Local and Global Behavior of Solutions of Quasilinear Equations of Emden-Fowler Type

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Abstract

We consider the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-1}u = 0$ for $p \leqq N$, $0 < p - 1 < q$. We study the isolated singularities and the behavior near infinity of nonradial positive solutions when $q < N(p-1)/(N-p)$, and give a complete classification of local and global radial solutions of any sign, for any q.

0. Introduction

In this article we study essentially the doubly nonlinear equation in a regular domain Ω of $\mathbb{R}^N$:

$$\Delta_p u + |u|^{q-1}u = 0, \tag{0.1}$$

where $p \leqq N$, $q > p - 1 > 0$, and $\Delta_p u$ is the p-Laplace operator:

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u). \tag{0.2}$$

When $p = 2$, equation (0.1) has been intensively studied. When $N > 2$, two critical values $N/(N-2)$ and $(N+2)/(N-2)$ appear. The first studies in the radial case are due to EMDEN; then FOWLER [7] [8] [9] gave existence results and a full classification of the global radial solutions in $\mathbb{R}^N$ or $\mathbb{R}^N/\{0\}$. Recently, NI, MCLEOD & SERRIN [16] give other methods to study equations of such a kind, using Pohojaev type identities and new techniques for oscillating solutions. In the nonradial case, the study of positive solutions near the origin is made by LIONS [15] when $q < N/(N-2)$, AVILES [2] when $q = N/(N-2)$; then GIDAS & SPRUCK [11] give local and global results when $q < (N+2)/(N-2)$; CAFFARELLI, GIDAS & SPRUCK [6] have just extended them to the critical case $q = (N+2)/(N-2)$. Nothing is known when q is greater.

In the general case $p > 1$, the first results in the radial case for positive solutions are due to NI & SERRIN [18] who pointed out the existence of the critical values

$$Q_1 = N(p-1)/(N-p), \quad Q_2 = (N(p-1)+p)/(N-p), \tag{0.3}$$

when $p < N$. Then GUEDDA & VERON [12] studied the global existence and the behavior near the origin for radial positive solutions and $q < Q_2$. They obtained some results also in the nonradial case when $q < Q_1$ under conditions of majorization or integrability of u near the origin.

In this paper we study first the isolated singularities and the behavior near infinity in the nonradial case, when $p < N$ and $q < Q_1$, or when $p = N$. Then we prove an exhaustive description of all the radial solutions of the equation, regular or singular near 0 or infinity, without any condition of sign, for any value of q.

In *Section 1* we deal with the nonradial case. We extend the results of BREZIS & LIONS [5] and LIONS [15]. Our first result is the following:

Suppose $p < N$ and $0 \in \Omega$. Let u be a continuous nonnegative function, p-superharmonic in $\Omega/\{0\}$, with $|\nabla u|^p$ and $\Delta_p u$ locally integrable in $\Omega/\{0\}$. Then $u^{p-1} \in M_{\mathrm{loc}}^{N/(N-p)}(\Omega)$, $|\nabla u|^{p-1} \in M_{\mathrm{loc}}^{N/(N-1)}(\Omega)$ and u satisfies

$$-\Delta_p u = g + \beta\,\delta_0, \tag{0.4}$$

in the sense of distributions in Ω, where $g \in L^1_{\mathrm{loc}}(\bar{\Omega})$, $\beta \geqq 0$ and δ_0 is the Dirac mass at the origin.

Then we deduce from [12] and [20] conclusions about singular solutions of equation (0.1):

Suppose $p < N$ and $q < Q_1$, or $p = N$. Let u be a nonnegative continuous solution of (0.1) in $\Omega/\{0\}$. Then either u is regular in Ω or there is an $\alpha > 0$ such that

$$\lim_{x \to 0} u(x)/\mu(x) = \alpha, \tag{0.5}$$

where μ is the fundamental p-harmonic function in $\mathbb{R}^N$; and u satisfies (0.4) with $g = u^q$ and $\beta = \alpha^{p-1}$.

Moreover we give estimates of $u - \mu$ extending those of [15].

Concerning the exterior problem, we prove the following:

Suppose $p < N$ and $q \leqq Q_1$, or $p = N$. Then any nonnegative continuous solution u of (0.1) in an exterior domain $|x| > R$ is identically zero.

This result was known for radial u from [12].

In the other sections we study the radial case. Equation (0.1) takes the radial form

$$r^{1-N}(r^{N-1}|u_r|^{p-2}u_r)_r + |u|^{q-1}u = 0, \tag{0.6}$$

for the function $r = |x| \mapsto u(r)$.

In *Section 2*, we verify that any solution is defined in $(0, +\infty)$ or $[0, +\infty)$ and give equivalent forms of the equation. The most useful change of variables, $u(r) = r^{-p/(q+1-p)}w(t)$ with $t = -\mathrm{Log}\, r$, reduces the equation to an autonomous one. We write it as a system:

$$\begin{aligned} w_t &= -pw/(q+1-p) + |y|^{(2-p)/(p-1)}\, y, \\ y_t &= -|w|^{q-1}\, w + (N - pq/(q+1-p))\, y. \end{aligned} \tag{0.7}$$

We give two energy functions fundamental for the study.

In *Section 3* we study the behavior of any solution near the origin when $p < N$, and q is not critical. The energy method appears to be more effective than phase plane techniques used in [12]. We also give properties of oscillating solutions with the technique of [16], extending the former results [9].

In *Section 4* we study the behavior near infinity ($p < N$, q not critical). We make another change of variables which reduces the study to the former one when q is greater than Q_1: let $\bar{u}(s) = -r^{N-1}\,|u_r|^{p-2}\, u_r$, with $s = r^{-\nu}/\nu$, $\nu = (N-p)(q-Q_1)/(p-1)(q+1)$; then $\lim_{r\to+\infty} s = 0$, and (0.5) is reduced to

$$s^{1-\bar{N}}(s^{\bar{N}-1}\,|\bar{u}_s|^{\bar{p}-2}\,\bar{u}_s)_s + |\bar{u}|^{\bar{q}-1}\,\bar{u} = 0, \tag{0.8}$$

where $\bar{p} = (q+1)/q$, $\bar{q} = 1/(p-1)$; $\bar{N} = \bar{p} + N/\nu q$ need not be an integer. Note in particular that when $q = 1$ and $p < 2$, equation (0.1) reduces in the radial case to a semilinear elliptic equation with linear principal part.

In *Section 5* we study the global regular or singular solutions ($p < N$, q not critical): what connexions are possible, what connexions do exist. The classification is similar to the one of FOWLER [9].

In *Section 6* we study the global behavior in the critical cases $q = Q_1$, $q = Q_2$.

In *Section 7* we study the case $p = N$.

1. Non-Radial Results

Let $N > 1$ be an integer. Set $B_R(x) = \{y \in \mathbb{R}^N \mid |y - x| < R\}$ for any $x \in \mathbb{R}^N$, $R > 0$, and $B_R = B_R(0)$, $B'_R = B_R/\{0\}$. Our main result is the following:

Theorem 1.1. *Let* $1 < p < N$ *and* $R > 0$. *Assume that* $u \in C^0(B'_R)$, $\nabla u \in L^p_{\mathrm{loc}}(B'_R)$, $\Delta_p u \in L^1_{\mathrm{loc}}(B'_R)$ *in the sense of* $D'(B'_R)$, *and*

$$u \geqq 0, \quad \Delta_p u \leqq 0 \quad \text{a.e. in } B_R. \tag{1.1}$$

Then $u^{p-1} \in M^{N/(N-p)}_{\mathrm{loc}}(B_R)$, $|\nabla u|^{p-1} \in M^{N/(N-1)}_{\mathrm{loc}}(B_R)$, *and there are a* $g \in L^1_{\mathrm{loc}}(B_R)$ *and a* $\beta \geqq 0$ *such that*

$$-\Delta_p u = g + \beta\,\delta_0 \quad \text{in } D'(B_R). \tag{1.2}$$

This result has been proved in [5] by linear methods when $p = 2$. In the general case we use nonlinear techniques introduced by SERRIN [19], [20], TRUDINGER [22], also used in GIDAS & SPRUCK [11], and the study of equations with second member in L^1, given for $p = 2$ by BENILAN, BREZIS & CRANDALL [4], and for $p \neq 2$ in [3]. We prove Theorem 1.1 in four steps.

Lemma 1.1. *Let* $g(x) = -\Delta_p u(x)$, a.e. *in* B_R. *Then* $g \in L^1_{loc}(B_R)$ *and for any* $\eta \in D(B_R)$, $0 \leqq \eta \leqq 1$, $\eta(x) = 1$ *near* $x = 0$,

$$\int_{B_R} g\eta^p\, dx \leqq \int_{B_R} |\nabla u|^{p-2} \nabla u \nabla(\eta^p)\, dx. \tag{1.3}$$

Moreover, for any $\varrho < R$, *there is a* $c_\varrho > 0$ *such that*

$$\int_{B_\varrho \cap \{k<u<k+\alpha\}} |\nabla u|^p\, dx \leqq c_\varrho \alpha \qquad \forall\, k \geqq 0, \qquad \forall\, \alpha > 0. \tag{1.4}$$

Proof. From the definition of g we have

$$\int_{B_R} g\phi\, dx = \int_{B_R} |\nabla u|^{p-2} \nabla u \nabla \phi\, dx, \tag{1.5}$$

for any $\phi \in W^{1,\infty}(B'_R)$ with compact support in B'_R. Set $p_{k,\alpha}(t) = \min((t-k)^+/\alpha, 1)$ for any $t \geqq 0$ and $k \geqq 0$, $\alpha > 0$. Let $0 < \varrho < R$ and $\varepsilon < \varrho/2$. Let $\eta \in D(B_R)$ be such that $0 \leqq \eta \leqq 1$, $\eta(x) = 1$ on B_ϱ, and $\zeta_\varepsilon = \xi_\varepsilon \eta$, with $\xi_\varepsilon \in C^\infty(B_R)$, $0 \leqq \xi_\varepsilon \leqq 1$, $\xi_\varepsilon(x) = 0$ if $|x| < \varepsilon$, $\xi_\varepsilon(x) = 1$ if $|x| > 2\varepsilon$, $|\nabla \xi_\varepsilon| \leqq C/\varepsilon$. Then the function $\phi = (1 - p_{k,\alpha}(u))\, \zeta_\varepsilon^p$ is admissible in (1.5), and we get

$$\begin{aligned}\int_{B_R} g(1 - p_{k,\alpha}(u))\, \zeta_\varepsilon^p\, dx &+ \frac{1}{\alpha} \int_{\{k<u<k+\alpha\}} |\zeta_\varepsilon \nabla u|^p\, dx \\ &= \int_{B_R} (1 - p_{k,\alpha}(u))\, |\nabla u|^{p-2} \nabla u \nabla(\zeta_\varepsilon^p)\, dx. \end{aligned}\tag{1.6}$$

Taking first $\alpha = 1$ and adding the equalities for integer $k = 0, 1, \ldots, n$, because $\sum_{k=0}^{n} (1 - p_{k,1}(t)) = (n + 1 - t)^+$ we get

$$\begin{aligned}\int_{\{u<n+1\}} g(n + 1 - u)\, \zeta_\varepsilon^p\, dx &+ \int_{\{u<n+1\}} |\zeta_\varepsilon \nabla u|^p\, dx \\ &\leqq \int_{\{u<n+1\}} (n + 1 - u)\, |\nabla u|^{p-2} \nabla u \nabla(\eta^p)\, dx \\ &\quad + p \int_{\{u<n+1\} \cap B_{2\varepsilon}} (n + 1 - u)\, |\zeta_\varepsilon \nabla u|^{p-1}\, |\nabla \xi_\varepsilon|\, dx. \end{aligned}\tag{1.7}$$

Now for any real $h > 0$ we have $n + 1 - u(x) > (n + 1)\, h/(h + 1)$ a.e. in $\left\{u < \dfrac{n+1}{h+1}\right\}$; hence, dividing by $n + 1$, we show that

$$\begin{aligned}&\frac{h}{h+1} \int_{\left\{u<\frac{n+1}{h+1}\right\}} g\zeta_\varepsilon^p\, dx + \frac{1}{n+1} \int_{\{u<n+1\}} |\zeta_\varepsilon \nabla u|^p\, dx \\ &\leqq \int_{\{u<n+1\}} \left(1 - \frac{u}{n+1}\right) |\nabla u|^{p-2} \nabla u \nabla(\eta^p)\, dx + p \int_{\{u<n+1\} \cap B_{2\varepsilon}} |\zeta_\varepsilon \nabla u|^{p-1}\, |\nabla \xi_\varepsilon|\, dx \\ &\leqq \int_{\{u<n+1\}} \left(1 - \frac{u}{n+1}\right) |\nabla u|^{p-2} \nabla u \nabla(\eta^p)\, dx \\ &\quad + (p - 1)\, \beta^{p'} \int_{\{u<n+1\}} |\zeta_\varepsilon \nabla u|^p\, dx + \beta^{-p} \int_{B_{2\varepsilon}} |\nabla \xi_\varepsilon|^p\, dx, \end{aligned}$$

for any $\beta > 0$, by the Hölder inequality. Now choosing an adequate β, we get

$$\frac{h}{h+1} \int_{\left\{u<\frac{n+1}{h+1}\right\}} g\zeta_\varepsilon^p \, dx + \frac{1}{2(n+1)} \int_{\{u<n+1\}} |\zeta_\varepsilon \nabla u|^p \, dx$$
$$\leqq \int_{\{u<n+1\}} \left(1 - \frac{u}{n+1}\right) |\nabla u|^{p-2} \nabla u \, \nabla(\eta^p) \, dx + c(n+1)^{p-1} C^p \varepsilon^{N-p},$$

where $c = c(N, p)$. Now we make successively $\varepsilon \to 0$, $n \to +\infty$, $h \to +\infty$. With Fatou's lemma we get $g\eta^p \in L^1(B_R)$ and it satisfies (1.3); hence $g \in L^1(B_\varrho)$, $g \in L^1_{\text{loc}}(B_R)$. Moreover we have, for any integer n,

$$\int_{\{u<n+1\}\cap B_\varrho} |\nabla u|^p \, dx \leqq 2(n+1) \int_{B_R} |\nabla u|^{p-1} |\nabla(\eta^p)| \, dx. \tag{1.8}$$

Now take any $k \geqq 0, \alpha > 0$ in (1.6). Then from (1.6) and (1.8),

$$\frac{1}{\alpha} \int_{\{k<u<k+\alpha\}} \zeta_\varepsilon |\nabla u|^p \, dx$$
$$\leqq \int_{B_R} |\nabla u|^{p-1} |\nabla(\eta^p)| \, dx + p \int_{\{u<k+\alpha\}\cap B_{2\varepsilon}} |\nabla u|^{p-1} |\nabla \zeta_\varepsilon| \, dx$$
$$\leqq \int_{B_R} |\nabla u|^{p-1} |\nabla(\eta^p)| \, dx + \tilde{c} C \varepsilon^{\frac{N-p}{p}} \left(\int_{B_R} |\nabla u|^{p-1} |\nabla(\eta^p)| \, dx\right)^{\frac{p-1}{p}}$$

where $\tilde{c} = \tilde{c}(N, p, k, \alpha)$. As $\varepsilon \to 0$, we get (1.4).

