A Proof of the Bieberbach Conjecture for the Sixth Coefficient

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Communicated by M. M. SCHIFFER

1. Introduction

A conjecture, attributed to BIEBERBACH [1], asserts that if

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

is analytic and univalent in the unit disk, then $|a_n|\leqq n$ with equality holding only for the Koebe function

$$K(z)=\frac{z}{(1-z)^2}=z+\sum_{n=2}^{\infty} n z^n$$

or one of its rotations. The conjecture was proved to be true for $n=2, 3$ and 4 by BIEBERBACH [1], LOEWNER [11] and GARABEDIAN & SCHIFFER [5], respectively. Alternate proofs for the case $n=4$ have been provided in the papers [19], [4] and [14]. Recent evidence in support of the conjecture has been obtained by GARABEDIAN, ROSS & SCHIFFER [4], GARABEDIAN & SCHIFFER [6] and BOMBIERI [2]. In these papers it was shown that $\operatorname{Re} a_n \leqq n$ if $f(z)$ is sufficiently close to $K(z)$ in various topologies. The author [14] proved that, at the Koebe point, these topologies are all equivalent.

It is the purpose of this paper to prove the Bieberbach Conjecture for the sixth coefficient.

Theorem 1. *If $f(z)$ is normalized, analytic and univalent in the unit disk, then $|a_6|\leqq 6$ with equality holding only for the Koebe function or one of its rotations.*

The proof of the above theorem uses the formulas of GARABEDIAN, ROSS & SCHIFFER [4] together with an observation of the author [15] that the Grunsky matrix of a slit mapping is unitary. Grunsky's inequality is based on the fact that a function defined by (1.1) is univalent in the unit disk if and only if the series

$$\log\frac{f(z)-f(\zeta)}{z-\zeta}=\sum_{m,n=0}^{\infty} d_{mn} z^m \zeta^n$$

converges for $|z|<1$, $|\zeta|<1$. GRUNSKY [7] showed that this is the case if and only if the linear transformation defined by the symmetric infinite matrix

$$C=(c_{mn}), \qquad c_{mn}=\sqrt{mn}\, d_{mn} \tag{1.2}$$

satisfies the inequality

$$(C x, \bar{x}) \leqq \|x\|^2 \tag{1.3}$$

for every square summable complex vector $x=(x_1, x_2, \ldots, x_n, \ldots)$. Here, and elsewhere, (x, y) denotes the inner product $\sum_n x_n \bar{y}_n$. SCHIFFER noted that it is a consequence of SCHUR's diagonalization theorem [21] that (1.3) is equivalent to the inequality

$$\|C x\|^2 \leqq \|x\|^2. \tag{1.4}$$

The author [15] noticed that it follows from an independent proof of JENKINS [8] that equality holds in (1.4) if and only if f defines a slit mapping. Here a slit mapping is one which maps the disk onto the complement of a set of measure zero. An immediate consequence is that f is a univalent slit mapping if and only if the Grunsky matrix C is unitary.

In his proof that $|a_2| \leqq 2$, BIEBERBACH [1] made use of the fact that if $f(z)$ is univalent in the unit disk, then so is a branch of $\sqrt{f(z^2)}$. GARABEDIAN, ROSS & SCHIFFER [4] observed that any even order coefficient a_{2n} of f can be expressed as a polynomial in the matrix $\check{C}_n=(c_{2j-1,\, 2k-1})$, $j, k=1, 2, \ldots, n$, of the Grunsky matrix C associated with $\sqrt{f(z^2)}$. When $2n=4$, they used this fact to prove that $|a_4| \leqq 4$. An expression for a_6 is given by (see [4, pp. 985–86])

$$\begin{aligned} a_6 = P_{10}(\check{C}_3) \equiv \frac{2}{5} c_{55} + \frac{8}{\sqrt{15}} c_{11} c_{35} + 6 c_{11}^2 c_{33} + \frac{12}{\sqrt{15}} c_{13} c_{15} \\ + \frac{14}{3} c_{11} c_{13}^2 + \frac{32}{\sqrt{3}} c_{11}^3 c_{13} - \frac{2}{5} c_{11}^5 . \end{aligned} \tag{1.5}$$

The above polynomial, however, is not unique since there is a relation

$$P_6(\check{C}_3) \equiv c_{15} + \frac{\sqrt{5}}{3} c_{11}^3 - \frac{\sqrt{5}}{3} c_{33} - \frac{\sqrt{15}}{3} c_{11} c_{13} \equiv 0 \tag{1.6}$$

among the elements of $\check{C}_3$. Note that the polynomials P_k are homogeneous of degree k in the sense that replacing $f(z)$ by $e^{-i\theta} f(e^{i\theta} z)$ brings out the factor $e^{ik\theta}$ in front of P_k. The above authors were then led to the representation

$$a_6 = P_{10}(\check{C}_3) - (\lambda c_{11}^2 + \mu c_{13}) P_6(\check{C}_3), \tag{1.7}$$

where λ and μ are Lagrange multipliers. A comparison of the above formula with one obtained by a different method then led to the choice

$$\lambda = -\frac{8}{\sqrt{5}}, \quad \mu = \frac{12}{\sqrt{15}}.$$

The authors then considered the problem of maximizing the above polynomial over the class of all 3×3 matrices which satisfy Grunsky's inequality (1.3). An application of Schur's diagonalization theorem, together with the maximum principle, then showed that in the extremal case the truncated matrix must be unitary. By determining the most general 3×3 symmetric unitary matrix, a bound was obtained for $\operatorname{Re} a_6$ in terms of a trigonometric polynomial in five real variables. Computing machine experiments gave considerable evidence that the polynomial has the desired bound.

The author [16] used the unitary property of the infinite matrix to show that, for suitably normalized slit mappings, $\operatorname{Re} a_6$ is bounded by a computable function of the single real variable $t = \operatorname{Re} a_2/2$. Computations of this function gave more evidence that $\operatorname{Re} a_6 \leqq 6$. The results of the latter study indicate that one should be able to prove

$$\operatorname{Re}(a_6 - 6) \leqq \alpha(\operatorname{Re} a_2 - 2)$$

where α is near 1/2. Since this would require considerably greater effort, with no new ideas, we content ourselves with estimates which are consistent with the objective of proving Theorem 1.

The present study uses a simplified version of the method used in [16]. We are led naturally to the formula (1.7) with

(1.8) $$\lambda = -\frac{8}{\sqrt{5}}, \quad \mu = 0.$$

It is convenient to introduce the identity

(1.9) $$\frac{c_{35}}{\sqrt{15}} = \frac{c_{17}}{\sqrt{7}} - \frac{c_{11} c_{15}}{\sqrt{5}} + \frac{c_{11}^2 c_{13}}{\sqrt{3}} - \frac{c_{13}^2}{3}$$

which is obtained by direct computation. The usefulness of (1.9) is a consequence of the fact that its imaginary part does not depend on the real parts of elements outside of the truncated matrix $\check{C}_3$. After omitting certain terms, which the unitary property of C shows are negative, the remaining bound is reduced to a polynomial in the real and imaginary parts of the first now of C by means of (1.6) and (1.9).

Perhaps a few remarks on the determination of the multipliers are in order. This requires a certain amount of guessing since the optimum choice requires a knowledge of the eigenvalues of certain matrices as functions of λ and μ. We therefore choose μ and λ so that reasonably good estimates are obtained for univalent functions whose coefficients are all real. The value $\lambda = -9/\sqrt{5}$ gives the best local estimate for the latter class. The choice $\lambda = -8/\sqrt{5}$, which turns out to be a stationary value of the discriminant of a certain quadratic form, gives a better estimate for functions with complex coefficients. By comparing our bound for $\operatorname{Re} a_6$ with the one obtained for $\operatorname{Re} a_4$ in [15], the value $\lambda = -8/\sqrt{5}$ suggests the beginning of a pattern which, at least as far as the local result is concerned, may extend to the higher even order coefficients. Having chosen $\lambda = -8/\sqrt{5}$, μ is chosen to optimize the second derivative of the estimate for a_6 at $t = 1$. This is accomplished by choosing $\mu = 0$. We shall not carry out a justification of the remarks of this paragraph. The interested reader may wish to do so himself by following the lines of §3 for an arbitrary polynomial of the form (1.7).

