_0\leqq\pi/2, \tag{3.17}$$

so that, in particular, θ_i $(i=1,2)$ satisfy (3.17_1). These restrictions are necessary in the argument which follows.

Making the logarithmic substitution

$$r=r_2e^{-\xi}, \qquad \theta(r)=\theta(r_2e^{-\xi})\overset{\text{def.}}{=}\Theta(\xi), \tag{3.18}$$

so that

$$\Theta'(\xi)=-r\,\theta'(r), \qquad \Theta''(\xi)=r^2\theta''(r)+r\,\theta'(r), \tag{3.19}$$

we may rewrite (3.15) in the form

$$\Theta''(\xi)=G(\xi,\Theta,\Theta')\overset{\text{def.}}{=}\frac{1}{f(\Theta)}\left[\frac{(k_{11}-k_{33})}{2}\sin 2\Theta\{\Theta'(\xi)\}^2 + \frac{1}{2}k_{11}\sin 2\Theta+h(\xi)\frac{(\cos 2\Theta-\cos 2\theta_0)}{g(\Theta)}\right], \tag{3.20}$$

where

$$h(\xi)\overset{\text{def.}}{=}\frac{\lambda_2 r_2 e^{-\xi}}{2}\left(\frac{a}{2}r_2^2e^{-2\xi}-b\right). \tag{3.21}$$

In terms of the new variable ξ, we are interested in the existence of solutions of (3.20) on the interval $[0,\xi_1]$ subject to the boundary conditions

$$\Theta(0)=\theta_2, \qquad \Theta(\xi_1)=\theta_1, \tag{3.22}$$

$\xi_1=\log_e(r_2/r_1)$ being the value of ξ corresponding to $r=r_1$.

Existence Theorem. *Let* θ_i $(i=1,2)$ *be constants which satisfy* (3.17_1). *Then there exists at least one solution* $\Theta(\xi)$ *of equation* (3.20) *which satisfies the boundary conditions* (3.22) *and is such that* $|\Theta(\xi)|\leqq\theta_0$.

Proof. Since f, f' and g are bounded functions of Θ, it follows from (3.20) that

$$\text{(i)} \qquad |\Theta''|\leqq\mathscr{A}|\Theta'|^2+\mathscr{B}, \tag{3.23}$$

where $\mathscr{A}$ and $\mathscr{B}$ are positive numbers. Also, recalling the inequalities (2.12_1), (3.17_2), we see that

$$\text{(ii)} \qquad G(\xi,\theta_0,0)\geqq 0, \quad G(\xi,-\theta_0,0)\leqq 0. \tag{3.24}$$

For a function G with the properties (i) and (ii), the scalar version of Theorem 5.1 [28, p. 433] ensures the existence of a solution $\Theta(\xi)$, which satisfies the conditions of the theorem.

Uniqueness Theorem. *Suppose that the constants which appear in equations* (2.6), (2.8) *and* (3.11) *satisfy**

$$\begin{aligned}
&\text{(a)}\ \mu_1<0, \quad \mu_5-\mu_2-\mu_6-\mu_3>-2\mu_1>0,\\
&\text{(b)}\ k_{11}\geqq k_{33}, \quad k_{11}\neq 0,\\
&\text{(c)}\ \lambda_2>0,\\
&\text{(d)}\ \lambda_2 r_2\left(\frac{|a|}{2}r_2^2+|b|\right)<\frac{k_{11}\cot 2\theta_0(\mu_6+\mu_3+\mu_4)^2}{g(\theta_0)}, \quad \theta_0\neq\pi/4.
\end{aligned} \tag{3.25}$$

Then there exists at most one solution $\Theta(\xi)$ *of equation* (3.20) *which satisfies the boundary conditions* (3.22) *and is such that* $|\Theta(\xi)|\leqq\theta_0$.

Proof. We first show that, for fixed (ξ,Θ'), the inequalities (3.25) are one possible set of sufficient conditions which ensure that G is a strictly increasing function of Θ on the interval $[-\theta_0,\theta_0]$. In view of (3.17_2) and (3.25c, d), we confine our attention to $0<\theta_0<\pi/4$.

