Stability, Instability and Center Manifold Theorem for Fully Nonlinear Autonomous Parabolic Equations in Banach Space

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Introduction

In this paper we generalize some classical results on stability for autonomous semilinear evolution equations to the fully nonlinear case.

The typical example we have in mind is the equation

$$u_t(t, x) = \phi(u(t, x), u_x(t, x), u_{xx}(t, x)), \quad t \geqq 0, \quad 0 \leqq x \leqq \pi, \tag{0.1}$$

which will be studied by abstract methods, reducing it to the equation

$$u'(t) = g(u(t)), \quad t \geqq 0 \tag{0.2}$$

where $g: D \to X$ is a regular function and D, X are Banach spaces with D continuously embedded in X.

Assuming that $g(0) = 0$, we will study stability, instability and saddle points of the zero solution of (0.2) by means of a linear approximation. To treat also some critical cases of stability, we will establish the existence of an attracting local center manifold for equation (0.2).

Our treatment follows closely the methods used in the semilinear case (see for instance [1], [4]), but extending these ideas is not trivial because of technical difficulties due to the fully nonlinear character of problem (0.2).

The main assumption on g is that the operator $A = g'(0): D \to X$ generates an analytic semigroup e^{tA} in X.

The usual method for studying the initial value problem by linearization, namely

$$\begin{aligned} u'(t) &= g(u(t)) = Au(t) + \psi(u(t)), \quad t \geqq 0, \\ u(0) &= u_0, \end{aligned} \tag{0.3}$$

would be to solve the integral equation

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} \psi(u(s))\, ds, \quad t \geqq 0$$

in the space $C([0, T]; D)$. This, however, does not work, in general, because the convolution term is not sufficiently smooth.

This difficulty can be overcome by replacing X by a suitable interpolation space Y between D and X, and using maximal regularity properties for the linear problem (see [2]):

$$\begin{aligned} v'(t) &= Av(t) + h(t), \quad t \geqq 0, \\ v(0) &= v_0 . \end{aligned} \tag{0.4}$$

Then it is possible to obtain a local solution of (0.3) provided Au_0 belongs to Y and g is sufficiently regular (see [2], [5]). Nevertheless, we are not able to extend to our case some of the usual properties of the solutions. In particular,

a) local estimates for the solution of problem (0.4), such as

$$\|Av(t)\|_Y \leqq C\left(\|Av_0\|_Y + \int_0^t \|h(s)\|_Y \, ds\right),$$

are not available; hence Gronwall's lemma cannot be used;

b) we have, in general,

$$\inf_{T>0} \sup_{0 \leqq t \leqq T} \left\| A \int_0^t e^{(t-s)A} h(s) \, ds \right\|_Y > 0$$

so that making the interval $[0, T]$ small does not help us in proving the local existence of the solution;

c) we are not able to prove existence in the large if an *a priori* bound on $\|Au(t)\|_Y$ is given: a stronger *a priori* estimate is necessary.

Section 1 contains preliminary results on linear problems in unbounded intervals. In Section 2 we study linearized stability and instability of the zero solution of (0.2). In Section 3 we prove that a local center manifold exists and we study some of its properties of regularity and attractivity. Finally, in Section 4 we use the center manifold to study two critical cases of stability, namely linearized stability and instability when 0 is a simple eigenvalue of A (while the remainder of the spectrum has negative real part) and existence and stability of periodic solutions of

$$u'(t) = f(\lambda, u(t)) \tag{0.5}$$

under the classical Hopf bifurcation assumptions.

Some applications to parabolic partial differential equations and systems are discussed in Sections 2 and 4.

1. Maximal Regularity for Linear Problems in Unbounded Intervals

Throughout the paper we assume that D and X are Banach spaces and D is continuously embedded in X. The norm in X is denoted by $\|\cdot\|$. By $\tilde{X}$ and $\tilde{D}$ we mean the usual complexification of X and D respectively. If $A : D \to X$ is a linear operator define $\tilde{A} : \tilde{D} \to \tilde{X}, \tilde{A}(x + iy) = Ax + iAy$.

We are here concerned with the problem

$$u'(t) = Au(t) + h(t),$$
$$u(0) = 0, \tag{1.1}$$

where the linear operator $A : D \to X$ satisfies the following assumption:

The resolvent set $\varrho(\tilde{A})$ of $\tilde{A}$ contains the sector

$$S = \{\lambda \in \mathbb{C};\, \lambda \neq \lambda_0,\, |\arg(\lambda - \lambda_0)| < \theta\}, \quad \lambda_0 \in \mathbb{R}, \quad \theta \in \left]\frac{\pi}{2}, \pi\right[$$
$$\text{and} \quad \sup_{\lambda \in S} |\lambda - \lambda_0|\, \|R(\lambda, \tilde{A})\|_{L(\tilde{X})} < +\infty. \tag{1.2}$$

Then A generates an analytic semigroup e^{tA} in X, not necessarily continuous at $t = 0$ (see [8]). Set $\bar{\lambda} = \sup\{\operatorname{Re}\lambda;\, \lambda \in \sigma(\tilde{A})\}$ and fix $\omega > \bar{\lambda}$; then there exist $M > 0$ and $N > 0$ such that

$$\|e^{tA}\|_{L(X)} \leqq Me^{\omega t}, \quad \|Ae^{tA}\|_{L(X)} \leqq \frac{N}{t}\, e^{\omega t} \quad \forall\, t > 0. \tag{1.3}$$

We shall use the following notation: if Z is a Banach space and $\eta \in \mathbb{R}$ we denote by $C_\eta([0, +\infty[; Z)$ (or $C_\eta(]-\infty, 0]; Z)$) the set of all f: $[0, +\infty[\to Z$ (or $]-\infty, 0] \to Z$) such that $t \to e^{\eta t} f(t)$ (or $t \to e^{-\eta t} f(t)$) is continuous and bounded. These spaces are endowed with the norms:

$$\|f\|_{C_\eta([0,+\infty[;Z)} = \sup_{t \geqq 0} \|e^{\eta t} f(t)\|_Z,$$
$$\|f\|_{C_\eta(]-\infty,0];Z)} = \sup_{t \leqq 0} \|e^{-\eta t} f(t)\|_Z.$$

We set also

$$C_\eta^1([0, +\infty[; Z) = \{f\colon [0, +\infty[\to Z;\, f, f' \in C_\eta([0, +\infty[; Z)\}.$$

Let us introduce the interpolation spaces that we shall use in the sequel: we first consider the case $\omega < 0$ (ω is given in (1.3)).

For $\alpha \in\,]0, 1[$ we set

$$D_A(\alpha, \infty) = \left\{x \in X;\, \sup_{\xi > 0} \xi^{1-\alpha} \|Ae^{\xi A}x\| < +\infty\right\},$$
$$\|x\|_{D_A(\alpha,\infty)} = \sup_{\xi > 0} \xi^{1-\alpha} \|Ae^{\xi A}x\|,$$

$$D_A(\alpha + 1, \infty) = \{x \in D;\, Ax \in D_A(\alpha, \infty)\},$$
$$\|x\|_{D_A(\alpha+1,\infty)} = \|Ax\|_{D_A(\alpha,\infty)}.$$

For $\omega \geqq 0$ we set $D_A(\alpha, \infty) = D_{A-2\omega}(\alpha, \infty)$, $D_A(\alpha + 1, \infty) = D_{A-2\omega}(\alpha + 1, \infty)$ and

$$\|x\|_{D_A(\alpha,\infty)} = \|x\|_{D_{A-2\omega}(\alpha,\infty)},$$

$$\|x\|_{D_A(\alpha+1,\infty)} = \|x\|_{D_{A-2\omega}(\alpha+1,\infty)}.$$

In the sequel, if no confusion ought result, we shall write $\|x\|_\alpha$ and $\|x\|_{\alpha+1}$ instead of $\|x\|_{D_A(\alpha,\infty)}$ and $\|x\|_{D_A(\alpha+1,\infty)}$. The closure of D in $D_A(\alpha,\infty)$ will be denoted by $D_A(\alpha)$; it can be shown that

$$D_A(\alpha) = \left\{x \in X;\ \lim_{\xi\to 0^+} \xi^{1-\alpha} A e^{\xi A} x = 0\right\}.$$

We denote by $D_A(\alpha+1)$ the closed subspace of $D_A(\alpha+1,\infty)$:

$$D_A(\alpha+1) = \{x \in D;\ Ax \in D_A(\alpha)\}.$$

Increasing, if necessary, the values of M and N that appear in (1.3), we may assume

$$\begin{aligned} \|e^{tA}\|_{L(Z)} &\leqq M e^{\omega t} \quad \forall\, t \geqq 0, \\ \|Ae^{tA}\|_{L(Z)} &\leqq \frac{N}{t} e^{\omega t} \quad \forall\, t > 0, \end{aligned} \tag{1.4}$$

where Z is any of the spaces X, $D_A(\beta,\infty)$, $D_A(\beta+1,\infty)$, $\beta \in\,]0,1[$.
Moreover for any $\beta \in\,]0,1[\, \cup\,]1,2[$ and for any $\eta \in \mathbb{R}$ there exists $C(\eta,\beta) \geqq 1$ such that for each $x \in D_A(\beta,\infty)$ we have

$$\frac{1}{C(\eta,\beta)} \|x\|_{D_{A+\eta}(\beta,\infty)} \leqq \|x\|_{D_A(\beta,\infty)} \leqq C(\eta,\beta)\, \|x\|_{D_{A+\eta}(\beta,\infty)}. \tag{1.5}$$

Now we can state a theorem of existence and exponential decay for the solution of (1.1).

Proposition 1.1. *Let A satisfy* (1.2), *and let M, N, ω be such that* (1.4) *holds. Assume $\eta + \omega < 0$, $\tau \in\,]0, +\infty]$ and let $h \in C([0,\tau[; D_A(\alpha))$ be such that $\sup_{0\leqq t<\tau} \|e^{\eta t} h(t)\|_\alpha < +\infty$. Then problem* (1.1) *has a unique solution*

$$u(t) = \int_0^t e^{(t-s)A} h(s)\, ds, \quad 0 \leqq t < \tau, \tag{1.6}$$

and there exists $K_1(\eta,\omega) > 0$ such that

$$\sup_{0\leqq t<\tau} \|e^{\eta t} u'(t)\|_\alpha + \sup_{0\leqq t<\tau} \|e^{\eta t} u(t)\|_{\alpha+1} \leqq K_1(\eta,\omega) \sup_{0\leqq t<\tau} \|e^{\eta t} h(t)\|_\alpha. \tag{1.7}$$

In particular, if $\tau = +\infty$ and $h \in C_\eta([0,+\infty[; D_A(\alpha))$, then $u \in C^1_\eta([0,+\infty[; D_A(\alpha)) \cap C_\eta([0,+\infty[; D_A(\alpha+1))$.

Proof. By [8, Th. 5.5], problem (1.1) has a unique solution u given by (1.6); u belongs to $C^1([0,T]; D_A(\alpha)) \cap C([0,T]; D_A(\alpha+1))$ for any $T \in\,]0,\tau[$. Thus we have only to prove estimate (1.7).