It will be observed that the point $\lambda = -8/\sqrt{5}$, $\mu = 0$ lies outside of the region for which the scheme of Garabedian, Ross & Schiffer [4, p. 987] gives a negative local variation. This in no way implies a contradiction since we are considering a smaller class of matrices. The fact that the relations (1.6) and (1.9) are valid in our class allows us to estimate certain quadratic forms over subspaces in which smaller bounds are obtained. Some of the bounds of this paper can be improved by allowing λ and μ to vary with t. The above choice, however, is adequate for the global theorem and tends to minimize the complexity of the computations.

For special results related to the sixth coefficient problem, the reader is referred to OZAWA [13], SCHIFFER [18] and JENKINS & OZAWA [9].

In §2 we prove some preliminary lemmas. The basic estimates for a_6 are obtained in §3, and in §4 are reduced to estimates depending only on $t=\operatorname{Re}a_2/2$. It turns out to be necessary to treat the cases $\operatorname{Re}c_{13}\geqq 0$ and $\operatorname{Re}c_{13}\leqq 0$ differently. In §6 it is shown that the bound for the former case is dominant. §5 is devoted to estimating quartic forms in terms of the eigenvalues of quadratic forms. In §7 the results of the previous sections are used to prove Theorem 1.

2. Preliminary Lemmas

If C is a matrix, define $\delta C = C - I$ where I is the identity. The elements of C are denoted by $c_{jk}=r_{jk}+is_{jk}$ where r_{jk} and s_{jk} are real. One of the essential tools of our investigation is provided by the following lemma. δC_j denotes the j-th row of the matrix δC.

Lemma 2.1. *A necessary and sufficient condition that an analytic function $f(z)$, normalized by* (1.1), *be a slit univalent mapping of the unit disk is that the Grunsky matrix C associated with f satisfy*

$$\delta r_{jk} = -\tfrac{1}{2}(\delta C_j, \delta C_k). \tag{2.1}$$

For a proof of the above lemma see [15].

On several occasions we shall need to estimate the largest eigenvalue of a "border" matrix. The following lemma gives bounds which are adequate for our purposes.

Lemma 2.2. *The largest eigenvalue of the quadratic form*

$$q(x,y) = A x^2 + 2x(B,y), \qquad A, x \in R_1, \qquad y, B \in R_n \tag{2.2}$$

is given by

$$\tau = \frac{A + [A^2 + 4\|B\|^2]^{\frac{1}{2}}}{2}. \tag{2.3}$$

If A and B are of class C^2 with respect to t and if

$$A'' \geqq 0, \quad (B, B'') \geqq 0, \quad [A^2 + 4\|B\|^2]^{\frac{1}{2}} \neq 0, \tag{2.4}$$

then τ is a convex function of t.

Proof. A simple computation shows that the characteristic polynomial of the real symmetric matrix associated with q is

$$(-z)^{n-1}(z^2 - Az - \|B\|^2)$$

from which it follows that the largest eigenvalue of q is given by (2.3). Assuming A and B to be of class C_2 with respect to t, we have

$$2\tau'' = A'' + \frac{AA'' + 4(B,B'') + \|A'\|^2 + 4\|B'\|^2}{[A^2 + 4\|B\|^2]^{\frac{1}{2}}} - \frac{[AA' + 4(B,B')]^2}{[A^2 + 4\|B\|^2]^{\frac{3}{2}}}. \tag{2.5}$$

An application of Schwarz' inequality yields

$$[AA'+4(B,B')]^2 \leqq (A^2+4\|B\|^2)((A')^2+4\|B'\|^2).$$

Substitution of the above inequality into (2.5) gives

$$2\tau'' \geqq A''\left(1+\frac{A}{(A^2+4\|B\|^2)^{\frac{1}{2}}}\right)+\frac{4(B,B'')}{(A^2+4\|B\|^2)^{\frac{1}{2}}}.$$

Hence, if (2.4) is satsfied, then $\tau'' \geqq 0$. This completes the proof of Lemma 2.2.

3. The Basic Inequalities

Let $f(z)$ be a univalent slit mapping of the unit disk and let C be the Grunsky matrix of $\sqrt{f(z^2)}$. It may be assumed, without loss of generality, that

$$a_6 \geqq 0 \quad \text{and} \quad |\operatorname{Arg} a_2| \leqq \pi/5. \tag{3.1}$$

Setting $t=r_{11}$ and using the fact that $c_{11}=a_2/2$, it follows from (3.1) that

$$0 \leqq t \leqq 1 \quad \text{and} \quad |s_{11}| \leqq (\tan \pi/5)\, t \leqq \tfrac{3}{4} t. \tag{3.2}$$

By using formula (1.7) with $\lambda=-8/\sqrt{5}$ and $\mu=0$, together with (3.1), we obtain

$$\begin{aligned}\delta a_6 = \operatorname{Re}(a_6-6) = \operatorname{Re}\Big\{&\frac{2}{5}(c_{55}-1)+\frac{8}{\sqrt{15}}c_{11}c_{35}+\frac{10}{3}(c_{11}^2c_{33}-1)\\ &+\frac{8}{\sqrt{5}}c_{11}^2c_{15}+\frac{12}{\sqrt{15}}c_{13}c_{15}+\frac{24}{\sqrt{3}}c_{11}^3c_{13}+\frac{14}{3}c_{11}c_{13}^2+\frac{34}{15}(c_{11}^5-1)\Big\}.\end{aligned} \tag{3.3}$$

It is easily verified that

$$\operatorname{Re}(c_{11}^2c_{33}-1)=t^2(r_{33}-1)+(t^2-1)-s_{11}^2r_{33}-2ts_{11}s_{33}. \tag{3.4}$$

The real and imaginary parts of (1.6) are next equated in order to obtain

$$r_{33}=\frac{3}{\sqrt{5}}r_{15}+t^3-3t^2s_{11}-\frac{3}{\sqrt{3}}tr_{13}+\frac{3}{\sqrt{3}}s_{11}s_{13} \tag{3.5}$$

and

$$s_{33}=\frac{3}{\sqrt{5}}s_{15}+3t^2s_{11}-s_{11}^3-\frac{3}{\sqrt{3}}ts_{13}-\frac{3}{\sqrt{3}}r_{13}s_{11}. \tag{3.6}$$

Substituting (3.5) and (3.6) into the third and fourth terms on the right side of the equality in (3.4) and then substituting the resulting expression into (3.3), we obtain

$$\begin{aligned}\delta a_6 = {}&\frac{2}{5}(r_{55}-1)+\frac{8t}{\sqrt{15}}r_{35}+\left(\frac{12}{\sqrt{3}}r_{13}+8t^2\right)\frac{r_{15}}{\sqrt{5}}+\frac{10}{3}(t^2-1)r_{33}\\ &+\frac{14}{3}tr_{13}^2+\frac{24t^3}{\sqrt{3}}r_{13}+\frac{34}{15}(t^5-1)+\frac{10}{3}(t^2-1)-46t^3s_{11}^2-\frac{14}{3}ts_{13}^2\\ &-\frac{8}{\sqrt{15}}s_{11}s_{35}-\frac{52t^2}{\sqrt{3}}s_{11}s_{13}-\frac{36t}{\sqrt{5}}s_{11}s_{15}-\frac{12}{\sqrt{15}}s_{13}s_{15}+28ts_{11}^4\\ &+\frac{14}{\sqrt{3}}s_{11}^3s_{13}-\frac{42t}{\sqrt{3}}r_{13}s_{11}^2-\frac{28}{3}r_{13}s_{11}s_{13}-18s_{11}^2\frac{r_{15}}{\sqrt{5}}.\end{aligned} \tag{3.7}$$

