Shocks and Rarefactions in Two Space Dimensions

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This is a note about the solution of Riemann-type initial value problems for a single conservation law in two space dimensions. By using specific examples we exhibit new phenomena in the qualitative structure of solutions of hyperbolic conservation laws. There are good existence and uniqueness theories for the solution of a single conservation law in several space dimensions based upon the viscosity method [6], but the method gives one little insight into the qualitative structure of the discontinuity set of solutions. In one space dimension, the qualitative shock phenomena which occur have been described by GELFAND [2] and BALLOU [1] and fit into the framework of regularity theorems [3, 4, 5]. The examples which we calculate here display new kinds of "rarefaction" phenomena which occur in the solution of equations in two space dimensions.

The equation we study is the conservation law

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x_1}(f_1(u))+\frac{\partial}{\partial x_2}(f_2(u))=0 \tag{1}$$

where $f=(f_1,f_2)\colon \mathbb{R}\to\mathbb{R}^2$ is a specified (non-linear) function. Global continuous solutions of the initial value problem will not exist for appropriate smooth initial data if f is not linear. Therefore, equation (1) is interpreted in the distributional sense; namely, a function $u\colon \mathbb{R}^2\times\mathbb{R}^+\to\mathbb{R}$ is a solution of (1) if u satisfies the following integral identity: for every C^∞ test function $\phi\colon \mathbb{R}^2\times\mathbb{R}^+\to\mathbb{R}$ having compact support

$$\int_{\mathbb{R}^2\times\mathbb{R}^+} u\frac{\partial\phi}{\partial t}+f_1(u)\frac{\partial\phi}{\partial x_1}+f_2(u)\frac{\partial\phi}{\partial x_2}=0. \tag{2}$$

VOLPERT's existence and uniqueness theory [6] applies to this initial value problem with initial data in the class BV if functions of bounded variation (in the sense of TONELLI-CESARO). He proves that the solutions of these initial value problems also lie in the class BV. If a function $g\colon \mathbb{R}^k\to\mathbb{R}$ is in the class BV, then the singular set of g has locally finite Hausdorff $(k-1)$ dimensional measure. Moreover, on the complement of some set of vanishing $(k-1)$ dimensional Hausdorff measure, the singular set of g has a well defined normal and g has well defined limit values from the two sides of the normal. We shall call these points *shock points* to distinguish them from other singular points of g.

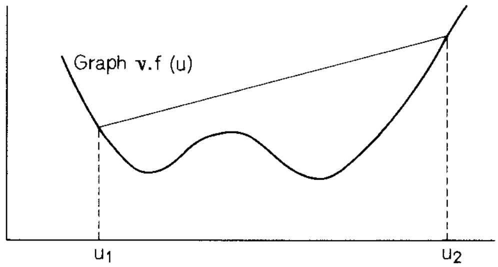

Fig. 1

A function of u: $\mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ in the class BV satisfies the integral equation (2) if it satisfies equation (1) at all regular points and if it satisfies the following *jump condition* at all shock points:

$$v \cdot \frac{f(u_2)-f(u_1)}{u_2-u_1}=0. \tag{3}$$

Here v is the normal to the discontinuity set, and u_1 and u_2 are the limit values of u on the two sides.

Let u: $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be a function in the class BV which satisfies the jump condition. We say u satisfies the *entropy condition* if at every shock point we have

$$v \cdot f(u) \leqq \frac{u-u_1}{u_2-u_1} f(u_2)+\frac{u_2-u}{u_2-u_1} f(u_1).$$

Here u_1 and u_2, $u_1 < u_2$, are the limiting values of u on the two sides of the shock point, v is the normal to the shock oriented toward the side of the shock set on which u_2 is the limit value of the solution, and $u_1 < u < u_2$. See Figure 1. VOLPERT proves that there is a unique global solution of equation (1) in the class BV which satisfies the jump and entropy condition at all shock points and has initial data ϕ: $\mathbb{R}^2 \to \mathbb{R}$ in the class BV.

The initial value problems we shall consider are of "Riemann type" with piecewise constant initial data having discontinuities on three rays with vertex at the origin. See Figure 2. To aid in the construction of solutions we now derive some general properties of such solutions.