Since, under our assumptions, both $\sin 2\Theta$ and $[f(\Theta)]^{-1}$ are strictly increasing functions of Θ on this interval, it suffices to show that

$$H(\xi,\Theta)\stackrel{\text{def.}}{=}\frac{1}{2}k_{11}\sin 2\Theta+h(\xi)\frac{(\cos 2\Theta-\cos 2\theta_0)}{g(\Theta)} \tag{3.26}$$

is a strictly increasing function of Θ for fixed ξ. We define

$$\begin{aligned}
K[h(\xi),\Theta]=\frac{\partial H}{\partial\Theta}=k_{11}\cos 2\Theta-\frac{h(\xi)\sin 2\Theta}{2[g(\Theta)]^2}\\
\cdot[4g(\Theta)+(\cos 2\Theta-\cos 2\theta_0)(2\mu_1\cos 2\Theta+\mu_5-\mu_2-\mu_6-\mu_3)],
\end{aligned} \tag{3.27}$$

* Assumption (a) is an additional constraint in the sense that it does not appear to follow from the inequalities (2.11). Since, as we shall see later, the value of b depends upon the radii of the tubes and the boundary conditions for $v(r)$, assumption (d) is a restriction on all these quantities.

and note that

$$K[h(\xi),\Theta]=K[-h(\xi),-\Theta], \quad K[h(\xi),0]=k_{11}>0. \tag{3.28}$$

Under assumption (a), the function appearing in the square brackets is an even positive function of Θ with minimum and maximum values occurring at 0 and θ_0 respectively. Thus, if either $0<\Theta\leqq\theta_0$ and $h(\xi)\leqq 0$ or $-\theta_0\leqq\Theta<0$ and $h(\xi)\geqq 0$, it is clear, since $k_{11}\cos 2\Theta$ is positive in each range, that K is always positive. In general, for any fixed ξ, $h(\xi)$ can be positive, negative or zero. In view of (3.28), to establish the result in the remaining cases we need only consider $h(\xi)>0$ and $0<\Theta\leqq\theta_0$. Now $g(\Theta)$ satisfies the inequalities

$$\mu_6+\mu_3+\mu_4<g(\Theta)<g(\theta_0). \tag{3.29}$$

Further, since $0<\theta_0<\pi/4$, $\cos 2\Theta$ is positive for $0<\Theta\leqq\theta_0$ with its minimum value occurring at θ_0. The maximum value of $\sin 2\Theta$ also occurs at θ_0. It now follows that the minimum value taken by K on the interval $[0, \theta_0]$, for fixed ξ, is

$$k_{11}\cos 2\theta_0-\frac{2h(\xi)\sin 2\theta_0\, g(\theta_0)}{(\mu_6+\mu_3+\mu_4)^2}, \tag{3.30}$$

which can be shown to be positive under assumption (d). We therefore see that, for fixed ξ, K is positive and hence, for fixed (ξ, Θ'), G is a strictly increasing function of Θ on the interval $[-\theta_0, \theta_0]$.

Suppose now that there exist two solutions $\Theta_1(\xi)$, $\Theta_2(\xi)$ of equation (3.20) which satisfy the boundary conditions (3.22) and are such that $|\Theta_i(\xi)|\leqq\theta_0$ $(i=1, 2)$. If $\Psi(\xi)=\Theta_1(\xi)-\Theta_2(\xi)$, then Ψ satisfies the boundary conditions

$$\Psi(0)=\Psi(\xi_1)=0 \tag{3.31}$$

and the differential equation

$$\Psi''(\xi)=G(\xi,\Theta_1,\Theta_1')-G(\xi,\Theta_2,\Theta_2'). \tag{3.32}$$

Equation (3.32) may be written in the form

$$\Psi''(\xi)=\widetilde{\left(\frac{\partial G}{\partial\Theta'}\right)}\Psi'(\xi)+\widetilde{\left(\frac{\partial G}{\partial\Theta}\right)}\Psi(\xi), \tag{3.33}$$

where the notation used in this equation is defined by

$$\tilde{\alpha}(\xi)=\int_0^1\alpha(\xi, s\Theta_1+(1-s)\Theta_2, s\Theta_1'+(1-s)\Theta_2')\,ds \tag{3.34}$$

[28, pp. 97 & 426].