It is convenient to introduce the vectors

$$U=\frac{1}{\sqrt{5}}\,\delta C_5+\frac{2t}{\sqrt{3}}\,\delta C_3+\left(\frac{3}{\sqrt{3}}\,r_{13}+2t^2-\alpha s_{11}^2\right)\delta C_1 \tag{3.8}$$

and

$$V=\frac{1}{\sqrt{3}}\,\delta C_3+2t\,\delta C_1. \tag{3.9}$$

Here α is a constant to be chosen later. It is a consequence of Lemma (2.1) that

$$\begin{aligned}
&\frac{2}{5}(r_{55}-1)+\frac{8t}{\sqrt{15}}\,r_{35}+\frac{8}{3}\,t^2(r_{33}-1)+\frac{8t^2}{\sqrt{5}}\,r_{15}+\frac{12}{\sqrt{15}}\,r_{13}r_{15}+\frac{16t^3}{\sqrt{3}}\,r_{13}\\
&\quad=-\|U\|^2-(14t-6)\,r_{13}^2+\frac{24}{\sqrt{3}}\,t^2(1-t)\,r_{13}+8t^4(1-t)\\
&\qquad+(20t-12)\frac{\alpha}{\sqrt{3}}\,r_{13}s_{11}^2-8\alpha t^2(1-t)\,s_{11}^2+2\alpha^2(1-t)\,s_{11}^4+\frac{4\alpha}{\sqrt{5}}\,r_{15}s_{11}^2
\end{aligned} \tag{3.10}$$

and

$$\frac{2}{3}\,t^2(r_{33}-1)+\frac{8t^3}{\sqrt{3}}\,r_{13}=-t^2\|V\|^2+8t^4(1-t). \tag{3.11}$$

By using $\alpha=9/2$ and substituting (3.10), (3.11) into (3.7) we obtain

$$\begin{aligned}
\delta a_6=&-\|U\|^2-t^2\|V\|^2+\frac{34}{15}(t^5-1)+\frac{10}{3}(t^2-1)+16t^4(1-t)\\
&-\left(\frac{28}{3}\,t-6\right)r_{13}^2+24t^2(1-t)\frac{r_{13}}{\sqrt{3}}-(36t^2+10t^3)\,s_{11}^2-\frac{14}{3}\,t s_{13}^2\\
&-\frac{8}{\sqrt{15}}\,s_{11}s_{35}-\frac{52t^2}{3}\,s_{11}s_{13}-\frac{36t}{\sqrt{5}}\,s_{11}s_{15}-\frac{12}{\sqrt{15}}\,s_{13}s_{15}\\
&+(40.5-12.5t)\,s_{11}^4+\frac{14}{\sqrt{3}}\,s_{11}^3s_{13}-(54-48t)\frac{r_{13}}{\sqrt{3}}\,s_{11}^2-\frac{28}{3}\,r_{13}s_{11}s_{13}.
\end{aligned} \tag{3.12}$$

At this point, it is convenient to introduce the linear forms

$$\xi_k=\sum_{j=1}^{k}\frac{t^{k-j}}{\sqrt{2j-1}}\,s_{1,2j-1},\qquad k=1,2,3, \tag{3.13}$$

$$\eta=\frac{s_{35}}{\sqrt{15}}+\frac{2t}{\sqrt{5}}\,s_{15} \tag{3.14}$$

and the nonlinear forms

$$\zeta_1=-\frac{2}{3}\,t s_{11}^3-\frac{9}{2}\,s_{11}^2\frac{s_{13}}{\sqrt{3}}-\frac{2t}{\sqrt{3}}\,r_{13}s_{11}+3\frac{r_{13}}{\sqrt{3}}\frac{s_{13}}{\sqrt{3}}, \tag{3.15}$$

$$\zeta_2=\frac{3}{\sqrt{3}}\,r_{13}s_{11}-\frac{9}{2}\,s_{11}^3, \tag{3.16}$$

$$\zeta_3=\frac{s_{11}^3}{3}+\frac{r_{13}}{\sqrt{3}}\,s_{11}. \tag{3.17}$$

The fact that $\sqrt{f(z^2)}$ is an odd function implies that $c_{jk}=0$ if j and k have opposite parity. By retaining the contributions of the imaginary parts of the first two non-vanishing components of U, the first three of V, and using (3.6), (3.8), (3.9), (3.13), (3.14), (3.15), we have

$$\begin{aligned}-\|U\|^2-t^2\|V\|^2 \leqq & -3(\eta+2t^3 s_{11}+\zeta_1)^2-5t^2\eta^2-(\xi_3+t\xi_2+\zeta_2)^2 \\ & -3t^2(\xi_3-\zeta_3)^2-t^2(\xi_2+t\xi_1)^2.\end{aligned} \tag{3.18}$$

It follows, by eliminating η, that

$$\begin{aligned}&-3(\eta+2t^3 s_{11}+\zeta_1)^2-5t^2\eta^2-8\eta s_{11} \\ &\leqq \frac{-15t^2}{3+5t^2}(2t^3 s_{11}+\zeta_1)^2+\frac{(48t^3+16)s_{11}^2+24\zeta_1 s_{11}}{3+5t^2}.\end{aligned} \tag{3.19}$$

As a consequence of (3.12), (3.13), (3.18), (3.19) and the inequalities

$$-(2t^3 s_{11}+\zeta_1)^2 \leqq -4t^6 s_{11}^2-4t^3 s_{11}\zeta_1,$$
$$-(\xi_3+t\xi_2+\zeta_2)^2 \leqq -(\xi_3+t\xi_2)^2-2(\xi_3+t\xi_2)\zeta_2,$$
$$-(\xi_3-\zeta_3)^2 \leqq -\xi_3^2+2\xi_3\zeta_3,$$

we arrive at the estimate

$$\begin{aligned}\delta a_6 \leqq & \frac{34}{15}(t^5-1)+\frac{10}{3}(t^2-1)+16t^4(1-t) \\ & -\left(\frac{28}{3}t-6\right)r_{13}^2+24t^2(1-t)\frac{r_{13}}{\sqrt{3}} \\ & +q_1(t;s)+s_{11}^2 q_2(t;s)+q_3(t;s)\frac{r_{13}}{\sqrt{3}}.\end{aligned} \tag{3.20}$$

Here the quadratic forms q_1, q_2 and q_3 are defined by

$$\begin{aligned}q_1= & -(\xi_3+t\xi_2)^2-3t^2\xi_3^2-t^2(\xi_2+t\xi_1)^2 \\ & -\left(36t^2+10t^3+\frac{60t^8-48t^3-16}{3+5t^2}\right)s_{11}^2-\frac{14}{3}t s_{13}^2-\frac{52t^2}{\sqrt{3}}s_{11}s_{13} \\ & -\frac{20t}{\sqrt{5}}s_{11}s_{15}-\frac{12}{\sqrt{15}}s_{13}s_{15},\end{aligned} \tag{3.21}$$

$$\begin{aligned}q_2= & \left(40.5-12.5t+18t^3+2t^4+\frac{8t(5t^5-2)}{3+5t^2}\right)s_{11}^2 \\ & +\left(14+18t+2t^3+\frac{54(5t^5-2)}{3+5t^2}\right)\frac{s_{11}s_{13}}{\sqrt{3}}+(9+2t^2)\frac{s_{11}s_{15}}{\sqrt{5}}\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}q_3= & \left(48t-54-12t^2+6t^4+\frac{24t(5t^5-2)}{3+5t^2}\right)s_{11}^2 \\ & -\left(28+12t-6t^3+\frac{36(5t^5-2)}{3+5t^2}\right)\frac{s_{11}s_{13}}{\sqrt{3}}+6(t^2-1)\frac{s_{11}s_{15}}{\sqrt{5}}.\end{aligned} \tag{3.23}$$