Remark. The estimate (1.4) is the keystone in proof of estimates in Marcinkiewics spaces for equation $-\Delta_p v = f \in L^1(B_R)$, $v \in W_0^{1,p}(B_R)$; see [3]. Here we need also some estimates of u; this is done in next lemma.

Lemma 1.2. *For any $\gamma \in (0, Q_1)$, $u^\gamma \in L^1_{\text{loc}}(B_R)$ and there is a $C = C(\gamma, N, p, u)$ such that, for any small σ,*

$$\int_{B_\sigma} u^\gamma \, dx \leqq C\sigma^{N - \frac{N-p}{p-1}\gamma}. \tag{1.9}$$

Proof. Here we use a test function ϕ introduced by SERRIN [19] to estimate the minimum of u on spheres of radius σ, and then the weak Harnack inequality. Let $C_1 = 2 \max_{|x|=R/2} u(x)$ and $\bar{u} = u - C_1$. For any fixed $\sigma \in (0, R/2)$, set $m(\sigma) = \min_{|x|=\sigma} \bar{u}(x)$. Suppose first that $m(\sigma) > 0$ and define

$$v(\sigma)(x) = \begin{cases} 0 & \text{if } \sigma < |x| < R/2 \text{ and } \bar{u}(x) \leqq 0, \text{ or if } |x| \geqq R/2, \\ \bar{u}(x) & \text{if } 0 \leqq \bar{u}(x) \leqq m(\sigma) \text{ and } \sigma < |x| < R/2, \\ m(\sigma) & \text{if } \bar{u}(x) > m(\sigma) \text{ and } \sigma < |x| < R/2, \text{ or if } |x| \leqq \sigma; \end{cases}$$

then $v(\sigma) \in C^0(B_R) \cap W^{1,p}(B_R)$. We take $\phi = v(\sigma) - m(\sigma)\,\eta$ in (1.5), where η is chosen as in Lemma 1. We get

$$\int_{B_R} |\nabla u|^{p-2} \nabla u \, \nabla(v(\sigma))\, dx + \int_{B_R} g(m(\sigma) - v(\sigma))\, dx = m(\sigma)\, K,$$

where K does not depend of σ:

$$K = \int_{B_R} |\nabla u|^{p-2} \nabla u \, \nabla \eta \, dx + \int_{B_R} g(1 - \eta)\, dx.$$

As $g(m(\sigma) - v(\sigma)) \geqq 0$, we get as in [19]

$$m(\sigma)\, K \geqq \int_{B_R} |\nabla u|^{p-2} \nabla u \, \nabla(v(\sigma))\, dx = \int_{B_R} |\nabla(v(\sigma))|^p \, dx$$
$$\geqq m(\sigma)^p \operatorname*{cap}_p B_\sigma = m(\sigma)^p \, \omega_N \left(\frac{N-p}{p-1}\right)^{p-1} \sigma^{N-p},$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$; hence, dividing by $m(\sigma) > 0$ and returning to u, we conclude that

$$\min_{|x|=\sigma} u(x) \leqq C_1 + C_2\, \sigma^{\frac{p-N}{p-1}} \qquad \forall\, \sigma \in (0, R/2), \tag{1.10}$$

where C_1 and C_2 do not depend up σ. Suppose now that $m(\sigma) \leqq 0$. Then (1.10) is trivial.

Now u is a weak supersolution of the p-Laplace equation in B'_R; hence from [22] it satisfies a weak Harnack inequality; for any $\gamma \in (0, Q_1)$, there is a $C = C(N, p, \gamma)$ such that, for any ball $B_{3\varrho}(x_0) \subset B'_R$,

$$\varrho^{-N/\gamma} \left(\int_{B_{2\varrho}(x_0)} u_\gamma \, dx \right)^{1/\gamma} \leqq C \min_{B_\varrho(x_0)} u(x).$$

Then there is another constant $C = C(N, p, \gamma)$ such that, for any $\sigma \in (0, R/2)$,

$$\sigma^{-N/\gamma} \left(\int_{3\sigma/4 < |x| < 5\sigma/4} u \, dx \right)^{1/\gamma} \leqq C \min_{|x|=\sigma} u(x). \tag{1.11}$$

Indeed we recover the annulus by n balls $B_{2\varrho}(x_i)$ where $\varrho = 7\sigma/24$, $|x_i| = \sigma$, $|x_i - x_{i-1}| < \varrho$, $u(x_0) = \min_{|x|=\sigma} u(x)$, and n is independent of σ; we prove easily that, for any $i = 1, \ldots, n$,

$$\varrho^{-N/\gamma} \left(\int_{B_{2\varrho}(x_i)} u^\gamma \, dx \right)^{1/\gamma} \leqq C^i (\omega_N)^{i/\gamma}\, u(x_0),$$

and hence we get (1.11). Now we get immediately, with another constant C,

$$\sigma^{-N/\gamma} \left(\int_{B_\sigma} u^\gamma \, dx \right)^{1/\gamma} \leqq C \min_{|x|=\sigma} u(x) \qquad \forall\, \sigma \in (0, R/3). \tag{1.12}$$

Then (1.9) follows from (1.10) and (1.12).

Lemma 1.3. *The function u satisfies equation* (1.2).

Proof. First we estimate $|\nabla u|^{p-1}$ near the origin. Let $\varrho < R$ and $\delta > 0$ be fixed. For any $\sigma < \varrho$ we have

$$\int_{B_\sigma} |\nabla u|^{p-1}\,dx = \int_{B_\sigma} \left|\frac{\nabla u}{(1+u)^{(\delta+1)/p}}\right|^{p-1} (1+u)^{(\delta+1)(p-1)/p}\,dx$$
$$\leq \left(\int_{B_\sigma} \frac{|\nabla u|^p}{(1+u)^{\delta+1}}\,dx\right)^{\frac{p-1}{p}} \left(\int_{B_\sigma} (1+u)^{(\delta+1)(p-1)}\,dx\right)^{\frac{1}{p}}.$$

Now from (1.4) we get

$$\int_{B_\sigma} \frac{|\nabla u|^p}{(1+u)^{\delta+1}}\,dx = \sum_{k=0}^{+\infty} \int_{\{k<u<k+1\}\cap B_\varrho} \frac{|\nabla u|^p}{(1+u)^{\delta+1}}\,dx \leq c_\varrho \sum_{k=0}^{+\infty} \frac{1}{(1+k)^{\delta+1}} < +\infty. \tag{1.13}$$

Take $\delta < p/(N-p)$; then $(\delta+1)(p-1) < Q_1$; hence by Lemma 1.2, $|\nabla u|^{p-1} \in L^1_{\mathrm{loc}}(B_R)$ and from (1.9) there is a C_δ such that, for small σ,

$$\int_{B_\sigma} |\nabla u|^{p-1}\,dx \leq C_\delta \sigma^{1-\delta(N-p)/p}. \tag{1.14}$$

Now since $|\nabla u|^{p-2}\nabla u \in L^1_{\mathrm{loc}}(B_R)$ we can define the distribution

$$T = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) - g \quad \text{in } D'(B_R).$$

Then, as in [5], we have $T = \sum_{|r|\leq m} \beta_r D^r \delta_0$. Let $\psi \in D(B_R)$ such that $(-1)^r (D^r\psi)(0) = \beta_r$ for every $|r| \leq m$, and $\psi_\varepsilon(x) = \psi(x/\varepsilon)$. Then

$$\langle T, \psi_\varepsilon\rangle = \sum_{|r|\leq m} \beta_r^2 \varepsilon^{-r} = \int_{B_R} |\nabla u|^{p-2}\nabla u \nabla\psi_\varepsilon\,dx - \int_{B_R} g\psi_\varepsilon\,dx.$$

As $g \geq 0$, we get from (1.14), for small ε,

$$\sum_{|r|\leq m} \beta_r^2 \varepsilon^{-r} \leq C_\delta \varepsilon^{-\delta(N-p)/p}. \tag{1.15}$$

Now $(\delta(N-p)/p) < 1$; hence $\beta_r = 0$ when $|r| \geq 1$. Finally, for any $\eta \in D(B_R)$, $0 \leq \eta \leq 1$, $\eta(x) = 1$ near the origin, we have

$$\langle T, \eta^p\rangle = \beta_0 = \int_{B_R} |\nabla u|^{p-2}\nabla u \nabla(\eta^p)\,dx - \int_{B_R} g\eta^p\,dx,$$

and hence $\beta_0 \geq 0$ from (1.13). □

Remark. The estimate (1.14) is not optimal. We prove in the next lemma that we can take $\delta = 0$ in (1.14).

Lemma 1.4. $u^{p-1} \in M^{N/(N-p)}_{\mathrm{loc}}(B_R)$ *and* $|\nabla u|^{p-1} \in M^{N/(N-1)}_{\mathrm{loc}}(B_R)$.

Proof. Here the difficulty comes from the fact that perhaps $u \notin L^1_{\mathrm{loc}}(B_R)$. Let $0 < \gamma < (p-1)/p$; as $u \in W^{1,p}_{\mathrm{loc}}(B'_R)$, we have by the chain rule $(1+u)^\gamma \in W^{1,p}_{\mathrm{loc}}(B'_R)$ and

$$\nabla((1+u)^\gamma) = \gamma(1+u)^{\gamma-1}\nabla u \quad \text{in } L^p_{\mathrm{loc}}(B'_R). \tag{1.16}$$

Now taking $\delta = p - 1 - p\gamma$ in (1.13) we get $\nabla((1+u)^\gamma) \in L^p_{\mathrm{loc}}(B_R)$. We have $\gamma < (N-1)(p-1)/(N-p)$; hence $(1+u)^\gamma \in L^1_{\mathrm{loc}}(B_R)$. Let θ be the gradient of $(1+u)^\gamma$ in $D'(B_R)$. Then

$$\theta = \gamma(1+u)^{\gamma-1}\nabla u + \sum_{|r|\leqq m} \alpha_r D^r \delta_0.$$

Defining ψ_ε as in Lemma 1.3, we get

$$\langle\theta, \psi_\varepsilon\rangle = \sum_{|r|\leqq m} \alpha_r^2 \varepsilon^{-r} + \int_{B_R} \gamma(1+u)^{\gamma-1}\nabla u\, \psi_\varepsilon\, dx = -\int_{B_R} (1+u)^\gamma \nabla\psi_\varepsilon\, dx;$$

hence, for small ε, from (1.9),

$$\sum_{|r|\leqq m} \alpha_r^2\varepsilon^{-r} \leqq C\left(\|(1+u)^{\gamma-1}\nabla u\|_{L^p(B_{R/2})}\, \varepsilon^{Np/(p-1)} + \varepsilon^{N-1-\frac{(N-p)\gamma}{p-1}}\right).$$

From the choice of γ we get $\alpha_r = 0 \;\; \forall\, |r| \leqq m$. Hence $(1+u)^\gamma \in W^{11}_{\mathrm{loc}}(B_R)$. Now we can adapt the proof of [3] to $(1+u)^\gamma$: for any $k \geqq 0, \alpha > 0$, we verify easily, since $\gamma < 1$, that

$$p_{k,\alpha}(1+u) \leqq p_{k^\gamma,(k+\alpha)^\gamma - k^\gamma}((1+u)^\gamma) \quad \text{a.e. in } B_R. \tag{1.17}$$