Since the function $\Psi(\xi)$ satisfies (3.31), on the interval $[0, \xi_1]$ Ψ can be zero, have at least one positive maximum or one negative minimum. We consider the possibility of a positive maximum. A negative minimum may be treated similarly by replacing Ψ by $-\Psi$. Suppose at $\xi=\xi_0$ there is a positive maximum, then at this point $\Psi'=0$ and $\Psi''<0$. However, from equation (3.33), since $\widetilde{(\partial G/\partial \Theta)}>0$ we see that when $\Psi'=0$, $\Psi''>0$, which is the required contradiction. We therefore conclude that there exists at most one solution having the required properties.

Although we have proved uniqueness under assumptions (3.25a to d), it is clear from the proof that they may be replaced by any conditions, which ensure that G is a strictly increasing function of Θ on the interval $[-\theta_0, \theta_0]$.

Finally, we consider the velocity field $v(r)$. Assuming that the fluid adheres to the tube and that the inner tube is stationary, it follows from (3.11) that

$$v(r)=-\frac{1}{2}a\int_{r_1}^{r}\frac{s\,ds}{g[\theta(s)]}+b\int_{r_1}^{r}\frac{ds}{s\,g[\theta(s)]}. \tag{3.35}$$

We now consider whether b can be chosen so that

$$\text{(A)}\ v(r_2)=0 \quad \text{and} \quad \text{(B)}\ v(r_2)=V, \tag{3.36}$$

where V is a constant.

(A) Since the solution (3.35) is to vanish at both boundaries, there must be at least one turning point on the interval $[r_1, r_2]$. Clearly, from equation (3.11), this requires that a and b have the same sign. In this case there exists one such positive turning point r_c given by

$$r_c^2=\frac{2b}{a}. \tag{3.37}$$

We also require that

$$r_1^2\leqq r_c^2\leqq r_2^2. \tag{3.38}$$

Since $g[\theta(s)]$ is positive, using a mean value theorem*, we have

$$v(r_2)=\left(-\frac{1}{2}a\eta_1+\frac{b}{\eta_2}\right)\int_{r_1}^{r_2}\frac{ds}{g[\theta(s)]}, \tag{3.39}$$

where $\eta_1, \eta_2\in[r_1, r_2]$. Thus, by choosing b such that $r_c^2=\eta_1\eta_2$, the solution (3.35) satisfies (3.36A) and (3.38).

(B) For (3.35) to satisfy $v(r_2)=V$ the appropriate choice of b is

$$\frac{2b}{a}=\frac{1}{\int_{r_1}^{r_2}\frac{ds}{s\,g[\theta(s)]}}\left\{\frac{2V}{a}+\int_{r_1}^{r_2}\frac{s\,ds}{g[\theta(s)]}\right\}. \tag{3.40}$$

Thus, given the geometry of the viscometer, the velocity V and the pressure gradient, in theory, b can be computed for all boundary velocities. In this case, there is no restriction placed upon the sign of a and b, although their signs do

* If f is a real continuous function on $[\alpha, \beta]$, and g is a non-negative function integrable over $[\alpha, \beta]$, then there exists an $\eta\in[\alpha, \beta]$ such that

$$\int_\alpha^\beta f(x)\,g(x)\,dx=f(\eta)\int_\alpha^\beta g(x)\,dx.$$

affect the bahaviour of (3.35). Alternatively, since the integrals are positive, the behaviour of the solution depends upon the sign of V/a. *If V/a is positive*, it can be shown, by the previous mean-value theorem, that $r_c \geqq r_1$. The solution (3.35) then either has a turning point or is monotonic depending upon whether $r_c < r_2$ or $r_c > r_2$. *If V/a is negative*, there are two possibilities depending upon the relative magnitudes of $|V/a|$ and $\frac{1}{2}\int_{r_1}^{r_2} s\,ds/g[\theta(s)]$. When

$$\left|\frac{V}{a}\right| > \frac{1}{2}\int_{r_1}^{r_2} \frac{s\,ds}{g[\theta(s)]} \tag{3.41}$$

the solution is monotonic but is of the opposite sign to a. In the opposite case, it can be shown that $r_c \leqq r_2$ and that the solution (3.35) is monotonic or has a turning point depending upon whether $r_c < r_1$ or $r_c > r_1$.

