A Mathematical Foundation for Thermodynamics

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I. General Theory of Actions on Systems

1. Introduction

In this essay we present and explore a mathematical foundation for thermodynamics which is simple in concept and sufficiently broad to support the many manifestations of the subject found in physics, including thermodynamical theories of "systems with memory". Our discussion splits naturally into two parts. Sections 1 through 8 comprise Part I and deal mainly with basic concepts and assumptions which appear to be present in all branches of thermodynamics; the theory developed in Part I permits us to discuss questions of existence, uniqueness, and regularity of energy and entropy functions. In Sections 9–13, *i.e.* Part II, we illustrate our general theory by developing a theory of "simple material elements" which contains as special cases the classical theories of elastic elements and viscous elements as well as recent theories of elements with internal variables and elements with fading memory. In Part II we show that many results which have been obtained from the Clausius-Duhem inequality are consequences of a weaker form of the Second Law of Thermodynamics.

It appears to us that all branches of thermodynamics employ, either explicitly or implicitly, the concepts of "state"[#] and "process". In the general theory of Part I, we take the collection Σ of states σ and the collection Π of processes P to be primitive concepts, and we define a *system* to be an ordered pair (Σ, Π), such that Σ is a Hausdorff space[##] and each element P of Π determines a continuous mapping $\sigma \mapsto P\sigma$ of an open subset $\mathscr{D}(P)$ of Σ into Σ (this mapping is called the "transformation induced by P"); we assume that the pair (Σ, Π) obeys two fundamental axioms: **(I)** for each σ in Σ, the set of states of the form $P\sigma$ with P in Π (*i.e.* the set of states "accessible from σ") is dense in Σ; **(II)** if P' and P'' are processes such that the range $\mathscr{R}(P')$ of the transformation $\sigma \mapsto P'\sigma$ induced by P' intersects the domain $\mathscr{D}(P'')$ of the transformation $\sigma \mapsto P''\sigma$ induced by P'', then Π contains a process, $P''P'$, whose induced transformation is the composition of $\sigma \mapsto P''\sigma$ and $\sigma \mapsto P'\sigma$.

An *action* $\mathscr{a}$ for a system assigns to a pair (P, σ) a number $\mathscr{a}(P, \sigma)$, referred to as *the supply of* $\mathscr{a}$ *on going from* σ *to* $P\sigma$ *via the process* P; according to our Definition 2.2., to be an action, the function $(P, \sigma) \mapsto \mathscr{a}(P, \sigma)$ must be continuous in σ and, in addition, additive on processes in the sense that whenever P can be represented as the successive application $P''P'$ of two processes P' and P'', $\mathscr{a}(P, \sigma)$ must be the sum of the supplies of $\mathscr{a}$ obtained by going from σ to $P'\sigma$ via P' and from $P'\sigma$ to $P\sigma$ via P'', *i.e.*

$$\mathscr{a}(P, \sigma) = \mathscr{a}(P'', P'\sigma) + \mathscr{a}(P', \sigma). \tag{1.1}$$

Thermodynamics is, at bottom, the theory of actions. In Definition 3.1 we say that an action $\mathscr{a}$ has the *Clausius property* at a state σ if $\mathscr{a}(P, \sigma)$ is approximately negative for every process P that is nearly cyclic at σ, *i.e.* if for each $\varepsilon > 0$, there is a neighborhood $\mathcal{O}_\varepsilon$ of σ such that

$$\mathscr{a}(P, \sigma) < \varepsilon$$

whenever P is a process with $P\sigma$ in $\mathcal{O}_\varepsilon$. The action $\mathscr{a}$ has the *conservation property* at σ if $\mathscr{a}(P, \sigma)$ is approximately zero for each process P that is nearly cyclic at σ, *i.e.* if for each $\varepsilon > 0$, there is a neighborhood $\mathcal{O}_\varepsilon$ of σ such that $P\sigma$ in $\mathcal{O}_\varepsilon$ implies

$$|\mathscr{a}(P, \sigma)| < \varepsilon.$$

It follows immediately from these definitions that if $\mathscr{a}$ has the $\left\{\begin{matrix}\text{Clausius} \\ \text{conservation}\end{matrix}\right\}$ property at σ, then $\mathscr{a}(P, \sigma)$ is $\left\{\begin{matrix}\text{not positive} \\ \text{zero}\end{matrix}\right\}$ whenever $\sigma = P\sigma$.

In Theorems 3.1 and 4.1 we show that if an action $\mathscr{a}$ has the $\left\{\begin{matrix}\text{Clausius} \\ \text{conservation}\end{matrix}\right\}$ property at one state in Σ, then the set of states at which $\mathscr{a}$ has the $\left\{\begin{matrix}\text{Clausius} \\ \text{conservation}\end{matrix}\right\}$ property is dense in Σ.

A real-valued function A on a dense subset of Σ is here called an *upper potential* for an action $\mathscr{a}$ if for each pair (σ_1, σ_2) of states in the domain of A, and for

[#] In continuum physics a "state" is determined by specifying values for the independent variables in the constitutive equations; for a material with fading memory, *cf.* [1964, 1, 2], the list of independent variables contains "histories".

[##] Whereas in examples Σ is often taken to be a subset of a finite or infinite dimensional normed vector space, an operation of "addition of states" is *not* required in our general theory.

each $\varepsilon>0$, there is a neighborhood $\mathcal{O}_\varepsilon$ of σ_2 such that

$$A(\sigma_2)-A(\sigma_1)>a(P,\sigma_1)-\varepsilon,$$

for every process P with $P\sigma_1$ in $\mathcal{O}_\varepsilon$. Hence, if A is an upper potential for a, and if σ_1 and σ_2 are in the domain of A, there holds

$$A(\sigma_2)-A(\sigma_1)\geqq a(P,\sigma_1), \tag{1.2}$$

for every P in Π with $P\sigma_1=\sigma_2$.

It is easily shown that if an action a has an upper potential A, then a has the Clausius property at every state in the domain of A. Converse theorems to this are more difficult to establish. The main result of Part I is Theorem 3.3 which asserts that if there is a state σ° at which an action a has the Clausius property, then a has an upper semicontinuous upper potential. Our proof of this theorem is constructive in that after defining, in terms of a and σ°, a set Σ° of states and an explicit real-valued function A° on Σ°, we demonstrate that Σ° is dense in Σ, that A° is an upper potential, and that A° is upper semicontinuous on all of Σ°. In Theorem 3.7 we show that the upper potential A° constructed in Theorem 3.3 has the following interesting property: If σ and σ' are two states with σ' in Σ°, and if σ is a limit of states σ_n obtained from σ' via processes P_n in which the supply of a is zero, then σ is in the domain Σ° of A° and $A(\sigma)$ is not less than $A(\sigma')$. The structure and closure properties of Σ° and the set $\mathscr{C}$ of states at which a has the Clausius property are discussed at length in Section 3.

We call a real-valued function A on a dense subset $\mathscr{A}$ of Σ a *potential** for an action a if for each pair (σ_1, σ_2) of states in $\mathscr{A}$ and each $\varepsilon>0$, there is a neighborhood $\mathcal{O}_\varepsilon$ of σ_2 such that $P\sigma_1$ in $\mathcal{O}_\varepsilon$ implies

$$\left|a(P,\sigma_1)-[A(\sigma_2)-A(\sigma_1)]\right|<\varepsilon.$$

Hence, when A is a potential for a, and σ_1 and σ_2 are in $\mathscr{A}$, we have

$$A(\sigma_2)-A(\sigma_1)=a(P,\sigma_1),$$

for every P in Π with $P\sigma_1=\sigma_2$. Clearly, if a has a potential, then a has the conservation property at every state in the domain $\mathscr{A}$ of A. In Section 4 we employ our main theorem, 3.3, to show that if a has the conservation property at a state, then a has a potential. In that section we show also that every potential is a continuous function and that two potentials for a given action can differ by at most a constant on the intersection of their domains.

In Sections 6 and 7, we suppose that the set Π of processes has a subset P for which the operation of successive application $(P'', P')\mapsto P''P'$ gives to P the structure of an Abelian semigroup which is an epimorphic image of either the additive semigroup, $\mathbb{N}$, of positive integers or the additive semigroup, $\mathbb{R}^{++}$, of positive real numbers. We write $P^{[\nu]}$ for the image in P of ν in $\mathbb{N}$ (or $\mathbb{R}^{++}$), and hence

$$P^{[\nu_2]}P^{[\nu_1]}=P^{[\nu_2+\nu_1]},$$

* The concept of "potential" defined here generalizes that used in the theory of vector fields and line integrals.

for all ν in $\mathfrak{J}$, where $\mathfrak{J}$ stands for $\mathbb{N}$ or $\mathbb{R}^{++}$. When there is at least one state σ' which is in $\bigcap_{\nu\in\mathfrak{J}} \mathscr{D}(P^{[\nu]})$ and is such that $\sigma = \lim_{\nu\to\infty} P^{[\nu]}\sigma'$ exists and is also in $\bigcap_{\nu\in\mathfrak{J}} \mathscr{D}(P^{[\nu]})$, we call P a *stagnating family of processes*, and we refer to σ as a *stagnant state* for P. In Section 6 we show that if P is a stagnating family, then for σ to be a stagnant state for P, it suffices that $P\sigma = \sigma$ for one process P in P, and if σ *is* a stagnant state for P, then $P\sigma = \sigma$ for *every* P in P. We use this observation to show that if σ is a stagnant state for a stagnating family P and if a is an action with the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at σ, then $a(P, \sigma)$ is $\left\{\begin{matrix}\text{not positive}\\ \text{zero}\end{matrix}\right\}$ for every P in P, and, moreover, for all distinct ν_1, ν_2 in $\mathfrak{J}$, the ratio

$$\frac{a(P^{[\nu_2]}, \sigma) - a(P^{[\nu_1]}, \sigma)}{\nu_2 - \nu_1}$$

equals $a(P^{[1]}, \sigma)$, a constant; if, in addition, P is an epimorphic image of $\mathbb{R}^{++}$ (*i.e.* if $\mathfrak{J}$ is not $\mathbb{N}$), then $a(P^{[\nu]}, \sigma)$ is a differentiable function of ν and

$$\frac{d}{d\nu}\, a(P^{[\nu]}, \sigma) = a(P^{[1]}, \sigma),$$

for all ν in $\mathbb{R}^{++}$. In more suggestive terms: if a has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at a stagnant state σ, then the rate of supply of a to maintain the state σ is constant and $\left\{\begin{matrix}\text{not positive}\\ \text{zero}\end{matrix}\right\}$.

We say that a state σ° is a *relaxed state for an action* a if there is a stagnating family P of process $P^{[\nu]}$, $\nu\in\mathfrak{J}$, such that σ° is in $\bigcap_{\nu\in\mathfrak{J}} \mathscr{D}(P^{[\nu]})$, and for some state σ in $\bigcap_{\nu\in\mathfrak{J}} \mathscr{D}(P^{[\nu]})$, *both* $P^{[\nu]}\sigma$ and $a(P^{[\nu]}, \sigma)$ have limits as $\nu\to\infty$, and, moreover,

$$\sigma^\circ = \lim_{\nu\to\infty} P^{[\nu]}\sigma;$$

the stagnating family P is said to *relax* σ to σ° for a. In other words, for σ° to be a relaxed state for a, σ° must be a stagnant state for some stagnating family P such that σ° is ultimately obtained via P with finite "total supply" of a. Our Theorem 7.1 shows that if P relaxes some state σ to σ° for a, then

$$a(P, \sigma^\circ) = 0,$$

for every P in P (whether or not a has the conservation property at σ°).

Our principal reason for introducing the apparatus required to define *relaxed states* at this level of generality is that the concept is employed in Theorems 7.2 to 7.4. In those theorems we assume that there is given an action a for which there are relaxed states, and we suppose that a has the Clausius property at one of its relaxed states, σ°. Theorem 7.2 states that the upper potential A° constructed in the proof of Theorem 3.3 is (with respect to pointwise partial order) the *smallest* element of the class of upper potentials which have common domain Σ° and vanish at σ°. Theorem 7.3 gives a condition under which the class of upper potentials defined in Theorem 7.2 has a *largest* element $\hat{A}^\circ$, which, like A°, can be exhibited explicitly. Theorem 7.4, a corollary to Theorems 7.2 and 7.3, gives a necessary and sufficient condition to have $\hat{A}^\circ = A^\circ$, *i.e.* for there to be precisely one upper potential A for $\mathscr{a}$ with domain Σ° and with $A(\sigma^\circ)=0$.

In Sections 5 and 8 we cast into terms closer to those employed in works on thermodynamics parts of the mathematical theory briefly described above and developed in detail in Sections 2, 3, 4, 6, and 7. Our version of the Laws of Thermodynamics is stated in Section 5, and we there gather together results which bear on the existence and analytic properties of energy and entropy functions. In Section 8 we discuss the ways the existence of relaxed states can limit the class of possible entropy functions.

In the last decade much progress has been made toward elevating the thermodynamics of space-filling bodies from an art to a mathematical science. In much of this research the emphasis has been laid upon the problem of finding the restrictions which the Second Law of Thermodynamics places upon constitutive assumptions, although important parts of the more recent work have dealt with the measure-theoretical foundations of continuum thermodynamics, with the implications of thermodynamics for wave propagation, and with the problem of giving a dynamical significance to the criteria for stability proposed by GIBBS and DUHEM. Most of the research dealing with the thermodynamic restrictions on constitutive assumptions has started with the Clausius-Duhem inequality,[#] which in integrated form is a special case of an inequality of the type (1.2) with the action $\mathscr{a}$ defined in terms of variables appearing in the constitutive equations.

In formulating the present general theory we have been influenced by the research based on the Clausius-Duhem inequality,[##] for that work shows what sort of special theories our mathematical foundation must be able to support.[###]

[#] Exceptions are the work of DAY [1968, 3] [1969, 1] and the earlier studies of UDESCHINI [1943, 1] and CAPRIOLI [1955, 1] in which entropy and stored energy are treated as derived rather than primitive quantities. See also the recent survey by DAY [1972, 1].

[##] Particularly the following: COLEMAN & NOLL [1963, 2], COLEMAN & MIZEL [1963, 1] [1964, 3] [1967, 2], COLEMAN [1964, 1, 2], COLEMAN & GURTIN [1967, 1], OWEN [1968, 4], and COLEMAN & OWEN [1970, 2].

[###] Some time after the research described in Part I was completed, J. C. WILLEMS' theory of "dissipative dynamical systems" appeared in this *Archive* [1972, 3]. WILLEMS' theory, although it has some formal features in common with that described here, employs thermodynamical assumptions much more restrictive than those which we consider. In our language, WILLEMS' "dissipative systems" have entropy functions which are *bounded above*. [See his Def. 1 and Thm. 1 in which his $-S_a$ and $-S$ play the roles of our entropy functions, *i.e.* upper potentials for a preassigned action $\mathscr{a}$ which he writes $-\int_{t_0}^{t_1} w(t)\,dt$.] WILLEMS shows for a system to be *dissipative*, in his sense, it is necessary and sufficient that there be for each state σ, a number $M(\sigma)$, such that

$$\mathscr{a}(P,\sigma) \leqq M(\sigma)$$

for *all* P in Π with σ in $\mathscr{D}(P)$; *i.e.* in WILLEMS' theory there is for each σ a finite upper bound $M(\sigma)$ for the supply of $\mathscr{a}$ on going to any state accessible from σ. Such a property for $\mathscr{a}$ is much stronger than the Clausius property, which does *not* imply the existence of an upper potential which is bounded above.

We do not believe that WILLEMS' theory is appropriate to the physical systems we have in mind. Even for an ideal gas the entropy is not bounded above, nor is the Helmholtz free energy bounded below.

We have also been influenced by some recent work of W. NOLL and W. A. DAY. The concept of a *system* as defined in Section 2 of Part I was suggested to us by W. NOLL's definition of a "material element", and our theory of stagnant states draws upon ideas in Section 12 of his paper [1972, 2] as well as ideas in a paper of COLEMAN [1962, 1]; our (non-thermodynamical) Theorem 6.2 follows NOLL's proof of his Theorem 12.1. The basic idea that, even for systems with memory, a thermodynamical inequality of the Clausius type should imply the existence of an entropy function is found in an important paper of W. A. DAY [1969, 1]. His "Thermodynamical Axiom" asserts that a certain sum (or integral) is not positive when evaluated on infinite processes which start and end at equilibrium states with the same strain and internal energy. In contrast, our formulation of the Second Law asserts that there is one (not necessarily equilibrium) state at which a preassigned action has the Clausius property, *i.e.* is approximately negative on processes which start at that state and return near to it. A principal difference between the theory we develop in Part I and that of DAY, and also that of NOLL, is that in accord with our search for simplicity and generality we employ concepts of process and state which do not require that processes be identified with functions on intervals or that states have "configurations".

2. Systems

The basic primitive objects of our theory are the collection Σ of the "states of a system" and the collection Π of the "processes of a system"; each process P determines a mapping ρ_P of states into states; ρ_P is called the "transformation induced by P." The pair (Σ, Π) is assumed to obey certain axioms which are rather weak; they assert that Σ possesses a Hausdorff topology which renders the mappings ρ_P continuous, that Π is large enough to insure that the set of states accessible from a given state via processes is "most of Σ" in the sense that it is dense in Σ, and that for each pair (P'', P') of processes such that the range of $\rho_{P'}$ reaches the domain of $\rho_{P''}$, the "successive application" of P'' and P', $P''P'$, is well defined, is again a process, and the transformation $\rho_{P''P'}$ induced by this process equals the composition of $\rho_{P''}$ and $\rho_{P'}$.

Formally, we lay down the following definition.

Definition 2.1. Let (Σ, Π) be an ordered pair in which Σ is a Hausdorff space, and Π is a set of objects each of which determines a continuous function ρ_P mapping a non-empty open subset $\mathscr{D}(P)$ of Σ onto a subset $\mathscr{R}(P)$ of Σ. If (Σ, Π) obeys the axioms **I** and **II** below, then (Σ, Π) is called a **system**; each element σ of Σ is called a **state**, each element P of Π is called a **process**, and ρ_P is called the **transformation induced by** P.

I. For each σ in Σ, the set

$$\Pi\sigma \overset{\text{def}}{=} \{\rho_P\sigma \mid P\in\Pi, \sigma\in\mathscr{D}(P)\} \tag{2.1}$$

is dense in Σ.

II. Let $\mathscr{P}$ be the subset of $\Pi\times\Pi$ comprised of all pairs (P'', P') such that the range of $\rho_{P'}$ meets the domain of $\rho_{P''}$, *i.e.*

$$\mathscr{P} = \{(P'', P')\in\Pi\times\Pi \mid \mathscr{D}(P'')\cap\mathscr{R}(P') \neq \phi\};$$

on $\mathscr{P}$ there is assigned a Π-valued function $(P'', P') \mapsto P''P'$ such that

$$\mathscr{D}(P''P') = \rho_{P'}^{-1}(\mathscr{D}(P'') \cap \mathscr{R}(P'))$$

and, for each σ in $\mathscr{D}(P''P')$,

$$\rho_{P''P'}\sigma = \rho_{P''}\rho_{P'}\sigma. \tag{2.2}$$

For each σ in Σ, we refer to $\Pi\sigma$, defined in (2.1), as the *set of states accessible from* σ. For the set of all ordered pairs (P, σ) with P in Π and σ in $\mathscr{D}(P)$, we use the symbol $\Pi \diamond \Sigma$.

Definition 2.2. An **action** $\mathscr{a}$ for a system (Σ, Π) is a real-valued function on $\Pi \diamond \Sigma$ obeying the following conditions of additivity and continuity.

(i) Additivity: if P' and P'' are in Π with (P'', P') in $\mathscr{P}$, and if σ is in $\mathscr{D}(P''P')$, then

$$\mathscr{a}(P''P', \sigma) = \mathscr{a}(P', \sigma) + \mathscr{a}(P'', \rho_{P'}\sigma). \tag{2.3}$$

(ii) Continuity: for each P in Π, the function $\mathscr{a}_P: \mathscr{D}(P) \to \mathbb{R}$, defined by

$$\mathscr{a}_P(\sigma) = \mathscr{a}(P, \sigma), \tag{2.4}$$

is continuous.

When discussing an action $\mathscr{a}$, if one writes $\mathscr{a}(P, \sigma)$, it is automatically understood that σ is in $\mathscr{D}(P)$, *i.e.* that (P, σ) is in $\Pi \diamond \Sigma$. We refer to $\mathscr{a}(P, \sigma)$ as *the supply of* $\mathscr{a}$ *on going from* σ *to* $\rho_P\sigma$ *via the process* P.

There are systems for which two processes P', P'' can induce the same transformation, *i.e.* can be such that $\rho_{P'} = \rho_{P''}$, but give different values to an action $\mathscr{a}$. Hence one must bear in mind the distinction between processes and their induced transformations. However, much of the present theory is more concisely described and more easily grasped if one employs a notation which does not render explicit this distinction. Throughout most of this study and, in particular, throughout the remainder of Part I, we simplify our notation by writing P for ρ_P, so that P represents both a process, *i.e.* an element of Π, and a transformation, *i.e.* a continuous Σ-valued function on a subset of Σ. In this notation, $\rho_P\sigma$ becomes $P\sigma$, and whenever we write $P\sigma$ it is to be understood that σ is in $\mathscr{D}(P)$. Thus for a pair (P'', P') in $\mathscr{P} \subset \Pi \times \Pi$, the symbol $P''P'$ can denote either a process or the transformation $\rho_{P''}\rho_{P'}$; of course, $P''P'\sigma$ can mean only $\rho_{P''}\rho_{P'}\sigma$. Whenever we write $P''P'$, we assume that $\mathscr{D}(P'') \cap \mathscr{R}(P')$ is not empty, or, equivalently, that (P'', P') is in $\mathscr{P}$. In accord with these conventions, the equation (2.3) can be written

$$\mathscr{a}(P''P', \sigma) = \mathscr{a}(P', \sigma) + \mathscr{a}(P'', P'\sigma).$$

In subsequent sections we shall refer frequently to the following direct consequence of the assumed continuity on $\mathscr{D}(P)$ of the function $\mathscr{a}_P$ and the transformation induced by P.

Remark 2.1. Let $\mathscr{a}$ be an action, σ° a given state, and $\mathscr{O}$ an open subset of Σ. If P is a process with $P\sigma^\circ$ in $\mathscr{O}$, then for every $\varepsilon > 0$ there is a neighborhood[#] $\hat{\mathscr{O}}$ of σ°

[#] Throughout this paper, whenever we refer to a "neighborhood of a state σ", it is to be understood that we mean an open subset of Σ containing σ, *i.e.* an *open neigborhood of* σ.

such that

$$\hat{\mathcal{O}} \subset \mathcal{D}(P) \quad \text{and} \quad P\hat{\mathcal{O}} \subset \mathcal{O},$$

while

$$a_P(\hat{\mathcal{O}}) \subset \mathcal{N}_\varepsilon(a_P(\sigma^\circ)),$$

where $P\hat{\mathcal{O}}$ is the set $\{P\sigma \mid \sigma \in \hat{\mathcal{O}}\}$, *i.e.* is the image of $\hat{\mathcal{O}}$ under the transformation ρ_P, and $\mathcal{N}_\varepsilon(a_P(\sigma^\circ))$ is the open interval $(a_P(\sigma^\circ)-\varepsilon,\ a_P(\sigma^\circ)+\varepsilon)$.

Proof. Let $\varepsilon>0$ be given. Recall that when we write $P\sigma^\circ$ we are automatically assuming that σ° is in $\mathcal{D}(P)$. If $P\sigma^\circ$ is in $\mathcal{O}$, then, by the continuity of a_P and P on $\mathcal{D}(P)$, and the fact that $\mathcal{D}(P)$ is open in Σ, there are neighborhoods $\mathcal{O}'$ and $\mathcal{O}''$ of σ° which are subsets of $\mathcal{D}(P)$ and such that $a_P(\mathcal{O}') \subset \mathcal{N}_\varepsilon(a_P(\sigma^\circ))$ and $P\mathcal{O}'' \subset \mathcal{O}$. The set $\mathcal{O}' \cap \mathcal{O}''$ obviously has the properties here required of $\hat{\mathcal{O}}$; q.e.d.

3. Actions with the Clausius Property

Definition 3.1. Let σ° be a state and let a be an action for a system (Σ, Π). If for each $\varepsilon>0$ there is a neighborhood $\mathcal{O}$ of σ° such that

$$P\in\Pi, \quad P\sigma^\circ\in\mathcal{O} \quad \Rightarrow \quad a(P,\sigma^\circ)<\varepsilon, \tag{3.1}$$

then we say that a has the **Clausius property** at σ°. If for each $\varepsilon>0$ there is a neighborhood $\mathcal{O}$ of σ° such that

$$P\in\Pi, \quad P\sigma^\circ\in\mathcal{O} \quad \Rightarrow \quad |a(P,\sigma^\circ)|<\varepsilon, \tag{3.2}$$

then a is said to have the **conservation property** at σ°.

In terms which are not quite precise, an action a has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at a state σ° if the supply of a on going from σ° to any state in a small neighborhood $\mathcal{O}$ of σ° is approximately $\left\{\begin{matrix}\text{negative}\\ \text{zero}\end{matrix}\right\}$ for all processes P which effect such a transition (even if $P=P''P'$ with $P'\sigma^\circ$ outside of $\mathcal{O}$). Suppose P is a process such that $P\sigma^\circ=\sigma^\circ$ (of course, there may not exist such a process); if a has the Clausius property at σ°, then $a(P,\sigma^\circ)$ is not positive; if a has the conservation property at σ°, then $a(P,\sigma^\circ)=0$.

If a is an action, σ° a state, and $\mathcal{O}$ a subset of Σ, then we write $a\{\sigma^\circ\to\mathcal{O}\}$ for the set of numbers $a(P,\sigma^\circ)$ obtained by letting P vary over the processes whose induced transformations take σ° into $\mathcal{O}$:

$$a\{\sigma^\circ\to\mathcal{O}\} \stackrel{\text{def}}{=} \{a(P,\sigma^\circ) \mid P\in\Pi,\ P\sigma^\circ\in\mathcal{O}\}. \tag{3.3}$$

Lemma 3.1. For an action a to have the Clausius property at a state σ°, it suffices that there exist a non-empty open set $\mathcal{O}$ of states such that the set $a\{\sigma^\circ\to\mathcal{O}\}$ is bounded above.

Proof. Suppose there is an $\mathcal{O}$ obeying the hypothesis, *i.e.* such that $\sup a\{\sigma^\circ\to\mathcal{O}\}=b<\infty$. Let ε be a positive number. By the definition of *supremum*, there is a $\bar{P}$

with $\bar{P}\sigma^\circ$ in $\mathcal{O}$ and

$$a(\bar{P}, \sigma^\circ) + \frac{\varepsilon}{2} > b. \tag{3.4}$$

In view of Remark 2.1, there exists a neighborhood $\hat{\mathcal{O}}$ of σ° such that

$$\bar{P}\hat{\mathcal{O}} \subset \mathcal{O} \tag{3.5}$$

and

$$a_{\bar{P}}(\hat{\mathcal{O}}) \subset \mathcal{N}_{\frac{\varepsilon}{2}}\big(a_{\bar{P}}(\sigma^\circ)\big). \tag{3.6}$$

The lemma will be proven if we show that for every P in Π obeying

$$P\sigma^\circ \in \hat{\mathcal{O}}, \tag{3.7}$$

there holds $a(P, \sigma^\circ) < \varepsilon$. Now, let P obey (3.7). By (3.4),

$$a(P, \sigma^\circ) + b < a(P, \sigma^\circ) + a(\bar{P}, \sigma^\circ) + \frac{\varepsilon}{2},$$

and, by (3.6) and (3.7),

$$a(\bar{P}, \sigma^\circ) - \frac{\varepsilon}{2} < a(\bar{P}, P\sigma^\circ).$$

Thus,

$$a(P, \sigma^\circ) + b < a(P, \sigma^\circ) + a(\bar{P}, P\sigma^\circ) + \varepsilon.$$

Clearly, $P\sigma^\circ \in \mathcal{R}(P) \cap \hat{\mathcal{O}} \subset \mathcal{R}(P) \cap \mathcal{D}(\bar{P})$, *i.e.* $\sigma^\circ \in \mathcal{D}(\bar{P}P)$, and (2.3) yields $a(\bar{P}P, \sigma^\circ) = a(P, \sigma^\circ) + a(\bar{P}, P\sigma^\circ)$; hence, the last inequality above can be written

$$a(P, \sigma^\circ) + b < a(\bar{P}P, \sigma^\circ) + \varepsilon. \tag{3.8}$$

On the other hand, (3.7) and (3.5) yield $\bar{P}P\sigma^\circ \in \bar{P}\hat{\mathcal{O}} \subset \mathcal{O}$, so that, by the definition of b,

$$a(\bar{P}P, \sigma^\circ) \leqq b. \tag{3.9}$$

The relations (3.8) and (3.9) yield $a(P, \sigma^\circ) < \varepsilon$; q.e.d.

Lemma 3.2. If a is an action with the Clausius property at a state σ°, then *every* state σ has a neighborhood $\mathcal{O}$ for which the set $a\{\sigma^\circ \to \mathcal{O}\}$ is bounded above.

Proof. If a has the Clausius property at σ°, then there exists a neighborhood $\mathcal{O}^\circ$ of σ° such that

$$\sup a\{\sigma^\circ \to \mathcal{O}^\circ\} < \tfrac{1}{2}, \tag{3.10}$$

where $\sup a\{\sigma^\circ \to \mathcal{O}^\circ]$ is the least upper bound for the supply of a on going from σ° into $\mathcal{O}^\circ$. Let σ be an arbitrary state, and let $\bar{P}$ be a fixed process for which $\bar{P}\sigma$ is in $\mathcal{O}^\circ$. (It follows from axiom **I** of Definition 2.1 that such a process $\bar{P}$ exists.) By Remark 2.1 there is a neighborhood $\mathcal{O}$ of σ for which

$$\bar{P}\mathcal{O} \subset \mathcal{O}^\circ \tag{3.11}$$

and

$$a_{\bar{P}}(\mathcal{O}) \subset \mathcal{N}_{\frac{1}{2}}\big(a_{\bar{P}}(\sigma)\big). \tag{3.12}$$

To prove the lemma we shall show that $\mathscr{a}\{\sigma^\circ \to \mathscr{O}\}$ has a finite upper bound. If $P\sigma^\circ$ is in $\mathscr{O}$, then σ° is in $\mathscr{D}(\bar{P}P)$, and, by (3.11), $\bar{P}P\sigma^\circ$ is in $\mathscr{O}^\circ$; hence (3.10) yields

$$\tfrac{1}{2} > \mathscr{a}(\bar{P}P, \sigma^\circ). \tag{3.13}$$

Since $P\sigma^\circ$ is in $\mathscr{O}$, we have, by (3.12),

$$\mathscr{a}(\bar{P}, P\sigma^\circ) > \mathscr{a}(\bar{P}, \sigma) - \tfrac{1}{2},$$

and (2.3) yields

$$\mathscr{a}(\bar{P}P, \sigma^\circ) - \mathscr{a}(P, \sigma^\circ) = \mathscr{a}(\bar{P}, P\sigma^\circ) > \mathscr{a}(\bar{P}, \sigma) - \tfrac{1}{2},$$

or, by (3.13),

$$1 - \mathscr{a}(\bar{P}, \sigma) > \mathscr{a}(P, \sigma^\circ). \tag{3.14}$$

Because $\bar{P}$ and σ do not depend on P, it follows from (3.14) that

$$\sup \mathscr{a}\{\sigma^\circ \to \mathscr{O}\} \leqq 1 - \mathscr{a}(\bar{P}, \sigma) < \infty; \tag{3.15}$$

q.e.d.

Lemma 3.3. Let σ° and σ be states, let $\mathscr{a}$ be an action with the Clausius property at σ°, and let $\mathfrak{S}(\sigma^\circ, \sigma)$ be the following class of open subsets of Σ:

$$\mathfrak{S}(\sigma^\circ, \sigma) = \{\mathscr{O} \mid \mathscr{O} \subset \Sigma,\ \mathscr{O} \text{ open},\ \mathscr{O} \text{ contains } \sigma,\ \sup \mathscr{a}\{\sigma^\circ \to \mathscr{O}\} < \infty\}. \tag{3.16}$$

If

$$m(\sigma^\circ, \sigma) \overset{\text{def}}{=} \inf_{\mathscr{O} \in \mathfrak{S}(\sigma^\circ, \sigma)} \sup \mathscr{a}\{\sigma^\circ \to \mathscr{O}\} \tag{3.17}$$

is finite, then $\mathscr{a}$ has the Clausius property at σ.[#]

Proof. Since $\mathscr{a}$ has, by hypothesis, the Clausius property at σ°, there exists a neighborhood $\mathscr{O}^\circ$ of σ° for which

$$\sup \mathscr{a}\{\sigma^\circ \to \mathscr{O}^\circ\} < \tfrac{1}{3}. \tag{3.18}$$

Assuming that $m(\sigma^\circ, \sigma)$ is finite, we shall show that the set $\mathscr{a}\{\sigma \to \mathscr{O}^\circ\}$ is bounded above; then application of Lemma 3.1 will yield this lemma. To this end, let P be a process with $P\sigma$ in $\mathscr{O}^\circ$. By Remark 2.1, there is a neighborhood $\hat{\mathscr{O}}$ of σ with $P\hat{\mathscr{O}} \subset \mathscr{O}^\circ$ and

$$\mathscr{a}(P, \hat{\mathscr{O}}) \subset \mathscr{N}_{\frac{1}{3}}(\mathscr{a}_P(\sigma)). \tag{3.19}$$

Because $\mathscr{a}$ has the Clausius property at σ°, Lemma 3.2 tells us that we can choose $\hat{\mathscr{O}}$ so that $\mathscr{a}\{\sigma^\circ \to \hat{\mathscr{O}}\}$ has a finite supremum, *i.e.* so that $\hat{\mathscr{O}}$ is an element of $\mathfrak{S}(\sigma^\circ, \sigma)$, and if we so choose $\hat{\mathscr{O}}$ there will exist, by the definition of *supremum*, a process $\bar{P}$ such that $\bar{P}\sigma^\circ$ is in $\hat{\mathscr{O}}$ and

$$\sup \mathscr{a}\{\sigma^\circ \to \hat{\mathscr{O}}\} < \mathscr{a}(\bar{P}, \sigma^\circ) + \tfrac{1}{3}. \tag{3.20}$$

When $m(\sigma^\circ, \sigma)$ is finite, we have, by (3.17) and (3.20),

$$m(\sigma^\circ, \sigma) + \mathscr{a}(P, \sigma) \leqq \sup \mathscr{a}\{\sigma^\circ \to \hat{\mathscr{O}}\} + \mathscr{a}(P, \sigma) < \mathscr{a}(\bar{P}, \sigma^\circ) + \mathscr{a}(P, \sigma) + \tfrac{1}{3}. \tag{3.21}$$

[#] The suprema shown in (3.16) and (3.17) are taken over all processes P such that $P\sigma^\circ$ is in $\mathscr{O}$, in accord with the definition (3.3) of the set $\mathscr{a}\{\sigma^\circ \to \mathscr{O}\}$.

Since, by (3.19),

$$\mathscr{a}(P, \sigma')+\tfrac{1}{3}>\mathscr{a}(P, \sigma),$$

for each σ' in $\hat{\mathcal{O}}$, and since $\bar{P}\sigma^\circ$ is in $\hat{\mathcal{O}}$, we have

$$\mathscr{a}(P, \bar{P}\sigma^\circ)+\tfrac{1}{3}>\mathscr{a}(P, \sigma),$$

which, in view of (3.21), (2.3), and the fact that σ° is in $\mathscr{D}(P\bar{P})$, yields

$$m(\sigma^\circ, \sigma)+\mathscr{a}(P, \sigma)<\mathscr{a}(\bar{P}, \sigma^\circ)+\mathscr{a}(P, \bar{P}\sigma^\circ)+\tfrac{2}{3}=\mathscr{a}(P\bar{P}, \sigma^\circ)+\tfrac{2}{3}. \tag{3.22}$$

As $\bar{P}\sigma^\circ\in\hat{\mathcal{O}}$ and $P\hat{\mathcal{O}}\subset\mathcal{O}^\circ$, the state $P\bar{P}\sigma^\circ$ is in $\mathcal{O}^\circ$, and thus, by (3.18),

$$\mathscr{a}(P\bar{P}, \sigma^\circ)<\tfrac{1}{3}. \tag{3.23}$$

The inequalities (3.22) and (3.23) imply that whenever $P\sigma$ is in $\mathcal{O}^\circ$, we have

$$\mathscr{a}(P, \sigma)<1-m(\sigma^\circ, \sigma).$$

Therefore,

$$\sup \mathscr{a}\{\sigma\rightarrow\mathcal{O}^\circ\}\leqq 1-m(\sigma^\circ, \sigma)<\infty,$$

and, by Lemma 3.1, $\mathscr{a}$ has the Clausius property at σ; q.e.d.

Suppose that $\mathscr{a}$ has the Clausius property at σ°. By Lemma 3.3, $\mathscr{a}$ has the property at every state in the set Σ° of states σ at which $m(\sigma^\circ, \sigma)$ [defined in (3.16) and (3.17)] is finite. We conjecture that there are systems and actions for which Σ° does not equal Σ,* and there may be cases in which Σ° does not contain every state at which $\mathscr{a}$ has the Clausius property. To prove one of our principal results, Theorem 3.1 below, we shall show, however, that Σ° does contain $\Pi\sigma^\circ$ and hence, by axiom I of Definition 2.1, Σ° is dense in Σ. Theorem 3.1 asserts that $\mathscr{a}$ has the Clausius property at either no state or an everywhere dense set of states. The observation that $\Pi\sigma^\circ$ is a subset of Σ°, employed to prove Theorem 3.1, will be strengthened later in this section when we examine in more detail the structure and closure properties of Σ°. [See Theorems 3.4 and 3.5 and Remark 3.2.]

Theorem 3.1. If the set of states at which an action has the Clausius property is not empty, then it must be dense in Σ.

Proof. Let $\mathscr{a}$ be an action, and suppose that $\mathscr{C}$, the set of states at which $\mathscr{a}$ has the Clausius property, is not empty. Let σ° be in $\mathscr{C}$, and put

$$\Sigma^\circ \overset{\text{def}}{=} \{\sigma \mid \sigma\in\Sigma, m(\sigma^\circ, \sigma)>-\infty\}, \tag{3.24}$$

where m is defined by (3.17). By Lemma 3.3, $\mathscr{a}$ has the Clausius property at each state in Σ°, *i.e.* $\Sigma^\circ\subset\mathscr{C}$. Let P be a fixed process and let $\mathcal{O}$ be in the class $\mathfrak{S}(\sigma^\circ, P\sigma^\circ)$, with $\mathfrak{S}$ as in (3.16). [Because $\mathscr{a}$ has the Clausius property at σ°, it follows from Lemma 3.2 that $\mathfrak{S}(\sigma^\circ, P\sigma^\circ)$ is not empty.] Since $P\sigma^\circ$ is in $\mathcal{O}$, the number $\mathscr{a}(P, \sigma^\circ)$ is in the set $\mathscr{a}\{\sigma^\circ\rightarrow\mathcal{O}\}$, defined by (3.3), and therefore

$$\sup \mathscr{a}\{\sigma^\circ\rightarrow\mathcal{O}\}\geqq\mathscr{a}(P, \sigma^\circ).$$

* In Theorem 13.2 we show that for "viscous elements" Σ° does equal Σ when $\mathscr{a}$ is identified with the action $\mathscr{s}$ of (9.7). We do not, however, believe that similar theorems hold for the general "elements with internal variables" and "elements with fading memory" described in Sections 11 and 12. For the elastic elements of Section 10, $\Pi\sigma^\circ$ equals Σ.