We first assume $\omega < 0$, $\eta = 0$. Then for $t \in [0, \tau[$ we have

$$\begin{aligned}\|u(t)\|_{\alpha+1} &= \sup_{\xi>0} \|\xi^{1-\alpha} A^2 e^{\xi A} u(t)\| \\ &\leqq \sup_{\xi>0} \left\| \xi^{1-\alpha} \int_0^t A e^{\frac{\xi+t-s}{2}A} \cdot A e^{\frac{\xi+t-s}{2}A} h(s)\, ds \right\| \\ &\leqq 2^{2-\alpha} N \sup_{0\leqq s<\tau} \|h(s)\|_\alpha \sup_{\xi>0} \int_0^\infty \frac{\xi^{1-\alpha}}{(\xi+s)^{2-\alpha}} ds \\ &= \frac{2^{2-\alpha}}{1-\alpha} N \sup_{0\leqq s<\tau} \|h(s)\|_\alpha,\end{aligned} \tag{1.8}$$

and the proposition is proved in this case. For general ω and η set $h_1(t) = e^{\eta t} h(t)$, $u_1(t) = e^{\eta t} u(t)$ so that

$$u_1(t) = \int_0^t e^{(t-s)(A+\eta)} h_1(s)\, ds.$$

Now, by (1.8) we get, for $t \in [0, \tau[$,

$$\|e^{\eta t} u(t)\|_{D_{A+\eta}(\alpha+1,\infty)} \leqq \frac{2^{2-\alpha}}{1-\alpha} N_1(\eta) \sup \|e^{\eta t} h(t)\|_{D_{A+\eta}(\alpha,\infty)}$$

where

$$N_1(\eta) = \sup_{t>0} t\, \|(A+\eta)\, e^{(A+\eta)t}\|_{L(X)}$$

and the conclusion follows by (1.5). □

Proposition 1.2. *Let A satisfy* (1.2) *and let M, N, ω be such that* (1.4) *holds. Let $k \in C_\eta(]-\infty, 0]; D_A(\alpha))$ with $\omega - \eta < 0$. Then the function*

$$v(t) = \int_{-\infty}^t e^{(t-s)A} k(s)\, ds$$

belongs to $C_\eta(]-\infty, 0]; D_A(\alpha+1))$ and there exists $K_2(\eta, \omega) > 0$ such that

$$\|v\|_{C_\eta(]-\infty,0];D_A(\alpha+1))} \leqq K_2(\eta, \omega)\, \|k\|_{C_\eta(]-\infty,0];D_A(\alpha))}. \tag{1.9}$$

Proof. Let us first consider, as before, the case $\omega < 0$, $\eta = 0$. Then, arguing as in Proposition 1.1., we find

$$\|v(t)\|_{\alpha+1} \leqq \frac{2^{2-\alpha}}{1-\alpha} N \|h\|_{C_0(]-\infty,0];D_A(\alpha))} \quad \forall\, t \leqq 0. \tag{1.10}$$

For general ω and η set $k_1(t) = e^{-\eta t} k(t)$, $v_1(t) = e^{-\eta t} v(t)$ so that

$$v_1(t) = \int_{-\infty}^t e^{(t-s)(A-\eta)} k_1(s)\, ds.$$

Now, by (1.10), we get

$$\| e^{-\eta t} v(t) \|_{D_{A-\eta}(\alpha+1,\infty)} \leqq \frac{2^{2-\alpha}}{1-\alpha} N_2(\eta) \sup_{t\leqq 0} \| e^{-\eta t} k(t) \|_{D_{A-\eta}(\alpha,\infty)}$$

where $N_2(\eta) = \sup_{t>0} t \, \|(A-\eta) e^{(A-\eta)t}\|_{L(X)}$.

By (1.5), the function $t \to e^{-\eta t} v(t)$ is bounded in $D_A(\alpha+1,\infty)$. In order to complete the proof we have only to remark that in any interval $[-T, 0]$ v is the uniform limit in $D_A(\alpha+1,\infty)$ of v_n, where

$$v_n(t) = \int_{-n}^{t} e^{(t-s)A} k(s) \, ds, \qquad n \in \mathbb{N}, \qquad n \geqq T,$$

and $v_n \in C([-T, 0]; D_A(\alpha+1))$ (see [8, Th. 5.5]). □

In the sequel, since we deal with stability, instability and saddle points, we shall need the usual splitting of the spectrum of $\tilde{A}$ into two parts, and so we shall assume, besides (1.2), that

$$\begin{gathered} \sigma(\tilde{A}) = \sigma_1(\tilde{A}) \cup \sigma_2(\tilde{A}) \quad \text{with} \\ \lambda_1 = \sup\{\operatorname{Re}\lambda; \lambda \in \sigma_1(\tilde{A})\} < \lambda_2 = \inf\{\operatorname{Re}\lambda; \lambda \in \sigma_2(\tilde{A})\}. \end{gathered} \tag{1.11}$$

We shall denote by P_2 the projection operator

$$P_2 = \frac{1}{2\pi i} \int_{C_2} R(\xi, \tilde{A}) \, d\xi \tag{1.12}$$

where C_2 is a suitable path around the bounded set $\sigma_2(A)$. We set $P_1 = 1 - P_2$, $X_1 = P_1(X)$, $X_2 = P_2(X)$, $\|x\|_{X_1} = \|x\|$, $\|x\|_{X_2} = \|x\|$, $A_1 = A/(D \cap X_1)$: $D \cap X_1 \to X_1$, $A_2 = A/X_2 : X_2 \to X_2$.

Then $A_2 \in L(X_2; D_A(\alpha+1))$ and $D_{A_1}(\alpha) = D_A(\alpha) \cap X_1$, $D_{A_1}(\alpha+1) = D_A(\alpha+1) \cap X_1$ with $\|x\|_{D_{A_1}(\alpha)} = \|x\|_\alpha$, $\|x\|_{D_{A_1}(\alpha+1)} = \|x\|_{\alpha+1}$. Let $\omega_1, \omega_2, \omega_3$ be such that

$$\lambda_1 < \omega_1 < \omega_2 < \lambda_2 < \lambda_3 < \omega_3$$

where $\lambda_3 = \sup\{\operatorname{Re}\lambda; \lambda \in \sigma_2(\tilde{A})\}$. There are $M_1, N_1 > 0$ such that

$$\| e^{tA_1} \|_{L(Z)} \leqq M_1 e^{\omega_1 t}, \qquad \| A_1 e^{tA_1} \|_{L(Z)} \leqq \frac{N_1}{t} e^{\omega_1 t}, \qquad t > 0, \tag{1.13}$$

where Z is any of the spaces X, $D_{A_1}(\beta,\infty)$, $D_{A_1}(\beta+1,\infty)$, $\beta \in]0, 1[$. Moreover there are $M_2, M_3 > 0$ such that

$$\begin{gathered} \| e^{tA_2} \|_{L(X_2; D_A(\alpha+1))} \leqq M_2 e^{\omega_2 t}, \qquad t \leqq 0, \\ \| e^{tA_2} \|_{L(X_2)} \leqq M_2 e^{\omega_2 t}, \qquad t \leqq 0, \end{gathered} \tag{1.14}$$

$$\| e^{tA_2} \|_{L(X_2; D_A(\alpha+1))} \leqq M_3 e^{\omega_3 t}, \qquad t \geqq 0. \tag{1.15}$$

Finally we set

$$p = \max\{\| P_2 \|_{L(D_A(\alpha), X_2)}, \| P_1 \|_{L(D_A(\alpha))}\}. \tag{1.16}$$

2. Linearized Stability and Instability

We consider here the nonlinear problem:

$$\begin{aligned} u'(t) &= g(u(t)), \quad t \geqq 0, \\ u(0) &= u_0 . \end{aligned} \tag{2.1}$$

We first recall a theorem of local existence and uniqueness (see [2], [5]). We assume that

$$\begin{aligned} &g \in C^1(D; X), A = g'(0) \text{ satisfies (1.2)}, g(0) = 0 \\ &\text{and } g \in C^1(D_A(\alpha + 1), D_A(\alpha)) \text{ for some } \alpha \in]0, 1[. \end{aligned} \tag{2.2}$$

We are interested only in solutions of (2.1) near 0; say in an open ball $B(0, R_0) \subset D_A(\alpha + 1)$ so chosen such that

i) For each $x \in B(0, R_0), g'(x): D_A(\alpha + 1) \to D_A(\alpha)$ satisfies (1.2) with X replaced by $D_A(\alpha)$,

ii) $\sup_{\|x\|_{\alpha+1} \leqq R_0} \|g'(x)\|_{L(D_A(\alpha+1); D_A(\alpha))} < +\infty$.

Now we can state

Proposition 2.1. *Assume* (2.2). *Then for any* $u_0 \in B(0, R_0) \subset D_A(\alpha + 1)$ *there exist* $\tau(u_0) > 0$ *and a unique function* $u = u(\cdot, u_0): [0, \tau(u_0)[\to B(0, R_0)$ *such that u belongs to* $C([0, \tau(u_0)[; D_A(\alpha + 1)) \cap C^1([0, \tau(u_0)[; D_A(\alpha))$ *and satisfies* (2.1). *If, in addition, there are* $\varepsilon > 0$, $K > 0$, $R_1 \in]0, R_0[$ *such that* $u(t, u_0) \in D_A(\alpha + \varepsilon + 1)$ *and* $\|u(t, u_0)\|_{\alpha+1} \leqq R_1$, $\|u(t, u_0)\|_{\alpha+1+\varepsilon} \leqq K \quad \forall\, t \in [0, \tau(u_0)[$, *then* $\tau(u_0) = +\infty$. □

We now prove a theorem of stability.

Theorem 2.2. *Assume* (2.2) *and suppose that* $\bar{\lambda} = \sup\{\operatorname{Re} \lambda; \lambda \in \sigma(\tilde{A})\} < 0$. *Then, if* $\eta + \bar{\lambda} < 0$, *there exists* $r > 0$ *such that if* $u_0 \in D_A(\alpha + 1)$ *and* $\|u_0\|_{\alpha+1} \leqq r$, *the solution* $u(\cdot, u_0)$ *of problem* (2.1) *is defined for all* $t \geqq 0$ *and belongs to* $C_\eta([0, +\infty[; D_A(\alpha + 1)) \cap C^1_\eta([0, +\infty[; D_A(\alpha))$. *Moreover*

$$\lim_{t \to +\infty} \|e^{\eta t} u(t, u_0)\|_{\alpha+1} = 0 \quad \text{uniformly with respect to } u_0 .$$

We say that 0 *is exponentially asymptotically stable in* $D_A(\alpha + 1)$.

Proof. Fix $\omega > \bar{\lambda}$ such that $\eta + \omega > 0$ and (1.4) holds. Let G be defined by:

$$\begin{gathered} G: B(0, R) \subset C^1_\eta([0, +\infty[; D_A(\alpha)) \cap C_\eta([0, +\infty[; D_A(\alpha + 1)) \\ \to C_\eta([0, +\infty[; D_A(\alpha)) \oplus D_A(\alpha + 1), \\ G(u) = (u' - g(u), u(0)) \end{gathered}$$

where $R < R_0$. It is easy to see that G is of class C^1. Moreover, by Proposition 1.1, the linear operator

$$G'(0)\, v = (v' - Av, v(0))$$

is an isomorphism. By the Implicit Function Theorem there exists $\bar{r}(\eta)$ such that for $\|u_0\|_{\alpha+1} \leqq \bar{r}(\eta)$ problem 2.1 has a unique solution $u \in C_\eta([0, +\infty[; D_A(\alpha+1)) \cap C^1_\eta([0, +\infty[; D_A(\alpha))$. Moreover, choosing $\varepsilon > 0$ such that $\eta + \omega + \varepsilon < 0$ we have

$$\lim_{t \to +\infty} \sup \{\|e^{\eta t} u(t, u_0)\|_{\alpha+1}; \|u_0\|_{\alpha+1} \leqq \bar{r}(\eta + \varepsilon)\} = 0.$$

In fact the Implicit Function Theorem implies uniqueness only for small solutions of (2.1); however, uniqueness in the large follows from Proposition 2.1. □

Let us suppose now that $\sigma(\tilde{A})$ contains some point with positive real part. In this case we can prove the following theorem of instability.