Combining the previous results, we conclude that, for flow between two coaxial circular tubes, there exist solutions $\theta(r)$ and $v(r)$ satisfying the required boundary conditions, provided we restrict our attention to orientations less than $\theta_0 \in [0, \pi/2]$. In the case when both tubes are stationary we also require a and b to have the same sign. Further, if the assumptions (3.25) are satisfied $\theta(r)$ is unique.

(ii) *Existence of solutions of* (3.15) *for a capillary tube of radius* r_2. The physical requirement of finite pressure and stresses throughout the region $0 \leqq r \leqq r_2$ leads us to put b identically zero and to consider the existence of solutions of (3.15) which are $0(r)$ as $r \to 0$. At present, it does not seem possible to establish the existence of a solution of the two point boundary-value problem with the above property by considering a solution $\Theta(\xi)$ of (3.20) on the interval $[0, \infty)$. We therefore adopt the approach given below and first consider the existence of such a solution in the neighbourhood of $r=0$.

Rearranging equation (3.15), we see that it may be rewritten in the form

$$r^2\theta''(r) = L(r\theta', \theta, r), \tag{3.42}$$

where

$$L(x,\theta,r) = -\frac{f'(\theta)}{2f(\theta)}x^2 - x + \frac{k_{11}}{2}\frac{\sin 2\theta}{f(\theta)} + \frac{1}{4}a\lambda_2\frac{(\cos 2\theta - \cos 2\theta_0)}{f(\theta)\,g(\theta)}r^3 \tag{3.43}$$

$$= \sum_{p=0}^{2}\sum_{q=0}^{\infty}\sum_{s=0}^{3}\frac{L_{pqs}}{p!\,q!\,s!}x^p\theta^q r^s, \tag{3.44}$$

and

$$L_{pqs} = \left\{\frac{\partial^{p+q+s}}{\partial x^p\,\partial\theta^q\,\partial r^s}L(x,\theta,r)\right\}_0. \tag{3.45}$$

The suffix 0 indicates that the partial derivative is evaluated at the point (0, 0, 0). In the analysis which follows, we require the results

$$\begin{aligned} &L_{100} = -L_{010} = -1\\ &L_{200} = L_{020} = L_{002} = L_{110} = L_{011} = L_{101} = L_{001} = L_{000} = 0. \end{aligned} \tag{3.46}$$

These are obtained by direct calculation.

Adopting FORSYTH's analysis [29, pp. 187–189], we assume a solution of the form

$$\theta(r)=\beta r+\gamma r^2+r u(r), \tag{3.47}$$

where the coefficients $\beta(=\theta'(0))$, γ are constants and $u(r)$ is $O(r^2)$ as $r\to 0$. We are required to show that, for equations (3.42) to (3.45), a sufficiently smooth function $u(r)$ exists and hence that (3.47) represents a possible solution.

Substituting (3.47) into (3.42) to (3.44), equating powers of r^2 and using (3.46), we find that

$$\gamma=0. \tag{3.48}$$

It then follows that our differential equation may be written as

$$r^3 u''(r)=-3r^2 u'(r)+\kappa r^3+r^2 \mathscr{G}(u'(r), u(r), r), \tag{3.49}$$

where κ is some constant. The function $r^2\mathscr{G}$ represents the terms remaining in (3.44) after the terms which are $O(r^3)$ and lower have been written out explicitly, taking into account (3.47) and (3.48). In particular, $\mathscr{G}$ does not contain linear terms in $u(r)$ or $u'(r)$. Since $u(r)$ is $O(r^2)$ as $r\to 0$, we can write

$$u(r)=r\,w(r), \tag{3.50}$$

where the function $w(r)$ vanishes when $r=0$. Then

$$y(r)=u'(r)=r\,w'(r)+w(r), \tag{3.51}$$

and since

$$u''(r)=y'(r), \tag{3.52}$$

the differential equation (3.49) is equivalent to the system

$$\begin{aligned} r\,y'(r)&=\kappa r-3y+\mathscr{G}(y, rw, r)\\ r\,w'(r)&=y-w, \end{aligned} \tag{3.53}$$