As this relation holds for every $\mathcal{O}$ in $\mathfrak{S}(\sigma^\circ, P\sigma^\circ)$, the number

$$m(\sigma^\circ, P\sigma^\circ) = \inf_{\mathcal{O}\in\mathfrak{S}(\sigma^\circ, P\sigma^\circ)} \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}\}$$

is finite. Thus $P\sigma^\circ$ is in Σ°, *i.e.* Σ° contains all states of the form $P\sigma^\circ$ with P a process. In other words,

$$\Pi\sigma^\circ \subset \Sigma^\circ \subset \mathscr{C},$$

and, since, by axiom **I** of Definition 2.1, $\Pi\sigma^\circ$ is dense in Σ, the set $\mathscr{C}$ is dense in Σ; q.e.d.

As a corollary to the proof just given we can assert

Remark 3.1. If an action has the Clausius property at a state σ°, then it has this property at all states accessible from σ°, *i.e.* at all states in the set $\Pi\sigma^\circ$.

The following lemma generalizes an important observation of DAY.[#]

Lemma 3.4. Let $\mathscr{a}$ be an action which has the Clausius property at σ_1 and σ_2, and let $\mathcal{O}$ be an open subset of Σ such that the sets $\mathscr{a}\{\sigma_1 \to \mathcal{O}\}$ and $\mathscr{a}\{\sigma_2 \to \mathcal{O}\}$ are both bounded above. For every $\varepsilon > 0$, there is a neighborhood $\mathcal{O}_2$ of σ_2 with $\mathscr{a}\{\sigma_1 \to \mathcal{O}_2\}$ bounded above and with

$$\sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\} + \sup \mathscr{a}\{\sigma_2 \to \mathcal{O}\} < \sup \mathscr{a}\{\sigma_1 \to \mathcal{O}\} + \varepsilon. \tag{3.25}$$

Proof. Let σ_1, σ_2, and $\mathcal{O}$ be as stated, and let $\varepsilon > 0$ be given. By the definition of supremum there is a process $\bar{P}$, with $\bar{P}\sigma_2 \in \mathcal{O}$, such that

$$\sup \mathscr{a}\{\sigma_2 \to \mathcal{O}\} < \mathscr{a}(\bar{P}, \sigma_2) + \frac{\varepsilon}{2}. \tag{3.26}$$

By Remark 2.1, a neighborhood $\mathcal{O}_2$ of σ_2 can be chosen so that $\bar{P}\mathcal{O}_2 \subset \mathcal{O}$ and

$$\mathscr{a}(\bar{P}, \mathcal{O}_2) \subset \mathscr{N}_{\frac{\varepsilon}{3}}(\mathscr{a}(\bar{P}, \sigma_2)). \tag{3.27}$$

Furthermore, since $\mathscr{a}$ has the Clausius property at σ_1, it follows from Lemma 3.2 that the set $\mathcal{O}_2$ can be chosen so that $\sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\}$ is finite. If P is a process for which $P\sigma_1$ is in $\mathcal{O}_2$, then σ_1 is in $\mathscr{D}(\bar{P}P)$ and by (2.3), (3.26), and (3.27), we have

$$\begin{aligned} \mathscr{a}(P, \sigma_1) + \sup \mathscr{a}\{\sigma_2 \to \mathcal{O}\} &< \mathscr{a}(P, \sigma_1) + \mathscr{a}(\bar{P}, \sigma_2) + \frac{\varepsilon}{2} \\ &< \mathscr{a}(P, \sigma_1) + \left[\mathscr{a}(\bar{P}, P\sigma_1) + \frac{\varepsilon}{3}\right] + \frac{\varepsilon}{2} = \mathscr{a}(\bar{P}P, \sigma_1) + \frac{\varepsilon}{3} + \frac{\varepsilon}{2}. \end{aligned} \tag{3.28}$$

Since here $P\sigma_1 \in \mathcal{O}_2$ and $\bar{P}\mathcal{O}_2 \subset \mathcal{O}$, we have $\bar{P}P\sigma_1 \in \mathcal{O}$, and hence

$$\mathscr{a}(\bar{P}P, \sigma_1) \leqq \sup \mathscr{a}\{\sigma_1 \to \mathcal{O}\}.$$

Thus, (3.28) implies that for every process P with $P\sigma_1$ in $\mathcal{O}_2$,

$$\mathscr{a}(P, \sigma_1) + \sup \mathscr{a}\{\sigma_2 \to \mathcal{O}\} < \sup \mathscr{a}\{\sigma_1 \to \mathcal{O}\} + \tfrac{5}{6}\varepsilon,$$

[#] [1969, 1], p. 91, Lemma 2, item (3).

and upon taking the supremum of the left side of this inequality over all such processes P we obtain

$$\sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\} + \sup \mathscr{a}\{\sigma_2 \to \mathcal{O}\} \leqq \sup \mathscr{a}\{\sigma_1 \to \mathcal{O}\} + \tfrac{5}{6}\varepsilon,$$

which yields (3.25); q.e.d.

Definition 3.2. Let $\mathscr{a}$ be an action for (Σ, Π). A real-valued function A is said to be an **upper potential** for $\mathscr{a}$ if

(1) the domain of A is a dense subset $\mathscr{A}$ of Σ, and

(2) whenever σ_1 and σ_2 are in $\mathscr{A}$, there is, for each $\varepsilon > 0$, a neighborhood $\mathcal{O}$ of σ_2 such that

$$P \in \Pi, \quad P\sigma_1 \in \mathcal{O} \quad \Rightarrow \quad A(\sigma_2) - A(\sigma_1) > \mathscr{a}(P, \sigma_1) - \varepsilon. \tag{3.29}$$

Theorem 3.2. If an action has an upper potential A, then it has the Clausius property at every state in the domain of A.

Proof. Let $\mathscr{a}$ be an action, and let A, with domain $\mathscr{A}$, be an upper potential for $\mathscr{a}$. Then, for each σ° in $\mathscr{A}$ and for each $\varepsilon > 0$, there is a neighborhood $\mathcal{O}$ of σ° such that (3.29) holds with $\sigma_1 = \sigma_2 = \sigma^\circ$. But when $\sigma_1 = \sigma_2 = \sigma^\circ$, (3.29) reduces to (3.1), and hence $\mathscr{a}$ has the Clausius property at σ°; q.e.d.

We are now ready to prove the main theorem of this section, which states that every action with the Clausius property has an upper semicontinuous upper potential.

Theorem 3.3. If there is a state at which an action has the Clausius property, then the action has an upper potential which is upper semicontinuous on its domain.

Proof. Suppose that $\mathscr{a}$ has the Clausius property at σ°, and let $m(\sigma^\circ, \cdot)$ be defined by (3.17) with $\mathfrak{S}$ as in (3.16). Let Σ° be the set defined in (3.24), and for each σ in Σ° put

$$A^\circ(\sigma) \overset{\text{def}}{=} m(\sigma^\circ, \sigma). \tag{3.30}$$

By the definition of Σ°, A° is a well-defined function with range in $\mathbb{R}$. In the proof of Theorem 3.1 we showed that Σ°, the domain of A°, is dense in Σ. We now show that A° has property (2) of Definition 3.2. Let σ_1 and σ_2 be given in Σ°, and let ε be any positive number. It follows from Lemma 3.3, that $\mathscr{a}$ has the Clausius property at σ_1 and σ_2. By the definition of *infimum*, there is a set $\mathcal{O}_2$ in the class of sets $\mathfrak{S}(\sigma^\circ, \sigma_2)$ such that

$$\sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_2\} < m(\sigma^\circ, \sigma_2) + \frac{\varepsilon}{2} = A^\circ(\sigma_2) + \frac{\varepsilon}{2}. \tag{3.31}$$

By Lemma 3.2, $\mathcal{O}_2$ can be chosen so that the set $\mathscr{a}\{\sigma_1 \to \mathcal{O}_2\}$ is bounded above. According to Lemma 3.4, σ_1 has a neighborhood $\mathcal{O}_1$ such that

$$\sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_2\} + \frac{\varepsilon}{2} > \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_1\} + \sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\}. \tag{3.32}$$

Moreover, by (3.17) and (3.30),

$$\sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_1\} \geqq m(\sigma^\circ, \sigma_1) = A^\circ(\sigma_1),$$

and, if we put this and (3.31) into (3.32), we obtain

$$A^\circ(\sigma_2) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > A^\circ(\sigma_1) + \sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\}. \tag{3.33}$$

Now, if $P\sigma_1$ is in $\mathcal{O}_2$, then

$$\mathscr{a}(P, \sigma_1) \leqq \sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\},$$

and (3.33) yields

$$A^\circ(\sigma_2) - A^\circ(\sigma_1) > \mathscr{a}(P, \sigma_1) - \varepsilon,$$

and hence A° is an upper potential. Now, let σ' in Σ° and $\varepsilon > 0$ be given. By (3.30) and (3.17), there is an $\mathcal{O}_\varepsilon$ in $\mathfrak{S}(\sigma^\circ, \sigma')$ such that

$$\varepsilon + A^\circ(\sigma') > \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_\varepsilon\}. \tag{3.34}$$

Of course, $\mathcal{O}_\varepsilon$ is open and contains σ'. We shall show that for each σ in $\mathcal{O}_\varepsilon \cap \Sigma^\circ$, there holds

$$\varepsilon + A^\circ(\sigma') > A^\circ(\sigma); \tag{3.35}$$

i.e. that A° is upper semicontinuous at σ' (in the topology induced on Σ° by Σ). To this end, let σ be in $\mathcal{O}_\varepsilon \cap \Sigma^\circ$, and note that since $\mathcal{O}_\varepsilon$ is in $\mathfrak{S}(\sigma^\circ, \sigma')$, $\sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_\varepsilon\}$ is finite, and, by (3.16), $\mathcal{O}_\varepsilon$ is in $\mathfrak{S}(\sigma^\circ, \sigma)$. By (3.17) and (3.30),

$$\sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_\varepsilon\} \geqq \inf_{\mathcal{O} \in \mathfrak{S}(\sigma^\circ, \sigma)} \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}\} = m(\sigma^\circ, \sigma) = A^\circ(\sigma)$$

which, in view of (3.34), yields (3.35); q.e.d.

If $\mathscr{a}$ has the Clausius property at σ°, the domain Σ° of the upper potential $A^\circ = m(\sigma^\circ, \cdot)$ contains $\Pi\sigma^\circ$ [see the proof of Theorem 3.1]; when $\Pi\sigma^\circ \neq \Sigma$, Σ° may be larger than $\Pi\sigma^\circ$. Indeed, Theorem 3.4 below tells us that if $\mathscr{a}$ has the Clausius property at σ°, the set Σ° contains all states which are limits of the form $\lim_{n\to\infty} P_n\sigma^\circ$, where P_n is a sequence of processes for which the values $\mathscr{a}(P_n, \sigma^\circ)$ of the action $\mathscr{a}$ have a limit; moreover, if the topology on Σ obeys a certain mild restriction met by all metric spaces, then Σ° is precisely the set of all states which are such limits, and $A^\circ(\sigma)$ equals a supremum of limits of numbers $\mathscr{a}(P_n, \sigma^\circ)$.

Theorem 3.4. Let $\mathscr{a}$ be an action which has the Clausius property at a state σ° in Σ. The set Σ° of (3.24) then contains every state σ for which there is a sequence $n \mapsto P_n$, with P_n in Π, such that

$$l = \lim_{n\to\infty} \mathscr{a}(P_n, \sigma^\circ) \tag{3.36}$$

exists and

$$\lim_{n\to\infty} P_n\sigma^\circ = \sigma. \tag{3.37}$$

Moreover, for each such state σ and sequence $n \mapsto P_n$,

$$l \leqq m(\sigma^\circ, \sigma). \tag{3.38}$$

If the topology of Σ satisfies the first axiom of countability, then for every element σ' of Σ° there is a sequence $n \mapsto \hat{P}_n$, with $\hat{P}_n$ in Π, such that

$$m(\sigma^\circ, \sigma') = \lim_{n\to\infty} \mathscr{a}(\hat{P}_n, \sigma^\circ), \qquad \lim_{n\to\infty} \hat{P}_n \sigma^\circ = \sigma'; \tag{3.39}$$

in this case

$$m(\sigma^\circ, \sigma') = \sup \lim_{n\to\infty} \mathscr{a}(P_n, \sigma^\circ), \tag{3.40}$$

where the supremum is taken over all sequences $n \mapsto P_n$ for which $\sigma' = \lim_{n\to\infty} P_n \sigma^\circ$ and for which the limit l in (3.36) exists [this supremum is a maximum, for it is achieved by the sequence $n \mapsto \hat{P}_n$ obeying (3.39)].

Proof. Let $n \mapsto P_n$ be such that l exists, and let σ be as in (3.37). Because $\mathscr{a}$ has the Clausius property at σ°, the set $\mathfrak{S}(\sigma^\circ, \sigma)$ of (3.16) is, by Lemma 3.2, not empty. Let $\mathcal{O}$ in $\mathfrak{S}(\sigma^\circ, \sigma)$ and $\varepsilon > 0$ be given. Then $\mathcal{O}$ is a neighborhood of σ, and, by (3.37) and (3.36),

$$P_n \sigma^\circ \in \mathcal{O} \quad \text{and} \quad l - \varepsilon \leqq \mathscr{a}(P_n, \sigma^\circ)$$

for at least one positive integer n. Hence

$$l - \varepsilon \leqq \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}\},$$

and as this holds for each $\mathcal{O}$ in $\mathfrak{S}(\sigma^\circ, \sigma)$, we have, by (3.17),

$$m(\sigma^\circ, \sigma) \geqq l - \varepsilon > -\infty;$$

i.e. σ is in Σ°. Furthermore, as $\varepsilon > 0$ is arbitrary, $m(\sigma^\circ, \sigma)$ obeys (3.38).

Suppose now that the topology of Σ obeys the first axiom of countability, let σ' in Σ° be given, and let $\{\mathcal{O}_k \mid k = 1, 2, \ldots\}$ be a neighborhood basis at σ'. By (3.17) there is, for each positive integer k, a set $\mathcal{O}^k$ in $\mathfrak{S}(\sigma^\circ, \sigma')$ such that

$$m(\sigma^\circ, \sigma') \leqq \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}^k\} < m(\sigma^\circ, \sigma') + \frac{1}{k}. \tag{3.41}$$

Clearly $\mathcal{O}^k \cap \mathcal{O}_k$ is in $\mathfrak{S}(\sigma^\circ, \sigma')$, and so also is

$$\hat{\mathcal{O}}^n \stackrel{\text{def}}{=} \bigcap_{k=1}^{n} (\mathcal{O}^k \cap \mathcal{O}_k).$$

Hence

$$\begin{aligned} m(\sigma^\circ, \sigma') &\leqq \sup \mathscr{a}\{\sigma^\circ \to \hat{\mathcal{O}}^n\} \\ &\leqq \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}^n\} < m(\sigma^\circ, \sigma') + \frac{1}{n}, \end{aligned} \tag{3.42}$$

because $\hat{\mathcal{O}}^n \subset \mathcal{O}^n$ and (3.41) holds for $k = n$. Since $\Pi \sigma^\circ$ is dense in Σ, for each positive integer n there is a $\hat{P}_n$ in Π with $\hat{P}_n \sigma^\circ \in \hat{\mathcal{O}}^n$; moreover $\hat{P}_n$ can be chosen so that

$$\sup \mathscr{a}\{\sigma^\circ \to \hat{\mathcal{O}}^n\} - \frac{1}{n} < \mathscr{a}(\hat{P}_n, \sigma^\circ). \tag{3.43}$$

By (3.17), (3.43), (3.3), and (3.42), we have

$$\begin{aligned} m(\sigma^\circ, \sigma') - \frac{1}{n} &\leqq \sup \mathscr{a}\{\sigma^\circ \to \hat{\mathcal{O}}^n\} - \frac{1}{n} \\ &< \mathscr{a}(\hat{P}_n, \sigma^\circ) \leqq \sup \mathscr{a}\{\sigma^\circ \to \hat{\mathcal{O}}^n\} \\ &< m(\sigma^\circ, \sigma') + \frac{1}{n}. \end{aligned}$$

Thus we can construct a sequence $n \mapsto \hat{P}_n$ with $\hat{P}_n$ in Π, $\hat{P}_n \sigma^\circ \in \hat{\mathcal{O}}^n$, and

$$m(\sigma^\circ, \sigma') - \frac{1}{n} < \mathscr{a}(\hat{P}_n, \sigma^\circ) < m(\sigma^\circ, \sigma') + \frac{1}{n}. \tag{3.44}$$

It is clear from (3.44) that the sequence $n \mapsto \hat{P}_n$ obeys $(3.39)_1$. Let $\mathcal{O}$ be a neighborhood of σ'; $\mathcal{O}$ contains, as a subset, an element $\mathcal{O}_N$ of the neighborhood basis $\{\mathcal{O}_k \mid k = 1, 2, \ldots\}$ at σ', and, by the definition of $\hat{\mathcal{O}}^N$, $\hat{\mathcal{O}}^N \subset \mathcal{O}_N$; furthermore, $\hat{\mathcal{O}}^n \subset \hat{\mathcal{O}}^N$ for all $n \geqq N$. Hence, for each $n \geqq N$, we have

$$\hat{P}_n \sigma^\circ \in \hat{\mathcal{O}}^n \subset \hat{\mathcal{O}}^N \subset \mathcal{O}_N \subset \mathcal{O},$$

and therefore the sequence $n \mapsto \hat{P}_n$ obeys $(3.39)_2$. When we proved (3.38), we showed that for *every* sequence $n \mapsto P_n$ obeying (3.36) and (3.37) with $\sigma = \sigma'$, there holds

$$\lim_{n \to \infty} \mathscr{a}(P_n, \sigma^\circ) \leqq m(\sigma^\circ, \sigma');$$

in view of this and $(3.39)_1$, the relation (3.40) is obvious; q.e.d.

Because of Theorem 3.4, the concept of "approach" defined in Definition 3.3, is important to our subject.

Definition 3.3. Let $\mathscr{a}$ be an action for (Σ, Π) and let σ° and σ be in Σ. If there exists a sequence $n \mapsto P_n \in \Pi$ such that $\sigma = \lim_{n \to \infty} P_n \sigma^\circ$ and the real sequence $n \mapsto \mathscr{a}(P_n, \sigma^\circ)$ converges, then we say that σ is **$\mathscr{a}$-approachable from** σ°. When a subset $\mathscr{S}$ of Σ contains all states which are $\mathscr{a}$-approachable from elements of $\mathscr{S}$ we say that $\mathscr{S}$ is **closed under $\mathscr{a}$-approach.**

Clearly, for any action $\mathscr{a}$ and any state σ, the set of states $\mathscr{a}$-approachable from σ contains $\Pi\sigma$, the set of states accessible from σ.

Theorem 3.4 tells us that *if $\mathscr{a}$ has the Clausius property at σ°, then the set Σ° of* (3.24) *contains all states $\mathscr{a}$-approachable from σ°*; if, in addition, the topology of Σ obeys the first axiom of countability, then Σ° is precisely the set of all states $\mathscr{a}$-approachable from σ° and the function $m(\sigma^\circ, \cdot): \Sigma^\circ \to \mathbb{R}$ is given by (3.40).

As was the case for the results preceding Theorem 3.4, the remaining discussion of this section does not require a topological axiom of countability.

Lemma 3.3 and the second sentence of Theorem 3.4 immediately yield the following theorem.

Theorem 3.5. The set of states at which an action $\mathscr{a}$ has the Clausius property is closed under $\mathscr{a}$-approach.

Less obvious, but also true, is

Theorem 3.6. If a is an action which has the Clausius property at a state σ°, then Σ°, the set defined in (3.24), is closed under a-approach.

Proof. Let σ° obey the hypothesis of the theorem, let σ' be in Σ°, and suppose that $\sigma\in\Sigma$ is a-approachable from σ', *i.e.* is such that there is a sequence $n\mapsto P_n\in\Pi$ for which $\lim_{n\to\infty} a(P_n,\sigma')$ exists and $\lim_{n\to\infty} P_n\sigma'=\sigma$. We must show that σ is in Σ°, and to this end we let $\mathcal{O}$ be an arbitrary set in $\mathfrak{S}(\sigma^\circ,\sigma)$. Since σ' is in Σ°, Lemma 3.3 tells us that a has the Clausius property at σ' as well as at σ°, and, by Lemma 3.2, there exists a neighborhood $\tilde{\mathcal{O}}$ of σ such that $a\{\sigma'\to\tilde{\mathcal{O}}\}$ is bounded above. Let $\hat{\mathcal{O}}=\mathcal{O}\cap\tilde{\mathcal{O}}$, and observe that the sets $a\{\sigma^\circ\to\hat{\mathcal{O}}\}$ and $a\{\sigma'\to\hat{\mathcal{O}}\}$ are both bounded above. By Lemma 3.4, σ' has a neighborhood $\mathcal{O}'$ such that

$$\sup a\{\sigma^\circ\to\mathcal{O}'\}+\sup a\{\sigma'\to\hat{\mathcal{O}}\}<\sup a\{\sigma^\circ\to\hat{\mathcal{O}}\}+\tfrac{1}{2}. \tag{3.45}$$

Because σ' is in Σ°, we have by (3.24) and (3.17),

$$-\infty<m(\sigma^\circ,\sigma')\leqq\sup a\{\sigma^\circ\to\mathcal{O}'\}. \tag{3.46}$$

Moreover, since $\sigma=\lim_{n\to\infty} P_n\sigma'$, and $l=\lim_{n\to\infty} a(P_n,\sigma')$ exists, there is a positive integer n such that $P_n\sigma'$ is in $\hat{\mathcal{O}}$ and

$$l<a(P_n,\sigma')+\tfrac{1}{2};$$

from this it follows that

$$l<\sup a\{\sigma'\to\hat{\mathcal{O}}\}+\tfrac{1}{2}. \tag{3.47}$$

On adding (3.46) to (3.47), we obtain, by (3.45), the inequality

$$m(\sigma^\circ,\sigma')+l-1<\sup a\{\sigma^\circ\to\hat{\mathcal{O}}\}.$$

Since $\hat{\mathcal{O}}\subset\mathcal{O}$, for each process P with $P\sigma^\circ\in\hat{\mathcal{O}}$, we have $P\sigma^\circ\in\mathcal{O}$, and therefore

$$\sup a\{\sigma^\circ\to\hat{\mathcal{O}}\}\leqq\sup a\{\sigma^\circ\to\mathcal{O}\}.$$

Thus,

$$\sup a\{\sigma^\circ\to\mathcal{O}\}>m(\sigma^\circ,\sigma')+l-1. \tag{3.48}$$

Because (3.48) holds for each set $\mathcal{O}$ in $\mathfrak{S}(\sigma^\circ,\sigma)$, and the right hand side of (3.48) is finite and independent of $\mathcal{O}$, it follows from (3.17) and (3.24) that σ is in Σ°; q.e.d.

It is a corollary to Theorem 3.6 that *whenever σ' is in Σ°, the set $\Pi\sigma'$ is a subset of Σ°.*

The following remark summarizes observations we have made about Σ°.

Remark 3.2. If an action a has the Clausius property at a state σ°, then the set Σ°, defined by (3.16), (3.17), and (3.24), enjoys the following properties:

(1) Σ° is dense in Σ, for Σ° contains each state a-approachable from σ° and hence each state accessible from σ°;

(2) Σ° is closed under a-approach;

(3) $\mathscr{a}$ has the Clausius property at each point of Σ° and hence at each state $\mathscr{a}$-approachable from σ°;

(4) there is defined on Σ° an upper semicontinuous function which is an upper potential for $\mathscr{a}$.

Theorem 3.7. Let σ°, σ', and σ be states, let $\mathscr{a}$ be an action which has the Clausius property at σ°, let Σ° be the set defined in (3.24), with m as in (3.17), and let $A^\circ\colon \Sigma^\circ \to \mathbb{R}$ be the upper potential defined for $\mathscr{a}$ by (3.30). If σ' is in Σ°, and there is a sequence $n \mapsto P_n \in \Pi$, such that

$$\lim_{n\to\infty} P_n \sigma' = \sigma \tag{3.49}$$

and, in addition,

$$\mathscr{a}(P_n, \sigma') = 0, \qquad \text{for } n = 1, 2, \ldots, \tag{3.50}$$

then not only is σ in Σ°, but

$$A^\circ(\sigma) \geqq A^\circ(\sigma'). \tag{3.51}$$

Proof. Let $n \mapsto P_n$ be as in (3.49) and (3.50). By item (2) of Remark 3.2, σ and $\sigma_n \overset{\text{def}}{=} P_n \sigma'$ (for every n) are in Σ°. By Definition 3.2 and the equation (3.50), there holds, for each n,

$$A^\circ(\sigma_n) \geqq A^\circ(\sigma') + \mathscr{a}(P_n, \sigma') = A^\circ(\sigma'). \tag{3.52}$$

By (3.49),

$$\lim_{n\to\infty} \sigma_n = \sigma$$

in the topology induced on Σ° by Σ, and, by Theorem 3.3, A° is upper semicontinuous at σ with respect to this topology. Hence, for each $\varepsilon > 0$, there is an N such that, for $n > N$,

$$A^\circ(\sigma) + \varepsilon > A^\circ(\sigma_n). \tag{3.53}$$

Employing (3.52) for an $n > N$, we obtain, in view of (3.53),

$$A^\circ(\sigma) + \varepsilon > A^\circ(\sigma'),$$

and as this holds for each $\varepsilon > 0$, the theorem is proven.

The argument employed to prove Theorem 3.7 yields also

Remark 3.3 Let $\mathscr{a}$ be an action which has the Clausius property at some state σ°, and let Σ° and A° be as in (3.24) and (3.30). *If* σ' in Σ° and $P^{(n)}$ in Π, $n = 1, 2, \ldots$, are such that the sequence $n \mapsto \sigma_n$, defined by

$$\sigma_1 = P^{(1)} \sigma', \quad \sigma_2 = P^{(2)} \sigma_1, \ \ldots, \ \sigma_n = P^{(n)} \sigma_{n-1}, \ \ldots,$$

has a limit σ in Σ and, moreover,

$$0 = \mathscr{a}(P^{(1)}, \sigma') = \mathscr{a}(P^{(2)}, \sigma_1) = \cdots = \mathscr{a}(P^{(n)}, \sigma_{n-1}) = \cdots,$$

then σ is in Σ° and

$$A(\sigma') \leqq A(\sigma_1) \leqq \cdots \leqq A(\sigma_n) \leqq A(\sigma_{n+1}) \leqq \cdots \leqq A(\sigma).$$

Proof. Since σ' is in Σ°, the state $\sigma_1 = P^{(1)}\sigma'$ is in Σ°, and, since $\mathscr{a}(P^{(1)}, \sigma') = 0$, Definition 3.2 yields

$$A(\sigma_1) - A(\sigma') \geqq \mathscr{a}(P^{(1)}, \sigma') = 0.$$

Similarly, for $n = 1, 2, \ldots,$

$$A(\sigma_{n+1}) - A(\sigma_n) \geqq \mathscr{a}(P^{(n+1)}, \sigma_n) = 0,$$

i.e. $A(\sigma_{n+1}) \geqq A(\sigma_n)$. Let us now define a sequence of processes P_n as follows:

$$P_1 = P^{(1)}, \quad P_2 = P^{(2)} P_1, \ \ldots, \ P_n = P^{(n)} P_{n-1}.$$

Since $P_n \sigma' = \sigma_n$, and, by (2.3),

$$\begin{aligned} \mathscr{a}(P_n, \sigma') &= \mathscr{a}(P^{(n)}, P_{n-1}\sigma') + \mathscr{a}(P_{n-1}, \sigma') \\ &= \mathscr{a}(P^{(n)}, \sigma_{n-1}) + \mathscr{a}(P_{n-1}, \sigma') \\ &= 0 + \mathscr{a}(P_{n-1}, \sigma'), \end{aligned}$$

i.e.

$$\mathscr{a}(P_n, \sigma') = \mathscr{a}(P_{n-1}, \sigma') = \cdots = \mathscr{a}(P_1, \sigma') = 0,$$

we conclude that the sequence $n \mapsto P_n$ obeys (3.49) and (3.50). Thus, σ is in Σ°, and $A(\sigma) \geqq A(\sigma')$. To show that $A(\sigma)$ dominates each of the numbers $A(\sigma_n)$, we note that since A is upper semicontinuous at σ, there is, for each $\varepsilon > 0$, an N such that for each integer $n > N$,

$$A(\sigma) + \varepsilon > A(\sigma_n),$$

and since $A(\sigma_n) \geqq A(\sigma_k)$ for $k \leqq n$, we have

$$A(\sigma) + \varepsilon > A(\sigma_n)$$

for *every positive integer n*; this can be true for every $\varepsilon > 0$ only if

$$A(\sigma) \geqq A(\sigma_n)$$

for $n = 1, 2, \ldots$; q.e.d.

The following observation, although not employed in subsequent sections, appears worth recording as an interesting generalization of a result of Day*.

Remark 3.4. If $\mathscr{a}$ is an action which has the Clausius property at σ_1 and σ_2, then for every $\varepsilon > 0$ there is a neighborhood $\mathscr{O}_1$ of σ_1 and a neighborhood $\mathscr{O}_2$ of σ_2, such that $\sup \mathscr{a}\{\sigma_1 \to \mathscr{O}_2\}$ is finite, $\sup \mathscr{a}\{\sigma_2 \to \mathscr{O}_1\}$ is finite, and

$$\sup \mathscr{a}\{\sigma_1 \to \mathscr{O}_2\} + \sup \mathscr{a}\{\sigma_2 \to \mathscr{O}_1\} < \varepsilon. \tag{3.54}$$

Proof. Let $\varepsilon > 0$ be given. Since $\mathscr{a}$ has the Clausius property at σ_1, there is, by definition, a neighborhood $\mathscr{O}_1$ of σ_1 such that

$$\sup \mathscr{a}\{\sigma_1 \to \mathscr{O}_1\} < \frac{\varepsilon}{3}. \tag{3.55}$$

By Lemma 3.2, since $\mathscr{a}$ has the Clausius property at σ_2, the set $\mathscr{O}_1$ can be chosen so that $\sup \mathscr{a}\{\sigma_2 \to \mathscr{O}_1\}$ is finite, and, by the definition of supremum, there then exists a process $\bar{P}$ for which

* [1969, 1], p. 91, Lemma 2, item (1).

$\bar{P}\sigma_2$ is in $\mathcal{O}_1$ and

$$\sup \mathscr{a}\{\sigma_2 \to \mathcal{O}_1\} < \mathscr{a}(\bar{P}, \sigma_2) + \frac{\varepsilon}{4}. \tag{3.56}$$

For this process $\bar{P}$, there is, by Remark 2.1, a neighborhood $\mathcal{O}_2$ of σ_2 such that $\bar{P}\mathcal{O}_2 \subset \mathcal{O}_1$ and

$$\mathscr{a}(\bar{P}, \mathcal{O}_2) \subset \mathcal{N}_{\frac{\varepsilon}{3}}(\mathscr{a}(\bar{P}, \sigma_2)). \tag{3.57}$$

Again by Lemma 3.2, since $\mathscr{a}$ has the Clausius property at σ_1, we can choose $\mathcal{O}_2$ such that $\sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\}$ is finite. If P is a process for which $P\sigma_1$ is in $\mathcal{O}_2$, then σ_1 is in $\mathscr{D}(\bar{P}P)$, $\bar{P}P\sigma_1$ is in $\mathcal{O}_1$, and, by (3.56), (3.57), (2.3), and (3.55),

$$\begin{aligned}
\mathscr{a}(P, \sigma_1) + \sup \mathscr{a}\{\sigma_2 \to \mathcal{O}_1\} &< \mathscr{a}(P, \sigma_1) + \mathscr{a}(\bar{P}, \sigma_2) + \frac{\varepsilon}{4} \\
&< \mathscr{a}(P, \sigma_1) + \left(\mathscr{a}(\bar{P}, P\sigma_1) + \frac{\varepsilon}{3}\right) + \frac{\varepsilon}{4} \\
&= \mathscr{a}(\bar{P}P, \sigma_1) + \frac{\varepsilon}{3} + \frac{\varepsilon}{4} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{4}.
\end{aligned}$$

Thus, for every process P with $P\sigma_1$ in $\mathcal{O}_2$,

$$\mathscr{a}(P, \sigma_1) < \frac{11}{12}\varepsilon - \sup \mathscr{a}\{\sigma_2 \to \mathcal{O}_1\},$$

and if we consider the supremum of the left side over all such processes P, we obtain

$$\sup \mathscr{a}\{\sigma_1 \to \mathcal{O}_2\} \leqq \frac{11}{12}\varepsilon - \sup \mathscr{a}\{\sigma_2 \to \mathcal{O}_1\},$$

which clearly implies (3.54); q.e.d.

The following theorem gives another generalization of DAY's Lemma 2 [1969, 1] and is closer to that lemma in form than our Lemma 3.4 and Remark 3.4.

Theorem 3.8. If $\mathscr{a}$ is an action with the Clausius property at σ°, then the function m defined in (3.17) has the following properties:

(i)

$$-\infty \leqq m(\sigma^\circ, \sigma^\circ) \leqq 0;$$

(ii) if σ_1 is such that $m(\sigma^\circ, \sigma_1)$ and $m(\sigma_1, \sigma^\circ)$ are finite, then

$$m(\sigma^\circ, \sigma_1) + m(\sigma_1, \sigma^\circ) \leqq 0;$$

(iii) if σ_1 and σ_2 are such that $m(\sigma^\circ, \sigma_1)$ and $m(\sigma_1, \sigma_2)$ are finite, then

$$m(\sigma^\circ, \sigma_1) + m(\sigma_1, \sigma_2) \leqq m(\sigma^\circ, \sigma_2).$$

Proof. To prove (i), we note that, when $\mathscr{a}$ has the Clausius property at σ°, there is, for each $\varepsilon > 0$, a neighborhood $\mathcal{O}_\varepsilon$ of σ° such that

$$\sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_\varepsilon\} \leqq \varepsilon. \tag{3.58}$$

It is obvious that $\mathcal{O}_\varepsilon$ is in the class $\mathfrak{S}(\sigma^\circ, \sigma^\circ)$ given by (3.16) [with σ set equal to σ°], and hence (3.58) and (3.17) yield

$$m(\sigma^\circ, \sigma^\circ) \leqq \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}_\varepsilon\} \leqq \varepsilon,$$

for each $\varepsilon > 0$, which implies (i). To prove (iii), we let σ°, σ_1, and σ_2 be such that α has the Clausius property at σ° and both $m(\sigma^\circ, \sigma_1)$ and $m(\sigma_1, \sigma_2)$ are finite. By Lemma 3.2, the class $\mathfrak{S}(\sigma^\circ, \sigma_2)$ is then not empty. For each $\hat{\mathcal{O}}$ in $\mathfrak{S}(\sigma^\circ, \sigma_2)$, there is, by Lemmata 3.2 and 3.3, a neighborhood $\mathcal{O}$ of σ_2 for which $\mathcal{O} \subset \hat{\mathcal{O}}$ and both $\sup \alpha\{\sigma^\circ \to \mathcal{O}\}$ and $\sup \alpha\{\sigma_1 \to \mathcal{O}\}$ are finite. Since α has the Clausius property at σ° and σ_1, it follows from Lemma 3.4 that, for each $\varepsilon > 0$, σ_1 has a neighborhood $\mathcal{O}_1$ with $\alpha\{\sigma^\circ \to \mathcal{O}_1\}$ bounded above and with

$$\sup \alpha\{\sigma^\circ \to \mathcal{O}_1\} + \sup \alpha\{\sigma_1 \to \mathcal{O}\} < \sup \alpha\{\sigma^\circ \to \mathcal{O}\} + \varepsilon. \tag{3.59}$$

Of course, $\mathcal{O}_1$ is in the class $\mathfrak{S}(\sigma^\circ, \sigma_1)$ and, as $\mathcal{O}$ is in $\mathfrak{S}(\sigma_1, \sigma_2)$, the inequality (3.59) implies the inequality

$$m(\sigma^\circ, \sigma_1) + m(\sigma_1, \sigma_2) < \sup \alpha\{\sigma^\circ \to \mathcal{O}\} + \varepsilon.$$

Clearly, $\sup \alpha\{\sigma^\circ \to \mathcal{O}\}$ does not exceed $\sup \alpha\{\sigma^\circ \to \hat{\mathcal{O}}\}$, and hence the inequality

$$m(\sigma^\circ, \sigma_1) + m(\sigma_1, \sigma_2) < \sup \alpha\{\sigma^\circ \to \hat{\mathcal{O}}\} + \varepsilon \tag{3.60}$$

holds for each $\hat{\mathcal{O}}$ in $\mathfrak{S}(\sigma^\circ, \sigma_2)$ and each $\varepsilon > 0$. The validity of (iii) follows immediately from (3.60) and the definition (3.17). It is obvious that (iii) and (i) together imply (ii); q.e.d.

4. Actions with the Conservation Property

In Definition 3.1 we defined *conservation property*. The definition was equivalent to

Definition 4.1. An action α has the **conservation property** at a state σ° if and only if both α and $-\alpha$ have the Clausius property at σ°.

We have here the following analogue of Theorems 3.1 and 3.5.

Theorem 4.1. If the set of states at which an action α has the conservation property is not empty, then it is dense in Σ and is closed under α-approach.

Proof. Let α be an action and suppose α has the conservation property at some state σ°. Let σ be α-approachable from σ°, and let $\ell = -\alpha$; then, clearly, ℓ is an action and by Definition 3.3, σ is ℓ-approachable from σ°. Of course, Definition 4.1 tells us that α and ℓ have the Clausius property at σ°, and by Theorem 3.5 both these actions have the Clausius property at σ. Thus α has the conservation property at σ, and it follows that the set $\mathscr{C}$ of states at which α has the conservation property is closed under α-approach. Moreover, since $\mathscr{C}$ contains all states α-approachable from σ°, $\mathscr{C}$ contains the set $\Pi\sigma^\circ$ which is dense in Σ; q.e.d.

If there is a state σ° at which an action α has the conservation property, then α has the Clausius property at σ°, and hence, by Theorem 3.3, α has an upper potential. The following theorem shows that the presence of the conservation property at σ° (as distinguished from the Clausius property) implies that two upper potentials for α must agree, modulo addition of a constant, on those states in their common domain which are α-approachable from σ°.

Theorem 4.2. Let α be an action, and let A_1 and A_2, with domains $\mathscr{A}_1$ and $\mathscr{A}_2$, be upper potentials for α. If α has the conservation property at σ°, then there is a number c such that

$$A_2(\sigma)=A_1(\sigma)+c \tag{4.1}$$

whenever σ is in $\mathscr{A}_1 \cap \mathscr{A}_2$ and is a-approachable from σ°.

Proof. Let σ' and σ'' be two arbitrary states which are in $\mathscr{A}_1 \cap \mathscr{A}_2$ and are a-approachable (and hence "$(-a)$-approachable") from σ°. The theorem will be proved if we show that

$$A_1(\sigma'')-A_1(\sigma')=A_2(\sigma'')-A_2(\sigma'). \tag{4.2}$$

Now, by hypothesis, both a and $-a$ have the Clausius property at σ°, and hence it follows from items (1) and (4) of Remark 3.2 that $-a$ has an upper potential, A_-, whose domain contains the states σ' and σ''. Let $\varepsilon>0$ be given. By Definition 3.2, σ'' has neighborhoods $\mathcal{O}_-$, $\mathcal{O}_1$, and $\mathcal{O}_2$ such that

$$A_-(\sigma'')-A_-(\sigma')>-a(P,\sigma')-\varepsilon, \tag{4.3a}$$

$$A_1(\sigma'')-A_1(\sigma')>a(P,\sigma')-\varepsilon, \tag{4.3b}$$

$$A_2(\sigma'')-A_2(\sigma')>a(P,\sigma')-\varepsilon, \tag{4.3c}$$

whenever P is such that $P\sigma'\in\mathcal{O}\overset{\text{def}}{=}\mathcal{O}_-\cap\mathcal{O}_1\cap\mathcal{O}_2$. As $\mathcal{O}$ is non-empty and open, it follows from axiom **I** of the definition of a system that there is at least one P with $P\sigma'\in\mathcal{O}$, and thus there exists a process P such that all the inequalities (4.3) hold. Similarly, there exists a process $\bar{P}$ such that

$$A_-(\sigma')-A_-(\sigma'')>-a(\bar{P},\sigma'')-\varepsilon, \tag{4.4a}$$

$$A_1(\sigma')-A_1(\sigma'')>a(\bar{P},\sigma'')-\varepsilon, \tag{4.4b}$$

$$A_2(\sigma')-A_2(\sigma'')>a(\bar{P},\sigma'')-\varepsilon. \tag{4.4c}$$

Upon adding the four inequalities (4.3a), (4.3c), (4.4a), and (4.4b), we find that

$$[A_2(\sigma'')-A_2(\sigma')]-[A_1(\sigma'')-A_1(\sigma')]>-4\varepsilon. \tag{4.5}$$

By adding (4.3a), (4.3b), (4.4a), and (4.4c), and multiplying the sum by -1, we obtain

$$[A_2(\sigma'')-A_2(\sigma')]-[A_1(\sigma'')-A_1(\sigma')]<4\varepsilon. \tag{4.6}$$

The inequalities (4.5) and (4.6) can hold for every $\varepsilon>0$ only if (4.2) holds; q.e.d.