In solving the initial value problem, we use the fact that our initial data is invariant under the dilations $x \to cx$, $c>0$. Equation (1) is also invariant under dilations. Hence the existence and uniqueness theorem implies that the solution of (1) with our piecewise constant initial data is invariant under the dilations $(x, t) \to (cx, ct)$, $c>0$. This means that the solution is constant along rays having the origin as vertex. Consequently, the solution can be immediately determined by its restriction to any plane $\mathbb{R}^2 \times \{t\}$, $t>0$. We shall describe the solution by determining its restriction to the plane $t=1$.

At regular points of a solution u of equation (1), the equation can be interpreted as the statement that u is constant in the direction

$$\frac{\partial}{\partial t}+a_1(u)\frac{\partial}{\partial x_1}+a_2(u)\frac{\partial}{\partial x_2}$$

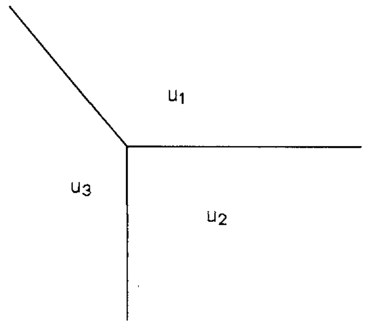

Fig. 2

with $a(u)=f'(u)$. If X is a vector field, we say that a function u is constant in the direction X if u is constant along the integral curves of X. Where u is smooth, it is constant along lines with direction

$$\frac{\partial}{\partial t}+a_1(u)\frac{\partial}{\partial x_1}+a_2(u)\frac{\partial}{\partial x_2}.$$

These lines are called *characteristics*. Our procedure for constructing solutions is to find the characteristics; more specifically, we determine the shocks at which characteristics end or begin.

If $a(u)$ is bounded, another property of solutions of equation (1) is that the value of the solution at a given point (x, t) depends only on the initial data inside the ball of radius ct centered at x, where c is a constant depending only on the function a. If $a(u)$ is not bounded, c may depend on the maximum and minimum values of the initial data. We use this result to assert that the value of the solution u far away from the origin in the plane $\mathbb{R}^2\times\{1\}$ can be determined by solving a Riemann problem in which the initial data has a single discontinuity occurring along a line. The solution of this simpler Riemann problem essentially reduces to the one dimensional case.

Assume that piecewise constant initial data are given with a single discontinuity on a line l through the origin. Let the two values of the initial data be ϕ_1 and ϕ_2, with $\phi_1<\phi_2$, and let v be a normal to the line oriented in the direction towards which ϕ_1 is the value of the initial data. We have already remarked that the existence and uniqueness theorem implies that the solution of equation (1) with this initial data is constant on rays having the origin as vertex. Now the group of translations along l acts as another group of symmetries leaving (1) and the initial data invariant. Consequently, the solution of this initial value problem is constant on lines parallel to l. Hence the discontinuities are planes containing l. Except perhaps for points of l, all singular points of the solution will be shock points. To find the shocks and the solution, we proceed below as if we were solving the one dimensional Riemann problem.

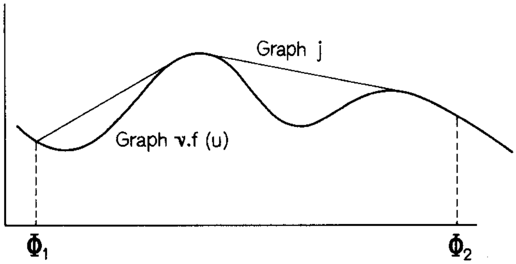

Fig. 3

Consider the graph of the function $v \cdot f\colon [\phi_1, \phi_2] \to \mathbb{R}$. From this graph, construct a new function $j\colon [\phi_1, \phi_2] \to \mathbb{R}$, with j defined by

$$j(u) = \limsup_{\phi_1 \leqq u_1 \leqq u \leqq u_2 \leqq \phi_2} \left\{ \frac{u_2 - u}{u_2 - u_1} f(u_1) + \frac{u - u_1}{u_2 - u_1} f(u_2) \right\}. \tag{4}$$

See Figure 3. There will be an open set $U \subset (\phi_1, \phi_2)$ on which $j(u) \neq v \cdot f(u)$. On each component of U, there will be a shock of the solution along the half plane containing l and

$$\frac{\partial}{\partial t} + \frac{f(u_2) - f(u_1)}{u_2 - u_1} \frac{\partial}{\partial x}.$$

Between each of these shocks the solution will take the value u on the half-plane containing l and $\frac{\partial}{\partial t} + a(u) \frac{\partial}{\partial x}$. Since j is a strictly concave function between the shocks determined above from U, the solution u is well defined between the shocks. Moreover, u has limit values from both sides of each shock and is such that both the jump conditions and the entropy conditions are satisfied. Note that the solution of the Riemann problem may have an infinite number of shocks, but only if $v \cdot f$ has an infinite number of inflection points in the interval $[\phi_1, \phi_2]$. The regions between shocks are called *rarefactions*.