Let $\varrho < R$ be fixed. By use of the injection of $W^{11}_0(B_\varrho)$ into $L^{N/(N-1)}(B_\varrho)$, there is a $\bar{c} = \bar{c}(N,p)$ such that, for large k,

$$\|p_{k,\alpha}(1+u)\|_{L^{N/(N-1)}(B_\varrho)} \leqq \bar{c}\,\|\nabla(p_{k^\gamma,(k+\alpha)^\gamma-k^\gamma}((1+u)^\gamma)\|_{L^1(B_\varrho)}.$$

Hence from (1.16)

$$\begin{aligned}
\|p_{k,\alpha}(1+u)\|_{L^{N/(N-1)}(B_\varrho)} &\leqq \frac{\bar{c}}{(k+\alpha)^\gamma - k^\gamma} \int_{B_\varrho \cap \{k<u<k+\alpha\}} |\nabla((1+u)^\gamma)|\, dx,\\
&\leqq \frac{\bar{c}\gamma}{(k+1)^{1-\gamma}((k+\alpha)^\gamma - k^\gamma)} \int_{B_\varrho\cap\{k<u<k+\alpha\}} |\nabla u|\, dx,\\
&\leqq \frac{\bar{c}}{\alpha} \int_{B_\varrho\cap\{k<u<k+\alpha\}} |\nabla u|\, dx,
\end{aligned}$$

if $\alpha < 1$. Then we deduce from (1.4) and [3] the estimates

$$\operatorname{meas}(\{u > k\} \cap B_\varrho) \leqq Ck^{N(1-p)/(N-p)}, \tag{1.18}$$

$$\operatorname{meas}(\{|\nabla u| > k\} \cap B_\varrho) \leqq Ck^{N(1-p)/(N-1)}, \tag{1.19}$$

where $C = C(N,p,\varrho)$, and hence the conclusion. □

In the case $p = N$ we have to make some restrictive hypothesis:

Proposition 1.1. *Let* $R > 0$. *Assume that* $u \in C^0(B'_R)$, $\nabla u \in L^N_{\rm loc}(B'_R)$, $\Delta_N u \in L^1_{\rm loc}(B'_R)$ *in the sense of* $\mathbf{D}'(B'_R)$, *and*

$$u \geqq 0,\ \Delta_N u \leqq 0 \quad \text{a.e. } in\ B_R. \tag{1.20}$$

Then $u^\gamma \in L^1_{\rm loc}(B_R)$ *for any* $\gamma > 0$, $|\nabla u|^\gamma \in L^1_{\rm loc}(B_R)$ *for any* $\gamma \in (0, N)$.

Moreover, if $\lim_{x\to 0} u = +\infty$, *then there are a* $g \in L^1(\Omega)$ *and a* $\beta \geqq 0$ *such that*

$$-\Delta_N u = g + \beta\,\delta_0 \quad in\ \mathbf{D}'(B_R). \tag{1.21}$$

Proof. Let $g(x) = -\Delta_N u(x)$, a.e. in B_R. Proceeding as in Lemma 1.2 and using the fact that

$$\operatorname*{cap}_N B_\sigma = \omega_N(\operatorname{Log}(R/\sigma))^{1-N},$$

(see [19]) we get instead of (1.9) the estimate

$$\min_{|x|=\sigma} u(x) \leqq C_1 + C_2\,|\operatorname{Log}\sigma| \qquad \forall\,\sigma \in (0, R/2). \tag{1.22}$$

Now the weak Harnack inequality is available for any $\gamma > 0$; hence (1.9) is replaced by the estimate

$$\int_{B_\sigma} u^\gamma\,dx \leqq C\sigma^N\,|\operatorname{Log}\sigma|^\gamma \tag{1.23}$$

for any small σ, where $C = C(\gamma, N, p, u)$, and hence $u^\gamma \in L^1_{\rm loc}(B_R)$. Now consider $u_k = \min(u, k)$ for any $k > 0$. Then a theorem of LINDQVIST & MARTIO [14] tells us that u_k is super p-harmonic in B'_R and satisfies, for any $x_0 \in B'_{R/2}$ and $0 < r < \varrho < |x_0|$,

$$\int_{B_r(x_0)} \left|\frac{\nabla u_k}{1+u_k}\right|^N dx \leqq (N/(N-1))^N\,\omega_{N-1}(\operatorname{Log}(\varrho/r))^{1-N}. \tag{1.24}$$

By Fatou's lemma we get the same estimate for u. For any $\sigma \in (0, R/3)$ we recover the annulus $\sigma/2 < |x| < 3\sigma/2$ by n balls $B_{3\sigma/4}(x_i)$ with $|x_i| = \sigma$, and make $r = 3\sigma/4$, $\varrho = 7\sigma/8$; hence

$$\int_{\sigma/2<|x|<3\sigma/2} \left|\frac{\nabla u}{1+u}\right|^N dx \leqq C, \tag{1.25}$$

and C does not depend on σ. Now for any $\gamma \in (0, N)$, by the Hölder inequality,

$$\int_{\sigma/2<|x|<3\sigma/2} |\nabla u|^\gamma\,dx$$
$$\leqq \left(\int_{\sigma/2<|x|<3\sigma/2} \left|\frac{\nabla u}{1+u}\right|^N dx\right)^{\gamma/N} \left(\int_{\sigma/2<|x|<3\sigma/2} (1+u)^{\gamma N/(N-\gamma)}\,dx\right)^{(N-\gamma)/N};$$

hence from (1.23), for small σ,

$$\int_{\sigma/2<|x|<3\sigma/2} |\nabla u|^\gamma \, dx \leqq C\sigma^{N-\gamma} |\mathrm{Log}\, \sigma|^\gamma .$$

By summation we get $|\nabla u|^\gamma \in L^1_{\mathrm{loc}}(B_R)$, with the estimate

$$\int_{B_\sigma} |\nabla u|^\gamma \, dx \leqq C\sigma^{N-\gamma} |\mathrm{Log}\, \sigma|^\gamma \tag{1.26}$$

for small σ, where $C = C(N, p, \gamma)$, $\gamma \in (0, N)$.

Now consider again the proof of Lemma 1.1. The estimate (1.26) does not allow us to go to the limit in (1.7) as ε goes to 0 for fixed n. Assume now that $\lim_{x\to 0} u(x) = +\infty$. Then, for fixed n, we get for ε small enough

$$\int_{\{u<n+1\}\cap B_{2\varepsilon}} (n+1-u)\, |\zeta_\varepsilon \nabla u|^{N-1} |\nabla \zeta_\varepsilon| \, dx = 0,$$

and hence we get the conclusions of Lemma 1.1 with $p = N$. In Lemma 1.3 we can replace (1.14) by

$$\int_{B_\sigma} |\nabla u|^{N-1} \, dx \leqq C\sigma \, |\mathrm{Log}\, \sigma|^{N-1}, \tag{1.27}$$

from (1.26); hence (1.15) is replaced by

$$\sum_{|r|\leqq m} \beta_r^2 \varepsilon^{-r} \leqq C \, |\mathrm{Log}\, \varepsilon|^{N-1}, \tag{1.28}$$

which proves that $\beta_r = 0$ when $|r| \geqq 1$; hence we get (1.21), with $\beta \geqq 0$ from (1.3). □

Remark. The estimate (1.23) is optimal because for the N-harmonic function $\bar{u} = \mathrm{Log}\,(1/|x|)$, $\sigma^{-N} |\mathrm{Log}\, \sigma|^{-\gamma} \int_{B_\sigma} \bar{u}^\gamma \, dx$ has a limit different from 0 as $\varepsilon \to 0$. Nevertheless the estimate (1.26) is not optimal, as $\sigma^{\gamma-N} \int_{B_\sigma} |\nabla \bar{u}|^\gamma \, dx$ has a limit as $\varepsilon \to 0$, for any $\gamma < N$. We could go to the limit in (1.7), had we proved an estimate of ∇u in $M^N_{\mathrm{loc}}(B_R)$.

Now we return to equation (0.1). Let μ be the fundamental p-harmonic function in $\mathbb{R}^N/\{0\}$:

$$\mu(x) = \begin{cases} C_{N,p} \, |x|^{(p-N)/(p-1)} & \text{for } 1 < p < N, \\ C_N \, \mathrm{Log}\,(1/|x|) & \text{for } p = N, \end{cases} \tag{1.29}$$

where

$$C_{p,N} = \frac{p-1}{N-p} (N\omega_N)^{-1/(p-1)}, \qquad C_N = (N\omega_N)^{-1(/N-1)}.$$

Theorem 1.2. *Let p and q be real, $1 < p \leqq N$, $p-1 < q < Q_1$, $p-1 < q < +\infty$ if $p = N$, and $R > 0$. Let $u \in C^0(B'_R)$ with $\nabla u \in L^p_{\mathrm{loc}}(B'_R)$ be a non-negative solution of the equation*

$$\Delta_p u + u^q = 0 \tag{1.30}$$

in $D'(B'_R)$. Then one of the two following possibilities must occur:

(i) $\lim_{x\to 0} u(x)$ *exists and the extended u is a solution of* (1.30) *in $D'(B_R)$;*

(ii) *there is an $\alpha > 0$ such that $\lim_{x\to 0} u(x)/\mu(x) = \alpha$, and u satisfies the equation*

$$\Delta_p u + u^q + \alpha^{p-1}\,\delta_0 = 0 \tag{1.31}$$

in $D'(B_R)$. Moreover we have the estimates

$$\lim_{\substack{x\to 0\\ x_i/|x|\to\xi_i}} |x|^{(N-1)/(p-1)}\, u_{x_i} = \alpha(N\omega_N)^{-1/(p-1)}\,\xi_i, \tag{1.32}$$

and for any $0 < |x| < r < R$,

$$-C \leqq u - \alpha\mu \leqq C(1 + |x|^{(N(p-2)+p-(N-p)q)/(p-1)})$$

$$\text{if } p < N,\ q \neq (N(p-2)+p)/(N-p),$$

$$-C \leqq u - \alpha\mu \leqq C(1 + |\mathrm{Log}\,|x||) \quad \text{if } p < N,\ q = (N(p-2)+p)/(N-p),$$

$$-C \leqq u - \alpha\mu \leqq C \quad \text{if } p = N, \tag{1.33}$$

with constants $C = C(N, p, q, r, u)$.

Proof. Equation (1.30) can be written in the form

$$\Delta_p u + d u^{p-1} = 0 \tag{1.34}$$

with $d = u^{q+1-p}$. By hypothesis the distribution $\Delta_p u$ in B'_R belongs to $L^\infty_{\mathrm{loc}}(B'_R)$; hence from Lemma 1.2 and Proposition 1.1 we have $u^\gamma \in L^1_{\mathrm{loc}}(B_R)$ for any $\gamma \in (0, Q_1)$ if $p < N$, for any $\gamma > 0$ if $p = N$. Since $q < Q_1$ if $p < N$, we can find a $\delta > 0$ such that $d \in L^{N/(p-\delta)}_{\mathrm{loc}}(B_R)$. In that case we can apply SERRIN's results [20]: either the singularity is removable and u can be extended as a solution of (1.30) in $D'(B_R)$, or there are C_1 and $C_2 > 0$ such that

$$C_1\mu(x) \leqq u(x) \leqq C_2\mu(x) \quad \text{near } 0.$$

In that case from Theorem 1.1 or Proposition 1.1 there is a $\beta > 0$ such that

$$\Delta_p u + u^q + \beta\,\delta_0 = 0$$

in $D'(B_R)$. GUEDDA & VERON [12] prove by scaling that there is an $\alpha > 0$ such that $\lim_{x\to 0} u(x)/\mu(x) = \alpha$. Now as in [12] we get (1.32) and (1.31). Let us prove (1.33). For any $\varepsilon > 0$ there is an $a > 0$ such that $u \geqq (\alpha - \varepsilon)\,\mu$ in B'_a. Let $0 < \varrho < \min(1, R)$. We have $-\Delta_p u \geqq -\Delta_p((\alpha - \varepsilon)\,\mu)$ in B'_R, and $u - (\alpha - \varepsilon)\,\mu \geqq -\alpha\mu(\varrho)$ for $|x| = \varrho$. From the maximum principle applied in any annulus $b < |x| < \varrho$ with $b < a$, we get

$$u \geqq (\alpha - \varepsilon)\,\mu - \alpha\mu(\varrho) \quad \text{in } B'_\varrho.$$

Letting ε to go 0 we get

$$u \geqq \alpha\mu - \alpha\mu(\varrho) \quad \text{in } B'_R.$$

Now there are a $K > 0$ and an $a > 0$ such that $u \leqq K\mu$ in B'_a and for $|x| = \varrho$. For any $\varepsilon > 0$ there is a unique radial function u_ε such that

$$\begin{gathered} \Delta_p u_\varepsilon + (K\mu)^q = 0 \quad \text{in } B'_\varrho, \\ \lim_{r \to 0} \mu^{-1} u = \lim_{r \to 0} \mu_r^{-1} u_r = \alpha + \varepsilon, \quad u_\varepsilon(\varrho) = k, \end{gathered} \tag{1.35}$$

with $k = \max_{|x|=\varrho} u$; it is given explicitly by

$$u_\varepsilon(r) = \begin{cases} k + (\alpha+\varepsilon)\dfrac{N-p}{p-1} C_{p,N} \displaystyle\int_r^\varrho s^{\frac{1-N}{p-1}} \left(1 + M_\varepsilon s^{N - \frac{N-p}{p-1}q}\right)^{\frac{1}{p-1}} ds & \text{if } p < N, \\ k + (\alpha+\varepsilon) C_N \displaystyle\int_r^\varrho s^{-1} \left(1 + N_\varepsilon \int_0^s t^{N-1} (\operatorname{Log}(1/t))^q \, dt\right)^{\frac{1}{N-1}} ds & \text{if } p = N, \end{cases}$$

where M_ε, N_ε have finite limits as $\varepsilon \to 0$. We get easily the estimates in B_ϱ:

$$|u_\varepsilon - (\alpha+\varepsilon)\mu| \leqq C(1 + r^N \mu^{q+1}) \quad \text{if } p < N \text{ and } q \neq (N(p-2)+p)/(N-p),$$

$$|u_\varepsilon - (\alpha+\varepsilon)\mu| \leqq C(1 + |\operatorname{Log} r|) \quad \text{if } p < N \text{ and } q = (N(p-2)+p)/(N-p),$$

$$|u_\varepsilon - (\alpha+\varepsilon)\mu| \leqq C \quad \text{if } p = N.$$

From the maximum principle we get $u \leqq u_\varepsilon$ in B'_ϱ. Letting ε go to 0, we get the estimates (1.33) with other constants C. □

We end this section with a theorem concerning the exterior problem, proved in the radial case in [12]; see also Section 4, Theorem 4.3.