Theorem 2.3. *Assume that* (2.2) *and* (1.11) *hold with* $\lambda_1 \leqq 0$ *and* $\lambda_2 > 0$. *Then* 0 *is unstable in* $D_A(\alpha+1)$; *that is, there exists a sequence* $\{u_{0n}\} \subset D_A(\alpha+1)$ *such that* $u_{0n} \to 0$ *in* $D_A(\alpha+1)$ *and*

$$\inf_{n \in \mathbb{N}} \sup_{t \in [0, \tau(u_{0n})[} \|u(t, u_{0n})\|_{\alpha+1} > 0. \tag{2.3}$$

Proof. We shall prove the existence of $v \in C^1_0(]-\infty, 0]; D_A(\alpha)) \cap C_0(]-\infty, 0]; D_A(\alpha+1))$ such that

$$\begin{aligned} &v' = g(v), \quad v \not\equiv 0, \\ &\lim_{t \to -\infty} v(t) = 0 \quad \text{in } D_A(\alpha+1). \end{aligned} \tag{2.4}$$

Then it will be sufficient to take $u_{0n} = v(-n)$, since $u(t, u_{0n}) = v(t-n)$ for $t \in [0, n]$.

To construct such a function let us consider the following integral equation:

$$\begin{aligned} v(t) = e^{tA_2} x_2 &+ \int_0^t e^{(t-s)A_2} P_2(g(v(s)) - Av(s))\, ds \\ &+ \int_{-\infty}^t e^{(t-s)A_1} P_1(g(v(s)) - Av(s))\, ds = (\Lambda v)(t) \end{aligned} \tag{2.5}$$

where $x_2 \in P_2(X)$.

Fix $0 < \omega_1 < \omega_2$ so as to make (1.13) and (1.14) hold. We shall find a fixed point of Λ in the set

$$Y_a = \{v \in C_\eta(]-\infty, 0]; D_A(\alpha+1)); \|e^{-\eta t} v(t)\|_{\alpha+1} \leqq a, t \leqq 0\} \tag{2.6}$$

where $\eta \in]\omega_1, \omega_2[$ and $a \in]0, R_0[$ are to be chosen later. Let us show that if $\|x_2\|$ and a are sufficiently small then $\Lambda(Y_a) \subset Y_a$. For any $v \in Y_a$ in fact, Λv is continuous with values in $D_A(\alpha+1)$ (see Proposition 1.2), and by (1.9), (1.14) and (1.16) we have

$$\|e^{-\eta t}(\Lambda v)(t)\|_{\alpha+1} \leqq M_2 \|x_2\| + p \left[\frac{M_2}{\omega_2 - \eta} + K_2(\eta, \omega_1)\right] aL(a)$$

with

$$L(\varrho) = \sup_{\|y\|_{\alpha+1} \leq \varrho} \|g'(y) - A\|_{L(D_A(\alpha+1), D_A(\alpha))}, \quad \varrho > 0. \tag{2.7}$$

Thus $\|e^{-\eta t}(\Lambda v)(t)\|_{\alpha+1} \leq a$, provided a and $\|x_2\|$ are sufficiently small. Furthermore for $v, \bar{v} \in Y_a$ we have

$$\|e^{-\eta t}((\Lambda v)(t) - (\Lambda \bar{v})(t))\|_{\alpha+1} \leq p\left[\frac{M_2}{\omega_2 - \eta} + K_2(\eta, \omega_1)\right] L(a)$$
$$\cdot \sup_{s \leq 0} \|e^{-\eta s}(v(s) - \bar{v}(s))\|_{\alpha+1}, \quad t \leq 0,$$

so that for a sufficiently small Λ is a contraction and so there exists a unique solution v of (2.5) in Y_a. It is easy to see that v satisfies (2.4). This concludes the proof. □

We shall show now that 0 is a saddle point. To this end we need an assumption slightly stronger than in Theorem 2.3, namely $\lambda_1 < 0$. We shall prove existence in the large of the solution of (2.1) for initial data lying on a stable local manifold $S \subset D_A(\alpha + 1)$, and existence in the large of a backward solution of (2.1) for initial data u_0 belonging to an unstable local manifold U. We shall denote again by $u(t, u_0)$ $(t \leq 0)$ the backward solution of (2.1), which exists by virtue of the following theorem.

Theorem 2.4. *Assume* (2.2) *and* (1.11) *with* $\lambda_1 < 0$, $\lambda_2 > 0$ *and fix* $\eta \in]0, -\lambda_1[$. *Then there exist* $r > 0$, $\varrho > 0$ *and two continuous functions*

$$h: B(0, r) \subset X_2 \to D_A(\alpha + 1),$$
$$k: B(0, r) \subset D_{A_1}(\alpha + 1) \to D_A(\alpha + 1),$$

such that, setting

$$U = \{h(x_2); x_2 \in B(0, r) \subset X_2\},$$
$$S = \{k(x_1); x_1 \in B(0, r) \subset D_{A_1}(\alpha + 1)\}, \tag{2.8}$$

we obtain the following conclusions:

i) *For any* $u_0 \in S$ *the solution* u *of* (2.1) *exists in the large; it belongs to* $C_\eta([0, +\infty[; D_A(\alpha + 1)) \cap C^1_\eta([0, +\infty[; D_A(\alpha))$; *and* $\|e^{\eta t} u(t, u_0)\|_{\alpha+1} \leq \varrho$, $t \geq 0$. *Conversely, if* u_0 *is such that* $\|P_1 u_0\|_{\alpha+1} \leq r$, $u(\cdot, u_0) \in C_\eta([0, +\infty[; D_A(\alpha + 1))$ *and* $\|e^{\eta t} u(t, u_0)\|_{\alpha+1} \leq \varrho$, $t \geq 0$, *then* u_0 *belongs to* S.

ii) *For any* $u_0 \in U$ *there exists a backward solution of* (2.1) $u \in C_\eta(]-\infty, 0]; D_A(\alpha + 1)) \cap C^1(]-\infty, 0]; D_A(\alpha))$ *and* $\|e^{-\eta t} u(t)_{\alpha+1}\| \leq \varrho$, $t \leq 0$. *Conversely, if* u_0 *is such that* $\|P_2 u_0\| \leq r$, *and a backward solution* u *of* (2.1) *exists and belongs to* $C_\eta(]-\infty, 0]; D_A(\alpha + 1))$, *with* $\|e^{-\eta t} u(t)\|_{\alpha+1} \leq \varrho$, $t \leq 0$, *then* u_0 *belongs to* U.

iii) S *(or* U*) is tangent to* X_1 *(or* X_2*) at the origin, that is* k *(or* h*) is differentiable at* 0 *with* $k'(0) = 0$ *(or* $h'(0) = 0$*).*

We shall say that 0 *is a saddle point in* $D_A(\alpha + 1)$.

Proof.

i) Let $x_1 \in D_{A_1}(\alpha + 1)$; consider the integral equation

$$z(t) = e^{tA_1} x_1 + \int_0^t e^{(t-s)A_1} P_1[g(z(s)) - A_1 z(s)]\, ds \\ - \int_t^{+\infty} e^{(t-s)A_2} P_2[g(z(s)) - A_2 z(s)]\, ds = (\Pi z)(t). \tag{2.9}$$

We will find a fixed point of Π in the set

$$Y_a = \left\{ z \in C_\eta([0, +\infty[; D_A(\alpha + 1)); \sup_{t \geq 0} \| e^{\eta t} z(t) \|_{\alpha+1} \leq a \right\}$$

for suitable a, η, x_1.

Fix ω_1, ω_2, η such that $\lambda_1 < \omega_1 < 0 < \omega_2 < \lambda_2$ and $\eta + \omega_1 < 0$. For any $z \in Y_a$, Πz is continuous with values in $D_A(\alpha + 1)$ and for $t \geq 0$ we have

$$\| e^{\eta t} (\Pi z)(t) \|_{\alpha+1} \leq M_1 \| x_1 \|_{\alpha+1} + b(\eta)\, a L(a)$$

where

$$b(\eta) = p \left[\frac{M_2}{\omega_2 + \eta} + K_1(\eta, \omega_1) \right]$$

and M_1, $L(a)$, $K_1(\eta, \omega_1)$, M_2 are given, respectively, by (1.13), (2.7), (1.7) and (1.14).

Moreover, for $z, \bar{z} \in Y_a$ we have

$$\| e^{\eta t}((\Pi z)(t) - (\Pi \bar{z})(t)) \|_{\alpha+1} \leq L(a)\, b(\eta) \sup_{s \geq 0} \| e^{\eta s} (z(s) - \bar{z}(s)) \|_{\alpha+1}.$$

Let $\varrho > 0$ be such that

$$L(\varrho)\, b(\eta) \leq \tfrac{1}{2}. \tag{2.10}$$

Then, for any $a \leq \varrho$ and $\| x_1 \|_{\alpha+1} \leq \dfrac{a}{2M_1}$, Π is a contraction of Y_a into itself and there exists a unique solution z of equation (2.9) in Y_a. Set $r = \dfrac{\varrho}{2M_1}$; since the mapping

$$B(0, r) \subset D_{A_1}(\alpha + 1) \times Y_\varrho \to C_\eta([0, +\infty[; D_A(\alpha + 1)),$$

$$(x_1, z) \to \Pi z$$

is continuous, the fixed point z depends continuously on x_1. Then, setting $k(x_1) = z(0)$ for $\| x_1 \|_{\alpha+1} \leq r$, we prove the first statement of i).

Now let u_0 be such that $\| P_1 u_0 \|_{\alpha+1} \leq r$. Then, if $\tau(u_0) = +\infty$ and $u(\cdot, u_0) \in C_\eta([0, +\infty[; D_A(\alpha + 1))$ with $\| u(\cdot, u_0) \|_{C_\eta([0,+\infty[; D_A(\alpha+1))} \leq \varrho$, we have

$$P_2 u(t) = e^{tA_2} P_2 u_0 + \int_0^t e^{(t-s)A_2} P_2(g(u(s)) - A u(s))\, ds.$$

Therefore,

$$P_2 u_0 = -\int_0^{+\infty} e^{-sA_2} P_2(g(u(s)) - Au(s))\, ds$$

so that $u \in Y_\varrho$ is a solution of $\Pi u = u$ with $x_1 = P_1 u(0)$ and then $u_0 = k(x_1) \in S$. This completes the proof of i); the proof of ii) is similar. In fact, the unstable manifold has been constructed in the proof of Theorem 2.3, and h can be defined by $h(x_2) = v(0)$, where v is the solution of equation (2.5).

iii) Let $a \leqq \varrho$, $\|x_1\|_{\alpha+1} \leqq \dfrac{a}{2M_1}$ and let z_{x_1} be the fixed point of Π in Y_a. Then, by (2.10)

$$\|z_{x_1}\|_{C_\eta([0,+\infty[;D_A(\alpha+1))} \leqq 2M_1 \|x_1\|_{\alpha+1}$$

so that

$$\|k(x_1) - x_1\|_{\alpha+1} = \left\| \int_0^{+\infty} e^{-sA_2} P_2(g(z_{x_1}(s)) - Az_{x_1}(s))\, ds \right\|_{\alpha+1}$$
$$\leqq \frac{2M_1 M_2 p}{\omega_2 + \eta} L(a) \|x_1\|_{\alpha+1}.$$

Since $\lim_{a \to 0} L(a) = 0$ the conclusion follows. A similar proof can be given for h. □

We finally remark that the results of Theorems 2.2, 2.3 and 2.4 can be easily extended to yield stability and instability for arbitrary stationary solutions of (2.1).