Definition 4.2. Let a be an action for a system (Σ, Π). A real-valued function A is called a **potential** for a if

(1) the domain of A is a dense subset $\mathscr{A}$ of Σ, and

(2) whenever σ_1 and σ_2 are in $\mathscr{A}$, there is, for each $\varepsilon>0$, a neighborhood $\mathcal{O}$ of σ_2 such that

$$P\in\Pi, \quad P\sigma_1\in\mathcal{O} \;\Rightarrow\; |A(\sigma_2)-A(\sigma_1)-a(P,\sigma_1)|<\varepsilon. \tag{4.7}$$

We have here the following obvious analogue of Theorem 3.2.

Theorem 4.3. If an action has a potential A, then it has the conservation property at every state in the domain of A.

Proof. If $\mathscr{a}$ is an action with potential A, and if σ° is a state in the domain of A, then for each $\varepsilon>0$, there is a neighborhood $\mathcal{O}$ of σ° such that (4.7) holds with $\sigma_1=\sigma_2=\sigma^\circ$; *i.e.*

$$P\in\Pi, \quad P\sigma^\circ\in\mathcal{O} \quad\Rightarrow\quad |\mathscr{a}(P,\sigma^\circ)|<\varepsilon,$$

which is (3.2); q.e.d.

Theorem 4.4. If there is a state σ° at which an action $\mathscr{a}$ has the conservation property, then it has a potential whose domain contains all states $\mathscr{a}$-approachable from σ° and is closed under $\mathscr{a}$-approach.

Proof. If $\mathscr{a}$ has the conservation property at σ°, then $\mathscr{a}$ and $-\mathscr{a}$ have the Clausius property at σ°. Since, by Definition 3.3, "$\mathscr{a}$-approach" is equivalent to "$(-\mathscr{a})$-approach", it here follows from items (1), (2), and (4) of Remark 3.2 that $\begin{Bmatrix}\mathscr{a}\\-\mathscr{a}\end{Bmatrix}$ has an upper potential $\begin{Bmatrix}A_+\\A_-\end{Bmatrix}$ whose domain $\begin{Bmatrix}\mathscr{A}_+\\\mathscr{A}_-\end{Bmatrix}$ contains all states $\mathscr{a}$-approachable from σ° and is closed under $\mathscr{a}$-approach; hence the set $\mathscr{A}^\circ\overset{\text{def}}{=}\mathscr{A}_+\cap\mathscr{A}_-$ also has these two properties. We write A for the restriction of A_+ to $\mathscr{A}^\circ$; to show that A is a potential for $\mathscr{a}$, we need only verify that A obeys item (2) of Definition 4.2, for it is obvious that $\mathscr{A}^\circ$ contains $\Pi\sigma^\circ$ and hence is dense in Σ. Let σ_1 and σ_2 be in $\mathscr{A}^\circ$, and let $\varepsilon>0$ be given. As A_+ and A_- are upper potentials for $\mathscr{a}$ and $-\mathscr{a}$, respectively, there are neighborhoods $\mathcal{O}_+$, $\mathcal{O}_-$ of σ_2 such that

$$A_+(\sigma_2)-A_+(\sigma_1)>\mathscr{a}(P,\sigma_1)-\varepsilon \tag{4.8}$$

and

$$A_-(\sigma_2)-A_-(\sigma_1)>-\mathscr{a}(P,\sigma_1)-\frac{\varepsilon}{3}, \tag{4.9}$$

for each process with $P\sigma_1\in\mathcal{O}\overset{\text{def}}{=}\mathcal{O}_+\cap\mathcal{O}_-$. Of course $\mathcal{O}$ is a neighborhood of σ_2, and, by axiom **I** for systems, there is at least one such process P. Similarly, there is at least one process $\bar{P}$ such that

$$A_+(\sigma_1)-A_+(\sigma_2)>\mathscr{a}(\bar{P},\sigma_2)-\frac{\varepsilon}{3},$$

$$A_-(\sigma_1)-A_-(\sigma_2)>-\mathscr{a}(\bar{P},\sigma_2)-\frac{\varepsilon}{3}.$$

On adding these last two inequalities to (4.9), we find that

$$A_+(\sigma_1)-A_+(\sigma_2)>-\mathscr{a}(P,\sigma_1)-\varepsilon. \tag{4.10}$$

In other words σ_2 has a neighborhood $\mathcal{O}$ such that for all P with $P\sigma_1\in\mathcal{O}$, (4.8) and (4.10) hold; but (4.8) and (4.10) together imply $|A(\sigma_2)-A(\sigma_1)-\mathscr{a}(P,\sigma_1)|<\varepsilon$; q.e.d.

Remark 4.1. When the topology of Σ obeys the first axiom of countability, if $\mathscr{a}$ has the conservation property at σ°, then σ° is $\mathscr{a}$-approachable from itself.

Proof. When the topology of Σ obeys the first axiom of countability, there exists a countable neighborhood basis $\{\mathcal{O}_n | n=1, 2, \ldots\}$ at σ°, and this basis can be chosen so that $\mathcal{O}_{n+1}$ is a subset of $\mathcal{O}_n$ for each n. By axiom **I** of the definition of a system there exists, for each n, a process P_n such that $P_n\sigma^\circ \in \mathcal{O}_n$. If the action $\mathscr{a}$ has the conservation property at σ°, then, by (3.2), for each positive integer k, there is an element $\mathcal{O}_{n_k}$ of the basis at σ° such that

$$P \in \Pi, \qquad P\sigma^\circ \in \mathcal{O}_{n_k} \quad \Rightarrow \quad |\mathscr{a}(P, \sigma^\circ)| < \frac{1}{k},$$

but $P_{n_k}\sigma^\circ$ is in $\mathcal{O}_{n_k}$, and hence $|\mathscr{a}(P_{n_k}, \sigma^\circ)| < \frac{1}{k}$, *i.e.* the sequence $k \mapsto \mathscr{a}(P_{n_k}, \sigma^\circ)$ converges to zero. Clearly the function $k \mapsto n_k$ occurring here can be chosen so that $n_{k+1} > n_k$, and by the properties of $\{\mathcal{O}_n | n=1, 2, \ldots\}$ this choice of $k \mapsto n_k$ yields

$$\lim_{k\to\infty} P_{n_k}\sigma^\circ = \sigma^\circ;$$

q.e.d.

It is clear from Remark 4.1 that when the first axiom of countability is assumed, Theorem 4.4 becomes

Remark 4.2. When the topology of Σ obeys the first axiom of countability, if $\mathscr{a}$ has the conservation property at σ°, then $\mathscr{a}$ has a potential whose domain contains σ° and is closed under $\mathscr{a}$-approach.

An argument similar[#] to that employed to prove Remark 4.1 can be used to establish

Remark 4.3. If the topology of Σ obeys the first axiom of countability, and if $\mathscr{A} \subset \Sigma$ is the domain of a potential for an action $\mathscr{a}$, then each state in $\mathscr{A}$ is $\mathscr{a}$-approachable from every state in $\mathscr{A}$.

The following lemma, which does not require a topological axiom of countability, shows how two potentials for a single action must be related at nearby points in their domains.

Lemma 4.1. Let A_1 and A_2, with domains $\mathscr{A}_1$ and $\mathscr{A}_2$, be potentials for a given action, and suppose that $\mathscr{A}_1 \cap \mathscr{A}_2$ is not empty. There then exists a number c such that, for each $\varepsilon > 0$, each state σ' in $\mathscr{A}_1$ has a neighborhood $\mathcal{O}' \subset \Sigma$ for which

$$\sigma'' \in \mathcal{O}' \cap \mathscr{A}_2 \quad \Rightarrow \quad |A_2(\sigma'') - A_1(\sigma') - c| < \varepsilon. \tag{4.11}$$

Proof. Let σ° be in $\mathscr{A}_1 \cap \mathscr{A}_2$, and denote by $\mathscr{a}$ the given action for which A_1 and A_2 are potentials. Put

$$c = A_2(\sigma^\circ) - A_1(\sigma^\circ). \tag{4.12}$$

[#] The main difference is that (4.7) should now be used where (3.2) occurred in the proof of Remark 4.1.

If σ' is in $\mathscr{A}_1$, then, by Definition 4.2, there is a neighborhood $\mathscr{O}'$ of σ' for which

$$P\sigma^\circ \in \mathscr{O}' \quad \Rightarrow \quad |A_1(\sigma') - A_1(\sigma^\circ) - \alpha(P, \sigma^\circ)| < \frac{\varepsilon}{2}. \tag{4.13}$$

Let σ'' be in $\mathscr{O}' \cap \mathscr{A}_2$. Again by Definition 4.2, there is a neighborhood $\mathscr{O}''$ of σ'' for which

$$P\sigma^\circ \in \mathscr{O}'' \quad \Rightarrow \quad |A_2(\sigma'') - A_2(\sigma^\circ) - \alpha(P, \sigma^\circ)| < \frac{\varepsilon}{2}. \tag{4.14}$$

Let P be such that $P\sigma^\circ$ is in $\mathscr{O}'' \cap \mathscr{O}'$ (by axiom **I** for systems, there is at least one such P in Π); then (4.12)–(4.14) yield

$$\begin{aligned} |A_2(\sigma'') - A_1(\sigma') - c| &= |A_2(\sigma'') - A_2(\sigma^\circ) - \alpha(P, \sigma^\circ) - [A_1(\sigma') - A_1(\sigma^\circ) - \alpha(P, \sigma^\circ)]| \\ &\leqq |A_2(\sigma'') - A_2(\sigma^\circ) - \alpha(P, \sigma^\circ)| + |A_1(\sigma') - A_1(\sigma^\circ) - \alpha(P, \sigma^\circ)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon; \end{aligned}$$

q.e.d.

Let A, with domain $\mathscr{A}$, be a potential for an action. If in Lemma 4.1 we put $A_2 = A_1 = A$, then we find that $c = 0$, and, further, if σ' is in $\mathscr{A}$, then, for each $\varepsilon > 0$, σ' has a neighborhood $\mathscr{O}' \subset \Sigma$ such that

$$|A(\sigma'') - A(\sigma')| < \varepsilon,$$

whenever σ'' is in $\mathscr{O}' \cap \mathscr{A}$. In other words, A is continuous with respect to the topology Σ induces on $\mathscr{A}$. Thus we have

Theorem 4.5. Every potential for an action is a continuous function.

Remark 4.4. If α is an action which has the conservation property at a state σ° and if A is an upper potential for α whose domain $\mathscr{A}$ contains the set $\Pi\sigma^\circ$, then $\bar{A}$, the restriction of A to $\Pi\sigma^\circ$, is a potential for α.

Proof. Since α has the conservation property at σ°, α has, by Theorem 4.4, a potential A_1 whose domain contains $\Pi\sigma^\circ$. Of course, A_1 is an upper potential for α, and, by Theorem 4.2, there is a number c such that

$$\bar{A}(\sigma) = A_1(\sigma) + c \tag{4.15}$$

for all σ in $\Pi\sigma^\circ$, the domain of $\bar{A}$. Of course $\Pi\sigma^\circ$ is dense in Σ. Since, for each $\varepsilon > 0$, A_1 obeys (4.7) for all σ_1 and σ_2 in $\Pi\sigma^\circ$, it follows from (4.15) that $\bar{A}$ does too, and, therefore, $\bar{A}$ is a potential for α.

The following theorem is a direct consequence of Lemma 4.1.

Theorem 4.6. If A_1 and A_2, with domains $\mathscr{A}_1$ and $\mathscr{A}_2$, are potentials for a given action, and if $\mathscr{A}_1 \cap \mathscr{A}_2$ is not empty, then there exists a number c such that

$$\sigma \in \mathscr{A}_1 \cap \mathscr{A}_2 \quad \Rightarrow \quad A_2(\sigma) = A_1(\sigma) + c.$$

Proof. The potentials A_1 and A_2 satisfy the hypothesis of Lemma 4.1. Hence there exists a number c such that if σ' is in $\mathscr{A}_1 \cap \mathscr{A}_2$, then, for each $\varepsilon > 0$, σ' has a neighborhood $\mathscr{O}' \subset \Sigma$ such that (4.11) holds. Let σ' be a fixed element of $\mathscr{A}_1 \cap \mathscr{A}_2$; then (4.11) yields, for every $\varepsilon > 0$,

$$|A_2(\sigma') - A_1(\sigma') - c| < \varepsilon; \tag{4.16}$$

for σ' certainly is in both $\mathscr{O}'$ and $\mathscr{A}_2$. But, since (4.16) holds for *every* $\varepsilon > 0$, we have $A_2(\sigma') - A_1(\sigma') - c = 0$ for the given σ' in $\mathscr{A}_1 \cap \mathscr{A}_2$; q.e.d.

Thus, *any two potentials for a given action can differ by no more than a constant on the intersection of their domains.*

5. The Laws of Thermodynamics

In each branch of thermodynamics one is concerned with a system (Σ, Π) having the structure required by the axioms **I** and **II** of Definition 2.1; on $\Sigma \diamond \Pi$ there are given two distinguished functions $\mathscr{e}$ and $\mathscr{s}$ which are actions in the sense of Definition 2.2 and which enter into two laws called the First and Second Laws of Thermodynamics:

1. **First Law.** There is a state σ° at which the action $\mathscr{e}$ has the conservation property.

2. **Second Law.** The action $\mathscr{s}$ has the Clausius property at σ°.

It follows immediately from these laws and Theorems 4.1 and 3.1 that the set $\Sigma_{\mathscr{e}}$ of states at which $\mathscr{e}$ has the conservation property and the set $\Sigma_{\mathscr{s}}$ of states at which $\mathscr{s}$ has the Clausius property are both everywhere dense in the set Σ of all states of the system. Moreover, by Theorem 4.1, $\Sigma_{\mathscr{e}}$ is closed under $\mathscr{e}$-approach and, by Theorem 3.5, $\Sigma_{\mathscr{s}}$ is closed under $\mathscr{s}$-approach.

By Theorem 4.4, there is a function E of state which is a potential for $\mathscr{e}$ in the sense of Definition 4.2. That is, the domain of E, $\mathscr{E}$, is dense in Σ, and whenever σ_1 and σ_2 are in $\mathscr{E}$, there is, for each $\varepsilon > 0$, a neighborhood $\mathscr{O}$ of σ_2 such that

$$\left|[E(\sigma_2) - E(\sigma_1)] - \mathscr{e}(P, \sigma_1)\right| < \varepsilon, \tag{5.1}$$

for all processes P which take σ_1 into $\mathscr{O}$. In particular, for processes P obeying $P\sigma_1 = \sigma_2$ with σ_1 and σ_2 in $\mathscr{E}$, there holds

$$E(\sigma_2) - E(\sigma_1) = \mathscr{e}(P, \sigma_1). \tag{5.2}$$

Theorem 4.4 tells us, further, that $\mathscr{e}$ has a potential E whose domain $\mathscr{E}$ is closed under $\mathscr{e}$-approach and contains all states $\mathscr{e}$-approachable from σ° (and hence all states accessible from σ°). By Theorem 4.6, any two potentials for $\mathscr{e}$ can differ by only a constant on the intersection of their common domains, and, by Theorem 4.5, every potential for $\mathscr{e}$ is a *continuous* function of state.

In thermodynamical theories one selects a potential function E for $\mathscr{e}$ and calls it the *energy function* for the system; the value of E at a state σ, $E(\sigma)$, is referred to as *the energy of the system in the state* σ. The energy function is chosen by normalizing E so that the system has zero energy in some preferred "reference state".

By Theorem 3.3 there is a function S of state, called an *entropy function*, which is an upper potential for $\mathfrak{s}$, in the sense of Definition 3.2. That is, the domain $\mathscr{S}$ of S is dense in Σ, and whenever σ_1 and σ_2 are in $\mathscr{S}$, there is, for each $\varepsilon>0$, a neighborhood $\mathcal{O}$ of σ_2 such that

$$S(\sigma_2)-S(\sigma_1)>\mathfrak{s}(P,\sigma_1)-\varepsilon \tag{5.3}$$

for all processes P taking σ_1 into $\mathcal{O}$. In particular, for processes P obeying $P\sigma_1=\sigma_2$, with σ_1 and σ_2 in $\mathscr{S}$, we have

$$S(\sigma_2)-S(\sigma_1)\geqq\mathfrak{s}(P,\sigma_1). \tag{5.4}$$

The entropy function S can be chosen so that

$$S(\sigma)=m(\sigma^\circ,\sigma), \tag{5.5}$$

where

$$m(\sigma^\circ,\sigma)=\inf_{\mathcal{O}\in\mathfrak{S}(\sigma^\circ,\sigma)}\sup\mathfrak{s}\{\sigma^\circ\to\mathcal{O}\} \tag{5.6}$$

with

$$\mathfrak{S}(\sigma^\circ,\sigma)=\{\mathcal{O}\mid\mathcal{O}\subset\Sigma,\ \mathcal{O}\text{ open},\ \sigma\in\mathcal{O},\ \sup\mathfrak{s}\{\sigma^\circ\to\mathcal{O}\}<\infty\}. \tag{5.7}$$

The domain Σ° of this entropy function $S=m(\sigma^\circ,\cdot)$ is the set of states σ for which the infimum in (5.6) exists; *i.e.*

$$\text{domain } S=\Sigma^\circ=\{\sigma\mid\sigma\in\Sigma,\ m(\sigma^\circ,\sigma)>-\infty\}, \tag{5.8}$$

and S is upper semicontinuous on Σ°. By Remark 3.2, the set Σ° is closed under $\mathfrak{s}$-approach and contains all states $\mathfrak{s}$-approachable from σ°. It follows from Remark 3.3 that S "increases monotonically to a limit under isolation" in the following sense: *If* σ and σ' are two states with σ' in Σ°, and *if* $n\mapsto\sigma_n$ is a sequence of states which has a limit σ in Σ and is such that, for some sequence $n\mapsto P_n$ of processes,

$$\sigma_1=P_1\sigma',\quad \sigma_2=P_2\sigma_1,\ \ldots,\ \sigma_n=P_n\sigma_{n-1},\ \ldots,$$

and

$$0=\mathfrak{s}(P_1,\sigma')=\mathfrak{s}(P_2,\sigma_1)=\cdots=\mathfrak{s}(P_n,\sigma_{n-1})=\cdots,$$

then not only is each state σ_n and the limiting state σ in Σ°, but further,

$$S(\sigma')\leqq S(\sigma_1)\leqq\cdots\leqq S(\sigma_n)\leqq S(\sigma_{n+1})\leqq\cdots\leqq S(\sigma). \tag{5.9}$$

6. Stagnant States

Definition 6.1. Let $\mathfrak{I}$ be either the set of positive integers $1,2,\ldots$ or the positive real axis $\mathbb{R}^{++}=(0,\infty)$. A subset P of Π of the form $\{P^{[\nu]}\mid\nu\in\mathfrak{I}\}$ is called a **stagnating family of processes** if it has the following three properties:

(1) the set $\mathscr{D}(\mathsf{P})\stackrel{\text{def}}{=}\bigcap_{\nu\in\mathfrak{I}}\mathscr{D}(P^{[\nu]})$ is not empty;

(2) if ν_1 and ν_2 are in $\mathfrak{I}$, then the pair $(P^{[\nu_2]},P^{[\nu_1]})$ is in $\mathscr{P}$, the process $P^{[\nu_2]}P^{[\nu_1]}$ is in P, and

$$P^{[\nu_2]}P^{[\nu_1]}=P^{[\nu_2+\nu_1]}; \tag{6.1}$$

(3) there is at least one state σ in $\mathscr{D}(\mathsf{P})$ for which the limit $\lim_{\nu\to\infty} P^{[\nu]}\sigma$ exists in the Hausdorff space Σ, and $\lim_{\nu\to\infty} P^{[\nu]}\sigma$ is in $\mathscr{D}(\mathsf{P})$; when σ is such a state, the state $\lim_{\nu\to\infty} P^{[\nu]}\sigma\in\mathscr{D}(\mathsf{P})$ is referred to as a **stagnant state for** P.

It is easily shown [see item (2) above and Definition 2.1] that, for a stagnating family $\{P^{[\nu]}\,|\,\nu\in\mathfrak{J}\}$,

$$P^{[\nu_2]^{-1}}\big(\mathscr{D}(P^{[\nu_1]})\cap\mathscr{R}(P^{[\nu_2]})\big)=P^{[\nu_1]^{-1}}\big(\mathscr{D}(P^{[\nu_2]})\cap\mathscr{R}(P^{[\nu_1]})\big)$$

for each pair of indices ν_1, ν_2 in $\mathfrak{J}$; here, of course, $P^{[\nu_1]}$ and $P^{[\nu_2]}$ denote the transformations induced by the indicated processes.

We write Σ_{P} for the set of all σ in $\mathscr{D}(\mathsf{P})$ for which

$$\mathsf{P}\sigma\stackrel{\text{def}}{=}\lim_{\nu\to\infty} P^{[\nu]}\sigma, \tag{6.2}$$

exists and is in $\mathscr{D}(\mathsf{P})$; the notation $\mathsf{P}\sigma$ emphasizes the fact that a stagnating family P induces a mapping of Σ_{P} into $\mathscr{D}(\mathsf{P})\subset\Sigma$; actually, as we shall show in Theorem 6.1, the mapping induced by P is an operator on Σ_{P} in the sense that it maps Σ_{P} into Σ_{P}. Of course, the function $\sigma\mapsto\mathsf{P}\sigma$ need not be continuous. We call Σ_{P} the *effective domain* of P. For the range of $\sigma\mapsto\mathsf{P}\sigma$ we write $\mathsf{P}\Sigma_{\mathsf{P}}$.

Theorem 6.1. If $\mathsf{P}=\{P^{[\nu]}\,|\,\nu\in\mathfrak{J}\}$ is a stagnating family of processes, then

$$\sigma\in\Sigma_{\mathsf{P}},\quad \nu\in\mathfrak{J}\quad\Rightarrow\quad P^{[\nu]}\sigma\in\Sigma_{\mathsf{P}},$$

and

$$\sigma\in\Sigma_{\mathsf{P}}\quad\Rightarrow\quad\mathsf{P}\sigma\in\Sigma_{\mathsf{P}};$$

that is, $P^{[\nu]}\,\Sigma_{\mathsf{P}}\subset\Sigma_{\mathsf{P}}$ for each $P^{[\nu]}$ in P, and $\mathsf{P}\Sigma_{\mathsf{P}}\subset\Sigma_{\mathsf{P}}$.

Proof. Let σ be in Σ_{P} and ν in $\mathfrak{J}$. To show that $P^{[\nu]}\sigma$ is in Σ_{P}, one must show that $\lim_{\nu'\to\infty} P^{[\nu']}P^{[\nu]}\sigma$ exists and lies in $\mathscr{D}(\mathsf{P})$. Item (2) of Definition 6.1 here yields $P^{[\nu']}P^{[\nu]}\sigma=P^{[\nu'+\nu]}\sigma=P^{[\nu]}P^{[\nu']}\sigma$, and, by the continuity of the transformation induced by $P^{[\nu]}$, we have

$$\lim_{\nu'\to\infty} P^{[\nu]}P^{[\nu']}\sigma=P^{[\nu]}\lim_{\nu'\to\infty} P^{[\nu']}\sigma,$$

where $\lim_{\nu'\to\infty} P^{[\nu']}\sigma$ exists and equals $\mathsf{P}\sigma$ which is in $\mathscr{D}(\mathsf{P})$, because σ is, by hypothesis, in Σ_{P}. Hence

$$\lim_{\nu'\to\infty} P^{[\nu']}P^{[\nu]}\sigma=\lim_{\nu'\to\infty} P^{[\nu]}P^{[\nu']}\sigma=P^{[\nu]}\,\mathsf{P}\sigma,$$

i.e. $\lim_{\nu'\to\infty} P^{[\nu']}P^{[\nu]}\sigma$ exists and equals $P^{[\nu]}\,\mathsf{P}\sigma$ with $\mathsf{P}\sigma$ in $\mathscr{D}(\mathsf{P})$. To show that $P^{[\nu]}\,\mathsf{P}\sigma$ is in $\mathscr{D}(\mathsf{P})$, we note that

$$\begin{aligned} P^{[\nu]}\,\mathsf{P}\sigma &= P^{[\nu]}\lim_{\nu'\to\infty} P^{[\nu']}\sigma \\ &= \lim_{\nu'\to\infty} P^{[\nu+\nu']}\sigma=\lim_{\nu'\to\infty} P^{[\nu']}\sigma=\mathsf{P}\sigma, \end{aligned}$$

with $\mathsf{P}\sigma$ in $\mathscr{D}(\mathsf{P})$. Thus $\lim_{\nu'\to\infty} P^{[\nu']}P^{[\nu]}\sigma\in\mathscr{D}(\mathsf{P})$, *i.e.* $P^{[\nu]}\sigma\in\Sigma_{\mathsf{P}}$.

To prove that $\mathsf{P}\sigma$ is in Σ_{P} when σ is in Σ_{P}, one must show that $\lim_{\nu\to\infty} P^{[\nu]}\mathsf{P}\sigma$ exists and lies in $\mathscr{D}(\mathsf{P})$. But for *each* ν in $\mathfrak{I}$, we have, as above, $P^{[\nu]}\mathsf{P}\sigma=\mathsf{P}\sigma$, *i.e.* $P^{[\nu]}\mathsf{P}\sigma$ is independent of ν, and hence $\lim_{\nu\to\infty} P^{[\nu]}\mathsf{P}\sigma$ exists and equals $\mathsf{P}\sigma$, which is in $\mathscr{D}(\mathsf{P})$, because σ is in Σ_{P}; q.e.d.

If P is a stagnating family of processes, then the states in $\mathsf{P}\Sigma_{\mathsf{P}}$ are the stagnant states for P. Thus σ is a stagnant state for $\mathsf{P}=\{P^{[\nu]}\,|\,\nu\in\mathfrak{I}\}$ if and only if σ has the form $\sigma=\lim_{\nu\to\infty} P^{[\nu]}\sigma'$ with $\sigma'\in\Sigma_{\mathsf{P}}$. In particular, each stagnant state is in $\mathscr{D}(\mathsf{P})$. To motivate the terms "stagnating" and "stagnant", we now show that, when regarded as a function on $\mathsf{P}\Sigma_{\mathsf{P}}$, each member of a stagnating family P is the identity operator,# and the function $\mathsf{P}\colon\Sigma_{\mathsf{P}}\to\Sigma_{\mathsf{P}}$, *i.e.* $\sigma\mapsto\mathsf{P}\sigma$, is idempotent.##

Theorem 6.2. If $\mathsf{P}=\{P^{[\nu]}\,|\,\nu\in\mathfrak{I}\}$ is a stagnating family of processes, and σ is in Σ, then the following statements are equivalent:

(a) σ is a stagnant state for P, *i.e.* $\sigma\in\mathsf{P}\Sigma_{\mathsf{P}}$;

(b) for every ν in $\mathfrak{I}$, $P^{[\nu]}\sigma=\sigma$;

(c) $\sigma\in\Sigma_{\mathsf{P}}$, and there is at least one number ν_0 in $\mathfrak{I}$ for which $P^{[\nu_0]}\sigma=\sigma$;

(d) $\sigma\in\Sigma_{\mathsf{P}}$, and $\mathsf{P}\sigma=\sigma$.

Proof.### To show (a) ⇒ (b), we observe that, according to (a), there is a σ' in Σ_{P} for which $\sigma=\lim_{\nu\to\infty} P^{[\nu]}\sigma'$, and therefore σ is in $\mathscr{D}(\mathsf{P})$; hence for an *arbitrary* element ν_0 of $\mathfrak{I}$, we have, by the continuity of the transformation induced by $P^{[\nu_0]}$ and by (6.1),

$$P^{[\nu_0]}\sigma=P^{[\nu_0]}\lim_{\nu\to\infty}P^{[\nu]}\sigma'=\lim_{\nu\to\infty}P^{[\nu_0]}P^{[\nu]}\sigma'$$
$$=\lim_{\nu\to\infty}P^{[\nu_0+\nu]}\sigma'=\sigma,$$

which, since ν_0 is arbitrary, is the same as (b). The implication (b) ⇒ (c) is trivial. To show (c) ⇒ (d) we note that if σ is in Σ_{P} and $P^{[\nu_0]}\sigma=\sigma$, then, by (6.1),

$$P^{[2\nu_0]}\sigma=P^{[\nu_0]}P^{[\nu_0]}\sigma=P^{[\nu_0]}\sigma=\sigma,$$

and, by induction, $P^{[n\nu_0]}\sigma=\sigma$ for all positive integers n; thus

$$\lim_{n\to\infty}P^{[n\nu_0]}\sigma=\sigma, \tag{6.3}$$

and because the limit (6.2) exists, (6.3) yields $\lim_{\nu\to\infty} P^{[\nu]}\sigma=\sigma$, which is the same as (d). It is obvious that (d) implies (a); q.e.d.

As corollaries to this theorem, we make the following two remarks.

Remark 6.1. If σ is a stagnant state for the stagnating family $\mathsf{P}=\{P^{[\nu]}\,|\,\nu\in\mathfrak{I}\}$, and if F is a mapping of $\mathscr{A}\subset\Sigma_{\mathsf{P}}$ into $\mathbb{R}$ with σ in $\mathscr{A}$, then the function f on $\mathfrak{I}$ defined by $f(\nu)=F(P^{[\nu]}\sigma)$ is a constant.

(a) ⇒ (b) in Theorem 6.2.

(a) ⇒ (d) in Theorem 6.2.

This proof is due to NOLL; it was employed by him in a less general context: [1972, 2], p. 23, Prop. 12.1.

Remark 6.2. A general system (Σ, Π) may have states σ such that $\sigma \notin \Pi\sigma$. However, for systems which have stagnating families of processes, we can assert that whenever σ is a stagnant state, $\sigma \in \Pi\sigma$. That is, since (a) implies (b) in Theorem 6.2, if σ is in $\mathsf{P}\Sigma_{\mathsf{P}}$ for some stagnating family P, then σ is accessible from itself via each process in the family P.

Theorem 6.3. Let σ be a stagnant state for a stagnating family $\mathsf{P}=\{P^{[\nu]} \mid \nu \in \mathfrak{I}\} \subset \Pi$. If a is an action for (Σ, Π), then $a_\sigma\colon \mathfrak{I} \to \mathbb{R}$, defined by

$$a_\sigma(\nu) = a(P^{[\nu]}, \sigma),$$

obeys the additivity relation

$$a_\sigma(\nu_1+\nu_2) = a_\sigma(\nu_1) + a_\sigma(\nu_2).$$

Proof. If ν_1 and ν_2 are in $\mathfrak{I}$, then, by application of (6.1), (2.3), and item (b) of Theorem 6.2, we obtain

$$\begin{aligned} a_\sigma(\nu_1+\nu_2) &\stackrel{\text{def}}{=} a(P^{[\nu_1+\nu_2]}, \sigma) = a(P^{[\nu_2]} P^{[\nu_1]}, \sigma) \\ &= a(P^{[\nu_1]}, \sigma) + a(P^{[\nu_2]}, P^{[\nu_1]}\sigma) \\ &= a(P^{[\nu_1]}, \sigma) + a(P^{[\nu_2]}, \sigma) \stackrel{\text{def}}{=} a_\sigma(\nu_1) + a_\sigma(\nu_2); \end{aligned}$$

q.e.d.

When a, σ, and P obey the hypothesis of Theorem 6.3, we call the number $a_\sigma(1) = a(P^{[1]}, \sigma)$ *the rate of supply of* a *to maintain the state* σ (by means of P).

If $\mathfrak{I} = \mathbb{R}^{++}$, and a_σ has some weak property of regularity, such as monotonicity, or even measurability, then additivity for a_σ implies linearity and hence differentiability with $\frac{d}{d\nu} a_\sigma(\nu) = a_\sigma(1) = \text{constant}$.

In Theorem 6.6 below we observe that if a, σ, and P are as in Theorem 6.3 but with $\mathfrak{I} = \mathbb{R}^{++}$, and if, in addition, a has the Clausius property at σ, then a_σ is differentiable, and, moreover, its constant derivative $a_\sigma(1)$ cannot be positive; *i.e.* one must "remove" a, rather than "supply" a to maintain the state σ.

Theorem 6.4. Let σ be a stagnant state for a stagnating family P, and let a be an action. If a has the Clausius property at σ, then

$$P \in \mathsf{P} \quad \Rightarrow \quad a(P, \sigma) \leqq 0. \tag{6.4}$$

If a has the conservation property at σ, then

$$P \in \mathsf{P} \quad \Rightarrow \quad a(P, \sigma) = 0. \tag{6.5}$$

Proof. If P is in P, then, since P is stagnating, and σ is in $\mathsf{P}\Sigma_{\mathsf{P}}$, Theorem 6.2 yields $P\sigma = \sigma$. Therefore, the paragraph following Definition 3.1 tells us that if a has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at σ, $a(P, \sigma)$ is $\left\{\begin{matrix}\text{not positive}\\ \text{zero}\end{matrix}\right\}$; q.e.d.

Theorem 6.5. Let a be an action, let $\mathsf{P}=\{P^{[\nu]} \mid \nu \in \mathfrak{I}\}$ be a stagnating family of processes, and let σ be a state in the effective domain, Σ_{P}, of P. Let ν in $\mathfrak{I}$ and $\varepsilon > 0$ be given. If a has the Clausius property at $\mathsf{P}\sigma$, then there is a number

$N = N(\nu, \varepsilon)$ such that

$$\mu \in \mathfrak{I}, \quad \mu > N \quad \Rightarrow \quad \mathscr{a}(P^{[\nu]}, P^{[\mu]}\sigma) < \varepsilon. \tag{6.6}$$

If $\mathscr{a}$ has the conservation property at $\mathsf{P}\sigma$, then there is an $N = N(\nu, \varepsilon)$ for which

$$\mu \in \mathfrak{I}, \quad \mu > N \quad \Rightarrow \quad |\mathscr{a}(P^{[\nu]}, P^{[\mu]}\sigma)| < \varepsilon. \tag{6.7}$$

Proof. If $\mathscr{a}$ has the Clausius property at $\mathsf{P}\sigma$, then Theorem 6.4 yields $\mathscr{a}_{P^{[\nu]}}(\mathsf{P}\sigma) \leqq 0$, *i.e.*

$$\mathscr{a}_{P^{[\nu]}}\big(\lim_{\mu \to \infty} P^{[\mu]}\sigma\big) \leqq 0. \tag{6.8}$$

By Definition 2.2, the function $\mathscr{a}_{P^{[\nu]}}: \mathscr{D}(P^{[\nu]}) \to \mathbb{R}$ is continuous, and hence (6.8) yields

$$\lim_{\mu \to \infty} \mathscr{a}_{P^{[\nu]}}(P^{[\mu]}\sigma) = \lim_{\mu \to \infty} \mathscr{a}(P^{[\nu]}, P^{[\mu]}\sigma) \leqq 0.$$

But this relation means that there exists an $N = N(\nu, \varepsilon)$ such that (6.6) holds. If $\mathscr{a}$ has the conservation property at $\mathsf{P}\sigma$, then application of this same argument to both $\mathscr{a}$ and $-\mathscr{a}$ yields (6.7); q.e.d.

If σ, $\mathscr{a}$, and P are as in Theorem 6.4, then they are as in Theorem 6.3, and $\mathscr{a}_\sigma$ is additive; moreover, when $\mathscr{a}$ has the Clausius property at σ, by (6.4), $\mathscr{a}_\sigma$ is non-increasing on the ordered set $\mathfrak{I}$. If $\mathfrak{I} = \mathbb{R}^{++}$, then a non-increasing additive function $\mathscr{a}_\sigma$ on $\mathfrak{I}$ is linear and hence differentiable with $\frac{d}{d\nu}\mathscr{a}_\sigma(\nu) = \mathscr{a}_\sigma(1)$. This proves

Theorem 6.6. Let σ be a stagnant state for a stagnating family $\mathsf{P} = \{P^{[\nu]} \mid \nu \in \mathfrak{I}\}$, and let $\mathscr{a}$ be an action. If $\mathscr{a}$ has the Clausius property at σ, then the function $\mathscr{a}_\sigma$, defined by

$$\mathscr{a}_\sigma(\nu) = \mathscr{a}(P^{[\nu]}, \sigma), \qquad \nu \in \mathfrak{I},$$

is such that

$$\frac{\mathscr{a}_\sigma(\nu_2) - \mathscr{a}_\sigma(\nu_1)}{\nu_2 - \nu_1} \equiv \mathscr{a}_\sigma(1) \leqq 0,$$

for all distinct ν_1, ν_2 in $\mathfrak{I}$; if, in particular, $\mathfrak{I} = \mathbb{R}^{++}$, then $\frac{d}{d\nu}\mathscr{a}_\sigma$ exists, and

$$\frac{d}{d\nu}\mathscr{a}_\sigma(\nu) \equiv \mathscr{a}_\sigma(1) \leqq 0.$$

If $\mathscr{a}$ has the conservation property at σ, then

$$\mathscr{a}_\sigma(1) = 0.$$

Thus *if $\mathscr{a}$ has the* $\left\{\begin{matrix}\textit{Clausius}\\ \textit{conservation}\end{matrix}\right\}$ *property at a stagnant state σ°, then the rate of supply of $\mathscr{a}$ to maintain the state σ° is* $\left\{\begin{matrix}\textit{not positive}\\ \textit{zero}\end{matrix}\right\}$.

Theorem 6.7. Let σ° be a stagnant state for a stagnating family $\mathsf{P} = \{P^{[\nu]} \mid \nu \in \mathfrak{I}\}$, and let $\mathscr{a}$ be an action. If $\mathscr{a}$ has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at σ°, then $\mathscr{a}$ has

$\left\{\begin{matrix}\text{an upper potential}\\ \text{a potential}\end{matrix}\right\}$, A, whose domain $\mathscr{A}$ contains σ°. Furthermore, for any such $\left\{\begin{matrix}\text{upper potential}\\ \text{potential}\end{matrix}\right\}$ A, with σ° in $\mathscr{A}$, $A(P^{[\nu]}\sigma^\circ)$, as a function on $\mathfrak{I}$, is a constant.

Proof. If α has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at σ°, then by $\left\{\begin{matrix}\text{Remark 3.2}\\ \text{Theorem 4.4}\end{matrix}\right\}$ α has $\left\{\begin{matrix}\text{an upper potential}\\ \text{a potential}\end{matrix}\right\}$, A, whose domain $\mathscr{A}$ contains the set $\Pi\sigma^\circ$. However, by Remark 6.2 we here have σ° in $\Pi\sigma^\circ$ and hence in $\mathscr{A}$. The constancy of $A(P^{[\nu]}\sigma^\circ)$ on $\mathfrak{I}$ follows trivially from Remark 6.1; q.e.d.