Theorem 1.3. *Assume* $1 < p \leqq N$, $p - 1 < q < Q_1$, $(p - 1 < q < +\infty$ *if* $p = N)$. *Let* $\Omega_R = \{x \in \mathbb{R}^N \mid |x| > R\}$ *and let* $u \in C^0(\Omega_R)$ *with* $\nabla u \in L^p_{\text{loc}}(\Omega_R)$ *be a nonnegative solution of equation* (1.30) *in* $D'(\Omega_R)$. *Then* $u \equiv 0$.

Proof. From TOLKSDORF [21] we have $u \in C^1(\Omega_R)$, as $u^q \in L^\infty_{\text{loc}}(\Omega_R)$. From (1.30) we have $\Delta_p u \in L^2_{\text{loc}}(\Omega_R)$, $-\Delta_p u \geqq 0$ a.e. in Ω_R. Suppose that u does not vanish identically on Ω; then u is positive everywhere in Ω_R, by the strong maximum principle (see [23], Theorem 5). Let $\varrho \in \mathbb{R}$ and $n \in \mathbb{N}$ be fixed, with $R < \varrho < n$. By minimization we construct a sequence $(u_{n,k})_{k \in \mathbb{N}}$ of radial functions satisfying $u_{n,0} \equiv 0$, and for any $k \geqq 1$,

$$\begin{aligned} -\Delta_p u_{n,k} &= |u_{n,k-1}|^{q-1} u_{n,k-1} \quad \text{for } \varrho < |x| < n, \\ u_{n,k}(x) &= m \quad \text{for } |x| = \varrho, \\ u_{n,k}(x) &= 0 \quad \text{for } |x| = n, \end{aligned} \tag{1.36}$$

where $m = \min_{|x|=\varrho} u > 0$. From [23] we have $u_{n,k} > 0$ for $\varrho < |x| < n$; from the classical maximum principle we get

$$u_{n,k} \leqq u_{n,k+1} \leqq u \quad \text{for } \varrho < |x| < n. \tag{1.37}$$

Now from (1.36) and (1.37), $(r^{N-1}\,|(u_{n,k})_r|^{p-2}\,(u_{n,k})_r)_{k\in\mathbb{N}}$ is equicontinuous on $[p, n]$. Hence $u_{n,k}$ converges in $C^1([\varrho, n])$ to a radial function u_n such that

$$0 < u_{n,k} \leqq u_n \leqq u \quad \text{for } \varrho < |x| < n, \tag{1.38}$$

and

$$\begin{aligned} -\Delta_p u_n &= u_n^q \quad \text{for } \varrho < |x| < n, \\ u_n(x) &= m \quad \text{for } |x| = \varrho, \\ u_n(x) &= 0 \quad \text{for } |x| = n. \end{aligned} \tag{1.39}$$

Consider now the sequence $(u_n)_{n\in\mathbb{N}}$ and let n go to infinity. From (1.38), (1.39), by extraction of a diagonal sequence, there is a subsequence $(u_\nu)_{\nu\in\mathbb{N}}$ converging in $C^1_{\text{loc}}([\varrho, +\infty))$ to a nonnegative radial function v; then v satisfies

$$\begin{aligned} -\Delta_p v &= v^q \quad \text{for } \varrho < |x|, \\ v(x) &= m \quad \text{for } |x| = \varrho. \end{aligned} \tag{1.40}$$

From [12] or from Section 4 such a solution cannot exist; hence we get a contradiction. □

2. First Properties in the Radial Case

From now on, N, p and q are reals such that $N \geqq p$ and $q > p - 1 > 0$. We study the equation

$$r^{1-N}(r^{N-1}\,|u_r|^{p-2}\,u_r)_r + |u|^{q-1}\,u = 0, \tag{2.1}$$

for a function $r > 0 \mapsto u(r)$.

Multiplying the equation by u_r we get an energy function, used for nonnegative u in [17]:

$$E(r) = \frac{|u_r|^p}{p'} + \frac{|u|^{q+1}}{q+1}, \tag{2.2}$$

which is nonincreasing: $E_r = -(N-1)\,|u_r|^p/r$.

Now remembering the change of variables introduced in [9], [12], for $p < N$, let

$$u(r) = \theta(x), \qquad x = r^{(p-N)/(p-1)}; \tag{2.3}$$

then

$$(|\theta_x|^{p-2}\,\theta_x)_x + \left(\frac{p-1}{N-p}\right)^p x^\sigma\,|\theta|^{q-1}\,\theta = 0, \tag{2.4}$$

where

$$\sigma = -p(N-1)/(N-p). \tag{2.5}$$

This is the Emden-Fowler equation when $p = 2$ and q is rational. Multiplying by θ_x we get another energy function:

$$F_p(x) = \frac{|\theta_x|^p}{p'} + \left(\frac{p-1}{N-p}\right)^p x^\sigma \frac{|\theta|^{q+1}}{q+1}, \tag{2.6}$$

nonincreasing since

$$(F_p)_x = \left(\frac{p-1}{N-p}\right)^p \sigma x^{\sigma-1} \frac{|\theta|^{q+1}}{q+1}.$$

When $p = N$ we replace (2.3) by

$$u(r) = \theta(x), \qquad x = -\operatorname{Log} r, \tag{2.7}$$

and get

$$(|\theta_x|^{N-2}\theta_x)_x + e^{-Nx}|\theta|^{q-1}\theta = 0; \tag{2.8}$$

the energy function is now

$$F_N(x) = \frac{|\theta_x|^N}{N'} + e^{-Nx}|\theta|^{q+1}, \tag{2.9}$$

nonincreasing since

$$(F_N)_x = -Ne^{-Nx}|\theta|^{q+1}.$$

Proposition 2.1. *For any* $r_0 > 0$, $u_0, u_1 \in \mathbb{R}$ *we have local existence and uniqueness of a solution* u *of* (2.1) *such that* $u(r_0) = u_0$, $u_r(r_0) = u_1$. *Each solution has a unique extension to* $(0, +\infty)$.

Proof. We write (2.1) as a system $(u_r, v_r) = f(r, u, v)$, where

$$f(r, u, v) = \left(r^{\frac{1-N}{p-1}}|v|^{\frac{2-p}{p-1}}v, \ -r^{N-1}|u|^{q-1}u\right).$$

Because f is continuous for $r \neq 0$, we get local existence from Peano's theorem. Because the energy functions E and F_p are nonincreasing, each solution has an extension to $[r_0, +\infty)$ and $(0, r_0]$; hence to $(0, +\infty)$. Obviously we get uniqueness when $p \leqq 2$ and $q \geqq 1$. If not, we get the uniqueness of the zero solution when $u_0 = u_1 = 0$ from the behavior of E and F_p; when $u_0 = 0 \neq u_1$ or $u_1 = 0 \neq u_0$, we compare directly two solutions as in [12]; we use the local Lipschitz properties of the functions $t \mapsto |t|^{q-1}t$, $t \mapsto |t|^{(2-p)/(p-1)}t$ in $(0, +\infty)$ to get uniqueness.

In the following sections we use essentially the classical change of variables that reduces (2.1) to an autonomous equation (see [9] [12]): let

$$\delta = p/(q+1-p), \qquad u(r) = r^{-\delta}w(t), \qquad t = -\operatorname{Log} r; \tag{2.10}$$

then (2.1) takes the equivalent forms

$$(|w_t + \delta w|^{p-2}(w_t + \delta w))_t - (N - \delta q)\,|w_t + \delta w|^{p-2}(w_t + \delta w) + |w|^{q-1} w = 0, \tag{2.11}$$

$$|w_t + \delta w|^{p-2}((p-1)\,w_{tt} - (N - \delta(q + p - 1))\,w_t - \delta(N - \delta q)\,w) + |w|^{q-1} w = 0. \tag{2.12}$$

Now let

$$y(t) = -r^{(\delta+1)(p-1)}\,|u_r|^{p-2}\,u_r; \tag{2.13}$$

then we write equation (2.11) as an autonomous system in two unknowns w and y:

$$\begin{aligned} w_t &= -\delta w + |y|^{(2-p)/(p-1)}\,y, \\ y_t &= -|w|^{q-1} w + (N - \delta q)\,y. \end{aligned} \tag{2.14}$$

Now we give two energy functions for this system. We set

$$\begin{aligned} A &= N - \delta(q+1) = ((N-p)\,q - (N(p-1)+p))/(q+1-p), \\ B &= N - \delta q = ((N-p)\,q - N(p-1))/(q+1-p). \end{aligned} \tag{2.15}$$

Proposition 2.2. *For any* $(w, y) \in \mathbb{R}^2$, *let*

$$\begin{aligned} V(w, y) &= \frac{|y|^{p'}}{p'} - \delta wy - A\,\frac{\delta^{p-1}}{p}\,|w|^p + \frac{|w|^{q+1}}{q+1}, \\ W(w, y) &= \frac{|y|^{p'}}{p'} - Bwy + A\,|B|^{\frac{1}{q}-1}\,B\,\frac{q}{q+1}\,|y|^{\frac{1}{q}+1} + \frac{|w|^{q+1}}{q+1}, \end{aligned} \tag{2.16}$$

and $V(t) = V(w(t), y(t))$, $W(t) = W(w(t), y(t))$. *Then*

$$V_t(t) = AX(t), \qquad W_t(t) = AZ(t), \tag{2.17}$$

where

$$X = (|\delta w|^{p-2}\,\delta w - y)\left(\delta w - |y|^{\frac{2-p}{p-1}}\,y\right) \geqq 0,$$

$$Z = \left(w - |By|^{\frac{1}{q}-1}\,By\right)(|w|^{q-1}\,w - By) \geqq 0.$$

Proof. From [1], one can obtain a Liapunov function L of an autonomous system $x_t = f(x, y)$, $y_t = g(x, y)$, such that $f(x, y) = 0$ is equivalent to $y = h(x)$, by

$$L(x, y) = \int_{h(x)}^{y} f(x, t)\,dt - \int_0^x g(t, h(t))\,dt;$$

hence we get (2.16) and (2.17) by computation. □

Remark. The sign of X and Z is due to the monotonicity of the functions $t \mapsto |t|^{\alpha-1} t$ for $\alpha = 1/(p-1)$ and $\alpha = q$. The existence of a certain symmetry for system (2.14) relative to q and $1/(p-1)$ and the link between the two energy functions will be pointed out in Section 4.

Remark. When $p = 2$, the function V is the energy function obtained by FOWLER [9] by multiplication of equation (2.11) by w_t; and $X = w_t^2$.

Remark. The constant solutions of (2.11): $w \equiv 0$, and, when B is positive, $w \equiv \pm\lambda$ where

$$\lambda = (\delta^{p-1} B)^{1/(q+1-p)}, \tag{2.18}$$

are essential in this study; they correspond to the particular solutions of (2.1) in form of a power of $r: u \equiv 0$, $u \equiv \pm\lambda r^{-\delta}$.

3. Behavior of Radial Solutions near the Origin ($p < N$)

In this section and up to Section 6 we suppose $p < N$. The critical values Q_1, Q_2 given in (0.3) appear naturally from Proposition 2.2 as $A = 0$ when $q = Q_2$, $B = 0$ when $q = Q_1$.

Here we study the behavior of solutions u of (2.1) near $r = 0$; this means the behavior of solutions w of (2.11) near $t = +\infty$. In Theorems 3.1, 3.2, 3.3, we extend the results of [12] relative to nonnegative solutions and $q < Q_2$ and FOWLER's results [8] [9] to the general case.