The inequality (3.20) will be used to derive the estimate $\delta a_6<0$ for $t\in[.85, 1)$. For intermediate values of t, different estimates are required. By equating imaginary parts in (1.9), we have

$$\text{(3.24)}\qquad \begin{aligned}\frac{s_{35}}{\sqrt{15}}=\frac{s_{17}}{\sqrt{7}}-\frac{t}{\sqrt{5}}s_{15}+\frac{t^2}{\sqrt{3}}s_{13}-\frac{2}{3}r_{13}s_{13}-\frac{1}{\sqrt{5}}s_{11}r_{15}\\ -\frac{1}{\sqrt{3}}s_{11}^2 s_{13}+\frac{2t}{\sqrt{3}}r_{13}s_{11}.\end{aligned}$$

By letting $\alpha=5/2$ and substituting (3.10), (3.11) and (3.24) into (3.7), neglecting the term $-\|U\|^2-t^2\|V\|^2$, we obtain the estimate (3.20) with the simpler expressions for the quadratic forms q_1, q_2 and q_3:

$$\text{(3.25)}\qquad \begin{aligned}q_1=-(20t^2+26t^3)s_{11}^2-\frac{14}{3}ts_{13}^2-\frac{8}{\sqrt{7}}s_{11}s_{17}-\frac{28t}{\sqrt{5}}s_{11}s_{15}\\ -\frac{60t^2}{\sqrt{3}}s_{11}s_{13}-\frac{12}{\sqrt{15}}s_{13}s_{15},\end{aligned}$$

$$\text{(3.26)}\qquad q_2=(12.5+15.5t)s_{11}^2+\frac{22}{\sqrt{3}}s_{11}s_{13},$$

$$\text{(3.27)}\qquad q_3=-(30+8t)s_{11}^2-\frac{12}{\sqrt{3}}s_{11}s_{13}.$$

The estimate obtained above will be used when $.25\leqq t\leqq .85$ and $r_{13}>0$.

When $.25\leqq t\leqq .85$ and $r_{13}<0$, let $\alpha=5/2$, substitute (3.10) and (3.24) into (3.7), and omit the term $2t^2(r_{33}-1)-\|U\|^2$. The result is an inequality which is obtained by adding the quantity

$$\text{(3.28)}\qquad -8t^4(1-t)+\frac{8t^3}{\sqrt{3}}r_{13}$$

to the right side of (3.20). q_1, q_2 and q_3 are again defined by (3.25), (3.26) and (3.27).

4. Reduction of the Number of Variables

It was shown in the previous section that δa_6 has an upper bound of the form

$$\text{(4.1)}\qquad \delta a_6\leqq P-\alpha r_{13}^2+\beta\frac{r_{13}}{\sqrt{3}}+q_1(s)+s_{11}^2 q_2(s)+q_3(s)\frac{r_{13}}{\sqrt{3}}$$

where P, α and β are polynomials in t and q_1, q_2 and q_3 are quadratic forms in $s=(s_{11}, s_{13}, s_{15}, s_{17})$ whose coefficients are rational functions of t. The purpose of this section is to reduce the above estimate to one depending only on t.

It follows from the fact that the first row of C has norm one that

$$\text{(4.2)}\qquad \|s\|^2\leqq 1-t^2-r_{13}^2\leqq 1-t^2.$$

Our goal is to obtain an upper bound for δa_6 subject to the restraints (4.2) and

$$\text{(4.3)}\qquad |s_{11}|\leqq \tfrac{3}{4}t.$$

The latter inequality is a consequence of the normalization (3.2).

Definition 4.1. If $F(s)$ is a quadratic or quartic form whose coefficients depend on t, we define

$$\rho(F) = \max \frac{F(s)}{\|s\|^2}, \quad \|s\|^2 \leqq 1 - t^2, \quad |s_{11}| \leqq 3t/4 \tag{4.4}$$

and

$$\sigma(F) = \max F(s), \quad \|s\|^2 \leqq 1 - t^2, \quad |s_{11}| \leqq 3t/4. \tag{4.5}$$

The quantities $\tilde{\rho}$ and $\tilde{\sigma}$ denote upper bounds for ρ and σ, respectively.

Lemma 4.1. *Let* $Q = q_1 + s_{11}^2 q_2$ *and suppose that* $\tilde{\rho}(Q) + \alpha > 0$. *Then*

$$\delta a_6 \leqq P + \tilde{\rho}(Q)(1 - t^2) + \frac{\gamma^2}{12(\alpha + \tilde{\rho}(Q))} \tag{4.6}$$

where

$$\gamma = \begin{cases} \beta + \tilde{\sigma}(q_3) & \text{if } r_{13} > 0, \\ \tilde{\sigma}(-q_3) - \beta & \text{if } r_{13} < 0. \end{cases} \tag{4.7}$$

Proof. It is an immediate consequence of (4.2) and (4.4) that

$$Q \leqq \tilde{\rho}(Q)(1 - t^2 - r_{13}^2). \tag{4.8}$$

By using (4.5) and (4.7), we have

$$(\beta + q_3) \frac{r_{13}}{\sqrt{3}} \leqq \frac{\gamma}{\sqrt{3}} |r_{13}|. \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.1), one obtains

$$\delta a_6 \leqq P + \tilde{\rho}(Q)(1 - t^2) - (\alpha + \tilde{\rho}(Q)) r_{13}^2 + \gamma \frac{|r_{13}|}{\sqrt{3}}.$$

The conclusion of Theorem 4.1 now follows from the assumption that $\alpha + \tilde{\rho}(Q) > 0$.

In the absence of the restriction (4.3), we have for a quadratic form $q(s)$

$$\sigma(q) = \tau(1 - t^2) \tag{4.10}$$

where τ is the largest eigenvalue of the symmetric matrix associated with q. The inequalities $s_{11}^2 \leqq 1 - t^2$ and $|s_{11}| \leqq 3t/4$ merge at $t = .8$. If the coefficient of s_{11}^2 is positive and $t \leqq .8$, more efficient estimates can be obtained by separating out some or all of the terms involving s_{11} and estimating them separately. The following lemma provides bounds which will be used to estimate border matrices.

Lemma 4.2. *If q is a quadratic form given by* (2.2) *then*

$$\sigma(q) \leqq \frac{3t}{4} \sqrt{A^2 + 4\|B\|^2} \sqrt{1 - t^2}, \tag{4.11}$$

$$\sigma(q) \leqq \frac{9}{16} t^2 |A| + \frac{3}{2} t \|B\| \sqrt{1 - t^2}, \tag{4.12}$$

and

$$\sigma(q) \leqq \frac{9}{16} t^2 |A| + \|B\| (1 - t^2). \tag{4.13}$$

The proof is a simple consequence of Schwarz' inequality, (4.2) and (4.3).

5. Estimates of Quartic Forms

This section is devoted to obtaining bounds for the quartic form

(5.1) $$Q = q_1 + s_{11}^2 q_2$$

where q_1 and q_2 are defined by (3.21), (3.22) or (3.25), (3.26). One method is to use the inequality

(5.2) $$Q \leqq q_1 + s_{11}^2 \tilde{\sigma}(q_2)$$

where $\tilde{\sigma}$ is one of the bounds (4.11)–(4.13), and then to estimate the largest eigenvalue of the quadratic form defined by the right side of (5.2). It is the nature of q_1 that the coefficient of s_{11}^2 is negative and large in magnitude compared to the other coefficients. It therefore seems reasonable that, for t in a neighborhood of one, better estimates can be found by obtaining a preliminary bound on the size of s_{11}^2 in order that

(5.3) $$Q \geqq M \|s\|^2, \quad \|s\|^2 \leqq 1 - t^2$$

where M is a tentative estimate for $\rho(Q)$.