For an incompressible simple fluid with fading memory, one can consider a material element which has, at time zero, a particular strain-energy history, *i.e.* a particular state σ. The state σ may be such that if one subsequently subjects the element to a shearing flow with constant rate of shear and removes heat at an appropriate constant rate, then, in the limit as $t\to\infty$, the state σ^∞ of the element will correspond to an isothermal steady viscometric flow. [Of course, the existence of a limiting state here requires that we identify the thermodynamic states of a material element with the "reduced states" discussed by NOLL#.]## During the approach to a steady viscometric flow, one is applying to the element, in each time interval $[0, t)$, a process $P_t \overset{\text{def}}{=} P^{[t]}$ in a stagnating family $\mathsf{P}=\{P^{[t]}\,|\,t\in\mathbb{R}^{++}\}$ with $\sigma^\infty=\lim_{t\to\infty} P_t\sigma=\mathsf{P}\sigma$; the state (or "reduced state") σ^∞ is a stagnant state for P. The viscometric flow corresponding to the reduced state σ^∞ is in the class of flows which COLEMAN### called "substantially stagnant motions".

Theorems 6.4 and 6.7 here tell us that if σ^∞ is a stagnant state for a stagnating family $\mathsf{P}=\{P_t\,|\,t\in\mathbb{R}^{++}\}$ and *if*, moreover, σ^∞ is in the common domain of the energy function E and the entropy function S mentioned in Section 5, *then* for each $P_t\in\mathsf{P}$,

$$e(P_t, \sigma^\infty)=0, \qquad s(P_t, \sigma^\infty)\leqq 0, \tag{6.9}$$

$$\frac{d}{dt}E(P_t\sigma^\infty)=0, \qquad \frac{d}{dt}S(P_t\sigma^\infty)=0, \tag{6.10}$$

and, by Theorem 6.6, $\frac{d}{dt}s(P_t, \sigma^\infty)$ exists and is a constant as P_t varies in P.#### The relations (6.9) have been used to show that the viscosity function of a simple incompressible fluid cannot be negative.#####

In the example in which $\sigma^\infty=\lim_{t\to\infty} P_t\sigma$ corresponds to a viscometric flow, we do not expect $s(P_t, \sigma)$ to have a limit as $t\to\infty$, and this is related to the fact that in that example the stagnant state σ^∞ is not a "relaxed state". It should be emphasized that although Theorem 6.7 yields an upper potential S^∞ for s with σ^∞ in its domain, it may turn out that σ^∞ is not in the domain $\mathscr{S}$ of some originally chosen entropy function S (albeit $\mathscr{S}$ must be dense in Σ). In other words, if σ° is in $\mathscr{S}$, even if $\mathscr{S}$ has the properties (1) and (2) given to Σ° by Remark 3.2, and, in addition $\sigma^\infty=\lim_{t\to\infty} P_t\sigma^\circ$, one cannot conclude that σ^∞ is in $\mathscr{S}$. However, as we point out in Section 7, if $\{P_t\,|\,t\in\mathbb{R}^{+}\}$ "relaxes σ° for s", *i.e.* is a stagnating family such that $\lim_{t\to\infty} P_t\sigma^\circ=\sigma^\infty$ and $\lim_{t\to\infty} s(P_t, \sigma^\circ)$

[1972, 2], § 10.

In a future work we shall treat formally and in more detail than here the theory of reduced states and stagnant states of incompressible simple fluids with memory. The thermodynamics of material elements with memory is discussed in Section 12 of this essay, but with an emphasis that does not require the introduction of symmetry groups and reduced states.

[1962, 1], § 5.

Cf. § 7 of COLEMAN [1962, 1].

[1962, 1], Remarks 6 and 7.

exists, then if $\mathscr{s}$ has the Clausius property at σ°, the entropy function S of (5.5)–(5.8) has σ^∞ in its domain Σ°.

Indeed, in thermodynamics one is often interested in stagnating families P of processes P_t for which, as $t \to \infty$, $\mathscr{e}(P_t, \sigma)$ and $\mathscr{s}(P_t, \sigma)$ both have limits for some initial states σ. In, for example, the theory of the thermodynamics of simple materials with fading memory, COLEMAN[#] discussed material elements which have, at time zero, an arbitrary strain-energy history, but from that time on have their strain and internal energy held constant, *i.e.* are subject to adiabatic "relaxation".[##] If for a stagnating family P and an action $\mathscr{a}$, there exist states σ for which the "limiting supply", $\lim_{t\to\infty} \mathscr{a}(P_t, \sigma)$, is finite, then, as we intimated above, one says that P is a *relaxing family of processes for* $\mathscr{a}$, and one refers to the stagnant states $\mathsf{P}\sigma$ as *relaxed states for* $\mathscr{a}$; if $\mathscr{a}(P_t, \sigma)=0$ for each state σ and process P_t in P, then P is called a *null family of processes for* $\mathscr{a}$. In Section 7 we develop a general theory of relaxing families of processes.

7. Relaxed States

Definition 7.1. Let $\mathscr{a}$ be an action and $P=\{P^{[\nu]} \mid \nu \in \mathfrak{J}\}$ be a stagnating family of processes for a system (Σ, Π). If there is a state σ in Σ_{P} for which the limit,

$$\lim_{\nu\to\infty} \mathscr{a}(P^{[\nu]}, \sigma),$$

exists, then P is called a **relaxing family of processes for** $\mathscr{a}$, $\sigma^\circ = \mathsf{P}\sigma = \lim_{\nu\to\infty} P^{[\nu]}\sigma$ is called a **relaxed state for** $\mathscr{a}$, and one says that P **relaxes** σ **to** σ° **for** $\mathscr{a}$.

Thus, σ° is a relaxed state for $\mathscr{a}$ if and only if σ° is a stagnant state for some stagnating family P which has a state σ in its effective domain Σ_{P} such that

$$\lim_{\nu\to\infty} \mathscr{a}(P^{[\nu]}, \sigma) \quad \text{exists and } \sigma^\circ = \mathsf{P}\sigma. \tag{7.1}$$

Clearly, if P relaxes σ to σ° for $\mathscr{a}$, then σ° is $\mathscr{a}$-approachable from σ, and in view of Remark 3.2 and Theorem 4.1 we can assert

Remark 7.1. If an action $\mathscr{a}$ has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at a state σ, and if P relaxes σ to σ' for $\mathscr{a}$, then $\mathscr{a}$ has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at σ'.

Theorem 7.1.[###] If $\mathscr{a}$ is an action and σ° a relaxed state for $\mathscr{a}$, then

$$\mathscr{a}(P, \sigma^\circ)=0 \tag{7.2}$$

whenever P is a member of a stagnating family which relaxes a state to σ° for $\mathscr{a}$.

Proof. Let $\mathsf{P}=\{P^{[\nu]} \mid \nu \in \mathfrak{J}\}$ be a family which relaxes σ to σ° for $\mathscr{a}$, and let $P^{[\nu_0]}$ be an arbitrary member of P. Since $\sigma^\circ = \mathsf{P}\sigma$, we have

$$\begin{aligned}\mathscr{a}(P^{[\nu_0]}, \sigma^\circ) &= \mathscr{a}(P^{[\nu_0]}, \mathsf{P}\sigma) = \mathscr{a}\big(P^{[\nu_0]}, \lim_{\nu\to\infty} P^{[\nu]}\sigma\big)\\ &= \lim_{\nu\to\infty} \mathscr{a}(P^{[\nu_0]}, P^{[\nu]}\sigma),\end{aligned}$$

[#] [1964, 1].

[##] As in an "adiabatic stress-relaxation experiment".

[###] Note that in Theorem 7.1, in contrast to the second part of Theorem 6.4, we do not assume that $\mathscr{a}$ has the conservation property at σ°.

by the continuity of $a_{P^{[\nu_0]}}$. However, since σ is in $\mathscr{D}(P^{[\nu_0+\nu]})=\mathscr{D}(P^{[\nu_0]}P^{[\nu]})$, the relations (2.3) and (6.1) yield

$$a(P^{[\nu_0]}, P^{[\nu]}\sigma)=a(P^{[\nu_0]}P^{[\nu]}, \sigma)-a(P^{[\nu]}, \sigma)=a(P^{[\nu_0+\nu]}, \sigma)-a(P^{[\nu]}, \sigma),$$

and hence

$$a(P^{[\nu_0]}, \sigma^\circ)=\lim_{\nu\to\infty}\left[a(P^{[\nu_0+\nu]}, \sigma)-a(P^{[\nu]}, \sigma)\right]. \tag{7.3}$$

Because P relaxes σ to σ°, $a(P^{[\nu]}, \sigma)$ has a limit as $\nu\to\infty$, and this implies that the limit on the right in (7.3) must be zero. Thus $a(P^{[\nu_0]}, \sigma^\circ)=0$, and, since $P^{[\nu_0]}$ is an arbitrary member of P, the theorem is proven.

Lemma 7.1.[#] Let σ° be a state and let a be an action with the Clausius property at σ°. For each state σ, let $\mathfrak{S}(\sigma^\circ, \sigma)$ and $m(\sigma^\circ, \sigma)$ be as in Lemma 3.3:

$$\mathfrak{S}(\sigma^\circ, \sigma)=\{\mathcal{O}\mid\mathcal{O}\subset\Sigma,\ \mathcal{O}\text{ open},\ \sigma\in\mathcal{O},\ \sup a\{\sigma^\circ\to\mathcal{O}\}<\infty\}, \tag{7.4}$$

$$m(\sigma^\circ, \sigma)=\inf_{\mathcal{O}\in\mathfrak{S}(\sigma^\circ, \sigma)}\sup a\{\sigma^\circ\to\mathcal{O}\}. \tag{7.5}$$

The set,

$$\Sigma^\circ=\{\sigma\mid\sigma\in\Sigma,\ m(\sigma^\circ, \sigma)>-\infty\}, \tag{7.6}$$

then contains all states of the form $\mathsf{P}\sigma'$, with σ' in Σ° and P a relaxing family for a. Furthermore, if σ° happens to be a relaxed state for a, then σ° is accessible from itself and

$$m(\sigma^\circ, \sigma^\circ)=0. \tag{7.7}$$

Proof. Let σ' be in Σ°. Each state of the form $\mathsf{P}\sigma'$, with P a relaxing family for a, is a-approachable from σ' and hence, by item (2) of Remark 3.2, is in Σ°. Thus the only thing requiring verification here is the last sentence of the lemma. Let σ° be a relaxed state for a. Then σ° is a stagnant state, and, by Remark 6.2, σ° is in $\Pi\sigma^\circ$ and hence in Σ°; furthermore, by Theorem 7.1 and item (b) of Theorem 6.2, there exists a process P for which (7.2) holds and $P\sigma^\circ=\sigma^\circ$. Hence, for every neighborhood $\mathcal{O}$ of σ°, we have $P\sigma^\circ$ in $\mathcal{O}$, and thus

$$\sup a\{\sigma^\circ\to\mathcal{O}\}\geqq a(P, \sigma^\circ)=0,$$

which, by (7.5), yields

$$m(\sigma^\circ, \sigma^\circ)\geqq 0. \tag{7.8}$$

However, since a has the Clausius property at σ°, for every $\varepsilon>0$ there is a neighborhood $\mathcal{O}$ of σ° for which

$$\sup a\{\sigma^\circ\to\mathcal{O}\}<\varepsilon; \tag{7.9}$$

by (7.4), each such $\mathcal{O}$ is in the class $\mathfrak{S}(\sigma^\circ, \sigma^\circ)$, and hence (7.5), (7.8), and (7.9)

$$0\leqq m(\sigma^\circ, \sigma^\circ)\leqq\varepsilon.$$

As this is true for every $\varepsilon>0$, $m(\sigma^\circ, \sigma^\circ)=0$; q.e.d.

[#] It is interesting to note that, when σ° is a relaxed state, the conclusion (i) of Theorem 3.8 reduces to the equation $m(\sigma^\circ, \sigma^\circ)=0$ which more closely resembles the conclusion (2) of DAY's Lemma 2 [1969, 1].

Employing Lemma 7.1 we now prove

Theorem 7.2. Let σ° be a relaxed state for an action $\mathscr{a}$. Suppose that $\mathscr{a}$ has the Clausius property at σ°, and consider the function $A^\circ\colon \Sigma^\circ \to \mathbb{R}$, defined by

$$A^\circ(\sigma) = m(\sigma^\circ, \sigma), \tag{7.10}$$

with $m(\sigma^\circ, \sigma)$ and Σ° given by (7.4)–(7.6). The function A° is an upper semicontinuous upper potential for $\mathscr{a}$, and the domain Σ° of A° not only contains $\Pi\sigma^\circ$ (which in this case includes σ°) but, in addition, is closed under $\mathscr{a}$-approach and hence is "closed under relaxation" in the sense that Σ° contains all states σ of the form $\sigma = \mathsf{P}\sigma'$ where σ' is in Σ° and P relaxes σ' to σ for $\mathscr{a}$. Furthermore,

$$A^\circ(\sigma^\circ) = 0. \tag{7.11}$$

If we let $\mathfrak{A}$ be the set of all upper potentials for $\mathscr{a}$ whose domains contain the set Σ° and which vanish at σ°, then A° is the least element of $\mathfrak{A}$ in the sense that

$$A \in \mathfrak{A}, \quad \sigma \in \Sigma^\circ \quad \Rightarrow \quad A(\sigma) \geqq A^\circ(\sigma). \tag{7.12}$$

Proof. Since $\mathscr{a}$ has the Clausius property at σ°, the proof of Theorem 3.3 tells us that the function A° defined by (7.10) is an upper potential for $\mathscr{a}$ and is upper semicontinuous. It follows from items (1) and (2) of Remark 3.2 that Σ° contains $\Pi\sigma^\circ$ and is closed under $\mathscr{a}$-approach and hence under relaxation. Furthermore, since σ° is a relaxed state for $\mathscr{a}$, σ° is, by Remark 6.2, in $\Pi\sigma^\circ$ and hence is in Σ°. By (7.7), $m(\sigma^\circ, \sigma^\circ) = 0$; *i.e.* A° obeys (7.11). It remains to show that (7.12) holds. To this end, let A be in $\mathfrak{A}$ and σ in Σ°. Then, by item (2) of Definition 3.2, there is, for each $\varepsilon > 0$, a neighborhood $\mathcal{O}$ of σ such that

$$A(\sigma) - A(\sigma^\circ) > \mathscr{a}(P, \sigma^\circ) - \varepsilon, \tag{7.13}$$

whenever $P\sigma^\circ$ is in $\mathcal{O}$. Therefore, since $A(\sigma^\circ) = 0$,

$$A(\sigma) \geqq \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}\} - \varepsilon. \tag{7.14}$$

Since (7.13) implies that $\sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}\}$ is finite, it follows from (7.14), (7.4), (7.5), and (7.10) that

$$A(\sigma) \geqq \inf_{\mathcal{O} \in \mathfrak{S}(\sigma^\circ, \sigma)} \sup \mathscr{a}\{\sigma^\circ \to \mathcal{O}\} - \varepsilon = m(\sigma^\circ, \sigma) - \varepsilon = A^\circ(\sigma) - \varepsilon,$$

and, because this holds for all $\varepsilon > 0$, we have $A(\sigma) \geqq A^\circ(\sigma)$; q.e.d.

As an immediate consequence of Theorems 3.2 and 7.2 we can assert that *when σ° is a relaxed state for $\mathscr{a}$, $\mathscr{a}$ has the Clausius property at σ° if and only if $\mathscr{a}$ has an upper potential whose domain contains σ°.*

Theorem 7.3. Let σ° be a relaxed state for an action $\mathscr{a}$. Suppose that $\mathscr{a}$ has the Clausius property at σ°, and let m and Σ° be as in (7.4)–(7.6). Suppose further that $m(\sigma, \sigma^\circ)$ is finite for every state σ for which $m(\sigma^\circ, \sigma)$ is finite, and consider the function $\hat{A}^\circ\colon \Sigma^\circ \to \mathbb{R}$, defined by

$$\hat{A}^\circ(\sigma) = -m(\sigma, \sigma^\circ). \tag{7.15}$$

$\hat{A}^\circ$ is an upper potential for $\mathscr{a}$ obeying

$$\hat{A}^\circ(\sigma^\circ)=0. \tag{7.16}$$

If we again let $\mathfrak{A}$ be the set of all upper potentials for $\mathscr{a}$ whose domains contain the set Σ° and which vanish at σ°, then $\hat{A}^\circ$ is the greatest element of $\mathfrak{A}$ in the sense that

$$A\in\mathfrak{A}, \quad \sigma\in\Sigma^\circ \quad \Rightarrow \quad A(\sigma)\leqq\hat{A}^\circ(\sigma). \tag{7.17}$$

Proof. By item (1) of Remark 3.2, Σ°, the domain of $\hat{A}^\circ$, is dense in Σ. Therefore, $\hat{A}^\circ$ is an upper potential for $\mathscr{a}$ if it obeys condition (2) of Definition 3.2. To see if that condition is obeyed, let σ_1 and σ_2 in Σ° and $\varepsilon>0$ be given. By Lemma 3.3 and the definition of Σ°, $\mathscr{a}$ has the Clausius property at σ_1 and σ_2, and, by Lemma 3.2, there is a neighborhood $\mathcal{O}^\circ$ of σ° for which

$$\sup \mathscr{a}\{\sigma_1\to\mathcal{O}^\circ\}<\infty \quad \text{and} \quad \sup \mathscr{a}\{\sigma_2\to\mathcal{O}^\circ\}<\infty. \tag{7.18}$$

By the hypothesis of the theorem, because σ_1 is in Σ°, we have not only $m(\sigma^\circ,\sigma_1)$ finite but also $m(\sigma_1,\sigma^\circ)$ finite, and by the definition [see (7.4) and (7.5)] of $m(\sigma_1,\sigma^\circ)$, we may choose $\mathcal{O}^\circ$ such that

$$m(\sigma_1,\sigma^\circ)>\sup \mathscr{a}\{\sigma_1\to\mathcal{O}^\circ\}-\frac{\varepsilon}{2}. \tag{7.19}$$

By (7.18), the states σ_1 and σ_2 and the open set $\mathcal{O}^\circ$ satisfy the hypothesis of Lemma 3.4. Thus there is a neighborhood $\mathcal{O}_2$ of σ_2 for which

$$\sup \mathscr{a}\{\sigma_1\to\mathcal{O}^\circ\}>\sup \mathscr{a}\{\sigma_1\to\mathcal{O}_2\}+\sup \mathscr{a}\{\sigma_2\to\mathcal{O}^\circ\}-\frac{\varepsilon}{2}.$$

In view of (7.15), (7.19), and this last inequality, we have

$$\begin{aligned}\hat{A}^\circ(\sigma_2)-\hat{A}^\circ(\sigma_1)&=-m(\sigma_2,\sigma^\circ)+m(\sigma_1,\sigma^\circ)\\ &>-m(\sigma_2,\sigma^\circ)+\sup \mathscr{a}\{\sigma_1\to\mathcal{O}^\circ\}-\frac{\varepsilon}{2}\\ &>-m(\sigma_2,\sigma^\circ)+\sup \mathscr{a}\{\sigma_1\to\mathcal{O}_2\}+\sup \mathscr{a}\{\sigma_2\to\mathcal{O}^\circ\}-\frac{\varepsilon}{2}-\frac{\varepsilon}{2},\end{aligned}$$

and since, by (7.4) and (7.5), $\sup \mathscr{a}\{\sigma_2\to\mathcal{O}^\circ\}\geqq m(\sigma_2,\sigma^\circ)$, there follows

$$\begin{aligned}\hat{A}^\circ(\sigma_2)-\hat{A}^\circ(\sigma_1)&>-m(\sigma_2,\sigma^\circ)+\sup \mathscr{a}\{\sigma_1\to\mathcal{O}_2\}+m(\sigma_2,\sigma^\circ)-\varepsilon\\ &=\sup \mathscr{a}\{\sigma_1\to\mathcal{O}_2\}-\varepsilon.\end{aligned}$$

However, for every P in Π with $P\sigma_1$ in $\mathcal{O}_2$, we have $\sup \mathscr{a}\{\sigma_1\to\mathcal{O}_2\}\geqq\mathscr{a}(P,\sigma_1)$, and hence the inequality above tells us that σ_2 has a neighborhood $\mathcal{O}_2$ such that

$$P\in\Pi, \quad P\sigma_1\in\mathcal{O}_2 \quad \Rightarrow \quad \hat{A}^\circ(\sigma_2)-\hat{A}^\circ(\sigma_1)>\mathscr{a}(P,\sigma_1)-\varepsilon;$$

i.e. condition (2) of Definition 3.2 *is* obeyed by $\hat{A}^\circ$, and $\hat{A}^\circ$ is an upper potential for $\mathscr{a}$. It is clear from (7.15) and (7.7) that $\hat{A}^\circ(\sigma^\circ)=0$, *i.e.* that $\hat{A}^\circ$ is in $\mathfrak{A}$, and the only thing remaining to show is that $\hat{A}^\circ$ is the largest element of $\mathfrak{A}$.

Let A in $\mathfrak{A}$, σ in Σ°, and $\varepsilon>0$ be given. Because A is an upper potential for a and has σ and σ° in its domain, there is a neighborhood $\mathcal{O}$ of σ° such that

$$A(\sigma^\circ)-A(\sigma)>a(P,\sigma)-\varepsilon, \tag{7.20}$$

whenever $P\sigma$ is in $\mathcal{O}$; but, since $A(\sigma^\circ)=0$, (7.20) becomes

$$-A(\sigma)>a(P,\sigma)-\varepsilon, \tag{7.21}$$

and this implies that $\sup a\{\sigma\to\mathcal{O}\}$ is finite. Indeed, (7.21) tells us that

$$-A(\sigma)\geqq\sup a\{\sigma\to\mathcal{O}\}-\varepsilon. \tag{7.22}$$

By (7.4), (7.5), (7.22), and (7.15),

$$-A(\sigma)\geqq m(\sigma,\sigma^\circ)-\varepsilon=-\hat{A}^\circ(\sigma)-\varepsilon, \tag{7.23}$$

and since $\varepsilon>0$ is arbitrary, this yields $\hat{A}^\circ(\sigma)\geqq A(\sigma)$; q.e.d.

Theorem 7.3 is less general than Theorem 7.2, because in Theorem 7.3 we have made the additional hypothesis $m(\sigma,\sigma^\circ)$ be finite whenever $m(\sigma^\circ,\sigma)$ is. This added condition is fulfilled if σ° is a-approachable from every σ in Σ°. That is, we have

Remark 7.2. Let σ° and σ be states, and let a be an action which has the Clausius property at σ°. If $m(\sigma^\circ,\sigma)$ is finite, and σ° is a-approachable from σ, then $m(\sigma,\sigma^\circ)$ is finite.

Proof. If $m(\sigma^\circ,\sigma)>-\infty$, then, by Lemma 3.3, a has the Clausius property at σ, and, if σ° is a-approachable from σ, there follows, by Theorem 3.4 [or item (1) of Remark 3.2], $m(\sigma,\sigma^\circ)>-\infty$; q.e.d.

As an immediate consequence of Theorems 7.2 and 7.3, we can assert

Remark 7.3. Let σ°, a, m, and $\mathfrak{A}$ be as in Theorem 7.3. Then the upper potentials A° and $\hat{A}^\circ$, defined by

$$A^\circ(\sigma)=m(\sigma^\circ,\sigma),\qquad \hat{A}^\circ(\sigma)=-m(\sigma,\sigma^\circ),$$

bound $\mathfrak{A}$ in the sense that

$$A\in\mathfrak{A},\qquad \sigma\in\Sigma^\circ\quad\Rightarrow\quad A^\circ(\sigma)\leqq A(\sigma)\leqq\hat{A}^\circ(\sigma).$$

It follows from this that if $m(\sigma^\circ,\sigma)=-m(\sigma,\sigma^\circ)$, then the upper and lower bounds A° and $\hat{A}^\circ$ are equal, and we have the following interesting theorem.

Theorem 7.4. Let a be an action which has the Clausius property at one of its relaxed states σ°, let m and Σ° be given by (7.4)–(7.6), and suppose that $m(\sigma,\sigma^\circ)$ is finite for every σ in Σ°.# The equation

$$m(\sigma^\circ,\sigma)+m(\sigma,\sigma^\circ)=0,\qquad \text{for all } \sigma\in\Sigma^\circ, \tag{7.24}$$

In view of Theorem 3.8, this hypothesis implies that $m(\sigma^\circ,\sigma)+m(\sigma,\sigma^\circ)$ is not positive.

gives a necessary and sufficient condition that $\mathscr{a}$ have exactly one upper potential A° such that

(1) the domain of A° is Σ°;

(2) $A(\sigma^\circ)=0$;

this upper potential obeys the formula

$$A(\sigma)=m(\sigma^\circ,\sigma)=-m(\sigma,\sigma^\circ).$$

The following remark draws together results about potentials stated in Theorems 4.3–4.6 and an observation about relaxed states made in Lemma 7.1.

Remark 7.4. Let σ° be a relaxed state for an action $\mathscr{a}$, and suppose that $\mathscr{a}$ has the conservation property at σ°. Then $\mathscr{a}$ has a potential A° whose domain $\mathscr{A}$ contains σ° and is closed under $\mathscr{a}$-approach (and hence is "closed under relaxation" *). The action $\mathscr{a}$ has the conservation property at every state in the set $\mathscr{A}$, and the potential A° is continuous on $\mathscr{A}$. Furthermore, A° can be chosen so that $A^\circ(\sigma^\circ)=0$, and, if this is done, A° is unique in the sense that every other potential for $\mathscr{a}$ which has σ° in its domain $\mathscr{A}'$ and vanishes at σ° must equal A on $\mathscr{A}\cap\mathscr{A}'$.

In the proofs of Theorems 7.2 and 7.3, the hypothesis that σ° is relaxed was used only to verify that σ° is in Σ° and that $A^\circ(\sigma^\circ)=\hat{A}(\sigma^\circ)=0$. If we replace that hypothesis by the assumptions that Σ° contains σ° and that (7.24) holds, then the proofs of Theorems 7.2 and 7.3 yield

Remark 7.5. Let m and Σ° be given by (7.4)–(7.6), and assume that σ° is in Σ°. The equation

$$m(\sigma^\circ,\sigma)+m(\sigma,\sigma^\circ)=0, \qquad \text{for all } \sigma\in\Sigma^\circ \tag{7.25}$$

is a *sufficient* condition that $\mathscr{a}$ have exactly one upper potential A such that

(1) the domain of A is Σ°, and

(2) $A(\sigma^\circ)=0$.

Definition 7.2. Let $\mathscr{a}$ be an action, and let $\mathsf{P}=\{P^{[\nu]}\mid\nu\in\mathfrak{J}\}$ be a stagnating family of processes. If $\sigma\in\Sigma_{\mathsf{P}}$ is such that $\mathscr{a}(P^{[\nu]},\sigma)\equiv 0$ for each $P^{[\nu]}\in\mathsf{P}$, then P is called a **null family of processes for $\mathscr{a}$ at σ.**

Clearly, if P is a null family for $\mathscr{a}$ at some σ in Σ, then P is a relaxing family for $\mathscr{a}$. A converse to this assertion is given in

Theorem 7.5. Let $\mathscr{a}$ be an action and P a relaxing family of processes for $\mathscr{a}$. If σ° is a relaxed state for $\mathscr{a}$ with respect to P, then P is a null family of processes for $\mathscr{a}$ at σ°.

In other words, a relaxing family P for $\mathscr{a}$ is a null family for $\mathscr{a}$ at each relaxed state in $\mathsf{P}\Sigma_{\mathsf{P}}$.

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* As explained in the statement of Theorem 7.2.

Proof. If P relaxes a state σ to σ° for a, then, by Definition 7.1, σ is in Σ_{P}, and $\sigma^\circ = \mathsf{P}\sigma$; hence Theorem 6.1 tells us that σ° is in Σ_{P}, while Theorem 7.1 yields $a(P^{[\nu]}, \sigma^\circ) = 0$ for each $P^{[\nu]}$ in P; *i.e.* P is a null family of processes for a at σ°; q.e.d.

The following corollary to Theorem 3.7 generalizes to our present context an observation made by Coleman in the theory of materials with fading memory.*

Theorem 7.6. Suppose a has the Clausius property at σ°, let Σ° and A° be as in (7.6) and (7.10), and let σ' be in Σ°. If P is a null family of processes for a at σ', then not only is $\mathsf{P}\sigma'$ in Σ°, but, moreover,

$$A^\circ(\mathsf{P}\sigma') \geqq A^\circ(\sigma').$$

8. State Functions in Thermodynamics

Returning to the formulation of thermodynamics discussed in Section 5, we now make the following definitions. A state σ° is here called a *relaxed state* if and only if there is a stagnating family $\mathsf{P} = \{P^{[\nu]} \mid \nu \in \mathfrak{I}\}$ which has a state σ in its essential domain Σ_{P} such that $\sigma^\circ = \mathsf{P}\sigma = \lim_{\nu \to \infty} P^{[\nu]}\sigma$ and *both*

$$\lim_{\nu \to \infty} e(P^{[\nu]}, \sigma) \quad \text{and} \quad \lim_{\nu \to \infty} s(P^{[\nu]}, \sigma) \tag{8.1}$$

exist. The set $\Pi\sigma^\circ$ is again called the *set of states accessible* from σ°, but we say that a set of states is *closed under approach*, only if it is closed under both s-approach and e-approach.

In this section we make the following assumption: *There exists at least one relaxed state σ°, and at this state the First and Second Laws hold;* i.e. *e has the conservation property at σ°, and s has the Clausius property at σ°.*

It now follows from Theorem 7.2 and Remark 7.4 that there is a set $\mathscr{T}$ which contains all states accessible from σ° (including σ°), is closed under approach, and is such that e has the conservation property and s has the Clausius property at every state in $\mathscr{T}$. Indeed, it is clear from Remark 3.2 and arguments employed to prove Theorems 4.1 and 4.4 that if we define $m^{(e)}$, $m^{(-e)}$, and $m^{(s)}$ by the relations (7.4) and (7.5) with a replaced, successively, by e, $-e$, and s, then we may choose $\mathscr{T}$ to be the set

$$\mathscr{T} = \{\sigma \mid \sigma \in \Sigma, \, m^{(i)}(\sigma^\circ, \sigma) \text{ finite for } i = e, -e, s\}. \tag{8.2}$$

We here make this choice for $\mathscr{T}$. It follows from Theorem 7.2 and Remark 7.4 (and the proofs of the several theorems behind Remark 7.4) that there exist functions E and S mapping $\mathscr{T}$ into $\mathbb{R}$ which obey

$$E(\sigma^\circ) = 0, \qquad S(\sigma^\circ) = 0, \tag{8.3}$$

and which are such that E is an energy function and S is an entropy function in the sense explained in Section 5, *i.e.* E is a potential for e, and S is an upper potential for s.

* [1964, 1], Remark 31, p. 38.

It follows from Theorems 4.5 and 4.6 that there is exactly one energy function E on $\mathscr{T}$ obeying $(8.3)_1$, and this energy function is continuous.

Let $\mathfrak{H}$ be the set of entropy functions on $\mathscr{T}$ which vanish at σ°. It is an immediate consequence of the Definition 3.2 that $\mathfrak{H}$ is a convex set, *i.e.*

$$S_1, S_2 \in \mathfrak{H}, \quad \alpha \in (0,1) \quad \Rightarrow \quad \alpha S_1 + (1-\alpha) S_2 \in \mathfrak{H}. \tag{8.4}$$

Theorem 7.2 gives us an extreme point for $\mathfrak{H}$. Indeed, by that theorem and its proof, $\mathfrak{H}$ contains an element S° which is the *smallest entropy function* in the sense that

$$S \in \mathfrak{H}, \quad \sigma \in \mathscr{T} \quad \Rightarrow \quad S(\sigma) \geqq S^\circ(\sigma). \tag{8.5}$$

The entropy function S° is determined by the action $\mathscr{s}$ through the formula

$$S^\circ(\sigma) = m^{\mathscr{s}}(\sigma^\circ, \sigma), \qquad \sigma \in \mathscr{T}. \tag{8.6}$$

Remark 7.2 here tells us that if the relaxed state σ° is $\mathscr{s}$-approachable from a state σ in $\mathscr{T}$, then $m^{\mathscr{s}}(\sigma, \sigma^\circ)$ is finite. By Theorem 7.3 and its proof, if $m^{\mathscr{s}}(\sigma, \sigma^\circ)$ is finite for each σ in $\mathscr{T}$, then the function $\hat{S}^\circ$, determined by $\mathscr{s}$ through the equation

$$\hat{S}^\circ(\sigma) = -m^{\mathscr{s}}(\sigma, \sigma^\circ), \qquad \sigma \in \mathscr{T}, \tag{8.7}$$

is in $\mathfrak{H}$ and is the *largest entropy function* in the sense that

$$S \in \mathfrak{H}, \quad \sigma \in \mathscr{T} \quad \Rightarrow \quad S(\sigma) \leqq \hat{S}^\circ(\sigma). \tag{8.8}$$

If $m^{\mathscr{s}}(\sigma, \sigma^\circ)$ is finite for each σ in $\mathscr{T}$, then, by Theorem 7.4, $\mathfrak{H}$ contains exactly one function if and only if the action $\mathscr{s}$ is such that

$$m^{\mathscr{s}}(\sigma^\circ, \sigma) + m^{\mathscr{s}}(\sigma, \sigma^\circ) = 0. \tag{8.9}$$

By Remark 3.3, even if (8.9) does not hold, the smallest entropy function S° of (8.6) "increases monotonically to a limit under isolation" as explained in the sentence containing equation (5.9).

II. Thermodynamics of Material Elements

Preface to Part II

The central goal of our research has been to render precise and develop concepts which are common to all branches of thermodynamics. At this point in our essay it remains for us to show that the mathematics developed in Part I is appropriate for describing physically useful systems and for finding the restrictions placed upon them by the laws of thermodynamics. We now (starting in Section 9) attempt to do this by developing a theory of simple material elements.# In this theory, we deduce, rather than assume, the existence of entropy as a function of state, in accord with the approach to thermodynamics advocated in Section 5. Indeed, in Theorem 9.3, which is a corollary to Theorem 3.3, we observe that the Second Law implies that every simple material element has an entropy function S which obeys an integrated form of the Clausius-Duhem inequality and which is

See Definition 9.1.

an upper semicontinuous function of state. Sections 10, 11, 12, and 13 are devoted to detailed discussions of the following important special classes of simple material elements: elastic elements, elements with internal variables, elements with fading memory, and viscous elements. We prove that every member of these classes of material elements furnishes a *definite example* of a *system* in the sense of Definition 2.1.[#] For each of the special classes, additional structure yields further regularity for S beyond semicontinuity. In Section 12, we show, for example, that for a material with fading memory S has "instantaneous derivatives" which obey the "generalized stress relation".[##]

9. Simple Material Elements

In the general theory of systems (Σ, Π), Σ is a Hausdorff space, and each process in Π induces a continuous mapping ρ_P of a subset $\mathscr{D}(P)$ of Σ onto a subset $\mathscr{R}(P)$ of Σ. One writes $\mathscr{P}$ for the subset of $\Pi \times \Pi$ consisting of pairs (P'', P') for which $\mathscr{D}(P'') \cap \mathscr{R}(P')$ is not empty, and one assumes that there is assigned a function $\mathscr{P} \to \Pi$, written $(P'', P') \mapsto P''P'$, such that $\rho_{P''P'}\sigma = \rho_{P''}\rho_{P'}\sigma$. The nature of the elements of Π is left unspecified, and Π has no structure other than that arising from the assumptions made in Definition 2.1 about the functions $P \mapsto \rho_P$ and $(P'', P') \mapsto P''P'$. Simple material elements, which we are about to define, are special systems for which each element of Π is a vector-valued function on an interval of the form $[0, t)$, $0 < t < \infty$. We shall often employ the same symbol, P_t, for an element of Π (*i.e.* a vector-valued function on $[0, t)$) and the Σ-valued function, $\rho_{P_t}: \mathscr{D}(P_t) \to \mathscr{R}(P_t)$, determined by it. When P_t is considered a Σ-valued function we shall write $P_t \sigma$ for its value at a state σ in $\mathscr{D}(P_t)$, *i.e.* $P_t \sigma \stackrel{\text{def}}{=} \rho_{P_t}\sigma$ (as in Part I); when, however, P_t is considered a vector-valued function on $[0, t)$, we shall place parentheses about its argument and write $P_t(\tau)$ for its value at a number τ in $[0, t)$.

We say that a vector-valued function P_t on $[0, t)$, with $t \in \mathbb{R}^{++} = (0, \infty)$, is *piecewise continuous* if the limit $P_t(\tau+)$ exists and equals $P_t(\tau)$ for each τ in $[0, \tau)$, the limit $P_t(\tau-)$ exists for each τ in $(0, t]$, and $P_t(\tau-) = P_t(\tau+)$ at all but a finite number of points in $(0, t)$. We are particularly interested in cases in which the range of P_t is a subset of the vector space

$$U = \operatorname{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}, \tag{9.1}$$

with $\mathscr{V}$ a real three-dimensional inner product space, and $\operatorname{Lin}(\mathscr{V})$ the space of tensors on $\mathscr{V}$ (i.e. linear mappings of $\mathscr{V}$ into $\mathscr{V}$).[###] For an inner product and norm on U we use

$$\begin{gathered}(\boldsymbol{B}, b, \boldsymbol{b}) \cdot (\boldsymbol{C}, c, \boldsymbol{c}) = \boldsymbol{B} \cdot \boldsymbol{C} + b c + \boldsymbol{b} \cdot \boldsymbol{c}, \\ |\boldsymbol{B}, b, \boldsymbol{b}| = \sqrt{(\boldsymbol{B}, b, \boldsymbol{b}) \cdot (\boldsymbol{B}, b, \boldsymbol{b})},\end{gathered} \tag{9.2}$$

[#] The proofs required to show this are in general non-trivial, for they rest upon modern results in the qualitative theory of ordinary and functional-differential systems.

[##] *Cf.* Coleman [1964, 1, 2].

[###] Later we use the symbols $\operatorname{Sym}(\mathscr{V})$ and $\operatorname{Lin}(\mathscr{V})^{++}$ to denote, respectively, the set of symmetric tensors in $\operatorname{Lin}(\mathscr{V})$ and the set of tensors in $\operatorname{Lin}(\mathscr{V})$ with positive determinant.

where $\boldsymbol{b}\cdot\boldsymbol{c}$ is the inner product of $\boldsymbol{b}$ and $\boldsymbol{c}$ as elements of $\mathscr{V}$, and $\boldsymbol{B}\cdot\boldsymbol{C}=\operatorname{trace}\boldsymbol{B}\boldsymbol{C}^T$ with $\boldsymbol{C}^T$ the transpose of $\boldsymbol{C}$.

A function mapping a finite interval into U is here said to be *piecewise differentiable* if it is continuous on its domain and differentiable at all but a finite number of points.