Example. Consider the problem

$$\begin{aligned} u_t(x,t) &= \phi(u(t,x), u_x(t,x), u_{xx}(t,x)), \quad t \geqq 0, \quad x \in [0,\pi], \\ u(t,0) &= u(t,\pi) = 0, \quad t \geqq 0, \\ u(x,0) &= u_0(x), \quad x \in [0,\pi], \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} &\phi : \mathbb{R}^3 \to \mathbb{R}, (p_1, p_2, p_3) \to \phi(p_1, p_2, p_3) \text{ is of class } C^\infty \\ &\text{and } \phi(0, p_2, 0) = 0 \quad \forall p_2 \in \mathbb{R}. \end{aligned} \tag{2.12}$$

We set $\phi_{p_i}(0,0,0) = \phi_i$ $(i = 1, 2, 3)$, $X = C([0,\pi])$ and

$$D = C_0^2([0,\pi]) = \{u \in C^2([0,\pi]);\, u(0) = u(\pi) = 0\}.$$

X and D are endowed with their usual norms. The function $g : D \to X$, $g(u) = \phi(u, u', u'')$, bellongs to $C^\infty(D; X)$ and $g'(0)\, v = \phi_1 v + \phi_3 v''$ $\forall v \in D$. We assume that $A = g'(0)$ is an elliptic operator; that is,

$$\phi_3 > 0. \tag{2.13}$$

Then A satisfies (1.2) (see [9]). For $\alpha \in]0, \frac{1}{2}[$ we have (see [2], [6])

$$D_A(\alpha) = h_0^{2\alpha}([0,\pi]) = \{u \in h^{2\alpha}([0,\pi]);\, u(0) = u(\pi) = 0\},$$

$$D_A(\alpha+1) = h_0^{2\alpha+2}([0,\pi]) = \{u \in h^{2\alpha+2}([0,\pi]);\, u(0) = u(\pi) = u''(0) = u''(\pi) = 0\}$$

where $h^{2\alpha}([0,\pi])$ is the closure of $C^1([0,\pi])$ in the $C^{2\alpha}$ norm. $h^{2\alpha}([0,\pi])$ consists of all the functions $u: [0,\pi] \to \mathbb{R}$ such that

$$\lim_{\tau\to 0} \sup_{\substack{0\leq x<y\leq\pi \\ y-x\leq\tau}} (y-x)^{-2\alpha} |u(y)-u(x)| = 0.$$

$h^{2\alpha+2}([0,\pi])$ is the set of the twice differentiable functions $u: [0,\pi]\to\mathbb{R}$ such that $u'' \in h^{2\alpha}([0,\pi])$. Then it is easy to verify that $g \in C^\infty(h^{2\alpha+2}([0,\pi]); h^{2\alpha}([0,\pi]))$. Since $\phi(0, p_2, 0) = 0 \;\forall\, p_2$, g belongs also to $C^\infty(D_A(\alpha+1), D_A(\alpha))$. The spectrum of A consists in the simple eigenvalues

$$\{\lambda_k = \phi_1 - k^2\phi_3;\, k = 1, 2, \ldots\}.$$

We can apply Theorems 2.2, 2.3, 2.4 to get the following result.

Proposition 2.5. *Assume* (2.12) *and* (2.13) *hold. Then*

i) *If* $\phi_1 < \phi_3$, 0 *is exponentially asymptotically stable in* $h_0^{2\alpha+2}([0,\pi])$.
ii) *If* $\phi_1 > \phi_3$, 0 *is unstable in* $h_0^{2\alpha+2}([0,\pi])$.
iii) *If* $\phi_1 > \phi_3$ *and* $\phi_1 - k^2\phi_3 \neq 0$, $k = 2, 3, \ldots$, 0 *is a saddle point in* $h_0^{2\alpha+2}([0,\pi])$. □

The critical case $\phi_1 = \phi_3$ will be studied in Section 4.

3. Center Manifold

Let us consider, as in Section 2, the initial value problem

$$\begin{aligned} u'(t) &= g(u(t)) \quad t \geqq 0, \\ u(0) &= u_0 \end{aligned} \tag{3.1}$$

where

$$\begin{gathered} g \in C^1(D; X),\, g(0) = 0,\, g'(0) = A : D \to X \text{ satisfies (1.2)} \\ \text{and } g \in C^1(D_A(\alpha+1), D_A(\alpha)). \end{gathered} \tag{3.2}$$

Moreover, let R_0 be defined as in Section 2. We shall construct here a local center manifold under the assumption that the points of the spectrum of $\tilde{A}$ have negative real part with the exception of a finite number of eigenvalues which have nonnegative real part and finite algebraic multiplicity. More precisely, we assume:

(1.11) holds,

$\sigma_2(\tilde{A}) = \{z_1, \ldots, z_k\}$ where $z_1, \ldots, z_k$ are eigenvalues with finite algebraic multiplicity, (3.3)

$$\lambda_1 = \sup\{\operatorname{Re}\lambda;\, \lambda\in\sigma_1(\tilde{A})\} < 0,$$

$$\lambda_2 = \min\{\operatorname{Re}\lambda;\, \lambda\in\sigma_2(\tilde{A})\} \geqq 0.$$

For convenience we shall transform problem (3.1) into a system which is equivalent to (3.1) for small solutions, namely,

$$\begin{aligned} x'(t) &= A_1 x(t) + f_1(x(t), y(t)), \quad x(0) = x_0, \\ y'(t) &= A_2 y(t) + f_2(x(t), y(t)), \quad y(0) = y_0, \end{aligned} \tag{3.4}$$

where

$$f_1 : D_{A_1}(\alpha + 1) \times X_2 \to D_{A_1}(\alpha), f_1(x, y) = P_1 g\left(x + \psi\left(\frac{y}{\varrho}\right) y\right) - A_1 x,$$

$$f_2 : D_{A_1}(\alpha + 1) \times X_2 \to X_2, f_2(x, y) = P_2 g\left(x + \psi\left(\frac{y}{\varrho}\right) y\right) - \psi\left(\frac{y}{\varrho}\right) A_2 y;$$

ϱ is a positive number to be chosen later, and $\psi : X_2 \to \mathbb{R}$ is a C^∞ function such that

$$\begin{gathered} \psi(x) = 1 \quad \text{for } \|x\| \leqq \frac{R_0}{2}, \quad \psi(x) = 0 \quad \text{for} \quad \|x\| \geqq R_0, \\ 0 \leqq \psi(x) \leqq 1. \end{gathered} \tag{3.5}$$

Clearly f_1 and f_2 have the same regularity as g.

Proposition 2.1 may be applied to system (3.4) to get existence and uniqueness of a local solution for initial data $(x_0, y_0) \in D_{A_1}(\alpha + 1) \times X_2$. In dealing with the applications, we shall be interested in small solutions of problem (3.1); therefore it will be sufficient to consider system (3.4) for small ϱ.

We shall find a finite-dimensional invariant manifold V for system (3.4), and we shall prove that it is attractive. Then the study of stability for system (3.4) will be reduced to a finite-dimensional problem.

Theorem 3.1. *Assume that* (3.2) *and* (3.3) *hold. There exists* $\varrho_1 > 0$ *such that, if* $\varrho \leqq \varrho_1$, *there exists a Lipschitz continuous function* $\gamma : X_2 \to D_{A_1}(\alpha + 1)$ *with* $\gamma(0) = 0$, *such that, if* $y_0 \in X_2$ *and* $x_0 = \gamma(y_0)$, *then the solution* (x, y) *of* (3.4) *is defined for all* $t \geqq 0$ *and* $x(t) = \gamma(y(t))$ $\forall\, t \geqq 0$.

Proof. We shall adapt the ideas from the semilinear case (see [4, Th. 6.1.2]). Fix $b > 0$ and set

$$\begin{aligned} Y = \{\gamma : X_2 \to D_{A_1}(\alpha + 1); \gamma(0) = 0, \|\gamma(y)\|_{\alpha+1} \leqq \varrho, \\ \|\gamma(y) - \gamma(\bar{y})\|_{\alpha+1} \leqq b \|y - \bar{y}\|\}, \end{aligned}$$

where $\varrho > 0$ is to be chosen later. Y is endowed with the topology of the sup norm

$$\|\gamma\|_\infty = \sup_{y \in X_2} \|\gamma(y)\|_{\alpha+1}$$

and it is closed with respect to this topology. Our goal now is to find a solution of (3.4) in the form $(\gamma(y(t)), y(t))$, for some $\gamma \in Y$. We shall first solve the system

$$z'(t) = A_2 z(t) + f_2(\gamma(z(t)), z(t)), \quad t \in \mathbb{R}, \qquad z(0) = y, \tag{3.7}$$

$$\gamma(y) = \int_{-\infty}^{0} e^{-sA_1} f_1(\gamma(z(s)), z(s))\, ds, \quad y \in X_2, \tag{3.8}$$

and then we shall prove that γ has all the required properties.

Concerning equation (3.7) we remark that for any $\gamma \in Y$, the solution $z = z(t, y, \gamma)$ is globally defined. Set now for $\varrho > 0$

$$L(\varrho) = \sup\left\{\left\|\frac{\partial f_1}{\partial x}(x, y)\right\|_{L(D_{A_1}(\alpha+1), D_{A_1}(\alpha))}, \left\|\frac{\partial f_1}{\partial y}(x, y)\right\|_{L(X_2, D_{A_1}(\alpha))}, \left\|\frac{\partial f_2}{\partial x}(x, y)\right\|_{L(D_{A_1}(\alpha+1), X_2)}, \left\|\frac{\partial f_2}{\partial y}(x, y)\right\|_{L(X_2)}; y \in X_2, \|y\| \leqq R_0 \varrho, x \in D_{A_1}(\alpha+1), \|x\|_{\alpha+1} \leqq \varrho\right\}. \tag{3.9}$$

Clearly $\lim_{\varrho \to 0} L(\varrho) = 0$. Let us choose ω_1 and ω_2 such that $\lambda_1 < \omega_1 < \omega_2 < 0$. Using the Gronwall lemma we get, for $t \leqq 0$, $y, \bar{y} \in X_2$, $\gamma, \bar{\gamma} \in Y$,

$$\|z(t, y, \gamma) - z(t, \bar{y}, \bar{\gamma})\| \leqq M_2 e^{-\mu t} \|y - \bar{y}\| + \frac{e^{-\mu t} - 1}{1 + b} \|\gamma - \bar{\gamma}\|_\infty, \tag{3.10}$$

where

$$\mu = \mu(\varrho) = M_2 L(\varrho)(1 + b) - \omega_2. \tag{3.11}$$

Let us consider now equation (3.8). For $\gamma \in Y$ set

$$(\Gamma\gamma)(y) = \int_{-\infty}^{0} e^{-sA_1} f_1(\gamma(z(s, y, \gamma)), z(s, y, \gamma))\, ds. \tag{3.12}$$

We shall prove that Γ is well defined, that $\Gamma(Y) \subset Y$ and that Γ is a contraction if ϱ is sufficiently small. Since $f_1(0, 0) = 0$, $\frac{\partial f_1}{\partial x}(0, 0) = 0$, $\frac{\partial f_1}{\partial y}(0, 0) = 0$ we have

$$\|f_1(\gamma(z(s, y, \gamma)), z(s, y, \gamma))\|_\alpha \leqq \varrho L(\varrho)(1 + R_0) \text{ for } s \leqq 0, \gamma \in Y, y \in X_2.$$

Hence there exists $\varrho_0 > 0$ such that for $\varrho \leqq \varrho_0$ formula (3.12) is meaningful and, by Proposition 1.2, $\|(\Gamma\gamma)(y)\|_{\alpha+1} \leqq \varrho$.