The symbols $\check{q}_k$, G, H and K are defined by

(5.4) $$q_k(s) = q_{k,11} s_{11}^2 + \check{q}_k(s),$$

(5.5) $$G = q_{2,11}(1-t^2), \quad H = -q_{1,11} - \tilde{\rho}(\check{q}_2)(1-t^2), \quad K = \tilde{\rho}(\check{q}_1) - M.$$

Let $\lambda-$ and $\lambda+$ $(\lambda- < \lambda+)$ be the roots of

(5.6) $$G z^2 - H z + K = 0.$$

Lemma 5.1. *Suppose that* $H^2 > 4GK$, $G > 0$, $H > 0$ *and* $K > 0$. *If* (5.3) *is satisfied at a point where* $s_{11}^2 \leqq \lambda_+ (1-t^2)$, *then*

(5.7) $$s_{11}^2 \leqq \lambda_- (1-t^2).$$

In particular, if $H > G + K$, *then* (5.7) *is valid.*

Proof. Let $s_{11} = \sqrt{1-t^2}\,\xi$ and $\gamma = \sqrt{1-t^2}(s_{13}, s_{15}, s_{17})$; hence $\xi^2 \leqq \lambda^+$ and $\xi^2 + |\gamma|^2 \leqq 1$. It is then easily demonstrated that

$$(Q(s) - M\|s\|^2)/(1-t^2) \leqq G\xi^4 - H\xi^2 + K.$$

The right side of the above inequality is negative if $\lambda- < \xi^2 < \lambda+$. It follows that if (5.3) is satisfied, then either $\xi^2 \leqq \lambda-$ or $\xi^2 \geqq \lambda+$. The latter possibility is precluded. The proof is completed by noting that if $H > G + K$, then $\lambda+ > 1$.

We next focus our attention on obtaining the preliminary information which is needed to apply Theorem 5.1 on the interval [.85, 1].

Lemma 5.2. *If* q_1 *and* q_2 *are defined by* (3.21) *and* (3.22), *and* $t \in [.85, 1]$, *then*

(5.8) $$\rho(\check{q}_1) \leqq 40t - 22 \quad \text{and} \quad \rho(\check{q}_2) \leqq 42t - 26.$$

Proof. In order to prove the estimate for $\rho(\check{q}_1)$, it is sufficient to verify that

(5.9) $$(40t - 22)(s_{11}^2 + s_{13}^2 + s_{15}^2) - \check{q}_1(s)$$

is positive definite. It is convenient to use the coordinates ξ_k defined by (3.13). We note that

(5.10) $$s_{1,2k-1}=\sqrt{2k-1}(\xi_k-t\xi_{k-1}), \quad \xi_0=0.$$

After substituting (5.10) into (5.9), we obtain the quadratic form

$$\sum_{j,k=1}^{3} A_{jk}\xi_j\xi_k$$

where

$$A_{11}=(40t-22)(1+3t^2)-38t^3-7t^4-3t^6,$$

$$A_{22}=(40t-22)(3+5t^2)+2t(1+t), \quad A_{33}=5(40t-22)+1+3t^2,$$

$$A_{12}=t[t^2+8-3(40t-22)], \quad A_{13}=4t, \quad A_{23}=t+6-5t(40t-22).$$

It will be shown that the determinants of the principle minors of the above matrix are positive. It is clear that $A_{33}>0$. We next compute

$$\Delta_2(t)=A_{22}A_{33}-A_{23}^2=7158-27{,}830t+26{,}053t^2+1366t^3-324t^4+600t^5.$$

It is easily verified that $\Delta_2'(t)>0$, $t\in[.85, 1]$ and that $\Delta_2(.85)>0$; hence $\Delta_2(t)>0$. As a consequence of convexity, we have $7t^4+3t^6\leqq 34t-24$, $t\in[.85, 1]$; hence

$$A_{11}\geqq(40t-22)(1+3t^2)-34t+24-38t^3.$$

It follows that $\Delta_3=\det(A_{jk})$ satisfies

$$\Delta_3\geqq\Phi(t)=14{,}316-12{,}712t+14{,}190t^2-390{,}254t^3$$
$$+904{,}110t^4-647{,}346t^5+139{,}861t^6-65{,}648t^7+49{,}197t^8.$$

One deduces from the inequality

$$\Phi^{(v)}(t)\geqq 10^7[(33t-17)t^2+(10t-8)]$$

that Φ''' is convex. It is next verified that $\Phi'''(.85)<0$, $\Phi'''(1)>0$, $\Phi''(.85)<0$ and $\Phi''(1)<0$. These inequalities, together with the convexity of Φ''', imply that Φ is concave. More computations give $\Phi(.85)>0$ and $\Phi(1)>0$; hence $\Phi(t)>0$, $t\in[.85, 1]$. This completes the proof of estimate for $\rho(\check{q}_1)$.

It can be shown that $q_{2,13}$, $q_{2,13}''$, $q_{2,15}$ and $q_{2,15}''$ are all positive on [.85, 1]. Hence, by Lemma 2.2, the largest eigenvalue, $\tau=\sqrt{q_{2,13}^2+q_{2,15}^2}$, of $\check{q}_2$ is convex. It follows, by computing τ at both endpoints, that $\tau\leqq 42t-26$, $t\in[.85, 1]$. This completes the proof of Lemma 5.2.

Lemma 5.3. *If* q_1 *and* q_2 *are defined by* (3.21) *and* (3.22), *and* $Q=q_1+s_{11}^2q_2\geqq(7.5-7t)\|s\|^2$, *then* $s_{11}^2\leqq .42\,(1-t^2)$.

Proof. An analysis of the proof of Lemma 5.2 shows that its conclusion is valid with $q_{1,11}$ and $q_{2,11}$ replaced by upper bounds. It is fairly easy to verify that $q_{1,11}''<0$ and $q_{2,11}''>0$ on [.85, 1]. Estimating $q_{1,11}$ by the tangent at .85 and $q_{2,11}$ by the chord between .85 and 1 we obtain

$$q_{1,11}\leqq 69-120t \quad\text{and}\quad q_{2,11}\leqq 7.5+43.5t, \quad t\in[.85, 1].$$

Using the above estimate, together with $M=7.5-7t$, in (5.5), we find that

$$G=(7.5+43.5t)(1-t^2), \quad H=(120t-69)-(42t-26)(1-t^2)$$

and $K=47t-29.5$. We have

$$H-G-K=85.5t^3-18.5t^2-12.5t-21.$$

The above expression is easily seen to be increasing and positive at .85. Hence $H>G+K$, $t\in[.85, 1]$. Putting $z=.42$ into (5.6), we obtain

$$-25.3134t^3+9.597t^2+21.9554t-10.117.$$

The above polynomial has a negative derivative and is negative at $t=.85$. It follows that .42 separates the roots of quadratic equation (5.6). This completes the proof.

We are now in a position to obtain a bound for $\rho(Q)$.