Definition 9.1. A **simple material element** $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ is a system (Σ, Π) for which

(1) each element of Π is a piecewise continuous function $P_t=(\boldsymbol{L}, h, \gamma)$ mapping an interval $[0, t)$, $t\in\mathbb{R}^{++}$, into the vector space U of (9.1); if P_t is in Π, then, for each τ in $(0, t)$, P_τ, the restriction of P_t to $[0, \tau)$, is in Π and $\mathscr{D}(P_\tau)\supset\mathscr{D}(P_t)$; if $\mathscr{D}(P_{t_2})\cap\mathscr{R}(P_{t_1})$ is not empty, *i.e.* if (P_{t_2}, P_{t_1}) is in $\mathscr{P}$, then the function $P_{t_1+t_2}\colon [0, t_1+t_2)\to U$, defined by

$$P_{t_1+t_2}(\tau)=\begin{cases} P_{t_1}(\tau), & \text{for } \tau\in[0, t_1),\\ P_{t_2}(\tau-t_1), & \text{for } \tau\in[t_1, t_2+t_1),\end{cases}$$

is in Π, has $\mathscr{D}(P_{t_1+t_2})=\rho_{P_{t_1}}^{-1}(\mathscr{D}(P_{t_2})\cap\mathscr{R}(P_{t_1}))$, and for each σ in $\mathscr{D}(P_{t_2+t_1})$ obeys the formula

$$P_{t_1+t_2}\sigma=P_{t_2}P_{t_1}\sigma;$$

the function $(P_{t_2}, P_{t_1})\mapsto P_{t_1+t_2}$ is the mapping $(P'', P')\mapsto P''P'$, from $\mathscr{P}$ into Π, which, by axiom **II** of Definition 2.1, must be assigned for a system;

(2) there is a continuous mapping $\sigma\mapsto {}_0\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta})$ which takes Σ into $\operatorname{Lin}(\mathscr{V})^{++}\times\mathbb{R}\times\mathscr{V}\subset U$; for each pair (P_t, σ) in $\Pi\diamond\Sigma$, the function $\tau\mapsto\sigma_\tau$ defined by

$$\sigma_\tau=\begin{cases}\sigma, & \text{for } \tau=0,\\ P_\tau\sigma, & \text{with } \quad P_\tau=P_t|_{[0,\tau)}, \quad \text{for } \tau\in(0, t],\end{cases} \tag{9.3}$$

is a piecewise continuous mapping of $[0, t]$ into Σ and is such that

$$\tau\mapsto {}_0\sigma_\tau=(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)) \tag{9.4}$$

is piecewise differentiable with a classical derivative $(\dot{\boldsymbol{F}}(\tau), \dot{e}(\tau), \dot{\boldsymbol{\beta}}(\tau))$ at each point τ of continuity of $P_t\colon [0, t)\to U$; moreover, for each σ in Σ, there is at least one P_t in Π such that σ is in $\mathscr{D}(P_t)$ and the function $\tau\mapsto\sigma_\tau$ of (9.3) is continuous at every τ in $[0, t]$;

(3) there are given continuous functions of state,

$$\begin{aligned} &\theta\colon \Sigma\to\mathbb{R}^{++},\\ &\boldsymbol{T}\colon \Sigma\to\operatorname{Sym}(\mathscr{V}),\\ &\boldsymbol{q}\colon \Sigma\to\mathscr{V},\end{aligned} \tag{9.5}$$

such that, for each pair (P_t, σ) in $\Pi\diamond\Sigma$, the relations

$$\begin{aligned} &\dot{\boldsymbol{F}}(\tau)=\boldsymbol{L}(\tau)\boldsymbol{F}(\tau),\\ &\dot{e}(\tau)=\boldsymbol{T}(\sigma_\tau)\cdot\boldsymbol{L}(\tau)+h(\tau),\\ &\dot{\boldsymbol{\beta}}(\tau)=\gamma(\tau),\end{aligned} \tag{9.6}$$

hold at each point of continuity of P_t, and the function $\mathscr{s}$ on $\Pi \diamond \Sigma$, defined by

$$\mathscr{s}(P_t, \sigma) = \int_0^t \frac{h(\tau)}{\theta(\sigma_\tau)}\, d\tau + \int_0^t \boldsymbol{q}(\sigma_\tau) \cdot \boldsymbol{\beta}(\tau)\, d\tau, \tag{9.7}$$

is an action; in (9.6) and (9.7), σ_τ is given by (9.3) with $P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma})$: $[0, t) \to U$, and $(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ is given by (9.4).

In the triplets $(\boldsymbol{L}, h, \boldsymbol{\gamma})$ which describe processes P_t, $\boldsymbol{L}$ stands for the velocity gradient, h is the rate at which heat is absorbed per unit mass $\left(\textit{i.e.}\ h = r - \frac{\operatorname{div} \boldsymbol{q}}{\mu}\right.$ with μ the mass density, $\boldsymbol{q}$ the heat flux vector, and r the rate of supply of heat by radiation$\Big)$, and $\boldsymbol{\gamma}$ is the material time-derivative of $-\frac{1}{\mu} \operatorname{grad} \frac{1}{\theta}$, with θ the absolute temperature. In the mapping $\sigma \to {}_0\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta})$, $\boldsymbol{F}$ is the *present* (or "instantaneous") value of the deformation gradient (relative to a preassigned reference), e the present value of the specific internal energy, and $\boldsymbol{\beta}$ the present value of the vector $-\frac{1}{\mu} \operatorname{grad} \frac{1}{\theta}$ $\left(\textit{i.e.}\ \boldsymbol{\beta} = \frac{1}{\mu\theta^2} \boldsymbol{g} \text{ with } \boldsymbol{g} = \operatorname{grad} \theta\right)$. The values of the state functions θ and $\boldsymbol{q}$ listed in (9.5) are, of course, the temperature and the heat flux vector; the value of $\boldsymbol{T}$ is the Cauchy stress divided by the mass density μ.

Remark 9.1. For each simple material element, the function $\mathscr{e}$, defined on $\Pi \diamond \Sigma$ by

$$\mathscr{e}(P_t, \sigma) = \int_0^t \boldsymbol{T}(\sigma_\tau) \cdot \boldsymbol{L}(\tau)\, d\tau + \int_0^t h(\tau)\, d\tau \tag{9.8}$$

with σ_τ as in (9.3), is an action which has the conservation property at every state in Σ. The potential E for $\mathscr{e}$, called the *energy function for the simple material element*, is defined on all of Σ, is unique to within a constant c, and has the form

$${}_0\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}) \quad \Rightarrow \quad E(\sigma) = e + c; \tag{9.9}$$

i.e. to within a constant c, $E(\sigma)$ is simply e.

Proof. Let σ', with ${}_0\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}')$, be in Σ, let $P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma})$ be in Π with σ' in $\mathscr{D}(P_t)$, let ${}_0\sigma_\tau = {}_0(P_\tau \sigma') = (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ as in (9.4), and note that because we have P_t piecewise continuous on $[0, t)$, $\boldsymbol{T}$ continuous on Σ, and $\tau \mapsto \sigma_\tau$ piecewise continuous on $[0, t]$, the integrals in (9.8) exist, and $\mathscr{e}(P_t, \sigma')$ is well defined. Since $\tau \mapsto e(\tau)$ is piecewise differentiable (and hence continuous), $(9.6)_2$ and (9.8) yield

$$\mathscr{e}(P_t, \sigma') = \int_0^t \dot{e}(\tau)\, d\tau = e(t) - e(0) = e(t) - e', \tag{9.10}$$

and, since e' depends continuously on σ', we may conclude that $\mathscr{e}_{P_t} = \mathscr{e}(P_t, \cdot)$ is continuous. For each pair (P_{t_2}, P_{t_1}) in $\mathscr{P}$, item (1) of Definition 9.1 gives us an explicit formula for $P_{t_2+t_1} = P_{t_2} P_{t_1}$, the value of the "successive application function" from $\mathscr{P}$ into Π, and this formula is clearly such that for each σ' in $\mathscr{D}(P_{t_2} P_{t_1})$, (9.10) yields

$$\begin{aligned} \mathscr{e}(P_{t_2} P_{t_1}, \sigma') &= \mathscr{e}(P_{t_1+t_2}, \sigma') = e(t_1 + t_2) - e' \\ &= [e(t_1) - e'] + [e(t_1 + t_2) - e(t_1)] \\ &= \mathscr{e}(P_{t_1}, \sigma') + e(P_{t_2}, P_{t_1}\sigma'), \end{aligned}$$

which has the form (2.3). Thus $\mathscr{e}$ is an *action* as defined in Definition 2.2. It follows from (9.10) that for each c in $\mathbb{R}$, the function E defined in (9.9) is such that whenever (P, σ') is in $\Pi \diamond \Sigma$, there holds

$$E(P\sigma') - E(\sigma') = \mathscr{e}(P, \sigma'). \tag{9.11}$$

By item (2) of Definition 9.1, E is continuous on Σ, and therefore each state σ'' has, for each $\varepsilon > 0$, a neighborhood $\mathcal{O}$ such that

$$P \in \Pi, \quad P\sigma' \in \mathcal{O} \quad \Rightarrow \quad |E(\sigma'') - E(P\sigma')| < \varepsilon.$$

On employing (9.11) to eliminate $E(P\sigma')$ from this last expression, we obtain

$$P \in \Pi, \quad P\sigma' \in \mathcal{O} \quad \Rightarrow \quad |E(\sigma'') - E(\sigma') - \mathscr{e}(P, \sigma')| < \varepsilon.$$

In view of Definition 4.2, this implies that E is a potential for $\mathscr{e}$, and, by Theorem 4.3, $\mathscr{e}$ has the conservation property at every state in the domain Σ of E. Furthermore, it follows from Theorem 4.6 that any potential A for $\mathscr{e}$ defined on all of Σ can differ by at most a constant from a given function E of the form (9.9); but this, in turn, implies that A also has the form (9.9); q.e.d.

For simple material elements the First Law of Thermodynamics is taken to be the assertion that the action $\mathscr{e}$, defined in (9.8), has the conservation property at at least one state. Thus, by Remark 9.1, we can assert

Remark 9.2. Every simple material element obeys the First Law of Thermodynamics.

Here the Second Law of Thermodynamics is taken to be as follows:

Second Law. There is at least one state σ° in Σ at which the action $\mathscr{s}$, defined in (9.7), has the Clausius property.

Although the First Law is automatically fulfilled by material elements obeying Definition 9.1, the Second Law is not, unless the state functions θ, $\boldsymbol{T}$, and $\boldsymbol{q}$ obey certain restrictions. The problem of finding these restrictions will be studied in detail in subsequent sections for various special cases of simple material elements, but to indicate how such restrictions come about, we now state two general theorems.

Theorem 9.1. Let $\sigma_{\circ}$ be a state of a simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, put ${}_0\sigma_{\circ} = (\boldsymbol{F}_{\circ}, e_{\circ}, \boldsymbol{\beta}_{\circ})$, and suppose that the action $\mathscr{s}$ of (9.7) has the Clausius property at $\sigma_{\circ}$. If Π contains a process $P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma})$ such that, for each τ in $(0, t)$, $h(\tau) = 0$ and $P_{\tau}\sigma_{\circ} = \sigma_{\circ}$, with P_{τ} the restriction of P_t to $[0, \tau)$, then

$$\boldsymbol{q}(\sigma_{\circ}) \cdot \boldsymbol{\beta}_{\circ} \leqq 0. \tag{9.12}$$

Proof. If $\mathscr{s}$ has the Clausius property at $\sigma_{\circ}$, and $P_t\sigma_{\circ} = \sigma_{\circ}$, then, by the discussion following Definition 3.1, $\mathscr{s}(P_t, \sigma_{\circ}) \leqq 0$; if, further, $h(\tau) = 0$ and $\sigma_{\tau} = P_{\tau}\sigma_{\circ} = \sigma_{\circ}$ for τ in $(0, t)$, then we have, by (9.7),

$$0 \geqq \int_0^t \boldsymbol{q}(\sigma_{\tau}) \cdot \boldsymbol{\beta}(\tau)\, d\tau = t\,\boldsymbol{q}(\sigma_{\circ}) \cdot \boldsymbol{\beta}_{\circ};$$

q.e.d.

Definition 9.2. Let P_t be a process of a simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, let σ be in $\mathscr{D}(P_t)$, and let σ_τ, for $\tau \in [0, t]$, be as in (9.3). If

$$\theta(\sigma_\tau) = \theta(\sigma), \qquad \text{for } \tau \in (0, t] \tag{9.13}$$

(*i.e.* if $\dot{\theta} \equiv 0$ on $[0, t)$), then P_t is said to be *isothermal when applied to* σ. If

$${}_0\sigma_\tau = (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{0}), \qquad \text{for } \tau \in [0, t] \tag{9.14}$$

(*i.e.* if $\gamma(\tau) \equiv 0$, and ${}_0\sigma$ has the form $(\boldsymbol{F}, e, \boldsymbol{\beta})$ with $\boldsymbol{\beta} = \boldsymbol{0}$), then P_t is *homothermal when applied to* σ. If

$$P_t \sigma = \sigma, \tag{9.15}$$

then P_t is *cyclic when applied to* σ.

Theorem 9.2. Let σ be a state of a simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, and suppose that the action $\mathfrak{s}$ of (9.7) has the Clausius property at σ. For each process P_t that is both cyclic and homothermal when applied to σ, there holds

$$\int_0^t \frac{\boldsymbol{T}(\sigma_\tau) \cdot \boldsymbol{L}(\tau)}{\theta(\sigma_\tau)} \, d\tau \geqq 0, \tag{9.16}$$

with σ_τ as in (9.3). Consequently, if P_t is cyclic, homothermal, and isothermal when applied to σ, the total work corresponding to P_t and σ, *i.e.* the number

$$w(P_t, \sigma) \stackrel{\text{def}}{=} \int_0^t \boldsymbol{T}(\sigma_\tau) \cdot \boldsymbol{L}(\tau) \, d\tau, \tag{9.17}$$

is not negative.

Proof. The discussion following Definition 3.1 tells us that, if P_t is cyclic when applied to σ, and if $\mathfrak{s}$ has the Clausius property at σ, there holds

$$0 \geqq \mathfrak{s}(P_t, \sigma),$$

and, by (9.7) and $(9.6)_2$, this inequality can be written

$$\begin{aligned} 0 &\geqq \int_0^t \left[-\frac{\boldsymbol{T}(\sigma_\tau) \cdot \boldsymbol{L}(\tau)}{\theta(\sigma_\tau)} + \dot{e}(\tau) \right] d\tau + \int_0^t \boldsymbol{q}(\sigma_\tau) \cdot \boldsymbol{\beta}(\tau) \, d\tau \\ &= \int_0^t -\frac{\boldsymbol{T}(\sigma_\tau) \cdot \boldsymbol{L}(\tau)}{\theta(\sigma_\tau)} \, d\tau + e(t) - e(0) + \int_0^t \boldsymbol{q}(\sigma_\tau) \cdot \boldsymbol{\beta}(\tau) \, d\tau. \end{aligned} \tag{9.18}$$

But, by (9.4), (9.3), and (9.15), we here have

$$(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t)) = {}_0P_t \sigma = {}_0\sigma = (\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0)),$$

and, in particular,

$$e(t) = e(0). \tag{9.19}$$

If P_t is homothermal when applied to σ, then (9.14) asserts that

$$\boldsymbol{\beta}(\tau) = \boldsymbol{0}, \qquad \text{for } \tau \in [0, t]. \tag{9.20}$$

The relations (9.18)–(9.20), obviously imply (9.16). If P_t is isothermal when applied to σ, then it follows from (9.13) and the positivity of the function θ that (9.16) is equivalent to the assertion that the number $w(P_t, \sigma)$ of (9.17) is not negative; q.e.d.

In the following remark we use the concept of "$\mathfrak{s}$-approach" rendered precise by Definition 3.3.

Remark 9.3. Every state of a simple material element is $\mathfrak{s}$-approachable from itself.

Proof. If σ is a state of a simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, then by item (2) of Definition 9.1 there is a P_t in Π which has σ in $\mathscr{D}(P_t)$ and is such that the function $\tau \mapsto \sigma_\tau$, defined in (9.3), is continuous on $[0, t]$. For this process P_t we put, for each positive integer n,

$$P_n \stackrel{\text{def}}{=} P_{t/n} = P_t|_{\left[0, \frac{t}{n}\right)};$$

and we observe that the continuity of $\tau \mapsto \sigma_\tau$ at $\tau = 0$ implies that

$$\lim_{n\to\infty} P_n \sigma = \sigma.$$

Moreover, the formula (9.7) for $\mathfrak{s}$, the piecewise continuity of the function $P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma})$, the continuity of the functions θ and $\boldsymbol{q}$ of (9.5), and the continuity of the function $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ of (9.4) imply that, as n increases without bound, the numbers $\mathfrak{s}(P_n, \sigma) = \mathfrak{s}(P_{t/n}, \sigma)$ not only have a limit, but

$$\lim_{n\to\infty} \mathfrak{s}(P_n, \sigma) = 0.$$

Hence, σ is $\mathfrak{s}$-approachable from σ; q.e.d.

In view of Remarks 9.3 and 3.2, Theorem 3.2, and observations made in Section 5 [see (5.4)], we have

Theorem 9.3. In order that a simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obey the Second Law of Thermodynamics, it is necessary and sufficient that $\mathfrak{s}$ have an upper potential S, called an *entropy function*, which is upper semicontinuous and whose domain is closed under $\mathfrak{s}$-approach and contains the state σ° occurring in the Second Law; if σ and σ' are in the domain of S, then

$$S(\sigma') - S(\sigma) \geqq \int_0^t \frac{h(\tau)}{\theta(\sigma_\tau)}\, d\tau + \int_0^t \boldsymbol{q}(\sigma_\tau) \cdot \boldsymbol{\beta}(\tau)\, d\tau, \tag{9.21}$$

for each P_t in Π with $\sigma' = P_t \sigma$. Of course, the states σ_τ and the vectors $\boldsymbol{\beta}(\tau)$ in (9.21) are given by (9.3) and (9.4) with $P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma})$: $[0, t) \to U$.

The relation (9.21) underlies much recent research[#] on the thermodynamics of simple materials. Various special cases of simple material elements can be

[#] See COLEMAN & NOLL [1963, 2], COLEMAN & MIZEL [1964, 3] [1967, 2], COLEMAN [1964, 1, 2] [1970, 1], COLEMAN & GURTIN [1965, 1, 2] [1966, 1] [1967, 1], OWEN [1968, 4], COLEMAN & OWEN [1970, 2], COLEMAN & DILL [1968, 1] [1973, 1], and the extensive list of references given by DAY [1972, 1]. In these works more is assumed about state functions than has been assumed so far here, and (9.21) is employed in differentiated form. For derivations of the differentiated form of (9.21) from a global form of the "Clausius-Duhem inequality", and for insight into the origin and early uses of (9.21), see DUHEM [1911, 1], TRUESDELL [1952, 1] [1961, 2], and TRUESDELL & TOUPIN [1960, 2].

obtained by imposing additional structure on Σ # and giving rules for calculating $P_t\sigma$. The most elementary examples of simple materials are elastic materials, and in the next section we shall show how the theory of elastic elements can be developed in our present framework. For elastic elements ${}_0\sigma=\sigma$, for each σ in Σ, and hence Σ is a subset of the finite-dimensional space $U=\text{Lin}(\mathscr{V})\oplus\mathbb{R}\oplus\mathscr{V}$.

10. Elastic Elements

Definition 10.1. A simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ is called an **elastic element** if

(1) the state space Σ is an open connected set of elements $(\boldsymbol{F}, e, \boldsymbol{\beta})$ of $\text{Lin}(\mathscr{V})^{++}\times\mathbb{R}\times\mathscr{V}$ (which is, in turn, an open connected subset of $U=\text{Lin}(\mathscr{V})\oplus\mathbb{R}\oplus\mathscr{V}$); for each fixed element $(\boldsymbol{F}_o, e_o, \boldsymbol{\beta}_o)$ of Σ, the set $\{\boldsymbol{F}_o\}\times\{e_o\}\times\mathscr{V}=\{(\boldsymbol{F}_o, e_o, \boldsymbol{\beta})\,|\,\boldsymbol{\beta}\in\mathscr{V}\}$ is contained in Σ; the topology of Σ is that induced by the natural topology of U, *i.e.* that arising from the inner product $(9.2)_1$; # # the mapping $\sigma\mapsto{}_0\sigma$ is taken to be the identity mapping:

$$ {}_0\sigma=\sigma; \tag{10.1} $$

(2) the functions θ and $\boldsymbol{q}$ of (9.5) are continuous, and $\boldsymbol{T}$ is continuously differentiable;

(3) let $P_t=(\boldsymbol{L}, h, \gamma)$ be a piecewise continuous function mapping an interval $[0, t)$, $t\in\mathbb{R}^{++}$, into the set U of (9.1), and let $\mathscr{D}(P_t)$ be the set of all states σ such that the differential equation

$$ \begin{aligned} \dot{\boldsymbol{F}}&=\boldsymbol{L}\boldsymbol{F},\\ \dot{e}&=\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta})\cdot\boldsymbol{L}+h,\\ \dot{\boldsymbol{\beta}}&=\gamma, \end{aligned} \tag{10.2} $$

with the initial condition

$$ (\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0))=\sigma, \tag{10.3} $$

has a solution# # # whose values $(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ lie in Σ for $\tau\in[0, t]$; if $\mathscr{D}(P_t)$ is not empty, then P_t is a member of Π, and for each σ in $\mathscr{D}(P_t)$ the state $P_t\sigma$ is the value at time t of the solution of (10.2) obeying (10.3):

$$ P_t\sigma\stackrel{\text{def}}{=}(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t)). \tag{10.4} $$

Note that so far Σ has only a topology; we have not yet assumed that Σ has a translation space or any manifold structure.

In applications one may, for example, take Σ to be a set of elements $(\boldsymbol{F}, e, \boldsymbol{\beta})$ of U with $\boldsymbol{F}$ varying over all of $\text{Lin}(\mathscr{V})^{++}$ and $\boldsymbol{\beta}$ varying over all of $\mathscr{V}$, while e has a lower bound $e_l(\boldsymbol{F})$ which depends continuously on $\boldsymbol{F}$; *i.e.* one may take, for some prescribed continuous function e_l: $\text{Lin}(\mathscr{V})^{++}\to\mathbb{R}$,

$$ \Sigma=\{(\boldsymbol{F}, e, \boldsymbol{\beta})\,|\,\boldsymbol{F}\in\text{Lin}(\mathscr{V})^{++}, e_l(\boldsymbol{F})<e<\infty,\ \boldsymbol{\beta}\in\mathscr{V}\}. $$

We of course mean a solution in the classical sense; such a solution $(\boldsymbol{F}, e, \boldsymbol{\beta})$ is continuous on $[0, t]$ and has a classical derivative $(\dot{\boldsymbol{F}}, \dot{e}, \dot{\boldsymbol{\beta}})$ at each point of continuity of $(\boldsymbol{L}, h, \gamma)$; wherever $(\dot{\boldsymbol{F}}, \dot{e}, \dot{\boldsymbol{\beta}})$ exists it obeys (10.2).

Thus if $P_t=(L, h, \gamma)$ is a process of an elastic element, then there is a point σ in Σ such that (10.2) has a solution (F, e, β) on $[0, t]$ obeying (10.3). Since we assume that $T: \Sigma \to \mathrm{Sym}(\mathscr{V})$ is continuously differentiable and that P_t, as a U-valued function on $[0, t)$, is piecewise continuous, the right-hand side of (10.2) obeys a local Lipschitz condition. It follows that the solution (F, e, β) is the only solution of (10.2) corresponding to the process P_t and the initial condition (10.3). Thus, the mapping $\sigma \mapsto P_t\sigma$ of (10.4) is well defined; *i.e.*, $P_t: \mathscr{D}(P_t) \to \Sigma$ is a function. Furthermore, because Σ is an open subset of U, the trajectory of the solution (F, e, β) forms the "center line" of a compact tube $\mathfrak{T}$ which has positive radius and lies in Σ at a positive distance from the complement of Σ in U. On $\mathfrak{T}$, the right-hand side of (10.2) has a single Lipschitz constant. Using these observations and standard techniques of the qualitative theory of differential equations, one can show that $\mathscr{D}(P_t)$ is an open subset of Σ and that the mapping $P_t: \mathscr{D}(P_t) \to \Sigma$ is continuous. It is evident that if τ is in $(0, t)$, and if P_τ is the restriction of P_t to $[0, \tau)$, then every σ in $\mathscr{D}(P_t)$ is in $\mathscr{D}(P_\tau)$.

Thus as the general theory of Part I requires, each element P_t of Π induces a transformation, *i.e.* determines a continuous Σ-valued function on a non-empty set $\mathscr{D}(P_t)$ which is open in Σ. In addition, the processes P_t of an elastic element and the function $\tau \mapsto \sigma_\tau = {}_0\sigma_\tau$ have the properties mentioned in item (2) of Definition 9.1. Moreover, when θ and q are continuous on $\Sigma \subset U$, the function $\mathfrak{s}: \Pi \diamond \Sigma \to \mathbb{R}$ of (9.7) is well defined and continuous in σ for each P_t in Π. It is easily verified that $\mathfrak{s}$ obeys the additivity relation (2.3); in other words, $\mathfrak{s}$ is an action, and since (10.2) implies (9.6), item (3) of Definition 9.1 is verifiable for an elastic element. Of course, for an elastic element the function $(P_{t_2}, P_{t_1}) \mapsto P_{t_2}P_{t_1} = P_{t_2+t_1}$ is defined as explained in item (1) of Definition 9.1; it is easily seen that this function does map $\mathscr{P}$ into Π and obeys axiom **II** of Definition 2.1. The requirement placed on a system (Σ, Π) in axiom **I** of Definition 2.1, i.e. that $\Pi\sigma \stackrel{\text{def}}{=} \{P_t\sigma \mid P_t \in \Pi, \sigma \in \mathscr{D}(P_t)\}$ be dense in Σ for each σ in Σ, is met here, for the following easy theorem shows that, for each state σ of an elastic element, $\Pi\sigma = \Sigma$.

Theorem 10.1. For each pair of states σ', σ'' of an elastic element there is a process P_t in Π such that

$$\sigma'' = P_t\sigma'.$$

Proof. Let $\sigma' = (F', e', \beta')$ and $\sigma'' = (F'', e'', \beta'')$ be two given elements of Σ. Since Σ is connected and open in $U = \mathrm{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}$, Σ is polygonally connected, and σ' and σ'' can be joined by a continuous path (F, e, β) in Σ formed from a finite number of segments on each of which $(\dot{F}, \dot{e}, \dot{\beta})$ is constant. Thus we have a function $(F, e, \beta): [0, t] \to \Sigma$ with $t > 0$ and such that

$$(F(0), e(0), \beta(0)) = (F', e', \beta') = \sigma', \tag{10.5}$$

$$(F(t), e(t), \beta(t)) = (F'', e'', \beta'') = \sigma'', \tag{10.6}$$

and the function $P_t = (L, h, \gamma): [0, t) \to U$, defined by

$$\begin{aligned} L &= \dot{F}F^{-1}, \\ h &= \dot{e} - T(F, e, \beta) \cdot (\dot{F}F^{-1}), \\ \gamma &= \dot{\beta}, \end{aligned} \tag{10.7}$$

is piecewise continuous. On comparing (10.7) and (10.5) with, respectively, (10.2) and (10.3), and noting that the trajectory of the function $(\boldsymbol{F}, e, \boldsymbol{\beta})$ lies in Σ, we see that the piecewise continuous function $P_t=(\boldsymbol{L}, h, \gamma)$ defined by (10.7) is a process in the sense of Definition 10.1, and, moreover, σ' is in $\mathscr{D}(P_t)$. Furthermore, comparison of (10.6) with (10.4) tells us that here $P_t\sigma'=\sigma''$; q.e.d.

In view of Theorem 10.1 and the observations made just before it, any object $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ which obeys items (1)–(3) of our Definition 10.1 is a *system* in general and, in particular, is an example of a *simple material element*; *i.e.* Σ and Π have the properties required by Definition 2.1, while $\Sigma, \Pi, \theta, \boldsymbol{T}$, and $\boldsymbol{q}$ have the properties required by Definition 9.1. That is,

Remark 10.1. Definition 10.1 is consistent with Definitions 9.1 and 2.1.

A particular elastic element is specified when $\Sigma, \Pi, \theta, \boldsymbol{T}$, and $\boldsymbol{q}$ are given. If one requires that an elastic element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obey the laws of thermodynamics, as stated at the beginning of Section 5, then θ, $\boldsymbol{T}$, and $\boldsymbol{q}$ must obey certain restrictions, which we shall discuss below.[#] As we observed in Remark 9.2, the First Law is here automatically satisfied. We shall here show that familiar restrictions on state functions follow from the Second Law. This is the case also in COLEMAN & NOLL's approach to the thermodynamics of elastic materials,[##] where entropy is taken to be a primitive concept.[###]

Theorem 10.2. If an action for an elastic element has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at one state in Σ, then it has the $\left\{\begin{matrix}\text{Clausius}\\ \text{conservation}\end{matrix}\right\}$ property at every state in Σ.

Proof. By Remark 3.1, an action has the Clausius property at a state σ only if it has that property at all states in $\Pi\sigma$, and Theorem 10.1 here yields $\Pi\sigma=\Sigma$. The analogous argument based on Theorem 4.1 here tells us that an action possessing the conservation property at one state has that property at all states; q.e.d.

For simple material elements in general, and hence for elastic elements in particular, the Second Law of Thermodynamics has been taken to be the assertion that the action $\mathfrak{s}$ defined in (9.7) has the Clausius property at a state in Σ. It follows from Theorem 10.2 that an elastic element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obeys the Second Law if and only if $\mathfrak{s}$ has the Clausius property at *every* state in Σ.

Theorem 10.3. If an elastic element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obeys the Second Law of Thermodynamics, then $\boldsymbol{q}$ must be such that for each state $(\boldsymbol{F}, e, \boldsymbol{\beta})$ in Σ,

$$\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta})\cdot\boldsymbol{\beta}\leqq 0. \tag{10.8}$$

Proof. Let $\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta})$ be an arbitrary state of the elastic element, and let t be a positive number. Clearly, the function $P_t=(\boldsymbol{L}, h, \gamma)$, defined by

$$P_t(\tau)=(\mathbf{0}, 0, \mathbf{0}), \qquad \text{for } \tau\in[0, t), \tag{10.9}$$

[#] See Theorems 10.3, 10.5, and 10.6, and Remark 10.3.

[##] [1963, 2]. See also COLEMAN & MIZEL [1964, 3].

[###] The theorems of this section generalize results obtained for elastic materials by CAPRIOLI [1955, 1] and DAY [1972, 1], who assumed, rather than showed, that $\boldsymbol{T}$ and θ are independent of $\boldsymbol{\beta}$.

is in Π, and σ is in $\mathscr{D}(P_t)$. For this process P_t, we have, for each τ in $(0, t)$,

$$h(\tau)=0 \quad \text{and} \quad P_\tau \sigma=\sigma, \tag{10.10}$$

where P_τ is the restriction of P_t to $[0, \tau)$. The relation (10.8) follows immediately from (10.10) and Theorem 9.1; q.e.d.

Theorem 10.1, Remark 3.2, and Theorem 3.2 here yield the following theorem which should be compared with Theorem 9.3.

Theorem 10.4. An elastic element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ satisfies the Second Law of Thermodynamics if and only if the action $\mathfrak{s}$ of (9.7) has an upper semicontinuous upper potential S whose domain is *all* of Σ. For each pair of states σ, σ' this entropy function S obeys the relation

$$S(\sigma')-S(\sigma) \geqq \int_0^t \frac{h(\tau)}{\theta(\sigma_\tau)} d\tau + \int_0^t \boldsymbol{q}(\sigma_\tau) \cdot \boldsymbol{\beta}(\tau) \, d\tau \tag{10.11}$$

for each process P_t with $P_t \sigma=\sigma'$. Here, again, σ_τ and $\boldsymbol{\beta}(\tau)$ are as in (9.3) and (9.4) with $P_t=(\boldsymbol{L}, h, \gamma)$: $[0, t) \rightarrow U$.

If S is differentiable on Σ, then (10.11) becomes equivalent to the assertion that the Clausius-Duhem relation,

$$\dot{S} \geqq \frac{h}{\theta} + \boldsymbol{q} \cdot \boldsymbol{\beta}, \tag{10.12}$$

holds on each solution of (10.2). In Theorems 10.5 and 10.7 we shall show that for an elastic element, the entropy function *is* differentiable and is unique to within a constant.

We say that a real-valued function f on Σ is *differentiable at* $\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta})$ if $U=\operatorname{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}$ contains an element $(\partial_{\boldsymbol{F}} f(\sigma), \partial_e f(\sigma), \partial_{\boldsymbol{\beta}} f(\sigma))$ obeying

$$\begin{aligned} f(\boldsymbol{F}+\boldsymbol{A}, e+a, \boldsymbol{\beta}+\boldsymbol{\alpha}) = {} & f(\boldsymbol{F}, e, \boldsymbol{\beta}) + \partial_{\boldsymbol{F}} f(\sigma) \cdot \boldsymbol{A} + \partial_e f(\sigma)\, a \\ & + \partial_{\boldsymbol{\beta}} f(\sigma) \cdot \boldsymbol{\alpha} + o(|(\boldsymbol{A}, a, \boldsymbol{\alpha})|) \end{aligned} \tag{10.13}$$

for those elements $(\boldsymbol{A}, a, \boldsymbol{\alpha})$ of U such that $(\boldsymbol{F}+\boldsymbol{A}, e+a, \boldsymbol{\beta}+\boldsymbol{\alpha})$ is in Σ; here $\partial_{\boldsymbol{F}} f(\sigma)$ is an element of $\operatorname{Lin}(\mathscr{V})$; $\partial_e f(\sigma)$ is in $\mathbb{R}$; and $\partial_{\boldsymbol{\beta}} f(\sigma)$ is in $\mathscr{V}$. We say that f is *continuously differentiable on* Σ if f is differentiable at each point in Σ, and the function $\sigma \mapsto (\partial_{\boldsymbol{F}} f(\sigma), \partial_e f(\sigma), \partial_{\boldsymbol{\beta}} f(\sigma))$ is a continuous mapping of Σ into U.

Theorem 10.5. If an elastic element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obeys the Second Law of Thermodynamics, then it has a continuously differentiable entropy function S which is defined on all of Σ and determines the state functions θ and $\boldsymbol{T}$ through the relations

$$\theta(\sigma)=[\partial_e S(\sigma)]^{-1}, \tag{10.14}$$

$$\boldsymbol{T}(\sigma)=-\theta(\sigma)[\partial_{\boldsymbol{F}} S(\sigma)] \boldsymbol{F}^T, \tag{10.15}$$

which hold for each state $\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta})$; furthermore, this entropy function is independent of $\boldsymbol{\beta}$, *i.e.* obeys the relation

$$S(\boldsymbol{F}, e, \boldsymbol{\beta})=S(\boldsymbol{F}, e, \mathbf{0}) \tag{10.16}$$

throughout Σ.

Proof. Consider the line integral

$$I(c)=\int_c \left[\frac{-\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta})\boldsymbol{F}^{T^{-1}}}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta})}\cdot d\boldsymbol{F}+\frac{1}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta})}\,de+\mathbf{0}\cdot d\boldsymbol{\beta}\right] \tag{10.17}$$

which is defined on each oriented polygonal curve c lying in

$$\Sigma\subset \mathrm{Lin}\,(\mathscr{V})^{++}\oplus\mathbb{R}\oplus\mathscr{V}.$$

Of course, if c^+ and c^- are two curves which are in the domain of I and which differ in orientation, and only in orientation, then

$$I(c^-)=-I(c^+). \tag{10.18}$$

Now, let σ be a fixed arbitrary element of Σ, and let c^+ be a closed oriented polygonal curve containing σ and lying in Σ. For each $t>0$ there is a continuous piecewise-linear parametrization $\tau\mapsto(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ of c^+, with domain $[0, t]$, obeying

$$\big(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t)\big)=\big(\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0)\big)=\sigma. \tag{10.19}$$

Corresponding to this parameterization we have a function $P_t=(\boldsymbol{L}, h, \boldsymbol{\gamma})\colon [0, t)\to U$, defined by (10.7). As we observed in the proof of Theorem 10.1, P_t is a process with $P_t\sigma=(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t))$, which here, by (10.19), reduces to $P_t\sigma=\sigma$. Employing the parameterization $\tau\mapsto(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$, we can write

$$I(c^+)=\int_0^t \left[\frac{-\boldsymbol{T}\big(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)\big)\boldsymbol{F}^{T^{-1}}(\tau)}{\theta\big(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)\big)}\cdot\dot{\boldsymbol{F}}(\tau)+\frac{\dot{e}(\tau)}{\theta\big(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)\big)}\right]d\tau,$$

and, by $(10.7)_2$, this becomes

$$I(c^+)=\int_0^t \frac{h(\tau)}{\theta\big(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)\big)}\,d\tau. \tag{10.20}$$

It is a consequence of Theorem 10.2 that if $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obeys the Second Law of Thermodynamics, then the action $\mathfrak{s}$ has the Clausius property at *every* point in Σ and, in particular, at the point σ. Since $P_t\sigma=\sigma$, we have, in view of (5.4), $\mathfrak{s}(P_t, \sigma)\leqq 0$, or, by (9.7),

$$\int_0^t \frac{h(\tau)}{\theta(\sigma_\tau)}\,d\tau+\int_0^t \boldsymbol{q}(\sigma_\tau)\cdot\boldsymbol{\beta}(\tau)\,d\tau\leqq 0,$$

where $\sigma_\tau=(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))=P_\tau\sigma$, with P_τ the restriction of P_t to $[0, \tau)$. Hence, (10.20) yields

$$I(c^+)\leqq-\int_0^t \boldsymbol{q}\big(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)\big)\cdot\boldsymbol{\beta}(\tau)\,d\tau. \tag{10.21}$$

Let

$$M(c^+)=\sup\{-\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta})\cdot\boldsymbol{\beta}\,|\,(\boldsymbol{F}, e, \boldsymbol{\beta})\in c^+\}. \tag{10.22}$$

Since c^+ is a compact subset of Σ, and $\boldsymbol{q}$ is continuous on Σ, $M(c^+)$ is finite. By Theorem 10.3, $M(c^+)$ is not negative. Obviously, $M(c^+)$ is independent of t and the parameterization of c^+, and (10.21) and (10.22) yield

$$I(c^+) \leqq t M(c^+).$$

Since this relation holds for each $t>0$, we have

$$I(c^+) \leqq 0. \tag{10.23}$$

If c^- differs from c^+ in orientation, and only in orientation, then the argument which gave (10.23), since it was independent of the orientation of c^+, gives also

$$I(c^-) \leqq 0. \tag{10.24}$$

But, by (10.18), $I(c^-) = -I(c^+)$, and hence (10.24) yields $-I(c^+) \leqq 0$, which is compatible with (10.23) only if

$$I(c^+) = 0. \tag{10.25}$$

As σ is an arbitrary state in Σ, and c^+ is an arbitrary closed, polygonal curve through σ, our proof of (10.25) shows that the line integral I vanishes on every closed, polygonal curve lying wholly in the open connected set

$$\Sigma \subset \operatorname{Lin}(\mathscr{V})^{++} \oplus \mathbb{R} \oplus \mathscr{V}.$$

Hence, by (10.17) and a well known fundamental theorem on the existence of potentials for vector fields, the function from Σ into $U = \operatorname{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}$ defined by

$$\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}) \mapsto \left(\frac{-\boldsymbol{T}(\sigma)}{\theta(\sigma)} \boldsymbol{F}^{T^{-1}}, \frac{1}{\theta(\sigma)}, \boldsymbol{0} \right)$$

has a potential in the classical sense; *i.e.* there exists a differentiable function $S: \Sigma \to \mathbb{R}$, obeying the relations

$$\begin{aligned} \partial_{\boldsymbol{F}} S(\sigma) &= -\boldsymbol{T}(\sigma) \boldsymbol{F}^{T^{-1}} / \theta(\sigma), \\ \partial_e S(\sigma) &= 1/\theta(\sigma), \\ \partial_{\boldsymbol{\beta}} S(\sigma) &= \boldsymbol{0}, \end{aligned} \tag{10.26}$$

which are equivalent to (10.14), (10.15), and (10.16). Since $\theta: \Sigma \to \mathbb{R}^{++}$ and $\boldsymbol{T}: \Sigma \to \operatorname{Sym}(\mathscr{V})$ are continuous functions, S is continuously differentiable on Σ.