where $\mathscr{G}$ does not contain linear terms in the variables r, w or y. This follows since originally $\mathscr{G}$ did not contain any linear terms in $u(r)$ or $u'(r)$. The equations (3.53) have the same form as the system considered by FORSYTH [29, pp. 45–52]. Further the roots of the corresponding critical quadratic (see [29], p. 45) are both negative integers. It therefore follows that, for sufficiently small r, there exists a unique power series solution $w(r)$ of (3.53) and hence from (3.47) a power series solution $\theta(r)$ in this region. In fact there are infinitely many such solutions corresponding to different assigned values of β.

We have shown that infinitely many solutions with the required physical properties exist in a neighbourhood of $r=0$. Further it can be shown that, provided these solutions remain bounded, their first and second derivatives are also bounded for $0\leq r\leq r_2$. It then follows* that they may be continued for any finite r.

At present it does not seem possible to consider the existence of a solution which has the required property at $r=0$ and also satisfies a specified boundary orientation at the capillary tube. It seems likely that some restriction on the boundary data may be necessary, since it is not clear that every solution of

* See Theorem 4.1 on p. 15 and the remarks on p. 19 of reference [30].

equation (3.15) has a finite derivative at $r=0$. It therefore seems possible to pick out a boundary orientation which gives rise to a solution with this undesirable physical property. However, the detailed analysis of this conjecture awaits further study. The velocity field for this flow follows immediately from equation (3.11) once the existence of $\theta(r)$ has been established.

4. Scaling Analysis

In a recent paper ERICKSEN [9] considered the flow of a given liquid crystal between two parallel flat plates, one moving with constant velocity and the other stationary, in which p, γ, $\boldsymbol{v}$ and $\boldsymbol{d}$ were allowed arbitrary spatial dependence. At the plates the fluid was assumed to adhere to the surfaces and the director was assumed to be given by constant vectors independent of the plate velocity and the gap width or, at most, varying with the product of these quantities. Further, ERICKSEN neglected the "molecular inertia".* By transforming the problem to that for a unit gap width, he was able to deduce, by scaling arguments, that a suitably defined apparent viscosity for this general flow is a function of the product of the plate velocity and the gap width.

We now adapt this analysis to the case of a steady flow of a given liquid crystal between two stationary coaxial circular cylinders of radii r_1, r_2 $(r_1<r_2)$, the flow being caused by a pressure gradient along their length. The variables γ, $\boldsymbol{v}$ and $\boldsymbol{d}$ depend upon r and ϕ and p depends upon r, ϕ and z. Following ERICKSEN, we neglect "molecular inertia". We assume that the fluid adheres to the cylinders so that

$$\boldsymbol{v}=\boldsymbol{0} \quad \text{at} \quad r=r_1 \quad \text{and} \quad r=r_2. \tag{4.1}$$

For the director we assume that

$$\boldsymbol{d}=\boldsymbol{d}_1 \quad \text{at} \quad r=r_1, \qquad \boldsymbol{d}=\boldsymbol{d}_2 \quad \text{at} \quad r=r_2, \tag{4.2}$$

where the vectors $\boldsymbol{d}_i$ $(i=1, 2)$ are given at the boundary. In view of our earlier assumptions these vectors are considered to be independent of the time and the axial distance so that the orientation of the preferred direction at the boundary can vary only with the azimuthal angle ϕ. They are also assumed to be independent of the pressure gradient along the tube.

Let R, Φ, Z and T be new variables defined by

$$r=r_2 R, \qquad \phi=\Phi, \qquad z=r_2 Z, \qquad t=kT, \tag{4.3}$$

where k is a positive constant. Here, we have used the outer radius r_2 to scale the spatial variables. We could, of course, use the inner radius r_1 in which case analogous results are obtained. This scaling therefore transforms our original problem, in which both radii varied, to one in which the outer cylinder is of unit radius and the inner has variable radius $R_1(=r_1/r_2)$. In the following, we do not scale the director. Denoting all quantities referred to this latter problem by a superscript 0, we have

$$\boldsymbol{v}=(r_2 k^{-1})\boldsymbol{v}^0, \qquad \boldsymbol{w}=k^{-1}\boldsymbol{w}^0 \tag{4.4}$$