Lemma 5.4. *If the hypothesis of Lemma 5.3 is satisfied, then*

$$\rho(Q)\leqq 7.5-7t, \quad t\in[.85, 1]. \tag{5.11}$$

Proof. As a consequence of Lemmas 5.2 and 5.3, it is sufficient to prove that the largest eigenvalue of the quadratic form

$$\tilde{q}=q_1+(.42q_{2,11}+42t-26)(1-t^2)s_{11}^2$$

is less than $7.5-7t$. It suffices to show that

$$(7.5-7t)\|s\|^2-\tilde{q}$$

is positive definite. It follows from (3.21) and (5.10) that the above quadratic form is equal to

$$\sum_{j,k=1}^{3} B_{jk}\xi_j\xi_k, \quad B_{jk}=B_{kj}$$

where

$$B_{11}=(7.5-7t)(1+3t^2)-q_{1,11}-(.42q_{2,11}+42t-26)(1-t^2)-38t^3-7t^4-3t^6,$$

$$B_{22}=2t(1+t)+(7.5-7t)(3+5t^2), \quad B_{33}=1+3t^2+5(7.5-7t),$$

$$B_{12}=t^3+8t-3t(7.5-7t), \quad B_{13}=4t, \quad B_{23}=t+6-5t(7.5-7t).$$

It is clear that $B_{33}>0$. We next compute

$$\Delta_2(t)=B_{22}B_{33}-B_{23}^2=830.25-1081t+501t^2-232t^3+118.5t^4-105t^5.$$

Note that

$$\Delta_2'(t)<-(1081-1002t)-(696-474t)t^2<0$$

and that $\Delta_2(1)>0$; hence $\Delta_2(t)>0$, $t\in[.85, 1]$. In order to bound $\Delta_3=\det(B_{jk})$ from below by a polynomial of lower degree, we compute

$$B_{11}=16.2t^6-4.856t^4-12.25t^3+49.4956t^2$$
$$-54.702t+11.96264+\frac{39.616t-2.41792}{3+5t^2}.$$

It is easily verified that $B_{11}(.85)>2$, $B'_{11}(.85)>34$ and $B''_{11}(t)>0$, $t\in[.85, 1]$; hence

$$B_{11}\geqq 34t-26.9, \quad t\in[.85, 1].$$

Using the above inequality, it can be shown that

$$\Delta_3\geqq\Psi(t)=-22{,}333.725+57{,}307.4t-71{,}161.525t^2$$
$$+99{,}502.05t^3-104{,}267.4t^4+44{,}783.5t^5-3686.5t^6-139t^7-3t^8.$$

It is next shown that, for $t\leqq 1$,

$$\Psi'''(t)\geqq 10^3(597-2503t-2687t^2-443t^3-34t^4-2t^5).$$

Using $t^k\leqq 1$, we then have

$$\Psi'''(t)\geqq 10^3(597-2503t-2208t^2)>0, \quad t\geqq .9.$$

By applying a similar argument to the interval [.85, .90], it is seen that $\Psi'''(t)>0$, $t\in[.85, 1]$. By computation, one shows that $\Psi''(t)<0$, $\Psi(.85)>0$ and $\Psi(1)>0$. It follows that $\Psi(t)>0$ on [.85, 1]. Hence the determinants of the principle minors of (B_{jk}) are all positive. This completes the proof of Lemma 5.4.

The problem of finding bounds for $\rho(Q)$ on the interval [.70, .85] is considered next.

Lemma 5.5. *If q_1 and q_2 are defined by* (3.25) *and* (3.26), *then*

(5.12) $$\rho(\check{q}_1)\leqq 25t-9.3 \quad \textit{and} \quad \rho(\check{q}_2)\leqq\frac{11}{\sqrt{3}}, \quad t\in[.70, .85].$$

Proof. The estimate for $\rho(\check{q}_2)$ is a simple consequence of the inequality between geometric and arithmetic means. The bound for $\rho(\check{q}_1)$ is obtained by demonstrating that

(5.13) $$(25t-9.3)\,\|s\|^2-\check{q}_1$$

is positive definite. After introducing the variables

(5.14) $$y_k=\sqrt{2k-1}\,s_{1,2k-1}, \quad k=1,2,3,4,$$

(5.13) becomes

$$(25t-9.3)\,y_1^2+(89t-27.9)\,y_2^2+5(25t-9.3)\,y_3^2+7(25t-9.3)\,y_4^2$$
$$+60t^2y_1y_2+28ty_1y_3+8y_1y_4+12y_2y_3.$$

Elimination of the variables y_3 and y_4 by completing the square shows that the above expression is bounded from below by

$$Dy_1^2+2Ey_1y_2+Fy_2^2$$

where

$$D=(15t-9.3)-\left(\frac{16}{7}+\frac{196t^2}{5}\right)\frac{1}{(25t-9.3)},$$

$$E=30t^2-\frac{84t}{5(25t-9.3)}$$

and

$$F=(89t-27.9)-\frac{36}{5(25t-9.3)}.$$

Estimating from above by the chord between .7 and .85 and from below by the tanget at .75 gives

$$.315-.28t \leqq \frac{1}{25t-9.3} \leqq .301-.255t, \qquad t\in[.7,.85].$$

By using the lower estimate in the expression for E and the upper estimate in the formulas for D and F, it is found that

$$DF-E^2 \geqq -297t^4-1006t^3+2650t^2-1677t+300 \equiv \Phi(t).$$

The polynomial $\Phi(t)$ is easily shown to satisfy $\Phi'''(t)<0$, $t\in[.70,.85]$, $\Phi''(.70)<0$, $\Phi(.70)>0$ and $\Phi(.85)>0$. It follows that $\Phi(t)>0$ on $[.70,.85]$. This completes the proof of Lemma 5.5.

Lemma 5.6. *If q_1 and q_2 are defined by* (3.25) *and* (3.26), *and $Q\equiv q_1+s_{11}^2 q_2 \geqq (11-10t)\|s\|^2$, $t\in[.70,.85]$, then $s_{11}^2 \leqq .4(1-t^2)$.*

Proof. By Lemma 5.5, (3.25) and (3.26), we see that it is sufficient to apply Lemma 5.1 with

$$G=(12.5+15.5t)(1-t^2), \qquad H=20t^2+26t^3-\frac{11}{\sqrt{3}}(1-t^2), \qquad K=35t-20.3.$$

Note that

$$H-G-K \geqq 41.5t^3+38.9t^2-50.5t+1.3$$

which is increasing and positive at $t=.73$; hence it is positive on $[.73,.85]$. When $t\leqq .73$ we use the inequality $s_{11}^2 \leqq 9t^2/16$ (see (4.3)) to deduce that $s_{11}^2 \leqq .65(1-t^2)$, $t\in[.70,.73]$. One notes that

$$G(.65)^2-H(.65)+K \leqq -23t^3-22t^2+42t-10.8$$

which is decreasing on $[.70,.73]$ and negative at $t=.70$. It follows that $\lambda+ \geqq .65$ on $[.70,.73]$. We next calculate

$$G(.4)^2-H(.4)+K \leqq -12.88t^3-12.544t^2+37.48t-15.756 \equiv \Phi(t).$$

The function Φ satisfies $\Phi'(t)<0$, $t\geqq .72$, and $\Phi(.72)<0$; hence $\Phi(t)<0$ for $t\in[.72,.85]$. It is easily shown that $|\Phi'(t)|<2$ on $[.70,.72]$ and that $\Phi(.70)<-.08$ from which it follows that $\Phi(t)<0$ on $[.70,.72]$. Hence $\lambda- <.40$. This completes the proof of Lemma 5.6.

Lemma 5.7. *Under the hypothesis of Lemma 5.6, we have*

$$\rho(Q) \leqq 11-10t, \qquad t\in[.70,.85]. \tag{5.15}$$

Proof. As a consequence of Lemmas 5.5 and 5.6, it is sufficient to prove that

$$(11-10t)\|s\|^2-q_1-\left\{.4(12.5+15.5t)+\frac{11}{\sqrt{3}}\right\}(1-t^2)s_{11}^2 \tag{5.16}$$

is positive definite. Estimating $1/(11-10t)$ from below by the tangent at .8 and from above by the chord between .7 and .85, we obtain

$$\frac{10t-5}{9} \leqq \frac{1}{11-10t} \leqq 1.25t-.125, \qquad t\in[.70,.85]. \tag{5.17}$$

By following the lines of the proof of Lemma 5.5, using the variables (5.14) together with (5.17), it is found that the quadratic form (5.16) is bounded from below by

$$(1.075-19.06t+55.85t^2-16.8t^3)\,y_1^2+2(11.34t^2+9.34t)\,y_1y_2+(37.5-25t)\,y_2^2.$$

The negative of the discriminant of the above quadratic form is bounded from below by

$$40-742t+2483t^2-2239t^3+291t^4$$

which is easily proved to be concave on the indicated interval and positive at its endpoints. This completes the proof of Lemma 5.7.