To see that S is an upper potential for s, *i.e.* an entropy function for $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, we observe first that S obviously satisfies the condition (1) of Definition 3.2. To verify condition (2), we let σ be an *arbitrary* state and $P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma})$ be an arbitrary process such that σ is in $\mathscr{D}(P_t)$. The supply of s in going from σ to $P_t \sigma$ via the process P_t is given by the formula (9.7). If we write, for each τ in $[0, t]$, $\sigma_\tau = (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ and take into account $(10.2)_{2\&1}$, then the equation (9.7) can be written in the form

$$\begin{aligned} s(P_t, \sigma) = \int_0^t &\left[\frac{-\boldsymbol{T}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)) \boldsymbol{F}^{T^{-1}}(\tau) \cdot \dot{\boldsymbol{F}}(\tau) + \dot{e}(\tau)}{\theta(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))} \right] d\tau \\ &+ \int_0^t \boldsymbol{q}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)) \cdot \boldsymbol{\beta}(\tau)\, d\tau, \end{aligned}$$

and, by (10.26), we have

$$\begin{aligned}\mathfrak{s}(P_t, \sigma) = \int_0^t [\partial_F S(F(\tau), e(\tau), \beta(\tau)) \cdot \dot{F}(\tau) + \partial_e S(F(\tau), e(\tau), \beta(\tau))\, \dot{e}(\tau) \\ + \partial_\beta S(F(\tau), e(\tau), \beta(\tau)) \cdot \dot{\beta}(\tau)]\, d\tau + \int_0^t q(F(\tau), e(\tau), \beta(\tau)) \cdot \beta(\tau)\, d\tau.\end{aligned} \tag{10.27}$$

Because S is differentiable on Σ, and the function $\tau \mapsto (F(\tau), e(\tau), \beta(\tau))$ is differentiable at all but a finite number of points in the interval $[0, t]$, we can apply the chain-rule to the composite function $S(F(\cdot), e(\cdot), \beta(\cdot))$ and cast (10.27) into the form

$$\begin{aligned}\mathfrak{s}(P_t, \sigma) &= \int_0^t \frac{d}{d\tau} S(F(\tau), e(\tau), \beta(\tau))\, d\tau + \int_0^t q(F(\tau), e(\tau), \beta(\tau)) \cdot \beta(\tau)\, d\tau \\ &= S(F(t), e(t), \beta(t)) - S(F(0), e(0), \beta(0)) \\ &\quad + \int_0^t q(F(\tau), e(\tau), \beta(\tau)) \cdot \beta(\tau)\, d\tau.\end{aligned} \tag{10.28}$$

By (10.3) and (10.4), equation (10.28) can be written

$$\mathfrak{s}(P_t, \sigma) = S(P_t \sigma) - S(\sigma) + \int_0^t q(\sigma_\tau) \cdot \beta(\tau)\, d\tau. \tag{10.29}$$

By Theorem 10.3, the integrand

$$q(\sigma_\tau) \cdot \beta(\tau) = q(F(\tau), e(\tau), \beta(\tau)) \cdot \beta(\tau)$$

is nowhere positive; therefore (10.29) yields

$$\mathfrak{s}(P_t, \sigma) \leqq S(P_t \sigma) - S(\sigma), \tag{10.30}$$

and, by the arbitrariness of σ and P_t, this relation holds for *every* σ in Σ and each P_t in Π which has σ in $\mathscr{D}(P_t)$. Let σ' be an arbitrary state in Σ and ε a given positive number. There exists, by the continuity of S on Σ, a neighborhood $\mathcal{O}$ of σ' for which

$$\sigma'' \in \mathcal{O} \quad \Rightarrow \quad S(\sigma'') - \varepsilon < S(\sigma'),$$

and hence

$$P_t \in \Pi, \quad P_t \sigma \in \mathcal{O} \quad \Rightarrow \quad S(\sigma') > S(P_t \sigma) - \varepsilon \quad \Rightarrow \quad S(\sigma') - S(\sigma) > S(P_t \sigma) - S(\sigma) - \varepsilon.$$

From this and (10.30) we conclude that, for each pair of states σ and σ' and each positive number ε, there is a neighborhood $\mathcal{O}$ of σ' for which

$$P_t \in \Pi, \quad P_t \sigma \in \mathcal{O} \quad \Rightarrow \quad S(\sigma') - S(\sigma) > \mathfrak{s}(P_t, \sigma) - \varepsilon.$$

Therefore, S satisfies item (2) in Definition 3.2 and is an upper potential for $\mathfrak{s}$; q.e.d.

As an immediate corollary to Theorem 10.5, we have

Theorem 10.6. For an elastic element $(\Sigma, \Pi, \theta, T, q)$ which obeys the Second Law, the functions θ and T are independent of β, *i.e.*

$$\theta(F, e, \beta) = \theta(F, e, 0), \qquad T(F, e, \beta) = T(F, e, 0), \tag{10.31}$$

throughout Σ.

It is clear that the equations (10.14)–(10.16) of Theorem 10.5 determine the differentiable function S to within a constant. Furthermore, employing (10.11) [*i.e.* (5.4)] and arguments given by COLEMAN & NOLL [1963, 2], one can show, for an elastic element, that *every differentiable* entropy function obeys the equations (10.14)–(10.16). Thus, the Second Law implies that a given elastic element has exactly one *differentiable* entropy function which is normalized, *i.e.* which takes a preassigned value at a prescribed "standard state". Actually, however, one can obtain a stronger result. The next theorem tells us that *an elastic element has no non-differentiable entropy functions*. We give two proofs for Theorem 10.7. The first employs Remark 7.5, and although longer than the second, may be of interest, for it shows that quantities of the type $\sup \mathscr{a}\{\sigma^\circ \to \mathscr{O}\}$ discussed at length in Section 3 and 7 can be calculated when the system is an elastic element and $\mathscr{a}$ is the action $\mathscr{s}$ of (9.7); the second proof is more elementary but does not illustrate as well the significance of Remark 7.5.

Theorem 10.7. Let σ° be a fixed state of an elastic element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, and let S° be a preassigned number. If the elastic element obeys the Second Law, then it has exactly one entropy function whose domain is Σ and for which

$$S(\sigma^\circ)=S^\circ. \tag{10.32}$$

Proof. Since an entropy function plus a constant is again an entropy function, the theorem will be proven if we can show that $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ has one and only one entropy function $\bar{S}$ on Σ obeying

$$\bar{S}(\sigma^\circ)=0. \tag{10.33}$$

Let $\hat{S}\colon \Sigma \to \mathbb{R}$ be a continuously differentiable entropy function of the type given to us by Theorem 10.5 and its proof, *i.e.* obeying (10.14)–(10.16) and (10.29). Clearly, the function $\bar{S}$ defined by

$$\bar{S}(\sigma)=\hat{S}(\sigma)-\hat{S}(\sigma^\circ), \qquad \sigma\in\Sigma, \tag{10.34}$$

is an entropy function on Σ and obeys (10.33). Thus, the elastic element has at least one entropy function $\bar{S}$ of the required type. To show that it has only one, we let σ' and σ'' be arbitrary elements of Σ and let $\mathscr{s}\{\sigma' \to \sigma''\}$ be the following subset of $\mathbb{R}$:

$$\mathscr{s}\{\sigma' \to \sigma''\}=\{\mathscr{s}(P,\sigma') \mid P\in\Pi, P\sigma'=\sigma''\}. \tag{10.35}$$

By Theorem 10.1, this set is not empty. Because $\hat{S}$ is an upper potential for $\mathscr{s}$, we have, by item (2) of Definition 3.2 [see (5.4)],

$$\mathscr{s}(P,\sigma')\leqq \hat{S}(\sigma'')-\hat{S}(\sigma'),$$

whenever $P\sigma'=\sigma''$, and hence the set $\mathscr{s}\{\sigma' \to \sigma''\}$ is bounded above with

$$\sup \mathscr{s}\{\sigma' \to \sigma''\}\leqq \hat{S}(\sigma'')-\hat{S}(\sigma'). \tag{10.36}$$

Furthermore, because $\hat{S}$ was chosen to obey (10.29), we have, for each process P with $P\sigma'=\sigma''$,

$$\mathscr{s}(P,\sigma')=\hat{S}(\sigma'')-\hat{S}(\sigma')+\int_0^t \boldsymbol{q}\big(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)\big)\cdot\boldsymbol{\beta}(\tau)\,d\tau, \tag{10.37}$$

where $(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ is the value at time $\tau \in [0, t]$ of the solution of (10.2) corresponding to the process P_t and the initial state σ'. Now, recall that Σ is polygonally connected, and let $\mathscr{c}$ be a polygonal curve which lies wholly in $\Sigma \subset U$ and has σ' and σ'' as end points. For each $t>0$, $\mathscr{c}$ has a continuous piecewise-linear parametrization, $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$, with domain $[0, t]$, and, as we saw in the proof of Theorem 10.1, each such parametrization of $\mathscr{c}$ determines a process P_t obeying $P_t \sigma' = \sigma''$; for this process (10.37) holds. When P_t is so determined,

$$\left| \int_0^t \boldsymbol{q}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)) \cdot \boldsymbol{\beta}(\tau)\, d\tau \right| \leqq t M(\mathscr{c}), \tag{10.38}$$

where

$$M(\mathscr{c}) = \sup \{ |\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta}) \cdot \boldsymbol{\beta}| \mid (\boldsymbol{F}, e, \boldsymbol{\beta}) \in \mathscr{c} \}$$

is finite, because $\mathscr{c}$ is compact, and $\boldsymbol{q}$ is continuous on Σ. Thus, for each $t>0$ and each process P_t obtained by parameterizing $\mathscr{c}$ with a parameter τ varying over $[0, t]$, we have, by (10.37) and (10.38),

$$|\mathfrak{s}(P_t, \sigma_1) - (\hat{S}(\sigma_2) - \hat{S}(\sigma_1))| \leqq \left| \int_0^t \boldsymbol{q}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)) \cdot \boldsymbol{\beta}(\tau)\, d\tau \right| \leqq t M(\mathscr{c}). \tag{10.39}$$

with $M(\mathscr{c})$ determined by the chosen curve $\mathscr{c}$. Clearly, (10.39) and (10.36) yield

$$\sup \mathfrak{s}\{\sigma' \to \sigma''\} = \hat{S}(\sigma'') - \hat{S}(\sigma'). \tag{10.40}$$

For each $\sigma' \in \Sigma$ and each open set $\mathcal{O} \subset \Sigma$, let

$$\mathfrak{s}\{\sigma' \to \mathcal{O}\} = \{\mathfrak{s}(P, \sigma') \mid P \in \Pi, P\sigma' \in \mathcal{O}\} \subset \mathbb{R}. \tag{10.41}$$

For σ' and σ'' in Σ we put, as in Lemma 3.3,

$$m(\sigma', \sigma'') = \inf_{\mathcal{O} \in \mathfrak{S}(\sigma', \sigma'')} \sup \mathfrak{s}\{\sigma' \to \mathcal{O}\}, \tag{10.42}$$

where $\mathfrak{S}(\sigma', \sigma'')$ is the following class of subsets of Σ:

$$\mathfrak{S}(\sigma', \sigma'') = \{\mathcal{O} \mid \mathcal{O} \subset \Sigma, \mathcal{O} \text{ open}, \sigma'' \in \mathcal{O}, \sup \mathfrak{s}\{\sigma' \to \mathcal{O}\} < \infty\}. \tag{10.43}$$

By Theorem 10.1 and the proof of Theorem 3.1 (or Lemma 7.1), m is well defined on $\Sigma \times \Sigma$, *i.e.* for $(\sigma', \sigma'') \in \Sigma \times \Sigma$, $m(\sigma', \sigma'') > -\infty$. If $\mathcal{O}$ is in $\mathfrak{S}(\sigma', \sigma'')$, then (10.35) and (10.41) yield

$$\mathfrak{s}\{\sigma' \to \mathcal{O}\} = \bigcup_{\sigma' \in \mathcal{O}} \mathfrak{s}\{\sigma' \to \sigma\},$$

and hence, by (10.40),

$$\sup \mathfrak{s}\{\sigma' \to \mathcal{O}\} = \sup \{\hat{S}(\sigma) \mid \sigma \in \mathcal{O}\} - \hat{S}(\sigma'); \tag{10.44}$$

as $\mathfrak{s}\{\sigma' \to \mathcal{O}\}$ is bounded above, so also is $\{\hat{S}(\sigma) \mid \sigma \in \mathcal{O}\}$. Since $\hat{S}$ is continuous on Σ, for each $\varepsilon > 0$ and $\mathcal{O}$ in $\mathfrak{S}(\sigma', \sigma'')$, there is an open neighborhood $\mathcal{O}_\varepsilon$ of σ'' such that $\mathcal{O}_\varepsilon \subset \mathcal{O}$ and

$$\sup \{\hat{S}(\sigma) \mid \sigma \in \mathcal{O}_\varepsilon\} < \hat{S}(\sigma'') + \varepsilon, \tag{10.45}$$

Clearly, $\mathcal{O}_\varepsilon$ is in $\mathfrak{S}(\sigma', \sigma'')$, and (10.45) yields

$$\inf_{\mathcal{O} \in \mathfrak{S}(\sigma', \sigma'')} \sup \{\hat{S}(\sigma) \mid \sigma \in \mathcal{O}\} \leqq \sup \{\hat{S}(\sigma) \mid \sigma \in \mathcal{O}_\varepsilon\} < \hat{S}(\sigma'') + \varepsilon.$$

As this holds for each $\varepsilon>0$, we have

$$\inf_{\mathcal{O}\in\mathfrak{S}(\sigma',\sigma'')} \sup\{\hat{S}(\sigma)\,|\,\sigma\in\mathcal{O}\}\leqq\hat{S}(\sigma''). \tag{10.46}$$

Furthermore, since each set $\mathcal{O}$ in $\mathfrak{S}(\sigma',\sigma'')$ contains σ'', we have also

$$\inf_{\mathcal{O}\in\mathfrak{S}(\sigma',\sigma'')} \sup\{\hat{S}(\sigma)\,|\,\sigma\in\mathcal{O}\}\geqq\hat{S}(\sigma''),$$

and comparison with (10.46) yields

$$\inf_{\mathcal{O}\in\mathfrak{S}(\sigma',\sigma'')} \sup\{\hat{S}(\sigma)\,|\,\sigma\in\mathcal{O}\}=\hat{S}(\sigma''),$$

or, by (10.44) and (10.42),

$$m(\sigma',\sigma'')=\hat{S}(\sigma'')-\hat{S}(\sigma'). \tag{10.47}$$

By interchanging the arbitrary states σ' and σ'' and repeating the argument which led to (10.47), we obtain

$$m(\sigma'',\sigma')=\hat{S}(\sigma')-\hat{S}(\sigma''),$$

and on adding this to (10.47) we find that

$$m(\sigma'',\sigma')+m(\sigma',\sigma'')=0$$

for *every* pair (σ',σ'') in $\Sigma\times\Sigma$. In particular, if σ° is the fixed state mentioned in the hypothesis of the theorem, we have

$$m(\sigma^\circ,\sigma)+m(\sigma,\sigma^\circ)=0$$

for every σ in Σ. Thus, the condition (7.25) of Remark 7.5 is here satisfied, and we conclude from that remark that there is exactly one upper potential for $\mathfrak{s}$ with domain Σ and which vanishes at σ°; *i.e.* $\bar{S}$, given by (10.34), is the only entropy function obeying (10.33); q.e.d.

Alternate proof of Theorem 10.7. Because the elastic element $(\Sigma,\Pi,\theta,\boldsymbol{T},\boldsymbol{q})$ obeys the Second Law, it is clear from Theorem 10.5 that the element has at least one entropy function on Σ which obeys (10.32). To show that there is only one, let S be *any* given entropy function for $(\Sigma,\Pi,\theta,\boldsymbol{T},\boldsymbol{q})$ with domain Σ, and let σ' and σ'' be two arbitrary states in Σ. Since Σ is polygonally connected, there is an oriented polygonal curve c^+ which joins σ' to σ'' and lies wholly in Σ. As we saw in the proof of Theorem 10.1, each continuous piecewise-linear parameterization of c^+, $\tau\mapsto(\boldsymbol{F}(\tau),e(\tau),\boldsymbol{\beta}(\tau))$ with $\tau\in[0,t]$, $t>0$, determines, through (10.7), a process $P_t=(\boldsymbol{L},h,\gamma)$ with $P_t\sigma'=\sigma''$. Since S is an upper potential for $\mathfrak{s}$ and has domain Σ, it follows from (5.4), (9.7), (9.3), (10.4), and $(9.6)_2$ that for the process P_t we have

$$\begin{aligned} S(\sigma'')-S(\sigma')&\geqq\int_0^t\frac{h(\tau)}{\theta(\sigma_\tau)}\,d\tau+\int_0^t\boldsymbol{q}(\sigma_\tau)\cdot\boldsymbol{\beta}(\tau)\,d\tau\\ &=\int_0^t\frac{-\boldsymbol{T}\big(\boldsymbol{F}(\tau),e(\tau),\boldsymbol{\beta}(\tau)\big)\boldsymbol{F}^T(\tau)^{-1}\cdot\dot{\boldsymbol{F}}(\tau)+\dot{e}(\tau)}{\theta\big(\boldsymbol{F}(\tau),e(\tau),\boldsymbol{\beta}(\tau)\big)}\,d\tau\\ &\quad+\int_0^t\boldsymbol{q}\big(\boldsymbol{F}(\tau),e(\tau),\boldsymbol{\beta}(\tau)\big)\cdot\boldsymbol{\beta}(\tau)\,d\tau\\ &=I(c^+)+\int_0^t\boldsymbol{q}\big(\boldsymbol{F}(\tau),e(\tau),\boldsymbol{\beta}(\tau)\big)\cdot\boldsymbol{\beta}(\tau)\,d\tau, \end{aligned} \tag{10.48}$$

with I the line integral defined by (10.17). The number

$$N(c^+)=\inf\{\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta})\cdot\boldsymbol{\beta} \mid (\boldsymbol{F}, e, \boldsymbol{\beta})\in c^+\} \tag{10.49}$$

is finite, because c^+ is compact, and $\boldsymbol{q}$ is continuous on Σ. The relations (10.48) and (10.49) yield

$$S(\sigma'')-S(\sigma')\geqq I(c^+)+tN(c^+), \tag{10.50}$$

where $S(\sigma'')$, $S(\sigma')$, $I(c^+)$, $N(c^+)$ do not depend on the parameterization of c^+. Since for every $t>0$ there is a continuous piecewise-linear parameterization of c^+ with domain $[0, t]$, (10.50) holds for every $t>0$, and we may conclude that

$$S(\sigma'')-S(\sigma')\geqq I(c^+). \tag{10.51}$$

If we now interchange σ' and σ'' and replace c^+ by the curve c^- which differs from c^+ only in orientation, then the argument which gave (10.51) gives

$$S(\sigma')-S(\sigma'')\geqq I(c^-),$$

but, since $I(c^-)=-I(c^+)$, this yields

$$S(\sigma'')-S(\sigma')\leqq I(c^+),$$

which is compatible with (10.51) only if

$$S(\sigma'')-S(\sigma')=I(c^+). \tag{10.52}$$

Thus, the difference in entropy for two arbitrary states σ' and σ'' does not depend on the choice of entropy function, and this implies that at most one entropy function S on Σ can satisfy (10.32); q.e.d.

Theorem 10.7 tells us that the entropy function of an elastic element is unique to within a constant; if we combine this fact with equations (10.40) and (10.35) of the first proof of the theorem, we can make the following assertion.

Remark 10.2. If $S: \Sigma\to\mathbb{R}$ is an entropy function for an elastic element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, then

$$S(\sigma'')-S(\sigma')=\sup\{s(P, \sigma') \mid P\in\Pi,\ P\sigma'=\sigma''\}, \tag{10.53}$$

for each pair (σ', σ'') in $\Sigma\times\Sigma$.

Remark 10.3.# It is a consequence of the "heat conduction inequality" (10.8) that

$$\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{0})=\boldsymbol{0} \tag{10.54}$$

whenever $(\boldsymbol{F}, \theta, \boldsymbol{0})$ is in Σ. (In other words, it follows from the Second Law that heat does not flow through an elastic element without a temperature gradient.)

Proof. By item (1) of Definition 10.1, if $(\boldsymbol{F}, \theta, \boldsymbol{0})$ is in Σ, so also is $(\boldsymbol{F}, e, \alpha\boldsymbol{b})$ for each α in $\mathbb{R}$ and each $\boldsymbol{b}$ in $\mathscr{V}$, and (10.8) yields

$$\boldsymbol{q}(\boldsymbol{F}, e, \alpha\boldsymbol{b})\cdot(\alpha\boldsymbol{b})\leqq 0.$$

Cf. Pipkin & Rivlin [1958, 1] and Coleman & Noll [1963, 2] whose proofs assume that $\boldsymbol{q}$ is differentiable; the present proof does not.

For α in $\mathbb{R}^{++}$ this reduces to

$$q(F, e, \alpha b) \cdot b \leqq 0,$$

and hence, by the continuity of q on Σ, we have, for each b in $\mathscr{V}$,

$$q(F, e, 0) \cdot b = \lim_{\alpha \to 0+} q(F, e, \alpha b) \cdot b \leqq 0.$$

But, since $q(F, \theta, 0)$ is independent of b, the inner product $q(F, \theta, 0) \cdot b$ can fail to be positive for each b in $\mathscr{V}$ only if $q(F, \theta, 0) = 0$; q.e.d.

Remark 10.4. Although we have not made use of the fact here, one can easily show that every state of an elastic element is *stagnant* in the sense in which the term is used in Section 6; furthermore, those states (F, e, β) for which $q(F, e, \beta) \cdot \beta = 0$ are *relaxed for* $\mathscr{s}$ in the sense of Definition 7.1.

The main results of this section may be summarized as follows.

Remark 10.5. An elastic element $(\Sigma, \Pi, \theta, T, q)$ satisfies the Second Law if and only if the action $\mathscr{s}$ of (9.7) has the Clausius property at every state in Σ. This condition is equivalent to the assertion that $\mathscr{s}$ has an upper potential S which is defined on all of Σ and is unique to within a constant. This entropy function for $(\Sigma, \Pi, \theta, T, q)$ is continuously differentiable, obeys (10.16), and determines the functions θ and T through the relations (10.14) and (10.15). Furthermore, in order for $(\Sigma, \Pi, \theta, T, q)$ to obey the Second Law it is necessary and sufficient that the state functions θ, T, and q satisfy both of the following two conditions:

(i) θ and T are independent of β, and there exists a differentiable function S of F and e such that, for each state (F, e, β),

$$\theta(F, e, \beta) = \theta(F, e) = [\partial_e S(F, e)]^{-1}$$

and

$$T(F, e, \beta) = T(F, e) = -\theta(F, e)[\partial_F S(F, e)] F^T;$$

(ii) for each state (F, e, β), q obeys the "heat conduction inequality",

$$q(F, e, \beta) \cdot \beta \leqq 0.$$

11. Elements with Internal Variables

Definition 11.1. A simple material element $(\Sigma, \Pi, \theta, T, q)$ is called an **element with internal variables** if

(1) there is given an open connected set Φ of elements (F, e, β, α) of $\mathrm{Lin}(\mathscr{V})^{++} \times \mathbb{R} \times \mathscr{V} \times \mathscr{W}$, with $\mathscr{W}$ a finite-dimensional vector space with an inner product "$\cdot$"; the topology of Φ is that induced by the natural topology of $\mathrm{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V} \oplus \mathscr{W} = U \oplus \mathscr{W}$; for each element $(F', e', \beta', \alpha')$ of Φ, the set $\{(F', e', \beta, \alpha') \mid \beta \in \mathscr{V}\}$ is a subset of Φ; if $\mathscr{A}$ is the set of all α' in $\mathscr{W}$ for which Φ has elements of the form (F, e, β, α'), then for each α' in $\mathscr{A}$, the set

$$\Sigma_{\alpha'} \stackrel{\text{def}}{=} \{(F, e, \beta, \alpha') \mid (F, e, \beta, \alpha') \in \Phi\} \tag{11.1}$$

is connected; the state space Σ is a subset of Φ as explained in items (3)–(4) below;

(2) the functions θ and $\boldsymbol{q}$ in (9.5) are defined and continuous on Φ, $\boldsymbol{T}$ is continously differentiable on Φ, and there is given a continuously differentiable function $\boldsymbol{f}$ mapping Φ into $\mathscr{W}$;

(3) if $(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')$ is in Φ, we write $\Psi_{(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')}$ for the set of piecewise continuous functions $P_t = (\boldsymbol{L}, h, \gamma)$ which map an interval $[0, t)$, with $t \in \mathbb{R}^{++}$, into U in such a way that the differential equation

$$\begin{aligned} \dot{\boldsymbol{F}} &= \boldsymbol{L}\boldsymbol{F}, \\ \dot{e} &= \boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}) \cdot \boldsymbol{L} + h, \\ \dot{\boldsymbol{\beta}} &= \gamma, \\ \dot{\boldsymbol{\alpha}} &= \boldsymbol{f}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}), \end{aligned} \tag{11.2}$$

has a solution $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ which obeys the initial condition

$$\big(\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0), \boldsymbol{\alpha}(0)\big) = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}') \tag{11.3}$$

and lies in Φ for all $\tau \in [0, t]$; the functions $\boldsymbol{T}$ and $\boldsymbol{f}$ are such that there exists a distinguished point $\sigma^\circ = (\boldsymbol{F}^\circ, e^\circ, \boldsymbol{\beta}^\circ, \boldsymbol{\alpha}^\circ)$ in Φ with the following property: for each neighborhood $\mathcal{O}$ of σ° in Φ, there is, for every $(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')$ in Φ an element P_t of $\Psi_{(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')}$ for which the solution of (11.2) and (11.3) has its final value $(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t), \boldsymbol{\alpha}(t))$ in $\mathcal{O}$;

(4) let Φ^{σ° be the subset of Φ which is comprised of all values of solutions of (11.2) with

$$\big(\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0), \boldsymbol{\alpha}(0)\big) = (\boldsymbol{F}^\circ, e^\circ, \boldsymbol{\beta}^\circ, \boldsymbol{\alpha}^\circ) = \sigma^\circ \tag{11.4}$$

and $P_t = (\boldsymbol{L}, h, \gamma)$ in Ψ_{σ°; the set Σ is the closure in Φ of Φ^{σ°; the topology of Σ is that induced by $U \oplus \mathscr{W}$;

(5)

$$\Pi = \bigcup_{\sigma \in \Sigma} \Psi_\sigma; \tag{11.5}$$

the domain $\mathscr{D}(P_t)$ of a process $P_t \in \Pi$ is the set of all elements σ of Σ for which P_t is in Ψ_σ; when σ is in $\mathscr{D}(P_t)$, the state $P_t\sigma$ is defined to be the value at time t of the solution $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ of (11.2) with $(\boldsymbol{L}, h, \gamma) = P_t$ and

$$\big(\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0), \boldsymbol{\alpha}(0)\big) = \sigma; \tag{11.6}$$

i.e.

$$P_t\sigma \stackrel{\text{def}}{=} \big(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t), \boldsymbol{\alpha}(t)\big). \tag{11.7}$$

Since we here assume that $\boldsymbol{T}\colon \Phi \to \mathrm{Sym}(\mathscr{V})$ and $\boldsymbol{f}\colon \Phi \to \mathscr{W}$ are continuously differentiable and that $P_t = (\boldsymbol{L}, h, \gamma)$ is piecewise continuous on $[0, t)$, the right-hand side of (11.2) is locally Lipschitzian. Thus, for each P_t in Π, when σ is in $\mathscr{D}(P_t)$, the equation (11.2) has only one solution obeying (11.6). (It has at least one solution, because $\sigma \in \mathscr{D}(P_t)$ implies $P_t \in \Psi_\sigma$.) It follows that the mapping $\sigma \mapsto P_t\,\sigma$ of (11.7) is well defined as a function from $\mathscr{D}(P_t) \subset \Sigma$ into Φ. Because Σ

is a subset of Φ, which is, in turn, an open subset of $U \oplus \mathscr{W}$, the trajectory of the solution $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ of (11.2) which obeys (11.6) (with $\sigma \in \Sigma$) forms the center line of a compact tube $\mathfrak{T}$ which has positive radius and lies in Φ at a positive distance from the complement of Φ in $U \oplus \mathscr{W}$. On $\mathfrak{T}$, the right-hand side of (11.2) has a single Lipschitz constant. One can use this observation to show that $\mathscr{D}(P_t)$ is an open subset of Σ and that the mapping $\sigma \mapsto P_t \sigma \in \Phi$ is continuous.

Now, let σ be in $\mathscr{D}(P_t)$ for a fixed process P_t in Π. If σ is in Φ^{σ°, *i.e.* is of the form $P_\tau \sigma^\circ$, then $P_t \sigma$ has the form $P_{t+\tau} \sigma^\circ$ for an appropriate process $P_{t+\tau}$, and hence $P_t \sigma$ is in Φ^{σ°, a subset of Σ. By item (4) above, each element of Σ is in Φ^{σ° or is, at worst, a limit point of Φ^{σ°. If σ is in Φ^{σ°, then as we have just seen, $P_t \sigma$ is in Σ. If σ is a limit of a sequence of points σ_n in Φ^{σ°, then all of these points σ_n with n greater than some integer are in the open set $\mathscr{D}(P_t)$, and for these points $P_t \sigma_n$ is, as we have observed, in Σ. By the continuity of P_t on $\mathscr{D}(P_t)$, $P_t \sigma_n$ converges to $P_t \sigma$. Thus, $P_t \sigma$ is in the closure of Σ, and hence in Σ. In other words, $\mathscr{R}(P_t)$, the range of the transformation induced by P_t, is a subset of Σ, as is required by Definition 2.1. The "successive application function" $(P_{t_2}, P_{t_1}) \mapsto P_{t_2} P_{t_1} = P_{t_2+t_1}$ is defined as explained in item (1) of Definition 9.1, and hence axiom **II** of Definition 2.1 holds here. To see that the remaining requirement placed on (Σ, Π) by Definition 2.1, *i.e.* axiom **I**, also holds here, let us note that if $\mathscr{O}$ is an arbitrary open subset of Σ, if σ is an element of Σ, and if $\mathscr{O}^\circ$ is a neighborhood of the distinguished state σ°, then by items (3)–(5) of Definition 11.1, there are processes P_t and P_τ in Π for which the state $\sigma' \stackrel{\text{def}}{=} P_t \sigma$ is in $\mathscr{O}^\circ$, and the state $\sigma'' = P_\tau \sigma^\circ$ is in $\mathscr{O}$; if we fix $\mathscr{O}$, σ, and P_τ, because of the continuity of P_τ we can choose $\mathscr{O}^\circ$ in such a way that P_τ maps $\mathscr{O}^\circ$ into $\mathscr{O}$, and then Π will contain a process $P_\tau P_t$ such that $P_\tau P_t$ is in $\mathscr{O}$; thus $\Pi \sigma \stackrel{\text{def}}{=} \{P\sigma \mid P \in \Pi, \sigma \in \mathscr{D}(P)\}$ is dense in Σ for each arbitrary σ in Σ; that is, axiom **I** holds here.

It is easily verified that a set Π of processes P_t, a set Σ of states σ, and a triple $(\theta, \boldsymbol{T}, \boldsymbol{q})$ of state functions (restricted from Φ to Σ) that obey items (1)–(5) of Definition (11.1) also obey items (1)–(3) of Definition 9.1.

In summary, we have

Remark 11.1. Definition 11.1 is consistent with Definitions 9.1 and 2.1.

A particular element with internal variables is specified when there are given, in accord with Definition 11.1, the following mathematical objects: the vector space $\mathscr{W}$, the set $\Phi \subset U \oplus \mathscr{W}$, the functions θ, $\boldsymbol{T}$, $\boldsymbol{q}$, and $\boldsymbol{f}$ on Φ, and the distinguished state σ°. Nevertheless, to emphasize that an element with internal variables is a simple material element, we usually refer to it as the "element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with internal variables".

The mapping $\sigma \mapsto {}_0\sigma$ introduced in item (2) of Definition 9.1 is the function $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}) \mapsto (\boldsymbol{F}, e, \boldsymbol{\beta})$, *i.e.* the perpendicular projection of $\Sigma \subset U \oplus \mathscr{W}$ into U. If σ is in $\Sigma_{\boldsymbol{\alpha}'}$, then, by (11.1), σ has the form $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}')$, which can be written $({}_0\sigma, \boldsymbol{\alpha}')$.

For simple material elements in general, the Second Law of Thermodynamics is taken to be the assertion that the action $\mathfrak{s}$ of (9.7) has the Clausius property at a state in Σ; for elements with internal variables we take this state to be the distinguished state σ° mentioned in items (3) and (4) of Definition 11.1. Thus, the Second Law here reads:

Second Law. The action $\mathscr{s}$ of (9.7) has the Clausius property at the state σ°.

It is worth noting that the set Φ^{σ° introduced in item (4) of Definition 11.1 equals the set $\Pi\sigma^\circ$ of states accessible from the distinguished state σ°:

$$\Phi^{\sigma^\circ} = \Pi\sigma^\circ \overset{\text{def}}{=} \{\sigma \mid \sigma = P\sigma^\circ, \ P\in\Pi\}. \tag{11.8}$$

Remark 3.1 here yields

Remark 11.2. If $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obeys the Second Law of Thermodynamics, then $\mathscr{s}$ has the Clausius property at each state σ in $\Pi\sigma^\circ = \Phi^{\sigma^\circ}$.

The following corollary to Theorem 9.1 shows that the Second Law restricts the choice of the function $\boldsymbol{q}$.

Theorem 11.1.[#] Let $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ be a state of an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with internal variables which obeys the Second Law of Thermodynamics. If $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is in $\Pi\sigma^\circ$ and is such that

$$\boldsymbol{f}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}) = \boldsymbol{0}, \tag{11.9}$$

then

$$\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}) \cdot \boldsymbol{\beta} \leqq 0. \tag{11.10}$$

Proof. If $\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is in $\Pi\sigma^\circ$, then by Remark 11.2, $\mathscr{s}$ has the Clausius property at σ. If this state σ also obeys (11.9), then each U-valued function $P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma})$ on $[0, t)$, with $t > 0$ and $\boldsymbol{L} \equiv \boldsymbol{0}$, $h \equiv 0$, $\boldsymbol{\gamma} \equiv 0$, is clearly in Ψ_σ, *i.e.* is a process, and for such a process P_t, we have $P_\tau\sigma = \sigma$ for $\tau \in (0, t)$. Thus, the state σ obeys the hypothesis of Theorem 9.1; q.e.d.

Although in this section we make no use of the general theory of stagnant states developed in Section 6, we should like to point out that a state obeys the condition (11.9) in the above theorem if and only if it is a stagnant state for a stagnating family $\{P_t \mid t \in \mathbb{R}^{++}\}$ of processes for which the functions $\boldsymbol{L}$, h, and $\boldsymbol{\gamma}$ vanish.

Theorem 11.2. Let σ be a state of an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with internal variables. In order that

$$\boldsymbol{f}(\sigma) = \boldsymbol{0}, \tag{11.11}$$

it is necessary and sufficient that σ be a stagnant state for the stagnating family $\{P_t \mid t \in \mathbb{R}^{++}\}$ of processes such that

$$P_t = (\boldsymbol{L}, h, \boldsymbol{\gamma}) \quad \text{with} \quad \boldsymbol{L} \equiv \boldsymbol{0}, \quad h \equiv 0, \quad \boldsymbol{\gamma} \equiv \boldsymbol{0} \quad \text{on } [0, t), \tag{11.12}$$

for every t in $\mathbb{R}^{++}$.

[#] *Cf.* COLEMAN & GURTIN [1967, 1], p. 601, eq. (5.20).

Proof. Suppose (11.11) holds. Consider the one-parameter family $\{P_t \mid t \in \mathbb{R}^{++}\}$ of processes obeying (11.12). This family clearly obeys item (2) of Definition 6.1. It follows from (11.11) that the constant function $\tau \mapsto \sigma$ is a solution of (11.2) for processes in this family, and hence the family also obeys items (1) and (3) of Definition 6.1, with the present given σ a state in $\mathscr{D}(\mathsf{P}) = \bigcap_{t>0} \mathscr{D}(P_t)$ such that $P_t \sigma = \sigma$, which implies that $\lim_{t\to\infty} P_t \sigma$ exists and is in $\mathscr{D}(\mathsf{P})$. Thus if (11.11) holds, the family $\{P_t \mid t \in \mathbb{R}^{++}\}$ obeying (11.12) is a stagnating family, and σ is one of its stagnant states.

Suppose now that σ is a stagnant state of the family $\{P_t \mid t \in \mathbb{R}^{++}\}$ of processes obeying (11.12). Then, by Theorem 6.2, $P_t \sigma = \sigma$ for each process obeying (11.12); *i.e.* when $\boldsymbol{L}$, h, and γ are identically zero, the equation (11.2) must have the constant function $\tau \mapsto \sigma$ as a solution, but this is possible only if $\boldsymbol{f}(\sigma) = \boldsymbol{0}$; q.e.d.

Since Theorem 9.3 holds for simple material elements in general, it holds for elements with internal variables in particular, and need not be repeated here. For elements with internal variables, one can go beyond Theorem 9.3 and use the Second Law to construct an entropy function which obeys analogues of the relations (10.14) and (10.15), which we have derived for elastic elements. Before doing this, however, we make a non-trivial remark and prove a needed lemma.

Remark 11.3. For each state $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')$ of an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with internal variables, the set

$$\Sigma_{\alpha'} = \{(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}') \mid (\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}') \in \Phi\} \tag{11.13}$$

is a subset of Σ. We call $\Sigma_{\alpha'}$ the *instantaneous space corresponding to* $\boldsymbol{\alpha}'$.

Proof. Let $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')$ be in Σ, let σ'' be in $\Sigma_{\alpha'}$, and let $\mathcal{O}''$ be an arbitrary open subset of Φ containing σ''. To prove that $\Sigma_{\alpha'}$ is a subset of Σ, we must show that σ'' is a limit point of elements of the set Φ^{σ° defined in item (4) of Definition 11.1, but to show this it suffices to prove that Σ has an open subset $\mathcal{O}'$ which contains σ' and is mapped into $\mathcal{O}''$ by some process P_t in Π. (For such a set $\mathcal{O}'$ certainly contains an element $\bar{\sigma}$ of Φ^{σ°, and as we observed in the second paragraph following Definition 11.1, if $\bar{\sigma}$ is in Φ^{σ°, then $P_t \bar{\sigma}$ is in Φ^{σ° as well as in $\mathcal{O}''$.) To this end, we let $\mathscr{c}$ be an oriented polygonal curve which lies in $\Sigma_{\alpha'}$ and joins σ' to σ''; we know that such a curve exists, because we assumed in item (1) of Definition 11.1 that Φ is open and $\Sigma_{\alpha'}$ is connected. For each $\delta > 0$, the union of all closed spheres which have radius δ and are centered at points on $\mathscr{c}$ forms a tube $\mathfrak{T}$ with $\mathscr{c}$ as its "center line". We choose $\delta > 0$ small enough that: (a) the closed sphere of radius δ about σ'' is contained in $\mathcal{O}''$, and (b) $\mathfrak{T}$ is contained in Φ. Clearly, $\mathfrak{T}$ is a compact set, and hence

$$M \stackrel{\text{def}}{=} \sup\{|\boldsymbol{f}(\sigma)| \mid \sigma \in \mathfrak{T}\} \tag{11.14}$$

is finite. Let $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\alpha}')$ be a piecewise linear parameterization of $\mathscr{c}$ with domain $[0, t]$, where t is chosen so that

$$tM < \delta, \tag{11.15}$$

and let $\tau \mapsto \boldsymbol{\alpha}(\tau)$ be the corresponding solution of the differential equation

$$\dot{\boldsymbol{\alpha}}(\tau)=\boldsymbol{f}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\alpha}(\tau)) \tag{11.16}$$

with the initial condition

$$\boldsymbol{\alpha}(0)=\boldsymbol{\alpha}'. \tag{11.17}$$

It is easily shown that (11.14) and (11.15) yield, for each τ in $[0, t]$,

$$|\boldsymbol{\alpha}(\tau)-\boldsymbol{\alpha}'|<\delta, \tag{11.18}$$

and hence

$$|(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\alpha}(\tau))-(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\alpha}')|<\delta. \tag{11.19}$$

(The estimate (11.18) assures the existence and uniqueness of $\tau \mapsto \boldsymbol{\alpha}(\tau)$ as a solution of (11.16) and (11.17) on $[0, t]$.) Let $P_t=(\boldsymbol{L}, h, \gamma)$ be defined on $[0, t)$ by the relations

$$\begin{aligned} &\boldsymbol{L}=\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \\ &h=\dot{e}-\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}) \cdot (\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}), \\ &\gamma=\dot{\boldsymbol{\beta}}, \end{aligned} \tag{11.20}$$

where the function $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\alpha}(\tau))$ is constructed as explained above. It is clear that P_t is piecewise continuous on $[0, t)$, and, by (11.20) and (11.16), $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\alpha}(\tau))$ is a solution of (11.2) with $(\boldsymbol{L}, h, \gamma)=P_t$; this solution obeys the initial condition

$$(\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0), \boldsymbol{\alpha}(0))=(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')=\sigma', \tag{11.21}$$

because here $\boldsymbol{F}$, e, and $\boldsymbol{\beta}$ come from a parameterization of $\mathscr{c}$, and $\boldsymbol{\alpha}$ obeys (11.17); furthermore, it follows from (11.19) that the trajectory of this solution lies in $\mathfrak{T}$ and hence in Φ. Thus, in view of items (3) and (5) of Definition 11.1, $P_t \in \Psi_{\sigma'} \subset \Pi$, *i.e.* P_t is a process. Furthermore, since the solution $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{\alpha}(\tau))$ of (11.2) corresponding to the process P_t and the initial condition (11.21) obeys, by (11.19),

$$|(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t), \boldsymbol{\alpha}(t))-(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t), \boldsymbol{\alpha}')|<\delta,$$

where

$$(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t), \boldsymbol{\alpha}')=\sigma'',$$

we may conclude that $P_t\sigma'$ is within distance δ of σ'' and hence is in $\mathcal{O}''$.