Now let $\gamma \in Y, y, \bar{y} \in X_2$, $s \leqq 0$; then by (3.10)

$$\|f_1(\gamma(z(s, y, \gamma)), z(s, y, \gamma)) - f_1(\gamma(z(s, \bar{y}, \gamma)), z(s, \bar{y}, \gamma))\|_\alpha \leqq M_2 L(\varrho)(1 + b) e^{-\mu s} \|y - \bar{y}\|.$$

Choose $\varrho_1 \in]0, \varrho_0]$ such that for $\varrho \leqq \varrho_1$ we have

$$\omega_1 + \mu < 0,$$

$$K_2(-\mu, \omega_1) M_2(1 + b) L(\varrho) \leqq b$$

where $K_2(-\mu, \omega_1)$ is given by Proposition 1.2 with A replaced by A_1. Then

$$\|(\Gamma\gamma)(y) - (\Gamma\gamma)(\bar{y})\|_{\alpha+1} \leqq b \|y - \bar{y}\|$$

so that $\Gamma(Y) \subset Y$. Finally, let $\gamma, \bar{\gamma} \in Y, y \in X_2, \; s \leqq 0$. By (3.10) we have

$$\|f_1(\gamma(z(s, y, \gamma)), z(s, y, \gamma)) - f_1(\bar{\gamma}(z(s, y, \bar{\gamma})), z(s, y, \bar{\gamma}))\|_\alpha$$
$$\leqq L(\varrho)\, e^{-\mu s} \|\gamma - \bar{\gamma}\|_\infty .$$

Therefore, choosing $\varrho_2 \in]0, \varrho_1]$ such that

$$L(\varrho)\, k_2(-\mu, \omega_1) \leqq \tfrac{1}{2} \quad \text{for } \varrho \leqq \varrho_2$$

we see that Γ is a $\frac{1}{2}$-contraction and there exists a unique fixed point γ of Γ in Y. For any $y_0 \in X_2$, set $x_0 = \gamma(y_0)$ and

$$x(t) = \gamma(z(t, y_0, \gamma)), y(t) = z(t, y_0, \gamma), \quad t \geqq 0;$$

then

$$x(t) = \int_{-\infty}^{0} e^{-sA_1} f_1(\gamma(z(t + s, y_0, \gamma)), z(t + s, y_0, \gamma))\, ds$$

$$= e^{tA_1} \gamma(y_0) + \int_0^t e^{(t-s)A_1} f_1(x(s), y(s))\, ds$$

so that $x \in C([0, +\infty[; D_{A_1}(\alpha + 1)) \cap C^1([0, +\infty[; D_{A_1}(\alpha))$ and (x, y) is the strict solution of system (3.4). □

In applications it is often necessary to use some properties of γ. These will be provided by the following theorem:

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1 assume that* $g \in C^k(D_A(\alpha + 1), D_A(\alpha))$, $k \geqq 2$. *Then there exists* $\varrho_k > 0$ *such that if* $\varrho \leqq \varrho_k$ *the function* γ *given by Theorem 3.1 is* $k - 1$ *times continuously differentiable and* $\gamma^{(k-1)}$ *is Lipschitz continuous.*

Sketch of the proof. The proof is similar to the preceding one, with the difference that now we have to look for a fixed point of the operator Γ defined by (3.12) in the set

$$Y_k = \{\gamma : X_2 \to D_{A_1}(\alpha + 1); \gamma(0) = 0, \|\gamma(y)\| \leqq \varrho,$$
$$\|\gamma^{(h)}(y)\| \leqq b_h, h = 1, \ldots, k - 1, \|\gamma^{(k-1)}(y) - \gamma^{(k-1)}(\bar{y})\| \leqq b_k \|y - \bar{y}\|\}.$$

Since Y_k is closed in the uniform topology, it is sufficient to show that Γ maps Y_k into itself if b_h, $h = 1, \ldots, k$ are suitably chosen. The proof is straightforward and relies on several applications of the Gronwall lemma and Proposition 1.2. □

We define now the center manifold for system (3.4):

$$V = \{(x, y) \in D_{A_1}(\alpha + 1) \times X_2; x = \gamma(y)\}. \tag{3.13}$$

Use of Theorem 3.1 reduces system (3.4) to the finite-dimensional equation

$$z'(t) = A_2 z(t) + f_2(\gamma(z(t)), z(t)), \quad t \geqq 0, \qquad z(0) = y_0, \tag{3.14}$$

provided $x_0 = \gamma(y_0)$.

We will show that V attracts the solutions of (3.4) for small initial data: more precisely, if $(x(t), y(t))$ is the solution of (3.4) in some interval $[0, \tau[$ and $\|x_0\|_{\alpha+1}$ is sufficiently small, then

$$\|x(t) - \gamma(y(t))\|_{\alpha+1} \leqq ce^{\omega_1 t} \|x_0 - \gamma(y_0)\|_{\alpha+1}, \quad \forall\, t \in [0, \tau[,$$

where $\omega_1 \in]\lambda_1, 0[$ and the constant c does not depend on x_0 and y_0. Therefore the study of the asymptotic behavior of the solutions of (3.4) will be reduced to a finite-dimensional problem.

Theorem 3.3. *Assume* (3.2) *and* (3.3) *hold. Then, for each* $\omega_1 \in]\lambda_1, 0[$ *there exists* $\bar{\varrho} > 0$ *such that for* $\varrho \leqq \bar{\varrho}$, $x_0 \in D_{A_1}(\alpha + 1)$, $y_0 \in X_2$ *with* $\|x_0\|_{\alpha+1} \leqq \frac{\varrho}{2M_1}$, *the solution* $(x(t), y(t))$ *of* (3.4) *satisfies*

$$\|x(t) - \gamma(y(t))\|_{\alpha+1} \leqq 2M_1 e^{\omega_1 t} \|x_0 - \gamma(y_0)\|_{\alpha+1}, \quad \forall\, t \in [0, \tau[. \tag{3.15}$$

Here $M_1 = M_1(\omega_1)$ *is given by* (1.13).

Proof. Let $\varrho \leqq \varrho_1$, where ϱ_1 is given by Theorem 3.1, so that there exists a center manifold for system (3.4).

First we shall show that, if ϱ and $\|x_0\|_{\alpha+1}$ are sufficiently small, then $\|x(t)\|_{\alpha+1} \leqq \varrho$ for each $t \in [0, \tau[$.

Let $\varrho' > 0$ be such that $L(\varrho') \leqq \dfrac{1}{2K_1(0, \omega_1)\,(1 + R_0)}$ where $L(\varrho)$ is defined in (3.9) and K_1 is given by Proposition 1.1 with A replaced by A_1. Assume $0 < \varrho < \varrho'$, $\|x_0\|_{\alpha+1} \leqq \frac{\varrho}{2M_1}$, $y_0 \in X_2$. To reach a contradiction, suppose that $I = \{t \in]0, \tau[;\, \|x(t)\|_{\alpha+1} > \varrho\}$ is not void, and set $t_0 = \inf I$. Since

$$x(t_0) = e^{t_0 A_1} x_0 + \int_0^{t_0} e^{(t_0 - s)A_1} f(x(s), y(s))\, ds$$

then

$$\begin{aligned} \varrho &\leqq M_1 \|x_0\|_{\alpha+1} + K_1(0, \omega_1) \sup_{0 \leqq s \leqq t_0} \|f(x(s), y(s))\|_\alpha \\ &\leqq \frac{\varrho}{2} + K_1(0, \omega_1)\,(1 + R_0)\, L(\varrho)\, \varrho < \varrho \end{aligned}$$

so that I is empty and the statement is proved. Now fix $\omega_2 \in]\omega_1, 0[$ and set

$$\zeta(t) = x(t) - \gamma(y(t)) = e^{tA_1} \zeta(0) + I_1(t) + I_2(t), \quad 0 \leqq t < \tau \tag{3.16}$$

where

$$I_1(t) = \int_0^t e^{(t-s)A_1} h_1(s, t)\, ds, I_2(t) = e^{tA_1} \int_{-\infty}^0 e^{-sA_1} h_1(s, t)\, ds, \quad 0 \leqq t < \tau,$$

and

$$h_1(s, t) = f_1(x(s), y(s)) - f_1(\gamma(w(s, t)), w(s, t)),$$

$$h_2(s, t) = f_1(\gamma(z(s)), z(s)) - f_1(\gamma(w(s, t)), w(s, t)).$$

Here $z(s) = z(s, y_0, \gamma)$ is the solution of (3.7) with $y = y_0$, and $w(s, t) = z(s - t, y(t), \gamma)$ is the solution of

$$\frac{d}{ds} w(s, t) = A_2 w(s, t) + f_2(\gamma(w(s, t)), w(s, t)), \quad s \in \mathbb{R},$$

$$w(t, t) = y(t).$$

Then the following estimates hold:

$$\| w(s, t) - y(s) \| \leqq M_2 L(\varrho) \int_s^t e^{\mu(\sigma - s)} \| \zeta(\sigma) \|_{\alpha+1}\, ds, \quad 0 \leqq s \leqq t < \tau, \quad (3.17)$$

$$\| z(s) - w(s, t) \| \leqq M_2 L(\varrho) \int_0^t e^{\mu(\sigma - s)} \| \zeta(\sigma) \|_{\alpha+1}\, ds, \quad s \leqq 0, 0 \leqq t < \tau, \quad (3.18)$$

where $\mu = \mu(\varrho)$ is given by (3.11) and $\omega_1 + \mu < 0$. (3.17) is an easy consequence of the estimate $\|x(t)\|_{\alpha+1} \leqq \varrho$ and of the Gronwall lemma. In order to obtain (3.18) it is sufficient to observe that

$$z(s) - w(s, t) = (z(s) - y(s)) + (y(s) - w(s, t))$$

$$= \int_0^t e^{(s-\sigma)A_2} k_1(\sigma, t)\, d\sigma + \int_s^0 e^{(s-\sigma)A_2} k_2(\sigma, t)\, d\sigma$$

with

$$k_1(s, t) = f_2(\gamma(w(s, t)), w(s, t)) - f_2(x(s), y(s)),$$

$$k_2(s, t) = f_2(\gamma(w(s, t)), w(s, t)) - f_2(\gamma(z(s)), z(s)).$$

Then (3.18) follows by (3.17) and the Gronwall lemma. Therefore

$$\| h_1(s) \|_\alpha \leqq M_2(1 + b)\, (L(\varrho))^2 \int_s^t e^{\mu(\sigma - s)} \| \zeta(\sigma) \|_{\alpha+1}\, d\sigma + L(\varrho) \| \zeta(s) \|_{\alpha+1},$$
$$0 \leqq s \leqq t < \tau, \quad (3.19)$$

$$\| h_2(s) \|_\alpha \leqq M_2(1 + b)\, (L(\varrho))^2 \int_0^t e^{\mu(\sigma - s)} \| \zeta(s) \|_{\alpha+1}\, d\sigma, \quad s \leqq 0 \leqq t < \tau. \quad (3.20)$$