Let us now return to "Riemann type" problems with initial data having three values ϕ_1, ϕ_2, and ϕ_3 in angular sectors centered at the origin. The general strategy will be first to solve the problem far away from the origin by using the solution of the Riemann problem considered above. This will be the correct solution outside some cone with vertex at the origin. Having solved the Riemann problem along each ray of discontinuity in the initial data, we then want to fit together (near the origin) the shocks of these three Riemann problems. Unfortunately, there is no reason to expect the shocks emanating from the three rays of discontinuity in the initial data to have a common line of intersection. The typical situation is depicted in Figure 4. There exists a tetrahedral cone with vertex at the origin and faces contained in what appear to be the shocks of the solution, but unfortunately the points of the interior of the cone are on the "wrong" side of each shock.

Something more subtle must be done to fit the shocks together near the origin. We are not able to give a satisfactory procedure for doing this, but can derive certain qualitative properties of the solution and compute two examples.

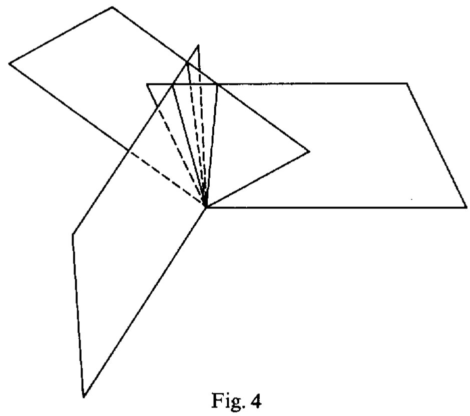

Fig. 4

The correct solution to the Riemann problem must agree with the values already found, except in some cone which intersects the plane $\mathbb{R}^2 \times \{0\}$ only at the origin. Consequently, any new shocks and characteristics that are introduced must begin either at the origin or at some positive time. Using the fact that the solution is constant along rays, we can derive equations which must be satisfied by such additional shocks and "rarefactions".

Suppose (x, t) lies on a shock which has limit values $u_1 < u_2$ on the two sides of the shock. The jump condition implies that the vector

$$\frac{\partial}{\partial t} + \frac{f_1(u_2) - f_1(u_1)}{u_2 - u_1} \frac{\partial}{\partial x_1} + \frac{f_2(u_2) - f_2(u_1)}{u_2 - u_1} \frac{\partial}{\partial x_2}$$

is tangent to the shock. The vector

$$t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

is also tangent to the shock because the shock consists of lines passing through the origin. Consequently, the vector

$$\left(\frac{x_1}{t} - \frac{f_1(u_2) - f_1(u_1)}{u_2 - u_1}\right) \frac{\partial}{\partial x_1} + \left(\frac{x_2}{t} - \frac{f_2(u_2) - f_2(u_1)}{u_2 - u_1}\right) \frac{\partial}{\partial x_2}$$

is tangent to the shock.

A similar argument gives a differential equation to be satisfied in a conical region in which the solution u is smooth. Let R be such a region and let $\mathscr{S} \subset R$ be a surface on which u is constant. $\mathscr{S}$ is conical; therefore

$$t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

is tangent to $\mathscr{S}$. In addition, $\mathscr{S}$ consists of characteristics. Hence

$$\frac{\partial}{\partial t} + a_1(u) \frac{\partial}{\partial x_1} + a_2(u) \frac{\partial}{\partial x_2}$$

is tangent to $\mathscr{S}$. We conclude that $\mathscr{S}$ is a subset of the plane spanned by these two vectors. In particular,

$$\left(\frac{x_1}{t}-a_1(u)\right)\frac{\partial}{\partial x_1}+\left(\frac{x_2}{t}-a_2(u)\right)\frac{\partial}{\partial x_2}$$

is tangent to $\mathscr{S}$.