We have just constructed a process P_t which takes σ' into $\mathcal{O}''$. Because P_t is a process of a system [see Remark 11.1 and Definition 2.1], the transformation induced by P_t is a continuous mapping of an open set $\mathscr{D}(P_t) \subset \Sigma$ into Σ. Therefore, σ' has a neighborhood $\mathcal{O}'$ in Σ such that $P_t\mathcal{O}' \subset \mathcal{O}''$; q.e.d.

Lemma 11.1. Let $\sigma'=(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')=({}_0\sigma', \boldsymbol{\alpha}')$ be a state of an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with internal variables, let $\Sigma_{\boldsymbol{\alpha}'}$ be the connected subset of Σ defined by (11.13), let $\mathscr{c}$ be an oriented polygonal curve which lies in $\Sigma_{\boldsymbol{\alpha}'}$ and has initial point σ', and let $I(\mathscr{c})$ be the following line integral:

$$I(\mathscr{c})=\int_{\mathscr{c}} \left[\frac{-\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}')\,\boldsymbol{F}^{T^{-1}}}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}')} \cdot d\boldsymbol{F}+\frac{1}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}')}\, de+\boldsymbol{0} \cdot d\boldsymbol{\beta}\right]. \tag{11.22}$$

For each $\varepsilon>0$, there is a process P_t in Π, with σ' in $\mathscr{D}(P_t)$ and

$$|\mathfrak{s}(P_t, \sigma')-I(c)|<\varepsilon. \tag{11.23}$$

Proof. For the given oriented curve c, we may construct a closed tube $\mathfrak{T}$ with center line c as described in the previous proof; the radius $\delta>0$ of $\mathfrak{T}$ (*i.e.* the radius of the closed spheres comprising $\mathfrak{T}$) may be chosen again small enough so that $\mathfrak{T}$ is contained in Φ. Now, let $\varepsilon>0$ be given, let l be the length of c, and put

$$M=\sup\{|f(\sigma)|\,|\,\sigma\in\mathfrak{T}\},$$

and

$$Q=\sup\{|q(F, e, \beta, \alpha)\cdot\beta|\,|\,(F, e, \beta, \alpha)\in\mathfrak{T}\}. \tag{11.24}$$

Since $\mathfrak{T}$ is compact, and f and q are continuous on Φ, the numbers M and Q are finite. Furthermore, the continuity on Φ of T and the positive function θ imply the existence of a number η such that $0<\eta\leqq\delta$ and

$$\left|\frac{T(\sigma_i)\,F_i^{T^{-1}}}{\theta(\sigma_i)}-\frac{T(\sigma_{ii})\,F_{ii}^{T^{-1}}}{\theta(\sigma_{ii})}\right|^2+\left|\frac{1}{\theta(\sigma_i)}-\frac{1}{\theta(\sigma_{ii})}\right|^2<\frac{\varepsilon^2}{4l^2} \tag{11.25}$$

whenever $\sigma_i=(F_i, e_i, \beta_i, \alpha_i)$ and $\sigma_{ii}=(F_{ii}, e_{ii}, \beta_{ii}, \alpha_{ii})$ are in $\mathfrak{T}$ and obey $|\sigma_i-\sigma_{ii}|<\eta$. Let $\mathfrak{T}'$ now be the tube, with centerline c, comprised of closed spheres of radius η but otherwise constructed as $\mathfrak{T}$, and let $\tau\mapsto(F(\tau), e(\tau), \beta(\tau), \alpha')$ be a parameterization of c with domain $[0, t]$, where t is chosen so that

$$tQ<\frac{\varepsilon}{2} \quad\text{and}\quad tM<\eta. \tag{11.26}$$

An argument given in the proof of Remark 11.3 here tells us that this parameterization of c determines a process P_t in Π such that σ' is in $\mathscr{D}(P_t)$ and, for each τ in $[0, t)$, if P_τ is the restriction of P_t to $[0, \tau)$, then $\sigma_\tau=P_\tau\sigma'$ is in $\mathfrak{T}'$ and

$$|\sigma_\tau-({}_0\sigma_\tau, \alpha')|<\eta. \tag{11.27}$$

[See (11.19), and note that $\sigma_\tau=({}_0\sigma_\tau, \alpha(\tau))$ and ${}_0\sigma_\tau=(F(\tau), e(\tau), \beta(\tau))$.] It follows from (9.7), (11.2), and (11.22) that

$$|\mathfrak{s}(P_t, \sigma')-I(c)|=\left|\int_0^t q(\sigma_\tau)\cdot\beta(\tau)\,d\tau-\int_0^t\frac{T(\sigma_\tau)\,F(\tau)^{T^{-1}}\cdot\dot{F}(\tau)-\dot{e}(\tau)}{\theta(\sigma_\tau)}\,d\tau\right.$$
$$\left.+\int_0^t\frac{T({}_0\sigma_\tau, \alpha')\,F(\tau)^{T^{-1}}\cdot\dot{F}(\tau)-\dot{e}(\tau)}{\theta({}_0\sigma_\tau, \alpha')}\,d\tau\right|.$$

Hence, by (11.24) and the triangle inequality, we have

$$\begin{aligned}|\mathfrak{s}(P_t, \sigma')-I(c)|\leqq Qt+\int_0^t\Bigg|&\left[\frac{T({}_0\sigma_\tau, \alpha')\,F(\tau)^{T^{-1}}}{\theta({}_0\sigma_\tau, \alpha')}-\frac{T(\sigma_\tau)\,F(\tau)^{T^{-1}}}{\theta(\sigma_\tau)}\right]\cdot\dot{F}(\tau)\\&+\left[\frac{1}{\theta({}_0\sigma_\tau, \alpha')}-\frac{1}{\theta(\sigma_\tau)}\right]\dot{e}(\tau)\Bigg|\,d\tau.\end{aligned} \tag{11.28}$$

Since (11.27) holds for all τ in $[0, t]$, we may employ (11.25) and Schwarz's inequality to obtain from (11.28),

$$|\mathscr{s}(P_t, \sigma') - I(c)| \leqq Q t + \int_0^t \left[\sqrt{\frac{\varepsilon^2}{4 l^2}} \times \sqrt{|\dot{F}(\tau)|^2 + |\dot{e}(\tau)|^2} \right] d\tau.$$

Because l is the length of c, *i.e.*

$$l = \int_0^t \sqrt{|\dot{F}(\tau)|^2 + |\dot{e}(\tau)|^2 + |\dot{\beta}(\tau)|^2}\, d\tau,$$

and t obeys $(11.26)_1$, the above inequality yields

$$|\mathscr{s}(P_t, \sigma') - I(c)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2l} \times l = \varepsilon;$$

q.e.d.

Theorem 11.3.# If an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with internal variables obeys the Second Law of Thermodynamics, then it has an upper semicontinuous entropy function S such that:

(1) the domain Σ° of S contains σ°, is closed under $\mathscr{s}$-approach, and is such that if $(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')$ is in Σ°, then $\Sigma_{\boldsymbol{\alpha}'}$, the instantaneous space corresponding to $\boldsymbol{\alpha}'$, is a subset of Σ°;

(2) at each point $\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ in Σ°, the partial derivatives $\partial_{\boldsymbol{F}} S(\sigma)$, $\partial_e S(\sigma)$, $\partial_{\boldsymbol{\beta}} S(\sigma)$ exist and obey the relations

$$\boldsymbol{T}(\sigma) = -\theta(\sigma)\, [\partial_{\boldsymbol{F}} S(\sigma)]\, \boldsymbol{F}^T, \tag{11.29}$$

$$\theta(\sigma) = [\partial_e S(\sigma)]^{-1}, \tag{11.30}$$

$$\boldsymbol{0} = \partial_{\boldsymbol{\beta}} S(\sigma). \tag{11.31}$$

The relations (11.29) and (11.30) imply that S determines the functions $\boldsymbol{T}$ and θ not only on Σ° but also on all of Σ. The relation (11.31) implies that S is independent of $\boldsymbol{\beta}$ on Σ°, *i.e.*

$$S(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}) = S(\boldsymbol{F}, e, \boldsymbol{0}, \boldsymbol{\alpha}) \tag{11.32}$$

for $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ in Σ°. Furthermore, $\boldsymbol{T}$ and θ are independent of $\boldsymbol{\beta}$ on Σ, *i.e.*

$$\boldsymbol{T}(\boldsymbol{F}, \theta, \boldsymbol{\beta}, \boldsymbol{\alpha}) = \boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{0}, \boldsymbol{\alpha}), \tag{11.33}$$

$$\theta(\boldsymbol{F}, \theta, \boldsymbol{\beta}, \boldsymbol{\alpha}) = \theta(\boldsymbol{F}, e, \boldsymbol{0}, \boldsymbol{\alpha}), \tag{11.34}$$

whenever $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is in Σ.

The set Σ° referred to here is that defined by the equations (5.6)–(5.8), and this set has the properties listed in Remark 3.2 (with $\mathscr{s}$ playing the role of $\mathscr{a}$); in particular, Σ° is dense in Σ, is closed under $\mathscr{s}$-approach, and contains each state $\mathscr{s}$-approachable from σ°. Since, by Remark 9.3, σ° is $\mathscr{s}$-approachable from itself, σ° is in Σ°. The statement that S is an *entropy function* for the element means that S is an upper potential for $\mathscr{s}$ and implies that S obeys the relation (9.21) for each pair of states σ, σ' in Σ° and each process P_t in Π with $\sigma' = P_t \sigma$.

For a derivation of (11.29)–(11.31) starting from the assumption that there exists a differentiable entropy function S on Σ, see Coleman & Gurtin [1967, 1], § 5.

To prove Theorem 11.3 we shall first show that when a state $\sigma' = ({}_0\sigma', \alpha')$ is in Σ°, $\Sigma_{\alpha'}$ is a subset of Σ°. We shall then define S on Σ° through the formula (5.5) and observe that, by the proof of Theorem 3.3, S is a semicontinuous upper potential for $\mathscr{s}$. We shall then prove that, for each state $\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ in Σ°, the line integral $I(\mathscr{c})$ of (11.22) vanishes on every closed polygonal curve $\mathscr{c}$ in Σ_{α}, and hence the restriction to Σ_{α} of the vector field $(-\boldsymbol{T}\boldsymbol{F}^{T^{-1}}/\theta, 1/\theta, \boldsymbol{0})$ has, in the classical sense, a potential $V_{\alpha}: \Sigma_{\alpha} \to \mathbb{R}$, which depends of course on α as a parameter. We shall prove, further, that the difference, $\bar{S}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$, between $S(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ and $V_{\alpha}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is independent of $\boldsymbol{F}$, e and $\boldsymbol{\beta}$ on Σ°, and this will yield (11.29)–(11.31) forthwith.

Proof of Theorem 11.3. Let Σ° be defined as in $(5.8)_2$:

$$\Sigma^\circ \stackrel{\text{def}}{=} \{\sigma \mid \sigma \in \Sigma,\ m(\sigma^\circ, \sigma) > -\infty\}; \tag{11.35}$$

here $m(\sigma^\circ, \sigma)$ is given by (5.6), *i.e.*

$$m(\sigma^\circ, \sigma) \stackrel{\text{def}}{=} \inf_{\mathscr{O} \in \mathfrak{S}(\sigma^\circ, \sigma)} \sup \mathscr{s}\{\sigma^\circ \to \mathscr{O}\} \tag{11.36}$$

with $\mathfrak{S}(\sigma^\circ, \sigma)$ as in (5.7). We have already observed that Σ° contains σ° and is closed under $\mathscr{s}$-approach. Now let $\sigma' = ({}_0\sigma', \alpha')$ be an arbitrary element of Σ°, and let σ'' be in $\Sigma_{\alpha'}$, *i.e.* have the form $\sigma'' = ({}_0\sigma'', \alpha')$. We want to show that σ'' is also in Σ°. To this end we let $\mathscr{c}$ be an oriented polygonal curve which lies in $\Sigma_{\alpha'}$ and joins σ' to σ''. (It follows from item (1) of Definition 11.1 that such curves exist.) By Lemma 11.1 and its proof, there is, for each positive integer n, a process $P^{(n)}$ in Π such that

$$|P^{(n)}\sigma' - \sigma''| < \frac{1}{n}$$

and

$$|\mathscr{s}(P^{(n)}, \sigma') - I(\mathscr{c})| < \frac{1}{n},$$

where I is the line integral defined in (11.22). The sequence $n \mapsto P^{(n)}$ clearly obeys the relations

$$\lim_{n \to \infty} P^{(n)}\sigma' = \sigma''$$

and

$$\lim_{n \to \infty} \mathscr{s}(P^{(n)}, \sigma') = I(\mathscr{c}),$$

and, by Theorem 3.6 or item (2) of Remark 3.2, the state σ'' is in Σ°, for σ'' is $\mathscr{s}$-approachable from σ'. Thus, the presence of $(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}')$ in Σ° implies that $\Sigma_{\alpha'}$ is a subset of Σ°. Let $S: \Sigma^\circ \to \mathbb{R}$ be defined as in (5.5), *i.e.*

$$S(\sigma) = m(\sigma^\circ, \sigma), \qquad \sigma \in \Sigma^\circ, \tag{11.37}$$

with m as in (11.36). As we showed when proving Theorem 3.3, S is an upper potential for $\mathscr{s}$ and is upper semicontinuous. Thus, S is an entropy function for $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, and its domain has the properties mentioned in item (1) of the present theorem.

Let $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{\alpha}') = ({}_0\sigma', \alpha')$ be again an arbitrary element of Σ°. By item (3) of Remark 3.2, the action $\mathscr{s}$ has the Clausius property at σ'. Therefore, for each

$\varepsilon>0$, there is a neighborhood $\mathcal{O}$ of σ' in Σ such that

$$\mathfrak{s}(P,\sigma')<\frac{\varepsilon}{2},$$

for each P in Π with $P\sigma'$ in $\mathcal{O}$. Let c^+ be a given closed oriented polygonal curve containing σ' and lying in $\Sigma_{\alpha'}$. By Lemma 11.1 and its proof, there is a process P in Π with σ' in $\mathcal{D}(P)$, $P\sigma'$ in $\mathcal{O}$, and

$$|\mathfrak{s}(P,\sigma')-I(c^+)|<\frac{\varepsilon}{2};$$

for this process P we have

$$I(c^+)<\mathfrak{s}(P,\sigma')+\frac{\varepsilon}{2}$$

and hence

$$I(c^+)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

As this is true for each $\varepsilon>0$, there holds $I(c^+)\leqq 0$. If we repeat this argument employing c^-, the closed curve which differs from c^+ only in orientation, we obtain $I(c^-)\leqq 0$, and since $I(c^-)=-I(c^+)$, we may conclude that $I(c^+)=0$. Thus, the line integral I vanishes on every closed oriented polygonal curve lying in $\Sigma_{\alpha'}$ and passing through σ'; because $\Sigma_{\alpha'}$ is polygonally connected, this implies, by a standard argument, that I vanishes on every closed oriented polygonal curve in $\Sigma_{\alpha'}$, including those which do not contain σ'. Hence, by (11.22) and the fundamental theorem on the existence of potentials for vector fields, the function

$$\sigma=(\boldsymbol{F},e,\boldsymbol{\beta},\boldsymbol{\alpha}')\mapsto\left(\frac{-\boldsymbol{T}(\sigma)}{\theta(\sigma)}\boldsymbol{F}^{T^{-1}},\frac{1}{\theta(\sigma)},\boldsymbol{0}\right)$$

is a U-valued function on $\Sigma_{\alpha'}$ which has a potential in the sense that there exists a differentiable function $V_{\alpha'}\colon\Sigma_{\alpha'}\to\mathbb{R}$ obeying

$$\begin{aligned}\partial_{\boldsymbol{F}}V_{\alpha'}(\sigma)&=-\boldsymbol{T}(\sigma)\boldsymbol{F}^{T^{-1}}/\theta(\sigma),\\ \partial_e V_{\alpha'}(\sigma)&=1/\theta(\sigma),\\ \partial_{\boldsymbol{\beta}}V_{\alpha'}(\sigma)&=\boldsymbol{0}.\end{aligned}\tag{11.38}$$

Let us define $\bar{S}\colon\Sigma^\circ\to\mathbb{R}$ by

$$\bar{S}(\sigma)=S(\sigma)-V_{\alpha}(\sigma),\tag{11.39}$$

for each $\sigma=(\boldsymbol{F},e,\boldsymbol{\beta},\boldsymbol{\alpha})$ in Σ°. Let $\boldsymbol{\alpha}'$ be such that there is a triplet $(\boldsymbol{F}',e',\boldsymbol{\beta}')$ for which the state $\sigma'=(\boldsymbol{F}',e',\boldsymbol{\beta}',\boldsymbol{\alpha}')=({}_0\sigma',\boldsymbol{\alpha}')$ is in Σ°. Then, as we have shown, $\Sigma_{\alpha'}\subset\Sigma^\circ$. Letting $\sigma''=(\boldsymbol{F}'',e'',\boldsymbol{\beta}'',\boldsymbol{\alpha}')=({}_0\sigma'',\boldsymbol{\alpha}')$ be an arbitrary element of $\Sigma_{\alpha'}$, we now show that $\bar{S}(\sigma'')=\bar{S}(\sigma')$. To this end, let c^+ be again an oriented polygonal curve in $\Sigma_{\alpha'}$ joining σ' to σ'', and let $\varepsilon>0$ be given. Because S is an upper potential for $\mathfrak{s}$ and has both σ' and σ'' in its domain, there is a neighborhood $\mathcal{O}''$ of σ'' such that

$$\mathfrak{s}(P,\sigma')<S(\sigma'')-S(\sigma')+\frac{\varepsilon}{2}\tag{11.40}$$

for each P in Π with $P\sigma'$ in $\mathcal{O}''$. (See Definition 3.2.) By Lemma 11.1 and its proof, there is a P in Π for which $P\sigma'$ is in $\mathcal{O}''$ and

$$|\mathfrak{s}(P,\sigma')-I(c)|<\frac{\varepsilon}{2},$$

with I the line integral (11.22). By the construction of $V_{\alpha'}$,

$$I(c)=V_{\alpha'}({}_0\sigma'',\alpha')-V_{\alpha'}({}_0\sigma',\alpha'),$$

and hence there is a process P for which (11.40) holds and for which there also holds

$$-\frac{\varepsilon}{2}<\mathfrak{s}(P,\sigma')-\big(V_{\alpha'}({}_0\sigma'',\alpha')-V_{\alpha'}({}_0\sigma',\alpha')\big). \tag{11.41}$$

Clearly, the relations (11.39)–(11.41) yield

$$-\varepsilon<S(\sigma'')-S(\sigma')-\big(V_{\alpha'}({}_0\sigma'',\alpha')-V_{\alpha'}({}_0\sigma',\alpha')\big)=\bar{S}(\sigma'')-\bar{S}(\sigma'),$$

and since $\varepsilon>0$ is arbitrary, we have

$$0\leqq\bar{S}(\sigma'')-\bar{S}(\sigma').$$

If, in the argument just given, we interchange roles of σ'' and σ' (and hence change c^+ to c^-), we obtain

$$0\geqq\bar{S}(\sigma'')-\bar{S}(\sigma'),$$

and thus $\bar{S}(\sigma'')=\bar{S}(\sigma')$ for each pair of states σ', σ'' in $\Sigma_{\alpha'}$; *i.e.* if α in $\mathscr{W}$ is such that $\Sigma_\alpha\cap\Sigma^\circ$ is not empty, then Σ_α is a subset of Σ° and *the restriction of* $\bar{S}$ *to* Σ_α *is a constant*. Hence the derivatives of $\bar{S}$ with respect to $\boldsymbol{F}$, e, and $\boldsymbol{\beta}$ exist at each point in Σ° and vanish:

$$\partial_{\boldsymbol{F}}\bar{S}\equiv\mathbf{0},\qquad \partial_e\bar{S}\equiv 0,\qquad \partial_{\boldsymbol{\beta}}\bar{S}\equiv\mathbf{0}\qquad \text{on } \Sigma^\circ. \tag{11.42}$$

Since V_α is differentiable on Σ_α, and since, by (11.39),

$$S(\sigma)=\bar{S}(\sigma)+V_\alpha(\sigma),\qquad \text{for } \sigma=(\boldsymbol{F},e,\boldsymbol{\beta},\boldsymbol{\alpha})\in\Sigma^\circ, \tag{11.43}$$

the equations (11.42) and (11.38) tell us that $\partial_{\boldsymbol{F}} S$, $\partial_e S$, and $\partial_{\boldsymbol{\beta}} S$ exist at each point in Σ° and obey the equations (11.29)–(11.31).

Knowledge of S and the relations (11.29) and (11.30) obviously yield knowledge of $\boldsymbol{T}$ and θ on Σ°. But Σ° is dense in Σ, and it has been assumed that $\boldsymbol{T}$ and θ are continuous on Σ (in fact, on Φ). Hence the restrictions of $\boldsymbol{T}$ and θ to Σ° have unique continuous extensions to Σ, and the relations (11.29) and (11.30) do indeed imply that S determines the functions $\boldsymbol{T}$ and θ on all of Σ. By item (1) of Definition 11.1, if $(\boldsymbol{F}',e',\boldsymbol{\beta}',\boldsymbol{\alpha}')$ is in $\Sigma_{\alpha'}$, then so also is $(\boldsymbol{F}',e',\boldsymbol{\beta},\boldsymbol{\alpha}')$ for all $\boldsymbol{\beta}$ in $\mathscr{V}$; hence if $(\boldsymbol{F}',e',\boldsymbol{\beta}',\boldsymbol{\alpha}')$ is in Σ°, then $(\boldsymbol{F}',e',\boldsymbol{\beta},\boldsymbol{\alpha}')$ is in Σ° for all $\boldsymbol{\beta}$, and (11.31) implies that (11.32) holds throughout Σ°. It is clear that (11.32), (11.29), and (11.30) imply that $\boldsymbol{T}$ and θ are independent of $\boldsymbol{\beta}$ on Σ°. To show that $\boldsymbol{T}$ and θ are independent of $\boldsymbol{\beta}$ on Σ, we let $\sigma'=(\boldsymbol{F}',e',\boldsymbol{\beta}',\boldsymbol{\alpha}')$ and $\sigma''=(\boldsymbol{F}',e',\boldsymbol{\beta}'',\boldsymbol{\alpha}')$ be two states in Σ which differ only in values of $\boldsymbol{\beta}$. Because Σ° is dense in Σ, there exists a sequence $n\mapsto\sigma^n$ of states $(\boldsymbol{F}^n,e^n,\boldsymbol{\beta}^n,\boldsymbol{\alpha}^n)$ in Σ° converging to σ'. For each of the vectors $\boldsymbol{\alpha}^n$, the corresponding instantaneous space, Σ_{α^n}, is a subset of Σ°, and since the property of belonging to sets of the form Σ_α is independent of the value of $\boldsymbol{\beta}$, we can assert that *each term of the sequence* $n \mapsto \hat{\sigma}^n \stackrel{\text{def}}{=} (\boldsymbol{F}^n, e^n, \boldsymbol{\beta}'', \boldsymbol{\alpha}^n)$ *is in* Σ°, and, obviously, the convergence of σ^n to σ' implies the convergence of $\hat{\sigma}^n$ to σ''. Thus, for each n,

$$\boldsymbol{T}(\hat{\sigma}^n) = \boldsymbol{T}(\sigma^n), \qquad \theta(\hat{\sigma}^n) = \theta(\sigma^n),$$

and by the continuity of $\boldsymbol{T}$ and θ on Σ,

$$\lim_{n\to\infty} \boldsymbol{T}(\sigma^n) = \boldsymbol{T}(\sigma'), \qquad \lim_{n\to\infty} \boldsymbol{T}(\hat{\sigma}^n) = \boldsymbol{T}(\sigma''),$$

$$\lim_{n\to\infty} \theta(\sigma^n) = \theta(\sigma'), \qquad \lim_{n\to\infty} \theta(\hat{\sigma}^n) = \theta(\sigma'').$$

Hence

$$\boldsymbol{T}(\sigma') = \boldsymbol{T}(\sigma''), \qquad \theta(\sigma') = \theta(\sigma''),$$

and we conclude that $\boldsymbol{T}$ and θ are independent of $\boldsymbol{\beta}$ on Σ; q.e.d.

12. Elements with Fading Memory

Let us review some concepts from the elementary theory[#] of fading memory formulated by COLEMAN & NOLL [1960, 1] [1961, 1] [1962, 2], COLEMAN [1964, 1, 2], and COLEMAN & MIZEL [1966, 2].

A real-valued, Lebesgue-measurable function k on $\mathbb{R}^{++} = (0, \infty)$ is called an *influence function with the relaxation property* if[##]

(i) $k(s) > 0$ a.e. on $\mathbb{R}^{++}$, and

$$\int_0^\infty k(s)\,ds < \infty; \tag{12.1}$$

(ii) the functions $\overline{K}$ and $\underline{K}$, defined by

$$\overline{K}(\sigma) \stackrel{\text{def}}{=} \operatorname*{ess.\,sup}_{s\in(0,\infty)} \frac{k(s+\sigma)}{k(s)}, \tag{12.2}$$

$$\underline{K}(\sigma) \stackrel{\text{def}}{=} \operatorname*{ess.\,sup}_{s\in(0,\infty)} \frac{k(s)}{k(s+\sigma)}, \tag{12.3}$$

have finite values on $\mathbb{R}^+ = [0, \infty)$;

(iii)

$$\sup_{[0,\infty)} \overline{K}(\sigma) < \infty. \tag{12.4}$$

If a real-valued function k on $(0, \infty)$ has properties (i) and (ii), then for it there exist positive numbers a, b, and c such that

$$a e^{-bs} < k(s) < c, \qquad \text{a.e. on } \mathbb{R}^{++}, \tag{12.5}$$

[#] We refer to this theory as the "elementary theory" of fading memory to distinguish it from COLEMAN & MIZEL's more recent abstract theory [1967, 2] [1968, 2] which employs general Banach function spaces whose norms need not have the representations (12.9) and $(12.13)_2$.

[##] See COLEMAN & MIZEL [1966, 2], particularly pp. 101 and 111. Of course, "a.e." and "ess. sup" here mean *almost everywhere* and *essential supremum*, with the underlying measure taken to be the Lebesgue measure.

and, moreover, as $s \to \infty$,

$$s k(s) \to 0 \quad \text{essentially.}^{*} \tag{12.6}$$

If k has properties (i) and (ii), then for k to have property (iii) it is sufficient (but not necessary) that k be monotone decreasing.[**]

The interested reader will have no difficulty in verifying that the following are examples of influence functions with the relaxation property:

$$k(s) = e^{-\gamma s}, \quad k(s) = \frac{1 + \nu \sin^2 s}{1 + s^2}, \quad \gamma > 0, \quad \nu \geqq 0.$$

Let $\mathbf{0}$ be the zero element of $U = \mathrm{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}$. For each function $\boldsymbol{\phi}: \mathbb{R}^{++} \to U$ and each $\nu \geqq 0$, we write $\mathsf{T}^{(\nu)}\boldsymbol{\phi}$ and $\mathsf{T}_{(\nu)}\boldsymbol{\phi}$ for the functions $\mathbb{R}^{++} \to U$ defined by[***]

$$\mathsf{T}^{(\nu)} \boldsymbol{\phi}(s) = \begin{cases} \mathbf{0}, & \text{for } s \in (0, \nu], \\ \boldsymbol{\phi}(s - \nu), & \text{for } s \in (\nu, \infty), \end{cases} \tag{12.7}$$

$$\mathsf{T}_{(\nu)} \boldsymbol{\phi}(s) = \boldsymbol{\phi}(s + \nu), \quad \text{for } s \in (0, \infty). \tag{12.8}$$

Now let there be assigned a fixed influence function k with the relaxation property and a fixed number p in $[1, \infty)$. We write $\mathscr{U}_r = \mathscr{U}_r(p, k)$ for the set of measurable functions $\boldsymbol{\phi}: \mathbb{R}^{++} \to U$ for which

$$\|\boldsymbol{\phi}\|_r^p \stackrel{\text{def}}{=} \int_0^\infty |\boldsymbol{\phi}(s)|^p k(s)\, ds \tag{12.9}$$

is finite. The normed space $\mathfrak{U}_r = \mathfrak{U}_r(p, k)$, with norm $\|\cdot\|_r$, obtained by considering the same those elements $\boldsymbol{\phi}, \boldsymbol{\psi}$ of $\mathscr{U}_r$ for which $\|\boldsymbol{\phi} - \boldsymbol{\psi}\|_r = 0$, is a separable Banach space. COLEMAN & MIZEL[****] showed that, granted condition (i) on k, the condition (ii) is equivalent to the assertion that for each $\nu \geqq 0$, $\mathsf{T}^{(\nu)}\boldsymbol{\phi}$ and $\mathsf{T}_{(\nu)}\boldsymbol{\phi}$ are in $\mathscr{U}_r$ whenever $\boldsymbol{\phi}$ is. They also showed that (i) and (ii) imply that (1)[*****] for each $\nu \geqq 0$, $\mathsf{T}^{(\nu)}$ and $\mathsf{T}_{(\nu)}$ are *bounded* linear operators on $\mathfrak{U}_r$, and (2)[******] the dependence of these operators on ν is continuous in the sense that for each $\boldsymbol{\phi}$ in $\mathfrak{U}_r$ and each $\delta \geqq 0$,

$$\lim_{\nu \to 0} \|\mathsf{T}^{(\nu + \delta)} \boldsymbol{\phi} - \mathsf{T}^{(\delta)} \boldsymbol{\phi}\|_r = 0, \tag{12.10}$$

$$\lim_{\nu \to 0} \|\mathsf{T}_{(\nu + \delta)} \boldsymbol{\phi} - \mathsf{T}_{(\delta)} \boldsymbol{\phi}\|_r = 0. \tag{12.11}$$

They showed further that, granted (i) and (ii), the condition (iii) is equivalent to the assertion that for each $\boldsymbol{\phi}$ in $\mathscr{U}_r$,

$$\lim_{\nu \to \infty} \|\mathsf{T}^{(\nu)} \boldsymbol{\phi}\|_r = 0. \tag{12.12}$$ [*******]

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[*] COLEMAN & MIZEL [1966, 2], Thm. 5, p. 106.

[**] [1966, 2], Corollary to Thm. 6, p. 112.

[***] *Cf.* [1966, 2], § 3.

[****] [1966, 2], Thm. 3, p. 101.

[*****] [1966, 2], Lemma 1, p. 95.

[******] [1966, 2], Remarks 3.2 and 4.3, pp. 97 and 105.

[*******] [1966, 2], Remark 6.4, p. 111.

If $\boldsymbol{\Phi}$ is a function on $\mathbb{R}^+ = [0, \infty)$, then we write ${}_r\boldsymbol{\Phi}$ for the restriction of $\boldsymbol{\Phi}$ to $\mathbb{R}^{++}$. Let $\mathscr{U} = \mathscr{U}(p, k)$ be the set of all functions $\boldsymbol{\Phi}: \mathbb{R}^+ \to U$ which have ${}_r\boldsymbol{\Phi}$ in $\mathscr{U}_r$, and note that the function $\|\cdot\|: \mathscr{U} \to \mathbb{R}^+$ defined by

$$\|\boldsymbol{\Phi}\| = |\boldsymbol{\Phi}(0)| + \|{}_r\boldsymbol{\Phi}\|_r = |\boldsymbol{\Phi}(0)| + \left[\int_0^\infty |\boldsymbol{\Phi}(s)|^p k(s)\, ds\right]^{\frac{1}{p}}, \tag{12.13}$$

is a semi-norm on $\mathscr{U}$. The normed space $\mathfrak{U}$, with norm $\|\cdot\|$, obtained by calling the same those elements $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$ of $\mathscr{U}$ with $\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\| = 0$ is clearly a separable Banach space; indeed[#]

$$\mathfrak{U} = U \oplus \mathfrak{U}_r. \tag{12.14}$$

In this section we discuss systems for which each state σ is an element of $\mathfrak{U}$, *i.e.* is an equivalence class of functions $\boldsymbol{\Phi}$ in $\mathscr{U}$ which are equivalent under $\|\cdot\|$. By (12.13) all the functions $\boldsymbol{\Phi}$ in a class σ have the same value $\boldsymbol{\Phi}(0)$ at 0. We call the vector $\boldsymbol{\Phi}(0) \in U$ the *present value* or *instantaneous value* of σ and use for it the symbol ${}_0\sigma$. Thus, each σ in $\mathfrak{U} = U \oplus \mathfrak{U}_r$ has a present value,

$${}_0\sigma = (\boldsymbol{F}, e, \boldsymbol{b}) \in U = \mathrm{Lin}(\mathscr{V}) \oplus \quad \oplus \mathscr{V}, \tag{12.15}$$

which is well defined and equals the projection of σ into U. The *past history* of $\sigma \in \mathfrak{U}$ is the projection of σ into $\mathfrak{U}_r$ and is written ${}_r\sigma$. This past history is just the equivalence class, under $\|\cdot\|_r$, of the restrictions to $\mathbb{R}^{++}$ of the functions $\boldsymbol{\Phi}$ in the class σ and is well defined. Obviously σ in $\mathfrak{U}$ is determined when ${}_0\sigma$ and ${}_r\sigma$ are given, and we have

$$\|\sigma\| = |{}_0\sigma| + \|{}_r\sigma\|_r \geqq |{}_0\sigma|. \tag{12.16}$$

If $\boldsymbol{f}$ maps an interval $\mathscr{I}$ into U, and if $\mathscr{I}$ contains the interval $(0, \tau)$, then we write $\mathfrak{d}\boldsymbol{f}^\tau$ for the function on $(0, \infty)$ defined by

$$\mathfrak{d}\boldsymbol{f}^\tau(s) \overset{\text{def}}{=} \begin{cases} \boldsymbol{f}(\tau - s), & \text{for } s \in (0, \tau], \\ 0, & \text{for } s \in (\tau, \infty); \end{cases} \tag{12.17}$$

when this function is in $\mathscr{U}_r$, we use symbol $\mathfrak{d}\boldsymbol{f}^\tau$ to denote also the corresponding element of $\mathfrak{U}_r$.

If $\boldsymbol{f}$: $[0, A] \to U$ is piecewise differentiable on $[0, A]$, then the function (12.17) is obviously in $\mathscr{U}_r$ for each τ in $(0, A]$.

Definition 12.1. A simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ is called an **element with fading memory** if

(1) there is a p in $[1, \infty)$ and an influence function k with the relaxation property such that Σ equals the subset of $\mathfrak{U}(p, k)$ formed from those functions $\boldsymbol{\Phi}$ in $\mathscr{U}(p, k)$ whose ranges are contained in Σ_i, a given open connected subset of $\mathrm{Lin}(\mathscr{V})^{++} \times \mathbb{R} \times \mathscr{V}$; the "instantaneous space" Σ_i is assumed to have the property that for each triplet $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ in Σ_i, the set $\{(\boldsymbol{F}', e', \boldsymbol{\beta}) \,|\, \boldsymbol{\beta} \in \mathscr{V}\}$ is contained in Σ_i; the topology of Σ is taken to be the metric topology induced on Σ by the norm topology of the Banach space $\mathfrak{U}(p, k)$ [whose norm $\|\cdot\|$

[#] The norm $\|\cdot\|$ defined in (12.13) is equivalent to the norm shown in equation (2.4) of [1966, 2].

is defined in (12.13)]; thus, for an element with fading memory, the states, *i.e.* the elements of Σ, are equivalence classes of functions taking $\mathbb{R}^+$ into Σ_i; each state σ has a well defined present value ${}_0\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}) \in \Sigma_i$ and a well defined past history ${}_r\sigma \in \mathfrak{U}_r$; we write

$$\sigma = ({}_0\sigma, {}_r\sigma) = (\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma) \in U \oplus \mathfrak{U}_r; \tag{12.18}$$

(2) the functions $\boldsymbol{T}$, $\boldsymbol{q}$, and $1/\theta$ of (9.5) are locally Lipschitz continuous on $\Sigma \subset \mathfrak{U}$;

(3) let $P_t = (\boldsymbol{L}, h, \gamma)$ be a piecewise continuous function mapping an interval $[0, t)$, $t \in \mathbb{R}^{++}$, into U, and consider the differential system

$$\begin{aligned} \dot{\boldsymbol{F}}(\tau) &= \boldsymbol{L}(\tau)\,\boldsymbol{F}(\tau), \\ \dot{e}(\tau) &= \boldsymbol{T}(\sigma_\tau)\cdot \boldsymbol{L}(\tau) + h(\tau), \\ \dot{\boldsymbol{\beta}}(\tau) &= \gamma(\tau); \end{aligned} \tag{12.19}$$

if σ is in Σ and A is in $(0, t]$, a function $\boldsymbol{f} = (\boldsymbol{F}(\cdot), e(\cdot), \boldsymbol{\beta}(\cdot))$ mapping $[0, A]$ into Σ_i is called *a solution up to A of* (12.19) *with initial state* σ provided (i) $\boldsymbol{f}$ is piecewise differentiable on $[0, A]$, (ii)

$$\boldsymbol{f}(0) = (\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0)) = {}_0\sigma, \tag{12.20}$$

and (iii) at each τ in $[0, A]$ at which P_t is continuous, the equation (12.19) holds where, for $\tau > 0$, σ_τ is given by the formulae

$$\begin{aligned} {}_0\sigma_\tau &= \boldsymbol{f}(\tau) = (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)), \\ {}_r\sigma_\tau &= \mathrm{T}^{(\tau)}{}_r\sigma + \mathfrak{d}\boldsymbol{f}^\tau, \end{aligned} \tag{12.21}$$

and for $\tau = 0$, $\sigma_\tau = \sigma$; let $\mathscr{D}(P_t)$ be the set of all states which are initial states for solutions up to t of the system (12.19); if $\mathscr{D}(P_t)$ is not empty, then P_t is a member of Π, and, for each σ in $\mathscr{D}(P_t)$,

$$P_t \sigma \overset{\text{def}}{=} \sigma_t, \tag{12.22}$$

where σ_t is defined by (12.21) with $\tau = t$ and with $\boldsymbol{f}$ a solution up to t of (12.19) with initial state σ.

The following remark follows immediately from theorems we give in a forthcoming paper [1974, 1] for this *Archive.*

Remark 12.1. If $P_t = (\boldsymbol{L}, h, \gamma)$ is in Π and σ is in $\mathscr{D}(P_t)$, then

(a) the solution $\boldsymbol{f}$ up to t of (12.19) with initial state σ is unique, and for this solution $\sup_{\tau \in [0, t]} \|\sigma_\tau\| < \infty$, where σ_τ is as in (12.21); moreover,

(b) there exists a $\delta = \delta(P_t, \sigma) > 0$ and an $M = M(P_t, \sigma) < \infty$ such that for each σ in Σ obeying $\|\sigma' - \sigma\| < \delta$, (12.19) has a solution $\boldsymbol{f}'$ up to t with initial state σ', and for this solution there holds, for each τ in $[0, t]$, the relation

$$\|\sigma'_\tau - \sigma_\tau\| \leqq M \|\sigma' - \sigma\|; \tag{12.23}$$

by (12.21) and (12.16), the relation (12.23) implies that

$$|f'(\tau)-f(\tau)|\leqq M\|\sigma'-\sigma\|,$$

i.e.

$$|F'(\tau)-F(\tau)|^2+|e'(\tau)-e(\tau)|^2+|\beta'(\tau)-\beta(\tau)|^2\leqq M^2\|\sigma'-\sigma\|^2. \quad (12.24)$$

It follows from the conclusion (a) of the above remark that for each P_t in Π the induced transformation $\sigma\mapsto P_t\sigma$, given by (12.22), is well defined as a Σ-valued function on $\mathscr{D}(P_t)$, and it follows from the conclusion (b) that this transformation $\sigma\mapsto P_t\sigma$ is continuous. The conclusion (b) also implies that the set $\mathscr{D}(P_t)$ is open in Σ. The successive application function, $(P_{t_2}, P_{t_1})\mapsto P_{t_2}P_{t_1}=P_{t_2+t_1}$, is defined as explained in item (1) of Definition 9.1, and, by item (3) of Definition 12.1, each element with fading memory obeys axiom **II** of Definition 2.1. The following remark tells us axiom **I** of that definition is also satisfied.