Proposition 3.1. *Let (w, y) be any solution of* (2.14). *If $q \leqq Q_2$, or if w is nonnegative, then (w, y) is bounded near $+\infty$.*

Proof. The case $w \geqq 0$ is classical; see [12] [17]. Now suppose $q \leqq Q_2$. From (2.12) we have $A \leqq 0$, and hence the function V defined in Proposition 2.2 is a nonincreasing one; hence for any $t_0 \leqq t < +\infty$,

$$\frac{|y(t)|^{p'}}{p'} - \delta w(t)\, y(t) + |A| \frac{\delta^{p-1}}{p} |w(t)|^p + \frac{|w(t)|^{q+1}}{q+1} \leqq V(t_0). \tag{3.1}$$

From the Young inequality we get

$$\frac{|w(t)|^{q+1}}{q+1} - \delta^p \frac{|w(t)|^p}{p} \leqq V(t_0); \tag{3.2}$$

hence w is bounded, as $q > p - 1$; then y is bounded from (3.1) as $p > 1$. □

Proposition 3.2. *Suppose $Q \neq Q_2$. Let w be any solution of* (2.11) *bounded near $+\infty$. Then $\lim_{t \to +\infty} w_t = 0$, w has a limit ℓ at $+\infty$, and*

$$\ell(|\ell|^{q+1-p} - \delta^{p-1} B) = 0. \tag{3.3}$$

Proof. From Proposition 2.2 the functions V and W are monotone. They have (finite) limits at $+\infty$. Indeed, if not, then $|y| \underset{t\to+\infty}{\to} +\infty$, and from (2.14) $|w_t| \underset{t\to+\infty}{\to} +\infty$, which is impossible. Now we have $A \neq 0$ from (2.15); hence from (2.3) the functions X and Z are integrable on any interval $[t_0, +\infty)$. We can suppose $w \not\equiv 0$. Hence $|w(t)| + |y(t)| > 0$ for any $t \in \mathbb{R}$ from Proposition 2.1.

Suppose that $p \leqq 2$. For any $1 < \varrho \leqq 2$, there is a c_ϱ such that

$$(|a|^{\varrho-2} a - |b|^{\varrho-2} b)(a - b) \geqq c_\varrho (a-b)^2 (|a| + |b|)^{\varrho-2}, \tag{3.4}$$

for any a and $b \in \mathbb{R}$ such that $|a| + |b| > 0$. Hence for any $t \geqq t_0$,

$$X(t) \geqq c_p w_t^2(t) (\delta |w(t)| + |y(t)|^{1/(p-1)})^{(p-2)}; \tag{3.5}$$

now $(\delta |w(t)| + |y(t)|^{1/(p-1)})^{2-p}$ is bounded in $[t_0, +\infty)$, hence $w_t \in L^2((t_0, +\infty))$. Now from (2.14) w_t and y_t are bounded and

$$w_{tt} = -\delta w_t + |y|^{(2-p)/(p-1)} y_t/(p-1),$$

hence w_{tt} is bounded. Then $w_t \underset{t\to+\infty}{\to} 0$ classically [9] [11]. Now the w-limit set is connected, and from 2.16

$$\lim_{t\to+\infty} V = \lim_{t\to+\infty} (|w|^{q+1}/(q+1) - BS^{p-1}|w|^p/p);$$

hence w has a limit ℓ at $+\infty$. From (2.14) we get

$$\lim_{t\to+\infty} y = \delta^{p-1} |\ell|^{p-2} \ell, \qquad \lim_{t\to+\infty} y_t = -|\ell|^{q-1} \ell + B\,\delta^{p-1} |\ell|^{p-2} \ell,$$

and hence $y_t \underset{t\to+\infty}{\to} 0$ and ℓ satisfies (3.1).

Suppose now that $p > 2$, and hence $\dfrac{1}{q} + 1 < 2$. Then from (3.4) with $\varrho = \dfrac{1}{q} + 1$, for any $t \geqq t_0$,

$$Z(t) \geqq c_\varrho y_t^2(t) (|w(t)|^q + |By(t)|)^{\frac{1}{q}-1}. \tag{3.6}$$

As above we get $y_t \in L^2((t_0, +\infty))$ and

$$y_{tt} = -(q-1)|w|^{q-1} w_t + By_t;$$

hence y_{tt} is bounded, $y_t \underset{t\to+\infty}{\to} 0$, y has a limit, and by (2.14) so does w. We conclude as above. □

Remark. When $p > 2$, w_{tt} is not bounded. We can also prove that $w_t \underset{t\to+\infty}{\to} 0$ using the Hölder continuity of w_t (see [10]) instead of the function W.

Now we consider three cases, according to the value of q.

Theorem 3.1. *Suppose* $q > Q_2$. *Let* u *be any solution of* (2.1), *with* $u \not\equiv 0$. *Then we have three possibilities near the origin*:

(i) $u \equiv \pm\lambda r^{-\delta}$,
(ii) *u is regular at* 0 $\left(\lim_{r\to 0} u = \alpha \neq 0, \lim_{r\to 0} r^{-1} |u_r|^{p-2} u_r = -|\alpha|^{q-1}\alpha/N\right)$,
(iii) r^δ *u is not bounded and u oscillates, intersecting the curves* $r \mapsto \pm\lambda r^{-\delta}$ *infinitely many times.*

Proof. From (2.15) we have $B > 0$ and so $w \equiv \pm\lambda$ are solutions of (2.11); also $A > 0$, and so the energy functions V and W are nondecreasing.

Consider first a nonconstant solution w bounded near $t = +\infty$. Then from Proposition 3.2, we get $\lim_{t\to+\infty} w = 0$ or $\pm\lambda$. Now the characteristic equation of the linearization of system (2.14) in $(\lambda, (\delta\lambda)^{p-1})$ is

$$t^2 - At + pB/(p-1) = 0 \tag{3.7}$$

(see [12]); hence this point is completely unstable. Then, as w is not constant, we have $\lim_{t\to+\infty} w = 0$. As a consequence we get $\lim_{t\to+\infty} W = 0$, because $\lim_{t\to+\infty} w_t = 0$ from Proposition 3.2. Hence W is not positive: for any real t,

$$\frac{|y(t)|^{p'}}{p'} - Bw(t)\,y(t) + AB^{\frac{1}{q}}\frac{q}{q+1}|y(t)|^{\frac{1}{q}+1} + \frac{|w(t)|^{q+1}}{q+1} \leqq 0,$$

and so

$$AB^{\frac{1}{q}-1}\frac{q}{q+1}|y(t)|^{\frac{1}{q}+1} \leqq w(t)\,y(t). \tag{3.8}$$

If $w(t) = 0$ for some t, then $y(t) = 0$ from (3.8) or (2.16), which is impossible from Proposition 2.1. Hence w has a constant sign, for example $w > 0$. From (3.8) there is a $c > 0$ such that, for any t,

$$|w_t(t) + \delta w(t)| \leqq cw(t)^{q/(p-1)}. \tag{3.9}$$

We conclude as in [12] that $w_t + \delta w > 0$, and $e^{\delta t}w$ is bounded near infinity. As it is nondecreasing, there is an $\alpha > 0$ such that $\lim_{t\to+\infty} e^{\delta t}w = \alpha$; hence $\lim_{r\to 0} u = \alpha$. Since u_r is bounded near $r = 0$ by (2.2), we get, by integration of (2.1), $\lim_{r\to 0} r^{-1}|u_r|^{p-2}u_r = -|\alpha|^{q-1}\alpha/N$; hence u satisfies (ii).

Consider now a solution w unbounded at $+\infty$. Then from Proposition 3.1, w does not have a constant sign for large t. As $|w| + |w_t| > 0$, there is an increasing sequence $t_n \to +\infty$ such that $w(t_n) = 0$, $w_t(t_n) \neq 0$, $w > 0$ on (t_{2n}, t_{2n+1}), $w < 0$ on (t_{2n+1}, t_{2n+2}). At any extremum s of w, we have from (2.12)

$$(p-1)\,w_{tt}(s) = \delta^{2-p}w(s)\,(\lambda^{q+1-p} - |w(s)|^{q+1-p}), \tag{3.10}$$

and

$$V(s) = \frac{|w(s)|^{q+1}}{q+1} - \frac{\delta^{p-1}}{p}B\,|w(s)|^p. \tag{3.11}$$

Since w is unbounded, there is a sequence of extrema $\sigma_m \to +\infty$ such that $w(\sigma_m) \to +\infty$, and so $V(\sigma_m) \to +\infty$ from (3.11); then, because of monotonicity, $\lim_{t \to +\infty} V = +\infty$. Let s_k be an extremum of w in (t_k, t_{k+1}); then $|w(s_k)| \to +\infty$ from (3.11), and so $|w(s_k)| > \lambda$ for large k; from (3.10), w has a maximum at s_{2n} and a minimum at s_{2n+1}. Hence s_k is unique in (t_k, t_{k+1}). The curve w oscillates around the axis $w = 0$, intersecting the lines $t \mapsto \pm\lambda$ twice on each interval (t_k, t_{k+1}) for large t. The curve u oscillates around the axis $u = 0$, intersecting $r \mapsto \pm\lambda r^{-\delta}$ twice. It has a unique extremum on each arch, as y has a unique zero $\tilde{\sigma}_k$ in (t_k, t_{k+1}) (since $y(t_k) \cdot y(t_{k+1}) < 0$ and $y_t(\tilde{\sigma}_k) \cdot w(\tilde{\sigma}_k) < 0$). □

Remark. If $\lim_{t \to +\infty} w = 0$, we cannot linearize the system (2.14) when $p > 2$ or $q < 1$. That is why we take another way: we use the energy function W to find a differential inequality (3.9) in w, which gives the behavior at $+\infty$.

Theorem 3.2. *Suppose $Q_1 < q < Q_2$. Let u be any solution of* (2.1), $u \not\equiv 0$. *Then we have three possibilities near the origin*:

(i) $u \equiv \pm\lambda r^{-\delta}$,

(ii) *u is regular at* 0 $\left(\lim_{r \to 0} u = \alpha \neq 0,\ \lim_{r \to 0} r^{-1} |u_r|^{p-2} u_r = -|\alpha|^{q-1} \alpha/N\right)$,

(iii) $r^\delta u \underset{r \to 0}{\to} \pm\lambda$ (*with $r^\delta u$ nonconstant*).

Proof. From (2.15) we have $B > 0$, and so $w \equiv \pm\lambda$ are constant solutions: also $A < 0$, and so V and W are nonincreasing functions. From Proposition 3.1 any solution w is bounded near $+\infty$. From Proposition 3.2 we have $\lim_{t \to +\infty} w = 0$ or $\pm\lambda$. Now from (3.7) the point $(\lambda, (\delta\lambda)^{p-1})$ is asymptotically stable; thus the case $\lim_{t \to +\infty} w = \pm\lambda$ is possible for nonconstant w.

If $\lim_{t \to +\infty} w = 0$, then $\lim_{t \to +\infty} W = 0$, and hence W is not negative: for any real t,

$$\frac{|y(t)|^{p'}}{p'} - Bw(t)\,y(t) - |A|\, B^{\frac{1}{q}} \frac{q}{q+1} |y(t)|^{\frac{1}{q}+1} + \frac{|w(t)|^{q+1}}{q+1} \geqq 0.$$

Now, from the Young inequality,

$$\left(\tfrac{1}{2} B^{\frac{1}{q}} |A| - \frac{q+1}{qp'} |y(t)|^{p'-1-\frac{1}{q}}\right) |y(t)|^{\frac{1}{q}+1} \leqq \left(1 + \frac{2B}{|A|}\right)^q \frac{|w(t)|^{q+1}}{q};$$

as $p' > 1 + 1/q$ and $\lim_{t \to +\infty} y(t) = 0$, there is a $c \geqq 0$, such that, for large t, w satisfies an inequality of type (3.9). Hence we conclude as in Theorem 3.1 that $\lim_{t \to +\infty} e^{\delta t} w = \alpha \neq 0$ and u is regular. □

Remark. This proof rests on the inequality $B > 0$, that is $q > Q_1$. If $B < 0$, we have necessarily $\lim_{t \to +\infty} w(t) = 0$ from Proposition 3.2, but we cannot use the functions V and W to conclude the proof. The linearization is not avalaible in the general case, and so we use an energy method in the change of variables (2.3).

Theorem 3.3. *Suppose* $q < Q_1$. *Let* u *be any solution of* (2.1), $u \not\equiv 0$. *Then we have two possibilities near the origin*:

(i) $\lim_{r \to 0} r^{(N-p)/(p-1)} u = \gamma \neq 0$,

(ii) u *is regular at* 0.

Proof. We study the behavior of the function θ defined in (2.3) near $+\infty$. Since the energy function F_p is a nonincreasing and nonnegative function, it is bounded near $+\infty$. Hence θ_x and $x^\sigma |\theta|^{q+1}$ are bounded from (2.6). Now integrating (2.4) we get for any $x \geqq 1$,

$$|\theta_x|^{p-2}\,\theta_x + \left(\frac{p-1}{N-p}\right)^p \int_1^x t^\sigma\,|\theta|^{q-1}\,\theta\,dt = |\theta_x(1)|^{p-2}\,\theta_x(1).$$

Now $x^\sigma |\theta|^q = O(x^{\sigma/(q+1)})$ and $\sigma + q + 1 = q - Q_1 < 0$, and so the integral is convergent and θ_x has a limit γ at $+\infty$. If $\gamma \neq 0$, then $\lim_{x \to +\infty} x^{-1}\,\theta = \gamma$, and u satisfies the condition (i) of the theorem.