Then, by (3.19) and Proposition 1.1, we have, for $0 \leqq t < \tau$,

$$\| e^{-\omega_1 t} I_1(t) \|_{\alpha+1} \leqq K_1 \left(-\omega_1, \frac{\omega_1 + \lambda_1}{2} \right) \sup_{0 \leqq s \leqq t} \| e^{-\omega_1 s} h_1(s) \|_\alpha$$

$$\leqq c_1(\varrho) \sup_{0 \leqq s \leqq t} \| e^{-\omega_1 s} \zeta(s) \|_{\alpha+1}$$

where $c_1(\varrho) = K_1(-\omega_1, (\omega_1 + \lambda_1)/2)\, L(\varrho) \left(1 + \dfrac{M_2(1+b)\, L(\varrho)}{|\omega_1 + \mu|} \right)$ and K_1 is given by Proposition 1.1 with $A = A_1$.

By Proposition 1.2 and (3.20) we have also, for $0 \leqq t < \tau$:

$$\| e^{-\omega_1 t} I_2(t) \|_{\alpha+1} \leqq K_2(-\mu, \omega_1)\, M_1 \sup_{s \leqq 0} \| e^{\mu s} h_2(s, t) \|_\alpha$$

$$\leqq c_2(\varrho) \sup_{0 \leqq s \leqq t} \| e^{-\omega_1 s} \zeta(s) \|_{\alpha+1}$$

where $c_2(\varrho) = K_2(-\mu(\varrho), \omega_1)\, M_1 M_2 (1 + b)\, (L(p))^2\, |\omega_1 + \mu(\varrho)|^{-1}$ and K_2 is given by Proposition 1.2 with $A = A_1$.

Let $\varrho'' > 0$ be such that $c_1(\varrho) + c_2(\varrho) < \frac{1}{2}$ $\forall\, \varrho \in]0, \varrho''[$. Then, for $0 < \varrho < \bar{\varrho} = \min\{\varrho_1, \varrho', \varrho''\}$ we get, from (3.16)

$$\| e^{-\omega_1 t} \zeta(t) \|_{\alpha+1} \leqq 2M_1 \| \zeta(0) \|_{\alpha+1}, \quad 0 \leqq t < \tau,$$

and the assertion of the theorem follows. □

We now give the following definition of stability. A compact subset $\Sigma \subset B(0, R_0) \subset D_A(\alpha + 1)$ is said to be *uniformly asymptotically stable* for the dynamical system defined by (3.1) if

i) for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\operatorname{dist}(u_0, \Sigma) \leqq \delta_\varepsilon$ then $\operatorname{dist}(u(t), \Sigma) \leqq \varepsilon$, $\forall\, t \in [0, \tau(u_0)[$;

ii) there is $\bar{\delta} > 0$ such that for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that if $\operatorname{dist}(u_0, \Sigma) \leqq \bar{\delta}$ and $t < \tau(u_0)$, $t \geqq T_\varepsilon$, then $\operatorname{dist}(u(t), \Sigma) \leqq \varepsilon$.

Obviously, the distance must be understood in $D_A(\alpha + 1)$.

We remark that if Σ is uniformly asymptotically stable and if u_0 is sufficiently close to Σ, then the orbit $\{u(t); t \in [0, \tau(u_0)[\}$ is bounded in $D_A(\alpha + 1)$. Unfortunately this does not imply existence in the large of the solution in $B(0, R_0) \subset D_A(\alpha + 1)$ and we cannot conclude that $\lim_{t \to \infty} \operatorname{dist}(u(t), \Sigma) = 0$. Nevertheless, such a conclusion can be derived by Proposition 2.1 if the norm $\| \cdot \|_{\alpha+1}$ is replaced by $\| \cdot \|_{\alpha+1-\delta}$, $0 < \delta < \alpha + 1$.

We shall now give another theorem on stability.

Theorem 3.4. *Assume* (3.2) *and* (3.3) *hold and that* $\Sigma \subset X_2$ *is a compact uniformly asymptotically stable set for the dynamical system defined by* (3.14), *with* $\varrho \leqq \bar{\varrho}$. *Assume also* $\|y\| \leqq \dfrac{\varrho}{8M_1 b}$ *for each* $y \in \Sigma$, *where* M_1 *is given by* (1.13) *and* b *is the Lipschitz constant of* γ. *Then the set*

$$\Sigma_V = \{(x, y) \in D_{A_1}(\alpha + 1) \times X_2 ; y \in \Sigma, x = \gamma(y)\}$$

is uniformly asymptotically stable for system (3.4).

Proof. For $\varepsilon > 0$, let $I(\Sigma, \varepsilon) = \{y \in X_2; \operatorname{dist}(y, \Sigma) \leqq \varepsilon\}$. By known results (see [4, Th. 4.2.1}] and [10]), for small ε there exists a Liapunov function $Q : I(\Sigma, \varepsilon) \to [0, +\infty[$ such that

$$|Q(y) - Q(\bar{y})| \leqq \|y - \bar{y}\| \quad \forall\, y, \bar{y} \in I(\Sigma, \varepsilon),$$

$$Q(z(t, y, \gamma)) \leqq e^{-t} Q(y) \text{ for each } t \geqq 0 \text{ such that } z(t, y, \gamma) \in I(\Sigma, \varepsilon),$$

$$\alpha\,(\operatorname{dist}(y, \Sigma)) \leqq Q(y) \leqq \operatorname{dist}(y, \Sigma),$$

where $\alpha(\cdot)$ is a positive, continuous, increasing function and $\alpha(0) = 0$. Let $\varepsilon \leqq \dfrac{\varrho}{8M_1 b}$. Following [4, Th. 6.1.4] we define a Liapunov function for system (3.4):

$$W(x, y) = Q(y) + p\,\|x - \gamma(y)\|_{\alpha+1}, \quad y \in I(\Sigma, \varepsilon), \quad x \in D_{A_1}(\alpha + 1),$$

where $p \geqq 1$ is a constant to be chosen later. Then we have

$$\beta\,(\operatorname{dist}((x, y), \Sigma_V)) \leqq W(x, y) \leqq p(1 + b) \operatorname{dist}((x, y), \Sigma_V) \tag{3.21}$$

where the distance is in the norm of $X_2 \times D_{A_1}(\alpha + 1)$ (which is equivalent to the norm of $D_A(\alpha + 1)$), and $\beta^{-1}(\xi) = \xi + (1 + b)\,\alpha^{-1}(\xi)$. Using the Gronwall inequality and (3.15), we find that, if $\|x_0\|_{\alpha+1} \leqq \dfrac{\varrho}{2M_1}$,

$$\|z(t, y_0, \gamma) - y(t)\| \leqq c_1 e^{c_2 t}\, \|x_0 - \gamma(y_0)\|_{\alpha+1} \tag{3.22}$$

where

$$c_1 = \frac{2M_1 M_3 L(\varrho)}{\omega_3 - \omega_1}, \quad c_2 = L(\varrho)\, M_3(1 + b) + \omega_3;$$

$\omega_3 > \sup\{\operatorname{Re}\lambda; \lambda \in \sigma(\tilde{A})\}$, M_3 is given by (1.14). For $t \geqq 0$ we have

$$W(x(t), y(t)) = Q(z(t, y_0, \gamma)) + [Q(y(t)) - Q(z(t, y_0, \gamma))] + p\,\|x(t) - \gamma(y(t))\|_{\alpha+1}.$$

For $T \geqq 0$ and $t \in [T, 2T]$ we have, by (3.22) and (3.15)

$$W(x(t), y(t)) \leqq e^{-T} Q(y_0) + [c_1 e^{c_2 t} + 2M_1 p e^{\omega_1 T}]\, \|x_0 - \gamma(y_0)\|_{\alpha+1}.$$

Now fix T so that

$$e^{-T} \leqq \tfrac{1}{2}, \quad 2M_1 e^{\omega_1 T} \leqq \tfrac{1}{4}, \quad \alpha^{-1}(e^{-T}\varepsilon) \leqq \frac{\varepsilon}{2}, \quad \frac{\varepsilon}{2c_1} e^{-c_2 T} \leqq \frac{\varrho}{4M_1}$$

and choose p so large that $c_1 e^{2c_2 T} \leqq \dfrac{p}{8}$; then

$$W(x(t), y(t)) \leqq \tfrac{1}{2}\, W(x_0, y_0) \quad \forall\, t \in [T, 2T]. \tag{3.23}$$

Finally, setting

$$U = \{(x, y) \in D_{A_1}(\alpha + 1) \times X_2;\ y \in I(\Sigma, \varepsilon), \|x - \gamma(y)\|_{\alpha+1} \leqq \frac{\varepsilon}{2c_1} e^{-c_2 T}\}$$

we have, for $(x_0, y_0) \in U$, $\|x_0\|_{\alpha+1} \leqq \frac{\varepsilon}{2c_1} e^{-c_2 T} + b\,\|y\| \leqq \frac{\varrho}{2M_1}$,

$$\begin{aligned}\operatorname{dist}(y(T), \Sigma) &\leqq \|y(T) - z(T, y_0, \gamma)\| + \operatorname{dist}(z(T, y_0, \gamma), \Sigma)\\ &\leqq c_1 e^{c_2 T}\|x_0 - \gamma(y_0)\|_{\alpha+1} + \alpha^{-1}(Q(z(T, y_0, \gamma)))\\ &\leqq \frac{\varepsilon}{2} + \alpha^{-1}(e^{-T}\varepsilon) \leqq \varepsilon\end{aligned}$$

and

$$\|x(T) - \gamma(y(T))\|_{\alpha+1} \leqq 2M_1 e^{\omega_1 T}\|x_0 - \gamma(y_0)\|_{\alpha+1} \leqq \frac{\varepsilon e^{-c_2 T}}{16 c_1}.$$

Therefore $(x(T), y(T))$ belongs to U, and we can repeat the previous argument to get

$$W(x(t), y(t)) \leqq 2^{-n}\, W(x_0, y_0) \qquad \forall\, t \in [nT, (n+1)T], \qquad n \in \mathbb{N},$$

and so $W(x(t), y(t)) \leqq 2e^{-\frac{t}{T}\log 2}\, W(x_0, y_0)$. This, together with (3.21), implies that Σ_V is uniformly asymptotically stable for system (3.4). □

4. Applications to Critical Cases of Stability

We first consider stability of the zero solution of equation (3.1) when 0 is a simple eigenvalue of $\tilde{A}$ and no other point of the spectrum lies on the imaginary axis. On the assumption (3.2), for any $u_0 \in B(0, R_0) \subset D_A(\alpha + 1)$ there exists a unique $u\colon [0, \tau(u_0)] \to B(0, R_0) \subset D_A(\alpha + 1)$ that solves the initial value problem (see Proposition 2.1)

$$\begin{aligned} u'(t) &= g(u(t)), \qquad t \geqq 0,\\ u(0) &= u_0. \end{aligned} \tag{4.1}$$

The main assumption here is

$$\text{(1.11) holds and } \sigma_2(\tilde{A}) \text{ consists in the simple eigenvalue } 0. \tag{4.2}$$

Then there exist $\xi_0 \neq 0$ such that $A\xi_0 = 0$ and $\phi_0 \in X^*$ such that $P_2 x = \langle x, \phi_0\rangle$ and $\langle \xi_0, \phi_0\rangle = 1$.