In a plane $t=$constant (say $t=1$), this gives us two vector fields relating equation (1) and the jump condition for a solution u. The entropy condition places additional restrictions on solutions. Let us now examine the nature of these restrictions. In a region R of the sort considered above, most of the characteristics cannot begin at the origin because the only set of characteristics which can emanate from a point are those which form a two dimensional set. Distinct characteristics with the same value of u must begin at different points. Consequently, most characteristics in R begin at some positive time. If a characteristic begins on a shock, the entropy condition implies that the characteristic will be tangent to the shock at its point of origin. Thus a shock from which characteristics emanate will be the envelope of the planes on which u is constant. The region R will be the union of portions of the tangent planes of the shock.

If the characteristics inside a region of rarefaction do not begin on a shock, then they must begin on a set of codimension two. This set must be a ray γ with vertex at the origin because the solution is invariant under dilations. R will be a union of portions of planes containing the ray γ. Besides these there are no other alternatives for the origin of characteristics in a region of rarefaction.

We summarize our discussion by listing several principles concerning the restriction of a solution to the plane $t=1$:

0: Almost all boundary points in a rarefaction region are non-singular points (in the sense of Hausdorff 1-dimensional measure).

1: Far away from the origin the solution is given by solving a Riemann problem with a single jump discontinuity along a line.

2: Inside each region of rarefaction, the curves $u=$constant are segments of lines passing through the points $a(u)$.

3: A shock curve with limit values u_1 and u_2 at the point x has tangent vector

$$x-\frac{f(u_2)-f(u_1)}{u_2-u_1}.$$

4: In a compact rarefaction region, each segment on which the solution is constant has one end which approaches the boundary tangentially or tends to a singular point of the boundary.

Using these principles we construct the solutions of two specific Riemann problems. The solution of each will be described by its restriction to the plane $t=1$. In each of these problems we have chosen the rays of discontinuity in the initial data to be the rays generated by $\frac{\partial}{\partial x}$, $-\frac{\partial}{\partial y}$, and $-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ and the three values of the initial data to be 0, 1, and -1.

Example 1. We consider the case $f(u)=(u^2, -1/3\,u^2)$. The initial data (given by Figure 5) is 0 in the angular sector from $\frac{\partial}{\partial x}$ to $-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$, 1 in the sector from

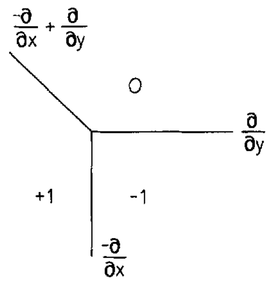

Fig. 5

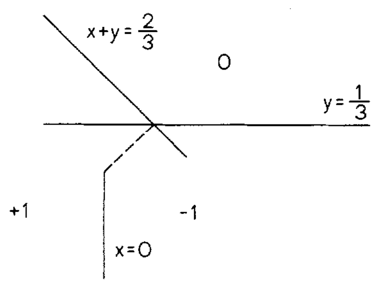

Fig. 6

$-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ to $-\frac{\partial}{\partial y}$, and -1 in the sector from $-\frac{\partial}{\partial y}$ to $\frac{\partial}{\partial x}$. From the rays of discontinuity in the initial data, we compute the shocks which arise. Along $\frac{\partial}{\partial x}$ we have

$$v=-dy, \quad v\cdot f=\tfrac{1}{3}u^2, \quad \text{and} \quad v\cdot\left(\frac{f(u_2)-f(u_1)}{u_2-u_1}\right)=-1/3.$$

Along $-\frac{\partial}{\partial y}$ we have

$$v=dx, \quad v\cdot f=u^2, \quad \text{and} \quad v\cdot\left(\frac{f(u_2)-f(u_1)}{u_2-u_1}\right)=0.$$

Along $-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ we have

$$v=\frac{1}{\sqrt{2}}(dx+dy), \quad v\cdot f=\frac{\sqrt{2}}{3}u^2, \quad \text{and} \quad v\cdot\left(\frac{f(u_2)-f(u_1)}{u_2-u_1}\right)=\frac{\sqrt{2}}{3}.$$

All three of the shocks determined in this manner satisfy the entropy condition. They intersect the plane $t=1$ along the lines $y=1/3$, $x=0$, and $x+y=2/3$ respectively. See Figure 6. To obtain the solution, note that

$$\frac{f(1)-f(-1)}{1-(-1)}=(0,0).$$

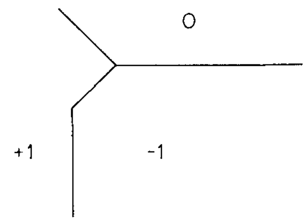

Fig. 7

This means that any plane through the origin might contain a conical shock by principle 3 above. Thus we insert the segment from the origin to the point $(\frac{1}{3}, \frac{1}{3})$ in Figure 6. The normal for the corresponding shock is $v = \frac{1}{\sqrt{2}}(dx - dy)$. Hence $v \cdot f = \frac{2\sqrt{2}}{3} u^2$ and the shock satisfies the entropy condition. The final solution restricted to the plane $t = 1$ is given by Figure 7.