If $\gamma = 0$, then by integration of (2.3), for any $x > 0$,

$$|\theta_x|^{p-2}\,\theta_x = \left(\frac{p-1}{N-p}\right)^p \int_x^{+\infty} t^\sigma\,|\theta|^{q-1}\,\theta\,dt; \tag{3.12}$$

hence $\theta_x = O(x^{(\sigma+q+1)/(p-1)(q+1)})$. Suppose that θ_x is not integrable. Then for any $k > 0$ such that $\theta_x = O(x^{-k})$ we have $k \leqq 1$; if $k = 1$, then for any $\varepsilon > 0$, $\theta = O(x^\varepsilon)$ at infinity. Hence from (3.12), taking $\varepsilon < Q_1/q$, we get $\theta_x = O(x^{(q\varepsilon - Q_1)/(p-1)})$ and get a contradiction with $\varepsilon < pQ_1/Nq$. Hence $k < 1$ and from (3.12) we have $\theta_x = O(x^{-(qk-(\sigma+q+1))/(p-1)})$. Let $k_0 = |\sigma + q + 1|/(p-1)(q+1)$ and $k_{n+1} = (qk_n - (\sigma + q + 1))/(p-1)$ for any $n \in \mathbb{N}$. Then $k_n < 1$, $k_n \geqq \left(\frac{q}{p-1}\right)^n k_0$; this is impossible because $q > p - 1$. Hence θ_x is integrable, θ has a limit at $+\infty$ and $x^{N/(N-p)}\theta_x$ has a limit from (3.12). Then u has a limit α and $r^{-1/(p-1)}u_r$ has a limit at the origin; we have $\alpha \neq 0$ from (2.3) as $u \not\equiv 0$, and hence u is regular. □

We end this section by the study of oscillating solutions in the case $q > Q_2$. The technique of the proof is essentially due to McLEOD, NI & SERRIN [16].

Theorem 3.4. *Let* $q > Q_2$ and *u be an oscillating solution near the origin. Let* $(r_n)_{n\in\mathbb{N}}$ *be the nonincreasing sequence of zeros of* u *in* (0.1] *and* $(\varrho_n)_{n\in\mathbb{N}}$ *be the sequence of its extrema* $(r_{n+1} < \varrho_n < r_n)$. *Then there are constants* $c_1, c_2, c_3 > 0$ *such that*

$$\lim_{n \to +\infty} \varrho_n^\alpha u(\varrho_n) = c_1, \qquad \lim_{n \to +\infty} r_n^{1+\alpha-\beta} u_r(r_n) = c_2, \tag{3.13}$$

$$\lim_{n \to +\infty} (r_{n+1}^\beta - r_n^\beta) = c_3, \tag{3.14}$$

where

$$\alpha = \frac{p(N-1)}{(p-1)\,q + 2p - 1}, \qquad \beta = \frac{(N-p)\,(Q_2 - q)}{(p-1)\,q + 2p - 1} = 1 - \alpha/\delta. \tag{3.15}$$

Proof. We make the change of variables, as $q \neq Q_2$,

$$u(r) = r^{-\alpha}v(\tau), \qquad \tau = r^{\beta}, \tag{3.16}$$

where α and β are defined in (3.15). We get a nonautonomous system of two equations to study near $\tau = +\infty$:

$$\begin{aligned} \beta v_\tau &= \alpha \frac{v}{\tau} + |z|^{(2-p)/(p-1)} z, \\ \beta z_\tau &= -|v|^{q-1} v - \alpha \frac{z}{\tau}. \end{aligned} \tag{3.17}$$

We consider a kind of energy function: for $\tau \geqq 1$, let

$$H(\tau) = \frac{|z|^{p'}}{p'} + \alpha \frac{vz}{\tau} + \frac{|v|^{q+1}}{q+1}; \tag{3.18}$$

then

$$\tau^2 H_\tau = -\alpha v z. \tag{3.19}$$

Hence, from the Young inequality,

$$\tau^2 |H_\tau| \leqq \frac{|z|^{p'}}{2p'} + 2^{p-1} |\alpha v|^p \leqq H + |2\alpha v|^p - \frac{|v|^{q+1}}{q+1}; \tag{3.20}$$

since $q > p - 1$, we get

$$\tau^2 |H_\tau| \leqq H + K, \tag{3.21}$$

where $K = (2\alpha)^{p\delta} p^{\delta}$. Then $e^{1/\tau}(H + K)$ is a nonincreasing and nonnegative function and it has a limit at infinity; hence H has a limit M at infinity. Now v and z are bounded from (3.18) (3.20), and hence

$$\lim_{s \to +\infty} \left(\frac{|z|^{p'}}{p'} + \frac{|v|^{q+1}}{q+1} \right) = M. \tag{3.22}$$

From Theorem 3.1, $r^{\delta}u$ is not bounded near 0; hence by (3.15) (3.16) $\tau^{\delta}v$ is not bounded at $+\infty$.

Suppose that $M = 0$. Then $\lim_{\tau \to +\infty} z = \lim_{\tau \to +\infty} v = 0$, and there is a $\tau_0 > 0$ such that $|vz| \leqq 1$ for $\tau \geqq \tau_0$. Then $\tau^2 |H_\tau| \leqq \alpha$, and so $\tau |H| \leqq \alpha$, and from (3.18),

$$|z|^{p'} + |v|^{q+1} \leqq 2m\alpha/\tau, \qquad m = q + 1 + p';$$

hence, from (3.19),

$$|H_\tau| \leqq \alpha(2m\alpha)^l/\tau^{l+2}, \qquad l = \frac{1}{q+1} + \frac{1}{p'},$$

and

$$|H| \leqq \alpha(2m\alpha)^l/\tau^{l+1}.$$

By induction we deduce easily the estimates for any integer k

$$|z|^{p'} + |v|^{q+1} \leqq (2m\alpha/\tau)^{1+l+\dots+l^k},$$
$$|H| \leqq \alpha(2m\alpha)^{l+\dots+l^k}/\tau^{1+l+\dots+l^k}.$$

Now we go to the limit as $k \to +\infty$ for any fixed $\tau \geqq \tau_0$; hence

$$|z|^{p'} + |v|^{q+1} \leqq (2m\alpha/\tau)^{\delta(q+1)}$$

since

$$(1-l)^{-1} = \delta(q+1).$$

This is impossible because $\tau^\delta v$ is not bounded. Hence $M > 0$. Returning to u in (3.22) we get from (3.16), (3.17),

$$\lim_{r\to 0}\left(\frac{|r^{1+\alpha-\beta}u_r|^p}{p'} + \frac{|r^\alpha u|^{q+1}}{q+1}\right) = M > 0,$$

and hence (3.13). At last, from (3.17) (3.22),

$$\lim_{\tau\to+\infty}\left(\frac{|\beta v_\tau|^p}{p'} + \frac{|v|^{q+1}}{q+1}\right) = M.$$

Let $\tau_n = r_n^\beta$; we deduce that

$$\tau_{n+1} - \tau_n \to c = 2\beta \int_0^{(M(q+1))^{1/(q+1)}} \frac{dv}{(p'(M-|v|^{q+1}/(q+1)))^{1/p}},$$

and hence (3.14). □

4. Behavior of the Solutions near Infinity ($p < N$)

We study the behavior of solutions u of (2.1) near $r = +\infty$, which means the behavior of solutions w of (2.11) near $t = -\infty$. We could have repeated the study of Section 3 by use of the energy functions V and W, but we prefer to introduce a new change of variables; it reduces the analysis to the former one when $Q > Q_1$; it also offers an interest in itself and shows the symmetry between w and y, V and W, q and $1/(p-1)$.

Proposition 4.1. *Let $q \neq Q_1$. Let u be any C^1 function on $(0, +\infty)$, and*

$$\bar{u}(s) = -r^{N-1}|u_r|^{p-2}u_r, \quad s = \frac{r^{-\nu}}{|\nu|}, \tag{4.1}$$

where

$$\nu = (N-p)(q-Q_1)/(q+1)(p-1); \tag{4.2}$$

then equation (2.1) *reduces to*

$$s^{1-\bar{N}}(s^{\bar{N}-1}|\bar{u}_s|^{\bar{p}-2}\bar{u}_s)_s + |\bar{u}|^{q-1}\bar{u} = 0, \tag{4.3}$$

where

$$\bar{p} = 1 + 1/q, \quad \bar{q} = 1/(p-1), \quad \bar{N} = \bar{p} + N/\nu q, \tag{4.4}$$

and

$$u(r) = -|\nu|^{\bar{N}-2} \nu s^{\bar{N}-1} |\bar{u}_s|^{\bar{p}-2} \bar{u}_s. \tag{4.5}$$

Moreover, let $\bar{\delta}, \bar{w}, \bar{y}, \bar{A}, \bar{B}$ *and the energy functions* $\bar{V}$ *and* $\bar{W}$ *associated to* $\bar{N}, \bar{p}, \bar{q}, \bar{u}$, *as* δ, w, y, A, B, V, W *are associated to* N, p, q, u:

$$\bar{\delta} = \bar{p}/(\bar{q} + 1 - \bar{p}), \quad \bar{u}(s) = s^{-\bar{\delta}}\bar{w}(\tau), \quad \tau = -\mathrm{Log}\, s \tag{4.6}$$

$$\bar{y}(\tau) = -s^{(\bar{\delta}+1)/(\bar{p}-1)} |\bar{u}_s|^{\bar{p}-2} \bar{u}_s, \tag{4.7}$$

$$\bar{A} = \bar{N} - \bar{\delta}(\bar{q} + 1), \quad \bar{B} = \bar{N} - \bar{\delta}\bar{q} \tag{4.8}$$

$$\bar{V}(\tau) = \frac{|\bar{y}|^{\bar{p}'}}{\bar{p}'} - \bar{\delta}\bar{w}\bar{y} - \frac{\bar{A}}{\bar{p}} \bar{\delta}^{\bar{p}-1} |\bar{w}|^{\bar{p}} + \frac{|\bar{w}|^{\bar{q}+1}}{\bar{q}+1} \tag{4.9}$$

$$\bar{W}(\tau) = \frac{|\bar{y}|^{\bar{p}'}}{\bar{p}'} - \bar{B}\bar{w}\bar{y} + \bar{A}\,|\bar{B}|^{\frac{1}{\bar{q}}-1} \bar{B} \frac{\bar{q}}{\bar{q}+1} |\bar{y}|^{\frac{1}{\bar{q}}+1} + \frac{|\bar{w}|^{\bar{q}+1}}{\bar{q}+1}. \tag{4.10}$$

Then we have the relations

$$\bar{w}(\tau) = |\nu|^{-\bar{\delta}} y(t), \quad \bar{y}(\tau) = |\nu|^{-(\delta+1)} \nu w(t), \tag{4.11}$$

$$\bar{V}(\tau) = |\nu|^{-\delta(q+1)} W(t), \quad \bar{W}(\tau) = |\nu|^{-\delta(q+1)} V(t). \tag{4.12}$$

Proof. We get the proposition by computation, using the relations

$$\bar{N} - 1 = (N + \nu)/\nu q, \quad \nu + (p - N)/(p-1) = -(N+\nu)/q,$$

$$\bar{\delta} = (N - \delta q)/\nu = \delta q - 1 = (q+1)\,\delta/p' = (q+1)(p-1)/(q-p+1), \tag{4.13}$$

$$\bar{A} = -A/\nu, \quad \bar{B} = \delta/\nu. \quad \square$$

Remark. Equation (4.3) has the same form as (2.1). Even if N is an integer, $\bar{N}$ need not be. Because $N > p$ we get $\bar{N} > \bar{p}$ when $\nu > 0$, that is $q > Q_1$; then s goes to 0 as r goes to $+\infty$. Hence we get the following results.

Theorem 4.1. *Suppose* $q > Q_2$. *Let* u *be any solution of* (2.1), $u \not\equiv 0$. *Then we have three possibilities near* $+\infty$:

(i) $u \equiv \pm\lambda r^{-\delta}$,

(ii) *u is regular at $+\infty$, that is*

$$\lim_{r\to+\infty} r^{(N-p)/(p-1)} u = c \neq 0, \qquad \lim_{r\to+\infty} r^{(N-1)/(p-1)} u_r = \frac{p-N}{p-1} c,$$

(iii) $r^{\delta} u \underset{r\to+\infty}{\to} \pm\lambda$ (*with* $r^{\delta}u$ *not constant*).