Theorem 4.1. *Assume that* (3.2) *and* (4.2) *hold and moreover that* $g \in C^4(D_A(\alpha + 1), D_A(\alpha))$ *and*

$$\langle g''(0)(\xi_0, \xi_0), \phi_0\rangle = 0. \tag{4.3}$$

Set

$$a = \langle g'''(0)(\xi_0, \xi_0, \xi_0), \phi_0\rangle - 3\langle g''(0)(\xi_0, A_1^{-1}P_1 g''(0)(\xi_0, \xi_0)), \phi_0\rangle. \tag{4.4}$$

Then, if $a < 0$, 0 *is asymptotically stable in* $D_A(\alpha + 1)$ *and, if* $a > 0$, 0 *is unstable.*

Proof. As in Section 3, we consider the system

$$\begin{aligned} x'(t) &= A_1 x(t) + f_1(x(t), y(t)), \quad x(0) = x_0, \\ y'(t) &= f_2(x(t), y(t)), \quad y(0) = y_0, \end{aligned} \tag{4.5}$$

where f_1 and f_2 are defined by (3.5) (with $A_2 = 0$), and $x_0 = P_1 u_0$, $y_0 = P_2 u_0$. Now we choose ϱ so small that the results of Theorems 3.1, 3.2, 3.3 are applicable. We remark that if (x, y) is a solution of (4.5) and $\|y(t)\| \leqq \frac{1}{2}\varrho R_0$, then $u(t) = x(t) + y(t)$ solves (4.1); conversely, if u is the solution of (4.1) and $\|P_2 u(t)\| \leqq \frac{1}{2}\varrho R_0$, then $(P_1 u, P_2 u)$ is the solution of (4.5).

From Theorems 3.1 and 3.2 there exists a center manifold for system (4.5):

$$V = \{(x, y) \in D_{A_1}(\alpha + 1) \times X_2;\, x = \gamma(y)\}$$

with $\gamma \in C^3(X_2, D_{A_1}(\alpha + 1))$.

By account of Theorems 3.3 and 3.4 our problem is reduced to studying the stability of 0 for the scalar equation

$$z'(t) = f_2(\gamma(z(t)), z(t)) = \psi(z(t)), \quad t \geqq 0. \tag{4.6}$$

By (3.8) we get

$$\gamma'(0) = 0, \quad \gamma''(0) = \int_{-\infty}^{0} e^{-sA_1} P_1 g''(0)\, ds = -A_1^{-1} P_1 g''(0).$$

It follows that

$$\begin{aligned} &\psi'(0) = 0,\ \psi''(0) = P_2 g''(0) = \langle g''(0)(\cdot,\cdot), \phi_0 > \xi_0, \\ &\psi'''(0) = P_2 g'''(0)(\cdot,\cdot,\cdot) + 3P_2 g''(0)(\cdot, \gamma''(0)(\cdot,\cdot)). \end{aligned}$$

Since $\psi'(0) = \psi''(0) = 0$, $\psi'''(0)(\xi_0, \xi_0, \xi_0) = a \neq 0$, the assertion of the theorem follows easily. □

Remark 4.1. Under the assumptions of Theorem 4.1, for $\|u_0\|_{\alpha+1}$ sufficiently small, the solution $u(t, u_0)$ is globally defined with values in $D_A(\alpha + 1 - \eta)$ if $0 < \eta < \alpha + 1$. Moreover

$$\lim_{t \to +\infty} \|u(t, u_0)\|_{\alpha+1-\eta} = 0$$

(see Proposition 2.1). □

An example. Let us consider again problem (2.11) with the notations of the example in Section 2. We assume (2.12) and (2.13) and we set

$$\phi_i = \frac{\partial \phi}{\partial p_i}(0,0,0), \quad \phi_{ij} = \frac{\partial^2 \phi}{\partial p_i\, \partial p_j}(0,0,0), \quad \phi_{ijk} = \frac{\partial^3 \phi}{\partial p_i\, \partial p_j\, \partial p_k}(0,0,0),$$
$$i, j, k = 1, 2, 3.$$

We assume also

$$\phi_1 = \phi_3 \tag{4.7}$$

so that (4.2) is satisfied. Set

$$\xi_0(x) = \sqrt{\frac{2}{\pi}} \sin x, \qquad 0 \leqq x \leqq \pi,$$

$$(P_2 u)(x) = \frac{2}{\pi} \int_0^\pi u(y) \sin y \, dy \sin x.$$

Now we have

$$g''(0)(\xi_0, \xi_0)(x) = \frac{2}{\pi} [(\phi_{11} - 2\phi_{13} + \phi_{33}) \sin^2 x + 2(\phi_{12} - \phi_{23}) \sin x \cos x],$$

$$0 \leqq x \leqq \pi,$$

and (4.3) is satisfied if and only if

$$\phi_{11} - 2\phi_{13} + \phi_{33} = 0. \tag{4.8}$$

Let us assume, for simplicity, that

$$\phi_{12} = \phi_{23}. \tag{4.9}$$

Then (4.4) reduces to

$$a = \langle g'''(0)(\xi_0, \xi_0, \xi_0), \phi_0 \rangle = \frac{3}{2\pi}(\phi_{111} + 3\phi_{133} - 3\phi_{311} - \phi_{333})$$

$$+ \frac{1}{\pi}(\phi_{122} - \phi_{322}). \tag{4.10}$$

Theorem 4.2. *Let* $\phi \in C^\infty(\mathbb{R}^3; \mathbb{R})$ *satisfy* (2.12), (2.13), (4.7), (4.8), (4.9). *Let* a *be given by* (4.10). *Then, if* $a < 0$, 0 *is asymptotically stable in* $h_0^{2+2\alpha}(0, \pi)$ *and, if* $a > 0$, 0 *is unstable.* □

We wish now to study existence and stability for small periodic solutions of the equation

$$u'(t) = f(\lambda, u(t)) \tag{4.11}$$

under the hypotheses

$$\left.\begin{array}{l} f(\lambda, 0) = 0,\ f \in C^1(]-1, 1[\times D; X) \text{ and } A = f_x(0,0): D \to X \text{ satisfies (1.2)}, \\ \pm i \text{ are simple eigenvalues of } \tilde{A},\ \sigma_2(\tilde{A}) = \{+i, -i\}, \\ f \in C^5(]-1, 1[\times D_A(\alpha+1); D_A(\alpha)). \end{array}\right\} \tag{4.12}$$

By standard arguments it can be shown that there exist $R_0 > 0$, $\lambda_0 \in]0, 1[$ such that for each $\lambda \in]-\lambda_0, \lambda_0[$ and $x \in B(0, R_0) \subset D_A(\alpha + 1)$, the operator $f_x(\lambda, x): D_A(\alpha + 1) \to D_A(\alpha)$ satisfies (1.2) (with X replaced by $D_A(\alpha)$) and $\sup\{\|f_x(\lambda, x)\|_{L(D_A(\alpha+1), D_A(\alpha))};\ |\lambda| < \lambda_0, \|x\|_{\alpha+1} < R_0\} < +\infty$. Moreover there exists a continuously differentiable path $]-\lambda_0, \lambda_0[\to \mathbb{C}, \lambda \to \alpha(\lambda) + i\beta(\lambda)$, such that

$$\alpha(\lambda), \beta(\lambda) \in \mathbb{R},$$
$$\alpha(0) = 0, \beta(0) = 1,$$

$\alpha(\lambda) \pm i\beta(\lambda)$ are simple isolated eigenvalues of the operator $\tilde{A}(\lambda)$, where $A(\lambda) = f_x(\lambda, 0)$.

The usual transversality assumption is

$$\alpha'(0) \neq 0. \tag{4.13}$$

Proceeding as in Section 3, we consider the system

$$\begin{aligned} x'(t) &= A_1 x(t) + f_1(\lambda, x(t), y(t)), \quad x(0) = x_0, \\ y'(t) &= A_2 y(t) + f_2(\lambda, x(t), y(t)), \quad y(0) = y_0, \end{aligned} \tag{4.14}$$

where

$$f_1 :]-1, 1[\times D_{A_1}(\alpha + 1) \times X_2 \to D_{A_1}(\alpha), \; f_1(\lambda, x, y) = P_1 f\left(\lambda, x + \psi\left(\frac{y}{\varrho}\right) y\right) - A_1 x$$

$$f_2 :]-1, 1[\times D_{A_1}(\alpha + 1) \times X_2 \to X_2,$$

$$f_2(\lambda, x, y) = P_2 f\left(\lambda, x + \psi\left(\frac{y}{\varrho}\right) y\right) - \psi\left(\frac{y}{\varrho}\right) A_2 y$$

and $\psi : X_2 \to \mathbb{R}$ is a C^∞ function satisfying (3.5); ϱ is a positive number to be chosen later.

We have now to generalize the results of Section 3 to the case of equations depending on a parameter λ. This generalization is quite standard and the proofs will be only outlined.

Theorem 4.3. *Assume* (4.12) *holds. There exists* $\bar{\varrho} > 0$ *and a four times continuously differentiable function* $\gamma :]-\bar{\varrho}, \bar{\varrho}[\times X_2 \to D_{A_1}(\alpha + 1)$, *with* $\gamma(\lambda, 0) = 0$, *such that if* $\varrho \leqq \bar{\varrho}, y_0 \in X_2, x_0 + \gamma(\lambda, y_0), \lambda \in]-\bar{\varrho}, \bar{\varrho}[$, *then the solution* (x, y) *of* (4.14) *is defined for all* $t \geqq 0$ *and* $x(t) = \gamma(\lambda, y(t)) \;\; \forall\, t \geqq 0$.