* The "molecular inertia" is the term used for the left hand side of equation (2.2_3).

so that

$$\operatorname{grad} \boldsymbol{v} = k^{-1} (\operatorname{grad} \boldsymbol{v})^0, \quad \operatorname{grad} \boldsymbol{d} = r_2^{-1} (\operatorname{grad} \boldsymbol{d})^0, \quad N = k^{-1} N^0. \tag{4.5}$$

Also, provided

$$p = r_2^{-2} p^0, \quad \gamma = r_2^{-2} \gamma^0, \quad k = r_2^2, \tag{4.6}$$

we have

$$\boldsymbol{\sigma} = r_2^{-2} \boldsymbol{\sigma}^0, \quad \boldsymbol{\pi} = r_2^{-1} \boldsymbol{\pi}^0, \quad \boldsymbol{g} = r_2^{-2} \boldsymbol{g}^0. \tag{4.7}$$

Let Q be the efflux emitted from the tubes in unit time, then

$$\begin{aligned} Q &= \int_{r_1}^{r_2} r\, dr \int_0^{2\pi} v_z(r, \phi)\, d\phi, \\ Q^0 &= \int_{R_1}^{1} R\, dR \int_0^{2\pi} v_z^0(R, \Phi)\, d\Phi. \end{aligned} \tag{4.8}$$

Using equations (4.3), (4.4) and $(4.6_{1,3})$, we obtain

$$\mathscr{P} = r_2^{-3} \mathscr{P}^0, \quad Q = r_2 Q^0, \tag{4.9}$$

where $\mathscr{P}$ denotes the pressure gradient along the tube.

Equations (2.2) form a system of seven partial differential equations for the seven unknowns p, γ, the components of the vector $\boldsymbol{v}$ and the two components of the unit vector $\boldsymbol{d}$. Using (4.3), (4.4) and (4.7), we obtain the corresponding system for the unknowns p^0, γ^0, $\boldsymbol{v}^0$ and $\boldsymbol{d}^0$. Moreover, this new system has precisely the same form as the original system. Provided that, for a fixed material and a fixed wall orientation, there exists a unique solution of this new system, Q^0, $\mathscr{P}^0$ and R_1 are uniquely related so that

$$Q^0 = \mathscr{I}(R_1, \mathscr{P}^0), \tag{4.10}$$

where $\mathscr{I}$ is some unknown function to be determined. Employing equations (4.9), we find

$$Q = r_2 \mathscr{I}(r_1/r_2, r_2^3 \mathscr{P}) \tag{4.11}$$

in the original problem. For the special case of a capillary tube of radius r_2, this reduces to

$$Q = r_2 \mathscr{I}(r_2^3 \mathscr{P}). \tag{4.12}$$

The analysis presented here essentially separates into two parts. First, we transformed the system of differential equations into an equivalent system and secondly, we related the measurable quantities, assuming that solutions of the equations existed. In the first part, the analysis goes through if p, γ, $\boldsymbol{v}$ and $\boldsymbol{d}$ are allowed arbitrary spatial and time dependence. Although we assumed that the vectors $\boldsymbol{d}_i$ $(i=1, 2)$ were independent of the pressure gradient, we now see, from equation (4.9_1), that it is possible to allow a dependence of the form $r_2^3 \mathscr{P}$. For the present analysis to proceed it is essential to neglect "molecular inertia". At the wall the "molecular inertia" vanishes since the vectors $\boldsymbol{d}_i$ $(i=1, 2)$ are assumed to be independent of time and the fluid velocity satisfies the boundary condition (4.1). However in the mainstream neglecting the "molecular inertia" may introduce a restriction.

The function $\mathscr{J}$ in the above is determined experimentally. For the case of the capillary tube if we plot Q/r_2 against $r_2^3\mathscr{P}$ for different values of r_2, and if the above theory is relevant, we should find that one curve results. The recent data obtained by FISHER & FREDRICKSON [15] agrees with this prediction when the wall orientation is perpendicular to the tube. However, in the case of parallel wall orientation, there is a discrepancy between the analysis presented here and the available experimental data.

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