Example 2. In this example no rarefaction takes place along the shocks far away from the origin, but rarefaction does occur in a cone with vertex at the origin. Here $f(u) = (\frac{1}{2}u^2, \frac{1}{3}u^3)$ and the initial data is again as in Example 1 and Figure 5.

The shocks arising from the rays of discontinuity in the initial data intersect the plane $t = 1$ along the lines $y = 1/3$, $x = 0$, and $x + y = 5/6$. See Figure 8. Each of these shocks satisfies the entropy condition. The line $y = 1/3$ also satisfies the jump condition as a conical shock from -1 to 1. However, it does not satisfy the entropy condition since $dy \cdot f = \frac{1}{3}u^3$ so that the segment joining the points $(-1, -1/3)$ and $(1, 1/3)$ lies on both sides of the graph of $dy \cdot f$.

To find the solution we assume that there will be a region of rarefaction with vertex at a point $p = (0, y_0)$; $y_0 > 0$ (in the plane $t = 1$). The value of y_0 is to be determined later. Inside the rarefaction region, the solution is to have the value u on lines joining $(0, y_0)$ with $a(u) = (u, u^2)$. Inside the rarefaction region u will vary from 1 to u_0, where u_0 is a value of u for which the points

$$(0, y_0), \quad \frac{f(u_0) - f(-1)}{u_0 + 1} \quad \text{and} \quad a(u_0)$$

are collinear. This line can then be a shock from u_0 both to -1 and to the limit value from inside the rarefaction region.

Finally, we compute the shock separating the rarefaction region from the region in which the solution is 0. This shock intersects the line $y = 1/3$ at a point q which depends upon y_0. The value of y_0 is determined by the condition that q lies on the line through $(0, y_0)$ and $a(u_0)$. This yields the diagram of the solution shown in Figure 9.

We now prove that y_0 can be chosen so that q lies on the line through $(0, y_0)$ and $a(u_0)$. The slope of the line dividing the rarefaction region and the region in which the solution is 1 is $(1 - y_0)$. The equation relating u_0 and y_0 is

$$y_0 = \tfrac{1}{3} - \tfrac{1}{3}(u_0 - 1)^2 .$$

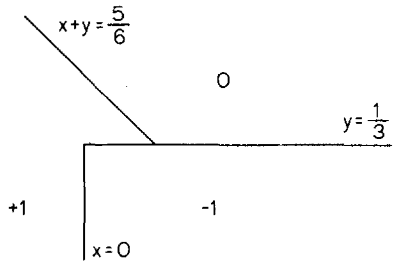

Fig. 8

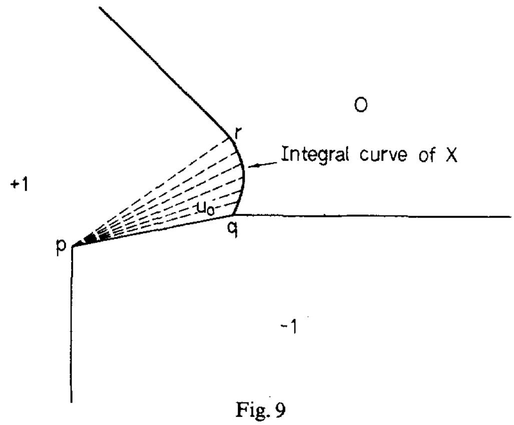

Fig. 9

The line through $(0, y_0)$ and $a(u_0)$ intersects the line $y=1/3$ at

$$\left(\frac{(u_0-1)^2}{4u_0-2}, 1/3\right).$$

On the interval $(\frac{1}{2}, 1)$, the function

$$i(u_0)=\frac{(u_0-1)^2}{4u_0-2}$$

is decreasing, with $i(1)=0$ and $i(u_0)\to\infty$ as $u_0\to\frac{1}{2}$. Since

$$i^{-1}(\tfrac{1}{2})=\frac{2-\sqrt{2}}{4}\cong 0.586$$

it follows that $u_0>0.58$. This imples that $y_0=1/3-1/3(u_0-1)^2>0.27$.