We turn now to the proof of the bound for $\rho(Q)$ on the interval [.25, .70].

Lemma 5.8. *If* q_1 *and* q_2 *are defined by* (3.25) *and* (3.26), *then*

$$\rho(Q)\leqq 4, \qquad t\in[.25,.70]. \tag{5.18}$$

Proof. We first prove that

$$\sigma(q_2)\leqq 14.8t, \qquad t\in[.25,.70]. \tag{5.19}$$

On the interval [.25, .56], the estimate (4.12) gives

$$14.8t-q_2\geqq 14.8t-\frac{225}{32}t^2-\frac{279}{32}t^3-5.5\sqrt{3}\,t\sqrt{1-t^2}\equiv\Phi(t).$$

A routine calculation shows that

$$\Phi''(t)=\frac{-225}{16}-\frac{837}{16}t+16.5\sqrt{3}\,\frac{1+2t^2/3}{(1-t^2)^{\frac{3}{2}}}\,t.$$

If $t\leqq .56$, it can be shown that

$$\frac{16.5\,3(1+2t^2/3)}{(1-t^{\,2})^{\frac{3}{2}}}\leqq 62.$$

It follows that $\Phi''(t)<0$ on [.25, .56], which together with $\Phi(.25)>0$, $\Phi(.56)>0$, yields the inequality (5.19). On the interval [.56, .70], the estimate (4.13) is used to obtain

$$14.8t-q_2\geqq 14.8t-\frac{225}{32}t^2-\frac{279}{32}t^3-\frac{11}{\sqrt{3}}(1-t^2)$$

which a concavity argument shows to be positive on [.56, .70].

Using (5.19) and again following the lines of the proof of Lemma 5.5, it is verified that

$$4\|s\|^2-q_1-s_{11}^2q_2$$

is bounded from below by a quadratic form, the negative of whose discriminant is greater than

$$\Psi(t)=34.96-129.608t-120.8t^2+660t^3-536t^4.$$

We note that

$$\Psi'(t) \leqq -129.608 + t\left[-2144\left(t - \frac{990}{2144}\right)^2 + 215.4\right].$$

Neglecting, for the moment, the squared term within the bracket, it is seen that $\Psi'(t) < 0$ if $t < .596$. When $t \geqq .596$, we have

$$\Psi'(t) \leqq -129.608 + 183.7\,t < 0 \quad \text{if } t \leqq .7.$$

The proof is completed by verifying that $\Psi(.7) > 0$.

6. Comparison of the Cases $r_{13} \leqq 0$ and $r_{13} \geqq 0$

In this section it will be shown that an estimate of δa_6 for the case $r_{13} \geqq 0$ always dominates a particular one for the case $r_{13} \leqq 0$. The estimate of $\sigma(q_3)$ which we shall use for the former case is given by the following lemma.

Lemma 6.1. *If* q_3 *is defined by* (3.23) *or* (3.27) *then*

$$\begin{aligned} \sigma(q_3) &\leqq .4(1-t^2), \quad t \in [.25, .85], \\ &\leqq (50t - 37.5)(1-t^2), \quad t \in [.85, 1]. \end{aligned} \tag{6.1}$$

Proof. By Lemma 2.2, the largest eigenvalue of the quadratic form (3.27) is given by

$$\frac{-(30+8t)+\sqrt{(30+8t)^2+48}}{2} = \frac{24}{(30+8t)+\sqrt{(30+8t)^2+48}} \leqq .4.$$

The proof of the first estimate is immediate. With q_3 given by (3.23), it is fairly easy to prove that

$$q''_{3,11} > 0 \quad \text{and} \quad q_{3,13}\, q''_{3,13} + q_{3,15}\, q''_{3,15} > 0, \qquad t \in [.85, 1].$$

Hence, by Lemma 2.2, the largest eigenvalue τ of q_3 is convex. Computations show that $\tau(.85) < 5$ and $\tau(1) < 12.5$; hence $\tau \leqq 50t - 37.5$. The proof is complete.

Lemma 6.2. *Suppose that in Theorem 4.1, the estimate of Lemma 6.1 is used for* $\tilde{\sigma}(q_3)$. *Then there is a bound for* $\tilde{\sigma}(-q_3)$ *such that the bound for* δa_6 *for the case* $r_{13} \geqq 0$ *is greater than the one for the case* $r_{13} \leqq 0$.

Proof. Consider first the case $.85 \leqq t \leqq 1$. As a consequence of (3.20) and (4.7), it is sufficient to prove that

$$\tau(q_3)(1-t^2) + 24t^2(1-t) \geqq \tau(-q_3)(1-t^2) - 24t^2(1-t) \tag{6.2}$$

where $\tau(q_3)$ and $\tau(-q_3)$ are the largest eigenvalues of q_3 and $-q_3$, respectively. By Lemma 2.2,

$$\tau(q_3) - \tau(-q_3) = q_{3,11}.$$

Hence, in order to prove (6.2), it is sufficient to demonstrate that

$$48t^2 + (1+t)\, q_{3,11} > 0, \qquad t \in [.85, 1].$$

The above quantity is easily shown to have a positive derivative on [.85, 1] and to be positive at .85. This completes the proof of (6.2).

We turn now to the case $.25 \leqq t \leqq .85$. It follows from (3.20), (3.28) and Theorem 4.1 that it is sufficient to prove

$$\text{(6.3)} \qquad \frac{[\sigma(-q_3)-24t^2(1-t)-8t^3]^2}{\frac{28}{3}t-6+\tilde{\rho}(Q)} \leqq \frac{[.4(1-t^2)+24t^2(1-t)]^2}{\frac{28}{3}t-6+\tilde{\rho}(Q)}.$$

Note that, in virtue of Lemmas 5.7 and 5.8, $28t/3-6+\tilde{\rho}(Q)>0$.

The case $.25 \leqq t \leqq .50$ is considered first. It follows from the estimate (4.12) that

$$\sigma(-q_3) \leqq 16.875t^2+4.5t^3+3\sqrt{3}\,t\sqrt{1-t^2};$$

hence

$$\begin{aligned}[.4(1-t^2)+24t^2(1-t)]-[\sigma(-q_3)-24t^2-8t^3]\\ \geqq .4-5.2t\sqrt{1-t^2}+30.725t^2-44.5t^3.\end{aligned}$$

By using the inequalities $1-t^2 \leqq 1$, $t \in [.25, .45]$ and $\sqrt{1-t^2} \leqq .9$, $t \in [.45, .50]$, one obtains on each of the intervals a positive concave lower bound for the right side of the above inequality. The validity of (6.3) follows.