Proof. From Proposition 4.1, our analysis is reduced to studying $\bar{u}$ near $s = 0$. Let $\bar{Q}_1 = \bar{N}(\bar{p} - 1)/(\bar{N} - \bar{p})$, $\bar{Q}_2 = (\bar{N}(\bar{p} - 1) + \bar{p})/(\bar{N} - \bar{p})$. From (4.4) we get

$$\bar{N} = (N - p)(q + 1)/((N - p)q - N(p - 1)) = (N - p)/(p - 1)\nu,$$

$$\bar{Q}_1 = 1/Q_1 = (N - p)/N(p - 1), \tag{4.14}$$

$$\bar{Q}_2 = ((N - p)(q + 1) - N(p - 1))/N(p - 1);$$

hence $\bar{q} = 1/(p - 1) > \bar{Q}_1$, and $q > Q_2$ is equivalent to $\bar{q} < \bar{Q}_2$. Hence from Theorem 3.2 we conclude that either $\bar{u} \equiv \pm\bar{\lambda}s^{-\bar{\delta}}$, where $\bar{\lambda} = (\bar{\delta}^{p-1}(\bar{N} - \bar{\delta}\bar{q}))^{1/(\bar{q}+1-\bar{p})}$, and hence (i) from (4.1) or (4.5), by use of the identity $\nu^{\bar{\delta}}\bar{\lambda} = (\lambda\delta)^{p-1}$; either $\lim_{s\to+\infty} \bar{u}s^{\bar{\delta}} = \pm\bar{\lambda}$ and $\bar{u}s^{\bar{\delta}}$ is not constant, and (iii) follows; or $\lim_{s\to 0} \bar{u} = \alpha$, $\lim_{s\to 0} s^{-1}|\bar{u}_s|^{\bar{p}-2}\bar{u}_s = -|\alpha|^{\bar{q}-1}\alpha/\bar{N}$, and hence (ii) follows by use of (4.14). □

Theorem 4.2. *Suppose* $Q_1 < q < Q_2$. *Let* u *be any solution of* (2.1), $u \not\equiv 0$. *Then we have three possibilities near* $+\infty$:

(i) $u \equiv \pm\lambda r^{-\delta}$,
(ii) *u is regular at* $+\infty$,
(iii) *$r^\delta u$ is not bounded and u oscillates, intersecting the curves* $r \mapsto \pm\lambda r^{-\delta}$ *infinitely many times.*

Proof. Now we have $\bar{q} > \bar{Q}_2$; applying Theorem 3.1 to $\bar{u}$ we get the conclusions by returning to u. □

Theorem 4.3. *Suppose* $q \leqq Q_1$. *Let* u *be any solution of* (2.1), $u \not\equiv 0$. *Then, when* $r \to +\infty$, *$r^\delta u$ is not bounded and u oscillates, intersecting the curves* $r \mapsto \pm r^{-\delta}$ *infinitely many times.*

Proof. Here we use the change of variables (2.10) and consider w near $t = -\infty$. Now V and W are nonincreasing functions. Suppose that w is bounded at $-\infty$. If $V \underset{t\to-\infty}{\to} +\infty$ or $W \underset{t\to-\infty}{\to} +\infty$, then $|y| \underset{t\to-\infty}{\to} +\infty$ from the Young inequality in (2.16); hence from (2.14) $|w_t| \underset{t\to-\infty}{\to} +\infty$, which is impossible. Hence V and W have (finite) limits at $-\infty$. As in Proposition 3.2 we conclude that $\lim_{t\to-\infty} w_t = 0$, $\ell = \lim_{t\to-\infty} w$ exists and $\ell(|\ell|^{q+1-p} - \delta^{p-1}B) = 0$, and hence $\ell = 0$ because $B \leqq 0$. Then $\lim_{t\to-\infty} V = \lim_{t\to-\infty} W = 0$, and hence V and W are not positive. From the Young inequality we get

$$0 \geqq \frac{|y|^{p'}}{p'} - \delta wy - A\,\delta^{p-1}|w|^p + \frac{|w|^{q+1}}{q+1}$$

$$\geqq -B\,\delta^{p-1}\frac{|w|^p}{p} + \frac{|w|^{q+1}}{q+1},$$

which is impossible as $B \leqq 0$ and $w \not\equiv 0$. Hence w is not bounded near $-\infty$. As in Theorem 3.1 we prove that w oscillates around the axis $w = 0$, intersecting the lines $t \mapsto \pm\lambda$ infinitely many times, and hence the conclusion follows. $\square$

As in the preceding section we can specify the oscillating solutions.

Theorem 4.4. *Let* $q < Q_2$ *and* u *be an oscillating solution near* $+\infty$. *Let* $(r_n)_{n\in\mathbb{N}}$ *be the nondecreasing sequence of zeros of* u *in* $[1, +\infty)$, *and let* $(\varrho_n)_{n\in\mathbb{N}}$ *be the sequence of its extrema* $(r_n < \varrho_n < r_{n+1})$. *Then the estimates* (3.13) *and* (3.14) *hold.*

Proof. In the change of variables (3.16) we have now $\beta > 0$, and hence $\tau \to +\infty$ as $r \to +\infty$, so that the proof of Theorem 3.4 holds. $\square$

5. Global Behavior ($p < N$)

Here we show all the possible connexions from $r = 0$ to $r = +\infty$ and prove their existence.

Theorem 5.1. *Let* $q > Q_2$. *Then, to within exchange of* u *and* $-u$, *each solution* $u \not\equiv 0$ *of* (2.1) *has one of the following forms*:

(i) $u \equiv \lambda r^{-\delta}$,

(ii) *u is regular at* 0 $\left(\text{with } \lim_{r\to 0} u = \alpha > 0\right)$, *$u$ is positive on* $(0, +\infty)$ *and* $\lim_{r\to+\infty} r^{\delta}u = \lambda$,

(iii) *u oscillates near* 0 *and* $\lim_{r\to+\infty} r^{\delta}u = \lambda$,

(iv) *u oscillates near* 0 *and is regular at* $+\infty$ $\left(\text{with } \lim_{r\to+\infty} r^{(N-p)/(p-1)}u = c > 0\right)$.

All of these solutions exist (for each α *and* $c > 0$).

Proof. The function $t \mapsto t^q$ is locally Lipschitz-continuous on $(0, +\infty)$. From [12] we conclude for each $\alpha > 0$ local existence near the origin of a regular solution u such that $\lim_{r\to 0} u = \alpha$. Using the same result for the function $\bar{u}$ defined in Proposition 4.1, we get for each $c > 0$ local existence near $+\infty$ of a solution u such that $\lim_{r\to+\infty} r^{(N-p)/(p-1)}u = c$. Moreover we have local existence near $+\infty$ of a u such that $\lim_{r\to+\infty} r^{\delta}u = \lambda$; indeed from (3.7) the point $(\lambda, (\delta\lambda)^{p-1})$ is asymptotically stable at $-\infty$.

Consider first a solution u with $\lim_{r\to 0} u = \alpha > 0$; then $\lim_{t\to+\infty} w = \lim_{t\to+\infty} w_t = 0$ and hence $\lim_{t\to+\infty} W = 0$. Now W is nondecreasing, hence $W \leqq 0$, and $w > 0$ on $\mathbb{R}$ (see the proof of Theorem 3.1). From Theorem 4.1, either $\lim_{r\to+\infty} r^{\delta}u = \lambda$ and hence u satisfies (ii), or $\lim_{r\to+\infty} r^{(N-p)/(p-1)}u = c$ and hence $\lim_{t\to-\infty} we^{-Bt/(p-1)} = c$; then $\lim_{t\to-\infty} w = 0$, $\lim_{t\to-\infty} y = 0$, from (2.16) and hence $\lim_{t\to-\infty} W = 0$; then W and w are identically zero from (2.16) and (2.17); this is a contradiction.

Consider now a solution $u = u_c$ such that $\lim_{r\to+\infty} r^{(N-p)/(p-1)} u = c > 0$; then from Theorem 3.1, by contradiction, u oscillates near 0, and hence u satisfies (iv). Moreover from uniqueness $w = w_c$ is characterized by $\lim_{t\to-\infty} we^{-Bt/(p-1)} = c$. Now for any solution w of (2.11), $t \mapsto w(t+K)$ is still a solution of (2.11) and hence

$$w_c(t) = w_1\left(t + \frac{p-1}{B}\operatorname{Log} c\right). \tag{5.1}$$

We now prove the existence of solutions u satisfying (iii): as w_1 oscillates near $+\infty$ and is positive near $-\infty$, let $(t_n)_{n\in\mathbf{N}}$ be the nondecreasing sequence of zeros of w_1 $(t_n \to +\infty)$, and $a_n = |w_{1t}(t_n)|$. Since V is a nondecreasing function, $(a_n)_{n\in\mathbf{N}}$ is a nondecreasing sequence and $a_0 > 0$ from uniqueness. From Proposition 2.1, there is a solution w of (2.11) such that $w(0) = 0$, $w_t(0) = a/2$. Then by construction there is no $c \neq 0$ such that $\lim_{t\to-\infty} we^{-Bt/(p-1)} = c$, as w cannot be a translate of w_1. Hence from Theorem 4.1 the corresponding solution u satisfies necessarily $r^\delta u \underset{r\to+\infty}{\to} \lambda$. □

Remark. We find again in Theorem 5.1 (ii) the existence of positive solutions $u \in C^1([0, +\infty))$ called "ground states", proved by FOWLER [9] when $p = 2$ and by NI & SERRIN [17] in the general case by a shooting method.

Remark. The functions w_c defined by (5.1) associated to the solutions u_c regular at $+\infty$ $\left(\lim_{r\to+\infty} r^{(N-p)/(p-1)} u_c = c\right)$ are the Emden functions relative to the case $q > Q_2$ introduced by FOWLER [9]. We have from (5.1) the relation

$$u_c(r) = c^{-(p-1)\delta/B} u_1(c^{-(p-1)/B} r). \tag{5.2}$$

Theorem 5.2. *Let $Q_1 < q < Q_2$. Then, to within exchange of u and $-u$, each solution $u \not\equiv 0$ of* (2.1) *has one of the following forms*:

(i) $u \equiv \lambda r^{-\delta}$,

(ii) *u is regular at* 0 $\left(\textit{with } \lim_{r\to 0} u = \alpha > 0\right)$ *and oscillates near* $+\infty$,

(iii) $\lim_{r\to 0} r^\delta u = \lambda$ *and u oscillates near* $+\infty$,

(iv) $\lim_{r\to 0} r^\delta u = \lambda$, *$u$ is positive in* $(0, +\infty)$ *and regular at* $+\infty$ $\left(\textit{with } \lim_{r\to+\infty} r^{(N-p)/(p-1)} u = c > 0\right)$.

All of these solutions exist (*for each α and* $c > 0$).

Proof. By the change of variable (4.1) we are reduced to study global solutions $\bar{u}$ with $\bar{q} > \bar{Q}_2$. Hence from Theorem 5.1, either $\bar{u} \equiv \bar{\lambda} s^{-\bar{\delta}}$ and hence $u \equiv \lambda r^{-\delta}$; either $\bar{u}$ is regular at $s = 0$, positive, and $\lim_{s\to+\infty} \bar{s}^{\bar{\delta}} \bar{u} = \lambda$ and hence u satisfies (iv). Either $\bar{u}$ oscillates near $s = 0$ and $\lim_{s\to+\infty} s^{\bar{\delta}} \bar{u} = \bar{\lambda}$ and hence u satisfies (iii), or $\bar{u}$ oscillates near $s = 0$ and is regular at $+\infty$, and so u satisfies (ii). All the solutions exist from Theorem 5.1. □

Remark. We find again in Theorem 5.2 the nonexistence of ground states proved in [12], [17], since any solution u regular at 0 necessarily oscillates at $+\infty$.

Remark. The solutions u^α regular at 0 $\left(\lim_{r\to 0} u^\alpha = \alpha\right)$ are associated to the functions w^α characterized by $\lim_{t\to+\infty} w^\alpha e^{\delta t} = \alpha$, oscillating near $-\infty$. They are the Emden functions relative to the case $Q_1 < q < Q_2$. We have the relations

$$w^\alpha(t) = w^1(t - \delta^{-1} \operatorname{Log} \alpha), \tag{5.3}$$

$$u^\alpha(r) = \alpha u^1(\alpha^{1/\delta} r). \tag{5.4}$$

Theorem 5.3. *Let* $q < Q_1$. *Then, to within exchange of* u *and* $-u$, *each solution* $u \not\equiv 0$ *of* (2.1) *has one of the following forms*:

(i) u *is regular at* 0 $\left(\lim_{r\to 0} u = \alpha > 0\right)$ *and oscillates near* $+\infty$,

(ii) $\lim_{r\to 0} r^{(N-p)/(p-1)} u = \gamma > 0$, *and* u *oscillates near* $+\infty$.

All of these solutions exist (*for each* $\alpha > 0$ *and* $\gamma > 0$).

Proof. Theorems 3.3 and 4.3 state that all the possible solutions have either the form (i) or the form (ii). Local existence of the two solutions near the origin is proved in [12]; hence the global existence from Proposition 2.1. □

6. The Critical Cases ($q = Q_1$, $q = Q_2$)

In the case $q = Q_2$ we can describe the global behavior of the solutions directly, as the energy functions V and W are constant.

Theorem 6.1. *Suppose* $q = Q_2$. *Then, to within exchange of* u *and* $-u$, *each solution* $u \not\equiv 0$ *of* (2.1) *has one of the following forms*:

(i) $u \equiv \lambda r^{-\delta} \equiv \lambda r^{(p-N)/p}$,

(ii) u *is positive in* $(0, +\infty)$ *regular at* 0 *and* $+\infty$, *and is given by*

$$u = c\left(r^{p/(p-1)} + \frac{1}{N}\left(\frac{p-1}{N-p}\right)^{p-1} c^{p^2/(N-p)}\right)^{(p-N)/p}, \quad c > 0, \tag{6.1}$$

(iii) u *is positive in* $(0, +\infty)$, *intersects the curve* $r \mapsto \lambda r^{-\delta}$ *infinitely many times and oscillates between two curves* $r \mapsto ar^{-\delta}$, $r \mapsto br^{-\delta}$ *where* a *and* b *satisfy* $0 < a < \lambda < b$.

(iv) u *oscillates in* $(0, +\infty)$, *intersects the curves* $r \mapsto \pm\lambda r^{-\delta}$ *infinitely many times and oscillates between two curves* $r \mapsto \pm br^{-\delta}$ *where* b *satisfies* $0 < \lambda < b$.

All of these solutions exist.

Proof. The existence of particular solutions of the form (6.1) was shown by Fowler [9] for $p = 2$, for general p by Guedda & Veron [12].