It does not seem possible to calculate explicitly the integral curve of the vector field determining the shock between the rarefaction region and the region in which the solution is 0, but we can make estimates of its behavior to prove that there is a value of y_0 for which

$$\left(\frac{(u_0-1)^2}{4u_0-2}, 1/3\right)$$

is on this integral curve.

If y_0 is specified, then the line from $(0, y_0)$ with slope $(1-y_0)$ intersects the line $x+y=5/6$ at the point

$$r=\left(\frac{5-6y_0}{6(2-y_0)}, \frac{5+y_0}{6(2-y_0)}\right),$$

which lies between the points $(\frac{1}{3}, \frac{1}{2})$ and $(\frac{3}{10}, \frac{8}{15})$. The shock will then contain the integral curve of the vector field $X(x,y)=(\frac{1}{2}u-x, \frac{1}{3}u^2-y)$, where (x,y) and u are related by the equation

$$\frac{y-y_0}{x}=\frac{u^2-y_0}{u}.$$

This equation yields

$$u=\frac{y-y_0}{2x}+\tfrac{1}{2}\sqrt{\left(\frac{y-y_0}{x}\right)^2+4y_0}.$$

We assert that X is never tangent to the line through $(0, y_0)$, (x, y) and (u, u^2) along the integral curve through r. Indeed this could happen only if the equation $\frac{y-\frac{1}{3}u^2}{x-\frac{1}{2}u}=\frac{y-u^2}{x-u}$ were also satisfied. Solving these equations yields $3y=(4x-u)u$ and $y-y_0=\frac{x(u^2-y_0)}{u}$. Hence $y_0=\frac{-u^2}{3}<0$. Therefore, X is not tangent to the line $u=$constant since $y_0>0$.

In particular, this implies that X is transverse to the segments in the rarefaction region on which u is constant. Moreover, the y component of X is $\frac{1}{3}u^2-y$, which is negative and bounded away from 0 if $u\in(\frac{1}{2}, 1)$ is bounded away from 1 and $y\geqq 1/3$. Along the line $x+y=5/6$, X is transverse and points down to the left. From these considerations we conclude that the integral curve of X intersects the line $y=1/3$ in the interval between $(0, 1/3)$ and $(\frac{1}{2}, 1/3)$.

Clearly X varies continuously with y_0. Hence there is a value of $y_0\in(0,3)$ such that the integral curve of X through r intersects the line $y=1/3$ at the point $(i(y_0), 1/3)$. We now insert this value of y_0 into our construction and thereby find the required solution.

The only thing which remains to be proved is that the entropy condition is satisfied on the shock separating the rarefaction region and the region in which the solution is zero. At each point along the shock the normal of the shock restricted to the plane $t=1$ is of the form $c(dx+b\,dy)$, where $c>0$ and $b<1$. Thus $v\cdot f=c(\frac{1}{2}u^2+\frac{1}{3}bu^3)$. Since the function $v\cdot f$ is convex on the interval $(0,1)$, the entropy condition requires only that

$$v\cdot\left(\frac{f(u)-f(0)}{u-0}\right)<v\cdot a(u) \quad \text{or} \quad \tfrac{1}{2}u+\frac{b}{3}u^2<u+bu^2.$$

This will be satisfied if $b>-3/4u$. The value of b is determined from the vector field X to be $b=-\frac{x-u/2}{y-u^2/3}$. Hence the entropy condition requires

$$\frac{x-u/2}{y-u^2/3}<3/4u.$$

Using this relation and the relation

$$X=\frac{y-y_0}{u^2-y_0}u,$$

we find that the entropy condition will be satisfied if $u^2>y$ along the shock. If it happened that $u^2=y$, then $x=u$. But $x<0.5<u$. We conclude that the entropy condition is satisfied along the shock separating the rarefaction region from the region in which the solution is 0. This completes the proof that the entropy condition is satisfied for our solution.

The above examples illustrate very clearly that solving the initial value problem for a single conservation law in more than one space dimension by geometric techniques involves complications not encountered in one space dimension. One would hope that for "generic" conservation laws and initial data, the solutions would be piecewise smooth. Even for piecewise constant initial data, however, this is not settled. Further barriers to proving such theorems arise from the apparent difficulty in reconciling the entropy conditions for shocks which lie in different directions.

Added in Proof. The author thanks B. KEYFITE for finding a mistake in his original calculations.

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