When $t \geqq .5$, the contribution of $8t^4(1-t)$ must be taken into account. Define

$$w=\frac{1}{6(1-t)}\left[\frac{28}{3}t-6+\rho(Q)\right]$$

and note that

$$8t^4(1-t) \geqq \frac{w[24t^2(1-t)]^2}{12\left[\frac{28}{3}t-6+\rho(Q)\right]}.$$

In order to prove (6.3), it is sufficient to establish that

$$\text{(6.4)} \qquad \sqrt{1+w}\,[24t^2(1-t)]-[\sigma(-q_3)-24t^2(1-t)-8t^3] \geqq 0.$$

Since $\tilde{\rho}(Q)=4$, $t \in [.25, .70]$ and $\tilde{\rho}(Q)=11-10t$, $t \in [.70, .85]$, we have

$$\text{(6.5)} \qquad \sqrt{1+w} \geqq \begin{cases} 1.37, & t \geqq .5, \\ 1.7, & t \geqq .65, \\ 1.94, & t \geqq .73, \\ 2.02, & t \geqq .76, \\ 2.06, & t \geqq .77. \end{cases}$$

We also have, as a consequence of (4.13), (4.11) and (2.3)

$$\text{(6.6)} \qquad \sigma(-q_3) \leqq \begin{cases} \frac{9}{16}t^2(30+8t)+\frac{6}{\sqrt{3}}(1-t^2), & t \in [.50, .77] \\ \frac{3}{4}t\sqrt{(30+8t)^2+48}\,\sqrt{1-t^2}, & t \in [.77, .80] \\ \frac{(30+8t)+\sqrt{(30+8t)^2+48}}{2}(1-t^2), & t \in [.80, .85]. \end{cases}$$

It is easily proved that

$$\sqrt{(30+8t)^2+48} \leqq 1.02(30+8t), \quad t \leqq .85 \tag{6.7}$$

and

$$\sqrt{1-t^2} \leqq \frac{5-4t}{3}, \quad t \in [.77, .80], \tag{6.8}$$

the latter inequality being obtained by using the tangent at .8 as an upper bound. By using (6.5)–(6.8), we obtain the following lower bounds for (6.4):

$$\begin{array}{ll} -3.466+43.471t^2-53.38t^3, & t\in[.50, .65] \\ -3.466+51.391t^2-61.3t^3, & t\in[.65, .73] \\ -3.466+57.151t^2-67.06t^3, & t\in[.73, .76] \\ -3.466+59.071t^2-68.98t^3, & t\in[.76, .77] \\ -38.25t+93.88t^2-57.28t^3, & t\in[.77, .80] \\ -30.3-8.08t+102.54t^2-56.16t^3, & t\in[.80, .85]. \end{array}$$

Each of the above polynomials in concave on and positive at the end points of the indicated interval. The proof of (6.4), and hence (6.3), follows.

7. Proof of Theorem 1

We begin by disposing of the case $0 \leqq t \leqq .25$. Since C is unitary, we have

$$|c_{jk}| \leqq 1 \quad \text{and} \quad |c_{13}c_{15}| \leqq \frac{|c_{13}|^2+|c_{15}|^2}{2} \leqq 1/2.$$

The inequality $|s_{11}| \leqq 3t/4$ (4.3) implies that

$$|c_{11}| \leqq \tfrac{5}{4}t.$$

After substituting the above inequalities into (1.7) with $\lambda = -8/\sqrt{5}$ and $\mu = 0$, we obtain

$$|a_6| \leqq \frac{2}{5}+\frac{6}{15}+\left(\frac{8}{\sqrt{15}}+\frac{14}{3}\right)\frac{5t}{4}+\left(\frac{10}{3}+\frac{8}{\sqrt{5}}\right)(5t/4)^2+\frac{24}{\sqrt{3}}(5t/4)^3+\frac{34}{15}(5t/4)^5.$$

The above polynomial is increasing and less than 6 at $t=1/4$. Hence $\delta a_6 < 6$ if $t \in [0, .25]$.

On the interval [.25, 1], it is a consequence of (3.20), (4.6), (4.7), (5.11), (5.15), (5.18), (6.1) and Lemma 6.2 that

$$\begin{aligned} \delta a_6 \leqq {} & \frac{10}{3}(t^2-1)+\frac{34}{15}(t^5-1)+16t^4(1-t) \\ & +\tilde{\rho}(Q)(1-t^2)+\frac{[24t^2(1-t)+\tilde{\sigma}(q_3)]^2}{12\left[\frac{28}{3}t+\tilde{\rho}(Q)-6\right]} \end{aligned} \tag{7.1}$$

where

(7.2) $$\tilde{\rho}(Q)=\begin{cases}7.5-7t, & t\in[.85, 1]\\ 11-10t, & t\in[.70, .85]\\ 4, & t\in[.25, .70]\end{cases}$$

and

(7.3) $$\tilde{\sigma}(q_3)=\begin{cases}(50t-37.5)(1-t^2), & t\in[.85, 1]\\ .4(1-t^2), & t\in[.25, .85].\end{cases}$$

By convexity we have

$$[24t^2+.4(1+t)]^2\leqq 16(21t-5), \qquad t\in[.25, .70].$$

Using the above inequality, together with (7.1), (7.2), (7.3), we obtain

(7.4) $$\delta a_6\leqq\frac{2}{3}(1-t^2)+\frac{34}{15}(t^5-1)+16t^4(1-t)+\frac{2(21t-5)}{14t-3}(t-1)^2.$$

The quantity $2(21t-5)/(14t-3)$ is increasing and hence can be shown to have the respective bounds 2.2 and 3 on the intervals [.25, .30] and [.30, .70]. Substituting these estimates into (7.4), we obtain on each interval a polynomial upper bound for δa_6. Each of these polynomials has a positive second derivative on the interval and is negative at its endpoints. It follows that $\delta a_6<0$ if $t\in[.25, .70]$.

On the interval [.70, .85], we have

$$\frac{[24t^2+.4(1+t)]^2}{4(15-2t)}\leqq 6.2$$

as a consequence of monotonicity. The above inequality, together with (7.1), (7.2), (7.3), implies that

$$\delta a_6\leqq\left(\frac{10}{3}-(11-10t)\right)(t^2-1)+\frac{34}{15}(t^5-1)+16t^4(1-t)+6.2(t-1)^2$$

for $t\in[.70, .85]$. The above polynomial is easily shown to be convex on [.70, .85] and negative at $t=.70$ and $t=.85$. It follows that $\delta a_6<0$ on [.70, .85].

Finally we come to the local estimate, $t\in[.85, 1]$. After substituting (7.2) and (7.3) into (7.1) and re-arranging terms, we obtain the inequality

(7.5) $$(28t+18)\delta a_6\leqq 46(t-1)+(t-1)^2\,\Psi(t)$$

where

$$\Psi(t)=80.2+41.6t-268t^2-568\frac{4}{15}t^3-384\frac{8}{15}t^4+[24t^2+(50t-37.5)(1+t)]^2.$$

The function $\Psi(t)$ has a positive second derivative on [.85, 1] and satisfies

$$\Psi(.85)\leqq 79.25, \qquad \Psi(.9)\leqq 369.02, \qquad \Psi(1)\leqq 1302;$$

hence

$$\Psi(t) \leq \begin{cases} 9329.8(t-.9)+369.02, & t\in[.90, 1] \\ 5795.4(t-.85)+79.25, & t\in[.85, .90]. \end{cases} \tag{7.6}$$

After putting (7.6) into (7.5), and replacing $1-t$ by s, we obtain

$$(28t+18)\,\delta a_6 \leq \begin{cases} -s[46-1302s+9329.8s^2], & s\in[0, .1] \\ -s[46-948.56s+5795.4s^2], & s\in[.1, .15]. \end{cases}$$

In the first estimate, the quadratic within the brackets is easily shown to have a negative discriminant. Hence $\delta a_6<0$ if $t\in[.90, 1)$. In the second estimate, the quadratic within the brackets can be shown to have a positive derivative for $s\geqq .1$ and to be positive at $s=.1$. Hence $\delta a_6<0$ if $t\in[.85, .90]$.

The previous considerations prove that $\delta a_6<0$ with equality only if $t=1$. It is well known that the latter possibility can occur only for the Koebe function. This completes the proof.

This research was supported by the National Science Foundation, Grant NSF GP-7662.

Acknowledgements. I would like to thank Professor ALAN PERLIS for having had the forsight to provide the Carnegie-Mellon Computer Library with a program of the Jacobi method for obtaining the eigenvalues of symmetric matrices. Thanks are also due to Mr. DAVID NICKERSON and Mr. MICHAEL ZIEGLER for coding the mathematics of [16] into machine language. I would also like to thank Professors P. R. GARABEDIAN, Z. NEHARI and M. M. SCHIFFER for their interest and encouragement.

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(Received July 30, 1968)