Proof. We shall use a standard device, setting

$$\hat{y} = (\lambda, y), \quad \|\hat{y}\| = |\lambda| + \|y\|$$

and writing the first equation of (4.14) as

$$\hat{y}'(t) = \hat{A}_2 \hat{y}(t) + \hat{f}_2(x(t), \hat{y}(t))$$

where

$$\hat{A}_2 \hat{y} = \hat{A}_2(\lambda, y) = (0, A_2 y);$$
$$\hat{f}_2(x, \hat{y}) = (0, f_2(\lambda, x, y)).$$

Furthermore writing

$$\hat{f}_1(x, \hat{y}) = (0, f_1(\lambda, x, y))$$

we get the following system, which is analogous to system (3.7)–(3.8):

$$\hat{z}'(t) = \hat{A}_2\hat{z}(t) + \hat{f}_2(\gamma(\hat{z}(t)), \hat{z}(t)), \quad t \in \mathbb{R},$$
$$\hat{z}(0) = \hat{y}, \tag{4.15}$$
$$\gamma(\hat{y}) = \int_{-\infty}^{0} e^{-sA_1}\hat{f}_1(\gamma(\hat{z}(s)), \hat{z}(s))\,ds, \quad \hat{y} \in]-\lambda_0, \lambda_0[\times X_2.$$

We have to find the solution in the set

$$\hat{Y} = \{\gamma :]-\lambda_0, \lambda_0[\times X_2 \to D_{A_1}(\alpha+1); \gamma(\lambda, 0) = 0, \|\gamma(y)\| \leqq \varrho,$$
$$\|\gamma^{(h)}(\hat{y})\| \leqq b_h, \quad h = 1, 2, 3, \quad \|\gamma^{(3)}(\hat{y}) - \gamma^{(3)}(\hat{y}')\| \leqq b_4 \|\hat{y} - \hat{y}'\|\}.$$

Now the proof that such a solution exists is based on arguments used in the proof of Theorems 3.1 and 3.2 and so it will be omitted. □

The proof of the following result is similar the proof of Theorem 3.3.

Theorem 4.4. *Assume* (4.12) *holds. For* $|\lambda| < \bar{\varrho}$, $x_0 \in D_{A_1}(\alpha+1)$, $y_0 \in X_2$ *let* $(x(t), y(t))$ *be the solution of* (4.14) *in some interval* $[0, \tau[$. *Then there exist* $\varrho_1 \in]0, \bar{\varrho}[$, $c_1, c_2 > 0$, $\omega_1 < 0$, *such that for* $\varrho < \varrho_1$, $|\lambda| < \varrho_1$ *and* $\|x_0\|_{\alpha+1} \geqq c_2\varrho$ *we have*

$$\|x(t) - \gamma(\lambda, y(t))\|_{\alpha+1} \leqq c_1 e^{\omega_1 t}\|x_0 - \gamma(\lambda, y_0)\|_{\alpha+1} \quad \forall\, t \in [0, \tau[. \quad \square \tag{4.16}$$

We now set

$$V_\lambda = \{(x, y) \in D_{A_1}(\alpha+1) \times X_2; x = \gamma(\lambda, y)\}$$
$$= \{u_0 \in D_A(\alpha+1); P_1u_0 = \gamma(\lambda, P_2u_0)\}$$

and assume that a compact set $\Sigma(\lambda) \subset X_2$ is uniformly asymptotically stable for the dynamical system defined by the equation

$$z'(t) = A_2z(t) + f_2(\lambda, \gamma(\lambda, z(t)), z(t)) = h(\lambda, z(t)) \tag{4.17}$$

for some $\lambda \in]-\bar{\varrho}, \bar{\varrho}[$ and $\varrho \in]0, \bar{\varrho}[$. Then, setting

$$\Sigma_V(\lambda) = \{(x, y) \in D_{A_1}(\alpha+1) \times X_2; y \in \Sigma(\lambda), x = \gamma(\lambda, y)\}$$

and arguing as in Theorem 3.4, we can see that if dist $(\Sigma_V(\lambda), 0)$ is sufficiently small, $\Sigma_V(\lambda)$ is uniformly asymptotically stable for system (4.14). Theorem 4.3 assures that V_λ is attracting (for initial data near 0). Hence if (4.11) has a small periodic solution u, then $u(t) \in V_\lambda$ for each t. Our problem is now reduced to looking for small periodic solutions of (4.17). We have

$$h \in C^4(]-\varrho_1, \varrho_1[\times X_2; X_2), h(\lambda, 0) = 0, h_x(0, 0) = A_2,$$

and upon setting

$$\hat{A}(\lambda) = h_x(\lambda, 0)$$

we see that there exists $\varrho_2 \in]0, \varrho_1]$ and a continuously differentiable function $]-\varrho_2, \varrho_2[\to \mathbb{C}$, $\lambda \to \hat{\alpha}(\lambda) + i\hat{\beta}(\lambda)$ such that

$$\hat{\alpha}(\lambda), \quad \hat{\beta}(\lambda) \in \mathbb{R},$$
$$\hat{\alpha}(0) = 0, \quad \hat{\beta}(0) = 1,$$
$$\hat{\alpha}(\lambda) \pm i\hat{\beta}(\lambda) \text{ are simple eigenvalues of } (\hat{A}(\lambda))^{\sim}.$$

It is easy to show that

$$\alpha'(0) \neq 0 \Leftrightarrow \hat{\alpha}'(0) \neq 0.$$

Then to study existence and stability of small periodic solutions of equation (4.11), it suffices to use the standard results for the two-dimensional case (see for instance [7]). In particular, conditions (4.12) and (4.13) ensure that there exist small periodic solutions of equation (4.17). More precisely, there exist $\sigma_0 > 0$, $r_0 > 0$, and C^3 functions $\lambda :]-\sigma_0, \sigma_0[\to \mathbb{R}$, $p :]-\sigma_0, \sigma_0[\to \mathbb{R}$, $z :]-\sigma_0, \sigma_0[\to B(0, r_0) \subset C^1(\mathbb{R}; X_2)$ with $\lambda(0) = 0$, $p(0) = 2\pi$, $z(0) = 0$, such that $z(\sigma)$ is $p(\sigma)$-periodic and not constant for $\sigma \neq 0$, and

$$\frac{d}{dt} z(\sigma)(t) = h(\lambda(\sigma), z(\sigma)(t)), \quad t \in \mathbb{R}, \sigma \in]-\sigma_0, \sigma_0[. \tag{4.18}$$

Concerning the stability of the orbit described by $z(\sigma)$, one can use the standard methods of [7].

Now using Theorem 4.3 and 4.4 we get the following result:

Theorem 4.5. *Assume that* (4.12) *and* (4.13) *hold. Then there exist* $\sigma_0 > 0$ *and* C^3 *functions* $\lambda :]-\sigma_0, \sigma_0[\to \mathbb{R}, p :]-\sigma_0, \sigma_0[\to \mathbb{R}$, $u :]-\sigma_0, \sigma_0[\to C(\mathbb{R}; D_A(\alpha + 1)) \cap C^1(\mathbb{R}; D_A(\alpha))$ *such that*

$$\lambda(0) = 0, \quad p(0) = 2\pi, \quad u(0) = 0,$$
$$\text{for } \sigma \neq 0, u(\sigma) \text{ is not constant and is } p(\sigma)\text{-periodic}, \tag{4.19}$$
$$\frac{d}{dt} u(\sigma)(t) = f(\lambda(\sigma), u(\sigma)(t)), \quad t \in \mathbb{R}, \quad \sigma \in]-\sigma_0, \sigma_0[.$$

Moreover, if for some $\sigma \in]\sigma_0, \sigma_0[$ *the set* $\Sigma = \{z(\sigma)(t) = P_2 u(\sigma)(t); t \in \mathbb{R}\}$ *is uniformly asymptotically stable in* X_2 *for the dynamical system defined by* (4.17), *then* $\Sigma_V = \{u(\sigma)(t); t \in \mathbb{R}\}$ *is uniformly asymptotically stable in* $D_A(\alpha + 1)$ *for the dynamical system defined by* (4.11). □

Another approach to the study of existence of small periodic solutions of (4.11) can be found in [3].

An Example. Consider the system

$$u_t(t, x) = \phi(\lambda, u(t, x), v(t, x), u_{xx}(t, x)), \quad t \in \mathbb{R}, \quad x \in [0, \pi],$$
$$v_t(t, x) = \psi(\lambda, u(t, x), v(t, x)), \quad t \in \mathbb{R}, \quad x \in [0, \pi], \tag{4.20}$$
$$u(t, 0) = u(t, \pi) = 0, \quad t \in \mathbb{R}.$$

Assume that

$$\begin{aligned} &\phi \in C^\infty(]-1,1[\times \mathbb{R}^3;\mathbb{R}), \quad \phi(\lambda,0)=0,\\ &\psi \in C^\infty(]-1,1[\times \mathbb{R}^2;\mathbb{R}), \quad \psi(\lambda,0)=0, \end{aligned} \tag{4.21}$$

Choose

$$X = C([0,\pi]) \oplus C_0^2([0,\pi]),$$

$$D = C_0^2([0,\pi])) \oplus C_0^2([0,\pi]),$$

and endow X and D with their natural norms. Setting

$$f\colon (]-1,1[\times D) \to X, \quad f\left(\lambda, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{bmatrix} \phi(\lambda,u,v,u'') \\ \psi(\lambda,u,v) \end{bmatrix}$$

we can write system (4.20) in the form (4.11). Clearly $f \in C^\infty(]-1,1[\times D;X)$ and moreover

$$A(\lambda)\begin{pmatrix} u \\ v \end{pmatrix} = f_x(\lambda,0)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \phi_1(\lambda)\,u + \phi_2(\lambda)\,v + \phi_3(\lambda)\,u'' \\ \psi_1(\lambda)\,u + \psi_2(\lambda)\,v \end{bmatrix}$$

where

$$\phi_i(\lambda) = \frac{\partial\phi}{\partial p_i}(\lambda,0),\ \psi_j(\lambda) = \frac{\partial\psi}{\partial p_j}(\lambda,0), \quad i=1,2,3, \quad j=1,2.$$

Assume

$$\phi_1(0) \neq 0. \tag{4.22}$$

Then, for $|\lambda|$ sufficiently small, the spectrum of $A(\lambda)$ consists in the solutions of the equations

$$\begin{aligned} &\xi^2 - \xi(-k^2\phi_3(\lambda) + \psi_2(\lambda) + \phi_1(\lambda)) - k^2\psi_2(\lambda)\,\phi_3(\lambda)\\ &\quad + \psi_2(\lambda)\,\phi_1(\lambda) - \phi_2(\lambda)\,\psi_1(\lambda) = 0, \quad k = 1,2,\dots \end{aligned}$$

Hence, if

$$\phi_3(0) > 0, \tag{4.23}$$

the operator $A = A(0)$ is elliptic and satisfies (1.2). Moreover, if

$$\begin{aligned} &(\psi_2(0))^2 + \phi_2(0)\,\psi_1(0) = -1,\\ &\phi_3(0) = \psi_2(0) + \phi_1(0), \end{aligned} \tag{4.24}$$

then A satisfies (4.12). Finally, the transversality condition (4.13) is satisfied if

$$\phi_3'(0) \neq \psi_2'(0) + \phi_1'(0). \tag{4.25}$$

For any $\alpha \in]0,\frac{1}{2}[$ we have

$$D_A(\alpha) = h_0^{2\alpha}([0,\pi]) \oplus C_0^2([0,\pi]),$$

$$D_A(\alpha+1) = h_0^{2\alpha+2}([0,\pi]) \oplus C_0^2([0,\pi])$$

($h_0^{2\alpha}([0, \pi])$ and $h_0^{2\alpha+2}([0, \pi])$ are defined in the preceding examples). It is easy to verify that

$$f \in C^\infty(]-1, 1[\times D_A(\alpha + 1); D_A(\alpha)).$$

Then, if (4.21), ..., (4.25) hold, there exist periodic solutions of (4.20). In order to study the stability of the periodic orbits given by Theorem 4.5, one can use the known results given in [7] for equation (4.17): one has to compute several derivatives of the functions h and x, and this can be done using (4.15) and (4.17).

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