Let $u \not\equiv 0$ be any solution of (2.1) and w, y be defined by (2.10) and (2.13). From (2.15) and (2.18) we have $A = 0$, $B = \delta = (N-p)/p$, $\lambda = \delta^\delta$. In particular $V = W$ and V is constant from (2.16): there is a $K \in \mathbb{R}$ such that

$$\frac{|y(t)|^{p'}}{p'} - \delta(wy)(t) + \frac{|w(t)|^{q+1}}{q+1} = K, \quad \forall\, t \in \mathbb{R}. \tag{6.2}$$

From the Young inequality we get

$$\frac{|w(t)|^{q+1}}{q+1} - \frac{|\delta w(t)|^p}{p} \leqq K, \quad \forall\, t \in \mathbb{R}. \tag{6.3}$$

Hence (w, y) is bounded on $\mathbb{R}$, and thence $r^\delta u$ is bounded in $(0, +\infty)$. For any $K \in \mathbb{R}$, let

$$\phi_K(x) = K + \frac{|\delta x|^p}{p} - \frac{|x|^{q+1}}{q+1} \quad \text{for any } x \in \mathbb{R}. \tag{6.4}$$

Then, on $(0, \lambda)$, ϕ_K increases from K to $\phi_K(\lambda) = K + \left(\frac{N-p}{p}\right)^{N-1} \Big/ (q+1)$; on $(\lambda, +\infty)$, ϕ_K decreases from $\phi_K(\lambda)$ to $-\infty$. Since $\phi_K(w(t)) \geqq 0$ on $\mathbb{R}$, we have necessarily $\phi_K(\lambda) \geqq 0$; hence

$$K \geqq K_m = -\left(\frac{N-p}{p}\right)^{N-1} \Big/ (q+1). \tag{6.5}$$

Now if w has a limit ℓ (at $\pm\infty$) then

$$\frac{|y(t)|^{p'}}{p'} - \delta\ell\, y(t) + \frac{|\ell|^{q+1}}{q+1} \to K$$

because y is bounded; since the ω-limit and α-limit sets are connected, y has a limit, and so does w_t. Then $w_t \to 0$ and hence $\ell = 0$ or $\ell = \pm\lambda$ from (2.12). If $|\ell| = \lambda$, then from (6.2) $\phi_K(\lambda) = 0$ and hence $K = K_m$. If $\ell = 0$ then $K = 0$. Now let us discuss the value of K:

If $K = K_m$, then necessarily $|w(t)| = \lambda$ for any t. Thus $w \equiv \pm\lambda$ and u (or $-u$) has the form (i).

If $K = 0$, ϕ_0 has three zeros: $0, \pm b_0$ with

$$b_0 = (N/(N-p))^{(N-p)/p^2}\, \lambda = ((q+1)/p)^{1/(q+1-p)}\, \lambda, \tag{6.6}$$

$$-b_0 < -\lambda < 0 < \lambda < b_0. \tag{6.7}$$

Now w has a constant sign: if there is a $t_0 \in \mathbb{R}$ such that $w(t_0) = 0$, then from (6.2) $y(t_0) = 0$ and hence $w_t(t_0) = 0$; this is impossible as $w \not\equiv 0$ from Proposition 2.1. Hence w is strictly monotone near $\pm\infty$, for $w(t) = b_0$ at any extremal point. Then from above w decreases to 0 as $t \to +\infty$. From [12] we deduce that $e^{\delta t} w$ is bounded at $+\infty$, while u is regular at 0. By uniqueness, proved in [12], u has the form (ii).

If $K_m < K < 0$, then ϕ_K has four zeros $\pm a_K$, $\pm b_K$ on $\mathbb{R}$, with

$$-b_0 < -b_K < -\lambda < -a_K < 0 < a_K < \lambda < b_K < b_0, \tag{6.8}$$

as $\phi_K(b_0) = K$; then from (6.3) w has a constant sign and w (or $-w$) satisfies $a_K \leqq w(t) \leqq b_K$ for any real t. Now w is not monotone for large $|t|$, as it has no limit. At any extremum t of w we get $\phi_K(t) = 0$ from (6.2) and hence $w(t) = a_K$ or b_K, and $w_{tt}(t) \neq 0$ from (3.10). Hence there is an increasing family $(t_n)_{n\in\mathbb{Z}}$ of extrema of w with $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to-\infty} t_n = -\infty$; w oscillates on $\mathbb{R}$ and $r^\delta u$ oscillates on $(0, +\infty)$ between a_K and b_K, and hence u satisfies (iii).

If $K > 0$, ϕ_K has two zeros $\pm b_K$, with

$$-b_K < -b_0 < -\lambda < 0 < \lambda < b_0 < b_K, \tag{6.9}$$

and w is not monotone for large $|t|$. Hence there is an increasing family $(t_n)_{n\in\mathbb{Z}}$ with $t_n \xrightarrow[n\to-\infty]{} -\infty$, $t_n \xrightarrow[n\to+\infty]{} +\infty$, of extrema of w. As above we get $w(t_n) = \pm b_K$. Then w oscillates on $\mathbb{R}$ between $-b_K$ and b_K, $r^\delta u$ oscillates on $(0, +\infty)$ between $-b_K$ and b_K, hence u satisfies (iv).

Solutions of the forms (iii) and (iv) do exist. More specifically, let any $a \in (0, \lambda)$ and $t_0 \in \mathbb{R}$; from Proposition 2.1 there is a solution w of (2.11) such that $w(t_0) = a$, $w_t(t_0) = 0$; the corresponding solution u has the form (iii). Let any $b \in (\lambda, +\infty)$ and $t_0 \in \mathbb{R}$; there is a solution w of (2.9) such that $w(t_0) = b$, $w_t(t_0) = 0$. If $b = b_0$, u has the form (ii); if $b < b_0$, u has the form (iii); if $b > b_0$, u has the form (iv). ☐

Remark. The functions u_c defined in (6.1) are regular at $+\infty$ $\left(\lim_{r\to+\infty} r^{(N-p)/(p-1)} u_c = c\right)$ and at 0 $\left(\lim_{r\to0} u = \alpha = \left(\frac{1}{N}\left(\frac{p-1}{N-p}\right)^{p-1}\right)^{\frac{p-N}{p}} c^{1-p}\right)$.

The functions w_c associated to u_c are the Emden functions relative to $q = Q_2$. They have a unique extremum equal to b_0. We have the relations

$$w_c(t) = w_1\left(t + \frac{p(p-1)}{N-p} \operatorname{Log} c\right), \tag{6.10}$$

$$u_c(r) = c^{1-p} u_1(c^{-p(p-1)/(N-p)} r). \tag{6.11}$$

Now we consider the case $q = Q_1$.

Theorem 6.2. *Suppose $q = Q_1$. Then, to within exchange of u and $-u$, each solution $u \not\equiv 0$ has one of the following forms:*

(i) *u is regular in 0 $\left(\lim_{r\to0} u = \alpha > 0\right)$ and oscillates near $+\infty$,*

(ii) $\lim_{r\to0} r\,|\operatorname{Log} r|^{1/p} u^{(p-1)/(N-p)} = \left(\frac{N-p}{p}\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{1/p}$ *and u oscillates near $+\infty$.*

All of these solutions exist (for each $\alpha > 0$). They intersect the curves $r \mapsto \pm r^{-\delta}$ infinitely many times.

Proof. From Theorems 4.3 and 4.4 we know the behavior near $+\infty$ of any solution u. VERON & GUEDDA [12] have described the behavior near 0 of any *positive* solution, in the form (i) or (ii). From [12] and Proposition 2.1 we have for each $\alpha > 0$ the existence of solutions u^α regular at 0, with $\lim_{r\to 0} u^\alpha = \alpha$. As in Theorem 5.2, Remark, the functions u^α and the Emden functions w^α relative to u^α satisfy the relations (5.3) and (5.4). Now let w be any solution of (2.11). Suppose that w does not have constant sign near $+\infty$. Then there is an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of zeros of w, with $\tau_0 \geqq 1$ and $\tau_n \to +\infty$. Now the energy functions V and W are nonincreasing, and

$$W(t) = \frac{|y(t)|^{p'}}{p'} + \frac{|w(t)|^{q+1}}{q+1} = \frac{|w_t(t) + \delta w(t)|^p}{p'} + \frac{|w(t)|^{q+1}}{q+1}.$$

From Propositions 3.1, 3.2, we have $\lim_{t\to+\infty} w_t = \lim_{t\to+\infty} w = 0$, as $B = 0$, and hence $\lim_{t\to+\infty} W = 0$ and $\lim_{n\to+\infty} w_t(\tau_n) = 0$. The Emden function w^1 oscillates near $-\infty$ and is positive near $+\infty$; let $(t_n)_{n\in\mathbb{N}}$ be the nonincreasing sequence of its zeros ($t_n \to -\infty$) and $a_n = |w_t^1(t_n)|$. As W is a nonincreasing function, $(a_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence, and $a_0 = w_t^1(t_0) > 0$ from uniqueness. On $(t_0, +\infty)$, w^1 has no zero; let s be the maximum point of w^1 on $(t_0, +\infty)$, unique from (2.12), and $M = w^1(s)$. Now there is a $k \in \mathbb{N}$ such that

$$|w_t(t)| \leqq a_0/2, \qquad |w(t)| < M/2 \qquad \forall\, t \geqq \tau_K,$$

and we can suppose $w_t(\tau_k) > 0$. Let w have its maximum in (τ_k, τ_{k+1}) at s_k, and $M_k = w(s_k)$. Because $M > M_k$ and $\lim_{t\to+\infty} w^1 = 0$, there is a $\sigma > s$ such that $w^1(\sigma) = M_k$ and $w_t^1(\sigma) < 0$. Let us compare w to a translate of w^1:

$$\overline{w}(t) = w_1(t - s_k + \sigma) \qquad \forall\, t \in \mathbb{R};$$

we have

$$(\overline{w} - w)(s_k) = 0, \qquad (\overline{w} - w)_t\,(s_k) < 0, \qquad (\overline{w} - w)(\tau_{k+1}) > 0,$$

and hence there is $\sigma_k \in (s_k, \tau_{k+1})$ such that $(\overline{w} - w)(\sigma_k) = 0$; then there are a θ and a $\tau \in (s_k, \sigma_k)$ such that

$$\overline{w}(\theta) = w(\tau), \qquad \overline{w}_t(\theta) = w_t(\tau).$$

By uniqueness we get

$$w(t) = \overline{w}(t - \tau + \theta) \qquad \forall\, r \in \mathbb{R};$$

this is impossible since $\overline{w}$ is positive near $+\infty$.

Hence any solution w of (2.11) is of one sign near $+\infty$; then the function u associated to w, or $-u$, has one of the forms (i) (ii).

Finally we prove the existence of solutions of the form (ii): for any $\theta_0 \in \mathbb{R}$ there is a solution w of (2.11) such that $w(\theta_0) = 0$, $w_t(\theta_0) = a_0/2$. Then w cannot be an Emden solution; the function u associated to w cannot have the form (i), and so it has the form (ii). □

7. Case $p = N$

In this section we consider the case $p = N > 1$ with $q > p - 1$. Proposition 2.1 is still true. We have now $A < 0$, $B < 0$ for any q.

Theorem 7.1. *Let* $p = N > 1$, $q > p - 1$. *Then, to within exchange of* u *and* $-u$, *each solution* $u \not\equiv 0$ *of* (2.1) *has one of the following forms*:

(i) u *is regular at* 0 $\left(\lim_{r\to 0} u = \alpha > 0\right)$ *and oscillates near* $+\infty$,

(ii) $\lim_{r\to 0} |\operatorname{Log} r|\, u = \gamma > 0$ *and* u *oscillates near* $+\infty$.

All of these solutions exist (for each $\alpha > 0$ *and* $\gamma > 0$).

Proof. The behavior near 0 and the existence of regular or singular positive solutions is proved in [12]. Now consider any solution u of (2.1) and function w associated to u by (2.10). Because $A < 0$, Propositions 3.1, 3.2 hold. Because $B < 0$, Theorem 4.3 holds and u oscillates near $+\infty$. Now to study the behavior near 0 we use the change of variables (2.7). Since the energy function F_N defined in (2.9) is nonincreasing, it is bounded near $+\infty$, hence $\theta = O(e^{Nx/(q+1)})$.

By integration of (2.8) we get, for any $x \geqq 1$,

$$|\theta_x|^{N-2}\,\theta_x + \int_1^x e^{-Nt}\,|\theta|^{q-1}\,\theta\,dt = |\theta_x(1)|^{N-2}\,\theta_x(1);$$

the integral is convergent, since $e^{-Nx}\,|\theta|^q = O(e^{-Nx/(q+1)})$, and hence θ_x has a limit γ at $+\infty$.

If $\gamma \neq 0$, then $\lim_{x\to+\infty} x^{-1}\theta = \gamma$ and hence u satisfies (i). If $\gamma = 0$ then by integration of (2.8), for any $x > 0$,

$$|\theta_x|^{N-2}\,\theta_x = \int_x^{+\infty} e^{-Nt}\,|\theta|^{q-1}\,\theta\,dt, \tag{7.1}$$

and $\theta_x = O(e^{-N'x/(q+1)})$. Then θ_x is integrable, θ has a limit α at $+\infty$ and $e^{N'x}\theta_x$ has a limit from (7.1); then $r^{-1/(N-1)}u_r$ has a limit, and $\lim_{r\to 0} u_r = 0$. We have $\alpha \neq 0$, for if $\alpha = 0$ then $u \equiv 0$ from (2.2); hence u is regular. □

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