Remark 12.2. For each pair of states σ', σ'' in Σ and each $\varepsilon>0$, there is a process P in Π such that σ' is in $\mathscr{D}(P)$ and

$$\|\sigma''-P\sigma'\|<\varepsilon. \quad (12.25)$$

Proof. Let $\boldsymbol{u}={}_0\sigma'$. Standard methods of the theory of $\mathscr{L}_p$-spaces can be used to show that there is a function $\boldsymbol{\Phi}\colon \mathbb{R}^+\to\Sigma_i$ which is in $\mathscr{U}_r$, is continuously differentiable, is equal to $\boldsymbol{u}$ at all points of $\mathbb{R}^+$ except for a set of the form $[0, N)$ with $0<N<\infty$, and, moreover, is such that

$$\boldsymbol{\Phi}(0)={}_0\sigma'' \quad \text{and} \quad \|\sigma-\sigma''\|=\|{}_r\sigma-{}_r\sigma''\|_r<\frac{\varepsilon}{2}, \quad (12.26)$$

where σ is the equivalence class in $\mathfrak{U}$ containing the function $\boldsymbol{\Phi}$. We write $\boldsymbol{u}^\dagger$ for both the constant function on $\mathbb{R}^{++}$ with value $\boldsymbol{u}$ and its equivalence class in $\mathfrak{U}_r$. Let M be choosen so that $M>N$ and

$$\|\mathsf{T}^{(M)}({}_r\sigma'-\boldsymbol{u}^\dagger)\|_r<\frac{\varepsilon}{2}. \quad (12.27)$$

[By (12.12) such a number M exists.] Let $\boldsymbol{f}=(\boldsymbol{F}(\cdot), e(\cdot), \boldsymbol{\beta}(\cdot))\colon [0, M]\to\Sigma_i$ be defined so that

$$\boldsymbol{f}(t)=\begin{cases}\boldsymbol{u}, & \text{for } t\in[0, M-N],\\ \boldsymbol{\Phi}(M-t), & \text{for } t\in(M-N, M].\end{cases} \quad (12.28)$$

Clearly, $\boldsymbol{f}$ is continuously differentiable on $[0, M]$, and, by (12.17) and (12.7),

$$\mathfrak{d}\boldsymbol{f}^M={}_r\boldsymbol{\Phi}-\mathsf{T}^{(M)}\boldsymbol{u}^\dagger. \quad (12.29)$$

We now let $P=(\boldsymbol{L}, h, \gamma)\colon [0, M)\to U$ be defined by

$$\begin{aligned}\boldsymbol{L}(\tau)&=\dot{\boldsymbol{F}}(\tau)\boldsymbol{F}(\tau)^{-1},\\ h(\tau)&=\dot{e}(\tau)-\boldsymbol{T}(\sigma_\tau)\cdot(\dot{\boldsymbol{F}}(\tau)\boldsymbol{F}(\tau)^{-1}),\\ \gamma(\tau)&=\dot{\boldsymbol{\beta}}(\tau),\end{aligned} \quad (12.30)$$

where, for each τ in $[0, M]$, σ_τ is determined by

$$\begin{aligned} {}_0\sigma_\tau &= (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)) = \boldsymbol{f}(\tau) \\ {}_r\sigma_\tau &= \mathsf{T}^{(\tau)}\, {}_r\sigma' + \mathfrak{d}\boldsymbol{f}^\tau = \mathsf{T}^{(\tau)}\, {}_r\sigma' + \mathsf{T}_{(M-\tau)}\, \mathfrak{d}\boldsymbol{f}^M. \end{aligned} \tag{12.31}$$

By the continuity of $\boldsymbol{f}$ and the continuity of the translation semi-groups $\mathsf{T}^{(\cdot)}$ and $\mathsf{T}_{(\cdot)}$ [as expressed in (12.10) and (12.11)], the mapping $\tau \mapsto \sigma_\tau$, defined by (12.31), is a continuous mapping of $[0, M)$ into $\Sigma \subset \mathfrak{U}$. Therefore, because $\boldsymbol{T}$ is continuous on Σ, and $\dot{\boldsymbol{f}} = (\dot{\boldsymbol{F}}(\cdot), \dot{e}(\cdot), \dot{\boldsymbol{\beta}}(\cdot))$ is continuous on $[0, M)$, the function P of (12.30) is continuous on its domain $[0, M)$. Furthermore, if we let P play the role of P_t in (12.19), then $\boldsymbol{f}$ becomes a solution up to M of (12.19) with initial state σ'. Thus, P is in Π, and σ' is in $\mathscr{D}(P)$. By (12.22), $P\sigma' = \sigma_\tau$ at $\tau = M$, and hence, in view of (12.31), (12.28), (12.26), and (12.29), we have

$$\begin{aligned} {}_0(P\sigma') &= {}_0\sigma_M = \boldsymbol{f}(M) = \boldsymbol{\Phi}(0) = {}_0\sigma'' \\ {}_r(P\sigma') &= {}_r\sigma_M = \mathsf{T}^{(M)}\, {}_r\sigma' + \mathfrak{d}\boldsymbol{f}^M = \mathsf{T}^{(M)}\, {}_r\sigma' + {}_r\sigma - \mathsf{T}^{(M)}\boldsymbol{u}^\dagger. \end{aligned} \tag{12.32}$$

It follows from (12.32), (12.26), and (12.27) that

$$\|\sigma'' - P\sigma'\| = \|{}_r\sigma'' - {}_r(P\sigma')\|_r \leqq \|{}_r\sigma'' - {}_r\sigma\|_r + \|\mathsf{T}^{(M)}({}_r\sigma' - \boldsymbol{u}^\dagger)\|_r < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;$$

q.e.d.

We shall refer frequently to the following easy proposition.#

Remark 12.3. Let (X, d) and (X', d') be metric spaces, and suppose that $f: X \to X'$ is locally Lipschitz continuous, *i.e.* such that for each x in X there is a neighborhood $\mathcal{N}(x)$ of x and a number $M(x)$ for which

$$z, y \in \mathcal{N}(x) \quad \Rightarrow \quad d'(f(z), f(y)) \leqq M(x)\, d(z, y).$$

Then, for each compact connected subset C of X, there are positive numbers $L(C)$ and $\delta(C)$ for which

$$d(x, C) < \delta(C), \quad d(y, C) < \delta(C) \quad \Rightarrow \quad d'(f(x), f(y)) \leqq L(C)\, d(x, y).$$

It is now obvious that a set Π of processes P_t, a set Σ of states σ, and a triple $(\theta, \boldsymbol{T}, \boldsymbol{q})$ of state functions that obey items (1)–(3) of Definition 12.1 also obey Definition 2.1, equation (9.6), and item (1) of Definition 9.1. Item (3) of Definition 12.1 and the proof of Remark 12.2 tell us that item (2) of Definition 9.1 is satisfied here. Moreover, since we have here assumed that $1/\theta$, $\boldsymbol{T}$, and $\boldsymbol{q}$ are locally Lipschitz continuous on Σ, we can verify easily, in view of Remarks 12.1 and 12.3 that the real-valued function $\mathfrak{s}(P_t, \cdot)$ given by (9.7) is continuous on $\mathscr{D}(P_t)$ for each P_t in Π. It is clear that $\mathfrak{s}$ has the property of additivity required by Definition 2.2. Hence $\mathfrak{s}$ is an action, and item (3) of Definition 9.1 holds here. In summary, we assert

Remark 12.4. Definition 12.1 is consistent with Definitions 9.1 and 2.1.

Cf. [1974, 1].

When we refer to "an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with fading memory", it is, of course, understood that the instantaneous space Σ_i has been assigned in advance and that enough is known to determine the function $\|\cdot\|$ shown in (12.13) (and hence the topology on Σ).

Elements with fading memory are simple material elements, and, therefore, for them the Second Law of Thermodynamics asserts that the action $\mathfrak{s}$ has the Clausius property at some state σ° in Σ. of course, Remark 3.1 here yields the following analogue of Remark 11.2.

Remark 12.5. If $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ obeys the Second Law of Thermodynamics, then there is a state σ° such that $\mathfrak{s}$ has the Clausius property at each state in $\Pi\sigma^\circ$.

Clearly, Theorem 9.3 holds for elements with fading memory. As in the case of elements with internal variables, for elements with fading memory one can go beyond Theorem 9.3 and construct an entropy function (*i.e.* an upper potential for $\mathfrak{s}$) which obeys analogues of the relations (10.14) and (10.15) derived previously for elastic elements; the analogues of (10.14) and (10.15) for elements with memory are called *generalized stress relations* #. With this goal in mind, we now prove

Lemma 12.1. Let $\sigma' = ({}_0\sigma', {}_r\sigma') = (\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma')$ and $\sigma'' = (\boldsymbol{F}'', e'', \boldsymbol{\beta}'', {}_r\sigma'')$, with ${}_r\sigma'' = {}_r\sigma'$, be states of an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with fading memory, let c be an oriented polygonal curve lying in the instantaneous space Σ_i and connecting $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ to $(\boldsymbol{F}'', e'', \boldsymbol{\beta}'')$, and let $I(c)$ be the line integral

$$I(c) = \int_c \left[-\frac{\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma')\, \boldsymbol{F}^{T^{-1}}}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma')} \cdot d\boldsymbol{F} + \frac{1}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma')} de + \boldsymbol{0} \cdot d\boldsymbol{\beta} \right]. \tag{12.33}$$

For each pair of positive numbers ν, ε, there is a process P_t in Π, with σ' in $\mathscr{D}(P_t)$, $\|\sigma'' - P_t \sigma'\| < \nu$, and

$$|\mathfrak{s}(P_t, \sigma') - I(c)| < \varepsilon. \tag{12.34}$$

Proof. Since c is a compact subset of U, the quantity

$$M = \sup\{|(\boldsymbol{F}, e, \boldsymbol{\beta})| \,|\, (\boldsymbol{F}, e, \boldsymbol{\beta}) \in c\} \tag{12.35}$$

is finite. For each $\delta > 0$, let

$$\mathfrak{T}_\delta = \{\sigma \,|\, \sigma \in \Sigma,\ {}_0\sigma \in c,\ \|{}_r\sigma - {}_r\sigma'\|_r < \delta\}. \tag{12.36}$$

The set

$$C = \{\sigma = ({}_0\sigma, {}_r\sigma) \,|\, {}_0\sigma \in c,\ {}_r\sigma = {}_r\sigma'\},$$

is clearly a connected compact subset of Σ, for it is the image in Σ, under a continuous mapping, of a connected compact set c in U. It therefore follows from Remark 12.3 and item (2) of Definition 12.1 that there are positive numbers L

Their existence was pointed out by Coleman [1964, 1, 2], who obtained them in a theory which started with the Clausius-Duhem inequality and an entropy functional with certain properties of differentiability, including the existence of "instantaneous derivatives". Our present approach to the generalized stress relations, although based on a different hypothesis, employs arguments foreshadowed in the Ph. D. thesis of W. A. Day, written in 1967 under the direction of M. E. Gurtin and published as [1968, 3].

and δ such that if g equals $\boldsymbol{T}$, $\boldsymbol{q}$, or $1/\theta$, then

$$|g(\sigma_i)-g(\sigma_{ii})|\leqq L\|\sigma_i-\sigma_{ii}\| \tag{12.37}$$

whenever σ_i and σ_{ii} are within distance δ of C. However, each element of $\mathfrak{T}_\delta$ is within distance δ of C, and hence (12.37) (with $g=\boldsymbol{T}, \boldsymbol{q}$, or $1/\theta$) holds whenever σ_i and σ_{ii} are in $\mathfrak{T}_\delta$. This implies that

$$Q=\sup\{|\boldsymbol{q}(\sigma)|\,|\,\sigma\in\mathfrak{T}_\delta\} \tag{12.38}$$

is finite. Now, let $\nu>0$ and $\varepsilon>0$ be given. As a consequence of (12.37) there exists an η in $(0,\delta]$ such that if $\sigma_i=(\boldsymbol{F}_i, e_i, \boldsymbol{\beta}_i, {}_r\sigma_i)$ and $\sigma_{ii}=(\boldsymbol{F}_{ii}, e_{ii}, \boldsymbol{\beta}_{ii}, {}_r\sigma_{ii})$ are in $\mathfrak{T}_\delta$ and have $\|\sigma_i-\sigma_{ii}\|<\eta$, then

$$\left|\frac{\boldsymbol{T}(\sigma_i)\boldsymbol{F}_i^{T^{-1}}}{\theta(\sigma_i)}-\frac{\boldsymbol{T}(\sigma_{ii})\boldsymbol{F}_{ii}^{T^{-1}}}{\theta(\sigma_{ii})}\right|^2+\left|\frac{1}{\theta(\sigma_i)}-\frac{1}{\theta(\sigma_{ii})}\right|^2<\frac{\varepsilon^2}{4l^2} \tag{12.39}$$

with l the length of c. By (12.1), (12.7), and (12.10), there is a $t>0$ such that

$$tQM<\frac{\varepsilon}{2}, \tag{12.40}$$

$$M\left[\int_0^t k(s)\,ds\right]^{\frac{1}{p}}<\frac{\eta}{2}, \tag{12.41}$$

and

$$\|\mathrm{T}^{(\tau)}{}_r\sigma'-{}_r\sigma'\|_r=\|\mathrm{T}^{(\tau)}{}_r\sigma'-\mathrm{T}^{(0)}{}_r\sigma'\|_r<\frac{\eta}{2} \tag{12.42}$$

for all τ in $(0,t]$. Let $\boldsymbol{f}=(\boldsymbol{F}, e, \boldsymbol{\beta})$ be a piecewise linear parameterization of c with domain $[0,t]$, and let $(\boldsymbol{L}, h, \gamma)$: $[0,t)\to U$ be given by

$$\begin{aligned}\boldsymbol{L}(\tau)&=\dot{\boldsymbol{F}}(\tau)\boldsymbol{F}(\tau)^{-1},\\ h(\tau)&=\dot{e}(\tau)-\boldsymbol{T}(\sigma_\tau)\cdot\boldsymbol{L}(\tau),\\ \gamma(\tau)&=\dot{\boldsymbol{\beta}}(\tau),\end{aligned} \tag{12.43}$$

where for each τ in $[0,t)$,

$$\begin{aligned}{}_0\sigma_\tau&=(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau)),\\ {}_r\sigma_\tau&=\mathrm{T}^{(\tau)}{}_r\sigma'+\mathfrak{d}\boldsymbol{f}^\tau.\end{aligned} \tag{12.44}$$

The formulae (12.43) and (12.44) imply that $(\boldsymbol{L}, h, \gamma)$ is piecewise continuous on $[0,t)$ and is such that $\boldsymbol{f}$ is a solution up to t of the differential system (12.19) with initial state σ'. Hence $P_t=(\boldsymbol{L}, h, \gamma)$ is a process, and σ' is in $\mathscr{D}(P_t)$. It follows from $(12.44)_2$ that

$$\|{}_r\sigma_\tau-{}_r\sigma'\|_r=\|\mathrm{T}^{(\tau)}{}_r\sigma'+\mathfrak{d}\boldsymbol{f}^\tau-{}_r\sigma'\|_r\leqq\|\mathrm{T}^{(\tau)}{}_r\sigma'-{}_r\sigma'\|_r+\|\mathfrak{d}\boldsymbol{f}^\tau\|_r,$$

and since $\boldsymbol{f}$ has its values in c, we have, by (12.42), (12.17), (12.9), (12.35), and (12.41),

$$\begin{aligned}\|{}_r\sigma_\tau-{}_r\sigma'\|_r&<\frac{\eta}{2}+\left[\int_0^\tau|\boldsymbol{f}(\tau-s)|^p k(s)\,ds\right]^{\frac{1}{p}}\leqq\frac{\eta}{2}+\left[\int_0^t M^p k(s)\,ds\right]^{\frac{1}{p}}\\ &<\frac{\eta}{2}+\frac{\eta}{2}=\eta.\end{aligned} \tag{12.45}$$

Note that $(12.44)_1$ and the definition of $\boldsymbol{f}=(\boldsymbol{F}, e, \boldsymbol{\beta})$ imply that

$$_0\sigma_t=(\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t))=(\boldsymbol{F}'', e'', \boldsymbol{\beta}''),$$

because (F'', e'', β'') is the end point of the curve c. Hence, in view of (12.16), the hypothesis that $_r\sigma''={}_r\sigma'$ and the inequality (12.45) yield

$$\|\sigma''-P_t\sigma'\|=\|\sigma''-\sigma_t\|=|_0\sigma''-{}_0\sigma_t|+\|_r\sigma''-{}_r\sigma_t\|_r=\|_r\sigma'-{}_r\sigma_t\|_r<\eta,$$

and if we choose η in $(0, \delta]$ to be equal to or less than ν, then we have $\|\sigma''-P_t\sigma'\|<\nu$. The values of the function $\tau\mapsto\sigma_\tau$ defined in (12.44) have their present values $_0\sigma_\tau$ in c and, by (12.45), their past histories $_r\sigma_\tau$ at distances less than η from $_r\sigma'$; *i.e.* for each τ in $[0, t)$, σ_τ is in the set $\mathfrak{T}_\eta\subset\mathfrak{T}_\delta$. Using this fact, we now estimate the number $|s(P_t, \sigma')-I(c)|$ for the process P_t which we have just constructed via (12.43). By (9.7), (12.33), and $(12.43)_2$, there holds

$$|s(P_t, \sigma')-I(c)|=\left|\int_0^t \boldsymbol{q}(\sigma_\tau)\cdot\boldsymbol{\beta}(\tau)\,d\tau-\int_0^t\frac{\boldsymbol{T}(\sigma_\tau)\boldsymbol{F}(\tau)^{T^{-1}}\cdot\dot{\boldsymbol{F}}(\tau)-\dot{e}(\tau)}{\theta(\sigma_\tau)}d\tau\right.$$
$$\left.+\int_0^t\frac{\boldsymbol{T}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), {}_r\sigma')\boldsymbol{F}(\tau)^{T^{-1}}\cdot\dot{\boldsymbol{F}}(\tau)-\dot{e}(\tau)}{\theta(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), {}_r\sigma')}d\tau\right|,$$

and, in view of (12.18) and $(12.44)_1$, we have

$$|s(P_t, \sigma')-I(c)|\leqq\int_0^t|\boldsymbol{q}(\sigma_\tau)|\,|\boldsymbol{\beta}(\tau)|\,d\tau$$
$$+\int_0^t\left|\left[\frac{\boldsymbol{T}({}_0\sigma_\tau, {}_r\sigma')\boldsymbol{F}(\tau)^{T^{-1}}}{\theta({}_0\sigma_\tau, {}_r\sigma')}-\frac{\boldsymbol{T}({}_0\sigma_\tau, {}_r\sigma_\tau)\boldsymbol{F}(\tau)^{T^{-1}}}{\theta({}_0\sigma_\tau, {}_r\sigma_\tau)}\right]\cdot\dot{\boldsymbol{F}}(\tau)\right.$$
$$\left.+\left[\frac{-1}{\theta({}_0\sigma_\tau, {}_r\sigma')}-\frac{-1}{\theta({}_0\sigma_\tau, {}_r\sigma_\tau)}\right]\dot{e}(\tau)\right|d\tau.$$

From this, by (12.35), (12.38), and Schwarz's inequality, we obtain

$$|s(P_t, \sigma')-I(c)|\leqq QMt$$
$$+\int_0^t\left[\left|\frac{\boldsymbol{T}({}_0\sigma_\tau, {}_r\sigma')\boldsymbol{F}(\tau)^{T^{-1}}}{\theta({}_0\sigma_\tau, {}_r\sigma')}-\frac{\boldsymbol{T}({}_0\sigma_\tau, {}_r\sigma_\tau)\boldsymbol{F}(\tau)^{T^{-1}}}{\theta({}_0\sigma_\tau, {}_r\sigma_\tau)}\right|^2\right.$$
$$\left.+\left|\frac{1}{\theta({}_0\sigma_\tau, {}_r\sigma')}-\frac{1}{\theta({}_0\sigma_\tau, {}_r\sigma_\tau)}\right|^2\right]^{\frac{1}{2}}[|\dot{\boldsymbol{F}}(\tau)|^2+|\dot{e}(\tau)|^2]^{\frac{1}{2}}\,d\tau,$$

and the inequalities (12.45), (12.39), and (12.40) now yield,

$$|s(P_t, \sigma')-I(c)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2l}\int_0^t[|\dot{\boldsymbol{F}}(\tau)|^2+|\dot{e}(\tau)|^2]^{\frac{1}{2}}\,d\tau\leqq\frac{\varepsilon}{2}+\frac{\varepsilon}{2l}\times l=\varepsilon,$$

which is (12.34); q.e.d.

Let Σ' be a subset of Σ such that whenever a state σ is in Σ', so also is every other state with the same past history $_r\sigma$. Following COLEMAN [1964, 1, 2], we say that a real-valued function f on Σ' has an *instantaneous derivative* at a state

$\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)$ in Σ' if U contains an element $(D_{\boldsymbol{F}} f(\sigma), D_e f(\sigma), D_{\boldsymbol{\beta}} f(\sigma))$ obeying

$$\begin{aligned} f(&\boldsymbol{F}+\boldsymbol{A}, e+a, \boldsymbol{\beta}+\boldsymbol{\alpha}, {}_r\sigma) \\ &= f(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma) + D_{\boldsymbol{F}} f(\sigma)\cdot \boldsymbol{A} + D_e f(\sigma)\, a + D_{\boldsymbol{\beta}} f(\sigma)\cdot\boldsymbol{\alpha} + o(|\boldsymbol{A}, a, \boldsymbol{\alpha}|) \end{aligned} \tag{12.46}$$

for all elements $(\boldsymbol{A}, a, \boldsymbol{\alpha})$ of U such that $(\boldsymbol{F}+\boldsymbol{A}, e+a, \boldsymbol{\beta}+\boldsymbol{\alpha})$ is in Σ_i; $D_{\boldsymbol{F}} f(\sigma)$ is an element of $\mathrm{Lin}(\mathscr{V})$, $D_e f(\sigma)$ is in $\mathbb{R}$, and $D_{\boldsymbol{\beta}} f(\sigma)$ is in $\mathscr{V}$.

We have now assembled the apparatus needed to derive the generalized stress relations from our formulation of the Second Law.

Theorem 12.1.[#] For an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with fading memory that obeys the Second Law of Thermodynamics, there is an upper semicontinuous entropy function S with the following properties:

(1) the domain Σ° of S contains σ°, is closed under $\mathfrak{s}$-approach, and is such that if σ' is in Σ°, then so also is every state σ with ${}_r\sigma = {}_r\sigma'$;

(2) at each state σ in Σ°, S has an instantaneous derivative

$$(D_{\boldsymbol{F}}\, S(\sigma), D_e\, S(\sigma), D_{\boldsymbol{\beta}}\, S(\sigma))$$

obeying the relations

$$\boldsymbol{T}(\sigma) = -\theta(\sigma)[D_{\boldsymbol{F}} S(\sigma)]\boldsymbol{F}^T, \tag{12.47}$$

$$\theta(\sigma) = [D_e S(\sigma)]^{-1}, \tag{12.48}$$

$$\boldsymbol{0} = D_{\boldsymbol{\beta}} S(\sigma). \tag{12.49}$$

The "generalized stress relations" (12.47) and (12.48) imply that S determines $\boldsymbol{T}$ and θ on all of Σ. The relation (12.49) implies that, for $(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)$ in Σ°,

$$S(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma) = S(\boldsymbol{F}, e, \boldsymbol{0}, {}_r\sigma). \tag{12.50}$$

Furthermore, throughout Σ, the functions $\boldsymbol{T}$ and θ are independent of the present value of $\boldsymbol{\beta}$ in the sense that

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{F}, \theta, \boldsymbol{\beta}, {}_r\sigma) &= \boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{0}, {}_r\sigma), \\ \theta(\boldsymbol{F}, \theta, \boldsymbol{\beta}, {}_r\sigma) &= \theta(\boldsymbol{F}, e, \boldsymbol{0}, {}_r\sigma), \end{aligned} \tag{12.51}$$

for each state $(\boldsymbol{F}, \theta, \boldsymbol{\beta}, {}_r\sigma)$.

Proof. Let Σ° be as in $(5.8)_2$:

$$\Sigma^\circ \overset{\text{def}}{=} \{\sigma \mid \sigma \in \Sigma, m(\sigma^\circ, \sigma) > -\infty\}; \tag{12.52}$$

here $m(\sigma^\circ, \sigma)$ is given by (5.6), *i.e.*

$$m(\sigma^\circ, \sigma) \overset{\text{def}}{=} \inf_{\mathcal{O}\in\mathfrak{S}(\sigma^\circ, \sigma)} \sup \mathfrak{s}\{\sigma^\circ \to \mathcal{O}\}, \tag{12.53}$$

with $\mathfrak{S}(\sigma^\circ, \sigma)$ as in (5.7). By Remarks 3.2 and 9.3, Σ° is dense in Σ, is closed under $\mathfrak{s}$-approach and contains σ°. Let $\sigma' = ({}_0\sigma', {}_r\sigma')$ be an arbitrary element of Σ°,

[#] For derivations of (12.47)–(12.50) starting from the assumption that there exists a Fréchet-differentiable entropy function S on Σ, see COLEMAN [1964, 1] and COLEMAN & MIZEL [1967, 2].

and let σ'' have the form $({}_0\sigma'', {}_r\sigma')$, *i.e.* be such that ${}_r\sigma'' = {}_r\sigma'$. Let c be an oriented polygonal curve which lies in Σ_i and joins ${}_0\sigma'$ to ${}_0\sigma''$. By Lemma 12.1, there is, for each positive integer n, a process $P^{(n)}$ in Π such that

$$\|P^{(n)}\sigma' - \sigma''\| < \frac{1}{n}$$

and

$$|\mathfrak{s}(P^{(n)}, \sigma') - I(c)| < \frac{1}{n},$$

where I is the line integral of (12.33). The sequence $n \mapsto P^{(n)}$ obeys the relations

$$\lim_{n\to\infty} P^{(n)}\sigma' = \sigma''$$

and

$$\lim_{n\to\infty} \mathfrak{s}(P^{(n)}, \sigma') = I(c),$$

and thus, by Theorem 3.6, the state σ'' is in Σ°. Hence the presence of σ' in Σ° implies that Σ° contains every state σ with ${}_r\sigma = {}_r\sigma'$. Let $S\colon \Sigma^\circ \to \mathbb{R}$ be defined as in (5.5):

$$S(\sigma) = m(\sigma^\circ, \sigma), \qquad \sigma \in \Sigma^\circ, \tag{12.54}$$

with m as in (12.53). As we showed when we proved Theorem 3.3, S is upper semicontinuous and is an upper potential for $\mathfrak{s}$, *i.e.* an entropy function.

We have just shown that $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ has an upper semicontinuous entropy function S obeying item (1) of our theorem. To show that S obeys item (2), we let $\sigma' = ({}_0\sigma', {}_r\sigma')$ be in Σ° and let $\varepsilon > 0$ be given. By item (3) of Remark 3.2, $\mathfrak{s}$ has the Clausius property at σ', and, therefore, there is a $\nu > 0$ such that

$$\mathfrak{s}(P, \sigma') < \frac{\varepsilon}{2} \tag{12.55}$$

for each P in Π with $\|\sigma' - P\sigma'\| < \nu$. Let I be the line integral (12.33), and let c^+ be an arbitrary closed polygonal curve in $\Sigma_i \subset U$ passing through the point ${}_0\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}')$. It follows from Lemma 12.1 that there is a P in Π with σ' in $\mathscr{D}(P)$, $\|\sigma' - P\sigma'\| < \nu$, and

$$|\mathfrak{s}(P, \sigma') - I(c^+)| < \frac{\varepsilon}{2}; \tag{12.56}$$

for P there holds, by (12.56),

$$I(c^+) < \mathfrak{s}(P, \sigma') + \frac{\varepsilon}{2},$$

and, by (12.55),

$$I(c^+) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As this last inequality holds for each $\varepsilon > 0$, we have $0 \geqq I(c^+)$, and the same argument applied to the curve c^- differing only in orientation from c^+ yields $0 \geqq I(c^-) = -I(c^+)$. Thus $I(c^+) = 0$; *i.e.*, I vanishes on each closed polygonal curve lying in Σ_i and passing through $(\boldsymbol{F}', e', \boldsymbol{\beta}')$. Since Σ_i is polygonally connected, this implies that I vanishes on every closed oriented polygonal curve in Σ_i, including those which do not contain σ'. Hence, by (12.33) and the fundamental theorem on the existence of potentials for vector fields, the function from Σ_i into U given by

$$(\boldsymbol{F}, e, \boldsymbol{\beta}) \mapsto \left(\frac{-\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma') \boldsymbol{F}^{T^{-1}}}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma')}, \frac{1}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma')}, \boldsymbol{0} \right)$$

has a potential in the sense that there exists a differentiable function $V_{{}_r\sigma'}: \Sigma_i \to \mathbb{R}$ such that

$$\begin{aligned} \partial_{\boldsymbol{F}} V_{{}_r\sigma'}(\boldsymbol{F}, e, \boldsymbol{\beta}) &= -\frac{\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma') \boldsymbol{F}^{T^{-1}}}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma')}, \\ \partial_e V_{{}_r\sigma'}(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \frac{1}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma')}, \\ \partial_{\boldsymbol{\beta}} V_{{}_r\sigma'}(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \boldsymbol{0}, \end{aligned} \tag{12.57}$$

for every triple $(\boldsymbol{F}, e, \boldsymbol{\beta})$ in Σ_i. Let us define $\bar{S}: \Sigma^\circ \to \mathbb{R}$ by

$$\bar{S}(\sigma) = S(\sigma) - V_{{}_r\sigma}(\boldsymbol{F}, e, \boldsymbol{\beta}), \qquad \sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma) \in \Sigma^\circ, \tag{12.58}$$

where $S: \Sigma^\circ \to \mathbb{R}$ is given by (12.54), and $V_{{}_r\sigma}: \Sigma_i \to \mathbb{R}$ obeys (12.57). Let ${}_r\sigma$ in $\mathfrak{U}_r$ be the past history of some state $\sigma = ({}_0\sigma, {}_r\sigma)$ in Σ°, and let $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma')$ and $\sigma'' = (\boldsymbol{F}'', e'', \boldsymbol{\beta}'', {}_r\sigma'')$ be two states in Σ such that ${}_r\sigma'' = {}_r\sigma' = {}_r\sigma$. By an argument given in the previous paragraph, σ' and σ'' are in Σ°, the common domain of S and $\bar{S}$. Let c^+ be an oriented polygonal curve in Σ_i joining $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ to $(\boldsymbol{F}'', e'', \boldsymbol{\beta}'')$, and let $\varepsilon > 0$ be given. Because S is an upper potential for $\mathfrak{s}$ and has both σ' and σ'' in its domain, there is, by (3.29), a neighborhood $\mathcal{O}''$ of σ'' such that, for each P in Π with $P\sigma'$ in $\mathcal{O}''$,

$$\mathfrak{s}(P, \sigma') < S(\sigma'') - S(\sigma') + \frac{\varepsilon}{2}. \tag{12.59}$$

Lemma 12.1 tells us that there exists a process P' with $P'\sigma'$ in $\mathcal{O}''$, with

$$|\mathfrak{s}(P', \sigma') - I(c)| < \frac{\varepsilon}{2},$$

and hence, in particular, with

$$-\frac{\varepsilon}{2} < \mathfrak{s}(P', \sigma') - I(c). \tag{12.60}$$

By the construction of $V_{{}_r\sigma'}$, (12.60) can be written

$$-\frac{\varepsilon}{2} < \mathfrak{s}(P', \sigma') - [V_{{}_r\sigma'}(\boldsymbol{F}'', e'', \boldsymbol{\beta}'') - V_{{}_r\sigma'}(\boldsymbol{F}', e', \boldsymbol{\beta}')],$$

and, by (12.59) and (12.58), we have, since ${}_r\sigma'' = {}_r\sigma'$,

$$-\varepsilon < \bar{S}(\sigma'') - \bar{S}(\sigma').$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$0 \leq \bar{S}(\sigma'') - \bar{S}(\sigma'),$$

and, by interchanging the roles of σ' and σ'' in this argument, we may conclude also that $0 \leqq \bar{S}(\sigma') - \bar{S}(\sigma'')$. Hence, if σ' and σ'' are in Σ° and are such that ${}_r\sigma' = {}_r\sigma'' = {}_r\sigma$, then $\bar{S}(\sigma'') = \bar{S}'(\sigma')$. In other words, $\bar{S}(\sigma)$ is independent of the present value of σ in the sense that for each pair of elements $(\boldsymbol{F}', e', \boldsymbol{\beta}')$, $(\boldsymbol{F}'', e'', \boldsymbol{\beta}'')$ of Σ_i,

$$\bar{S}(\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma) = \bar{S}(\boldsymbol{F}'', e'', \boldsymbol{\beta}'', {}_r\sigma), \tag{12.61}$$

whenever ${}_r\sigma$ is the past history of a state in the domain Σ° of $\bar{S}$. In the terminology of (12.46), the relation (12.61) states that $\bar{S}$ has an instantaneous derivative at each state in its domain, and this derivative is zero:

$$D_{\boldsymbol{F}}\bar{S} = \boldsymbol{0}, \quad D_e\bar{S} = 0, \quad D_{\boldsymbol{\beta}}\bar{S} = \boldsymbol{0}.$$

Moreover, the function $S: \Sigma^\circ \to \mathbb{R}$, since it obeys (12.58), is the sum of $\bar{S}$ and a function V defined on Σ° by

$$V(\sigma) = V_{{}_r\sigma}(\boldsymbol{F}, e, \boldsymbol{\beta}), \quad \sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma) \in \Sigma^\circ.$$

Clearly V has an instantaneous derivative given by

$$D_{\boldsymbol{F}} V(\sigma) = \partial_{\boldsymbol{F}} V_{{}_r\sigma}(\boldsymbol{F}, e, \boldsymbol{\beta}), \quad \text{etc.},$$

and hence S has an instantaneous derivative obeying

$$D_{\boldsymbol{F}} S = D_{\boldsymbol{F}}\bar{S}(\sigma) + D_{\boldsymbol{F}} V(\sigma) = \boldsymbol{0} + \partial_{\boldsymbol{F}} V_{{}_r\sigma}(\boldsymbol{F}, e, \boldsymbol{\beta}), \quad \text{etc.}$$

It follows from this and the relations (12.57) that S obeys item (2) of our theorem.

In view of the relations (12.47) and (12.48), the function S determines the restrictions of $\boldsymbol{T}$ and θ to Σ°. By item (2) of Definition 12.1, $\boldsymbol{T}$ and θ are continuous on Σ, and, therefore, since Σ° is dense in Σ, S determines $\boldsymbol{T}$ and θ on the entire state space Σ. For each $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ in Σ_i, item (1) of Definition 12.1 tells us that all triples of the form $(\boldsymbol{F}', e', \boldsymbol{\beta})$, with $\boldsymbol{\beta}$ in $\mathscr{V}$, are in Σ_i. In view of this, the sentence containing (12.50) is obviously true. It is clear that (12.50), (12.47), and (12.48) imply that (12.51) holds for each triple $(\boldsymbol{F}, e, \boldsymbol{\beta})$ in Σ_i and past history ${}_r\sigma$ in $\mathfrak{U}_r$ such that the state $(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)$ is in Σ°. If $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ and ${}_r\sigma'$ are such that $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma')$ is in Σ, then there exists a sequence $n \mapsto \sigma^n = (\boldsymbol{F}^n, e^n, \boldsymbol{\beta}^n, {}_r\sigma^n)$ of states in the set Σ° for which $\lim_{n\to\infty} \sigma^n = \sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma')$. (This is so because Σ° is dense in Σ.) Item (1) of Definition 12.1 and item (1) of the present theorem imply that $n \mapsto \hat{\sigma}^n \overset{\text{def}}{=} (\boldsymbol{F}^n, e^n, \boldsymbol{0}, {}_r\sigma^n)$ is again a sequence of states in Σ°. The convergence of the sequence $n \mapsto \sigma^n$ to $(\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma')$ implies the convergence of $n \mapsto \hat{\sigma}^n$ to $(\boldsymbol{F}', e', \boldsymbol{0}, {}_r\sigma')$, and the continuity of $\boldsymbol{T}$ on Σ then yields the relations

$$\boldsymbol{T}(\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma') = \lim_{n\to\infty} \boldsymbol{T}(\sigma^n), \quad \boldsymbol{T}(\boldsymbol{F}', e', \boldsymbol{0}, {}_r\sigma') = \lim_{n\to\infty} \boldsymbol{T}(\hat{\sigma}^n). \tag{12.62}$$

For each n, the states σ^n and $\hat{\sigma}^n$ lie in Σ°, and $(12.51)_1$ holds throughout Σ°; we therefore can write

$$\boldsymbol{T}(\sigma^n) = \boldsymbol{T}(\boldsymbol{F}^n, e^n, \boldsymbol{\beta}^n, {}_r\sigma^n) = \boldsymbol{T}(\boldsymbol{F}^n, e^n, \boldsymbol{0}, {}_r\sigma^n) = \boldsymbol{T}(\hat{\sigma}^n).$$

Hence, by (12.62), we have

$$\boldsymbol{T}(\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma') = \boldsymbol{T}(\boldsymbol{F}', e', \boldsymbol{0}, {}_r\sigma'),$$

and, of course, the same argument holds for the function θ; *i.e.* (12.51) holds throughout Σ; q.e.d.

The theorem just proven is the main result of this section; it shows that the principal results of COLEMAN's theory of the thermodynamics of materials with memory # hold also here.

It follows from (12.1) and (12.13) that if $(\boldsymbol{F}, e, \boldsymbol{\beta})$ is in U, then the constant function $(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger)$ with value $(\boldsymbol{F}, e, \boldsymbol{\beta})$, *i.e.*

$$\boldsymbol{F}^\dagger(s) = \boldsymbol{F}, \quad e^\dagger(s) = e, \quad \boldsymbol{\beta}^\dagger(s) = \boldsymbol{\beta}, \quad s \in \mathbb{R}^+, \tag{12.63}$$

is in $\mathscr{U}$. Furthermore, by item (1) of Definition 12.1, if $(\boldsymbol{F}, e, \boldsymbol{\beta})$ is in Σ_i, then the element of $\mathfrak{U}$ corresponding to the equivalence class $(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger)$ is in Σ, *i. e.* is a state. The functions $\boldsymbol{T}^\circ$, θ°, $\boldsymbol{q}^\circ$ defined on Σ_i by

$$\begin{aligned} \boldsymbol{T}^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \boldsymbol{T}(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger), \\ \theta^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \theta(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger), \\ \boldsymbol{q}^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \boldsymbol{q}(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger), \end{aligned} \tag{12.64}$$

are called *equilibrium response functions*, for they give $\boldsymbol{T}$, θ, $\boldsymbol{q}$ when $\boldsymbol{F}$, e, and $\boldsymbol{\beta}$ have been held constant for all time.

In the theory of COLEMAN referred to above it is shown that the equilibrium response functions $\boldsymbol{T}^\circ$, θ°, $\boldsymbol{q}^\circ$ have certain properties enjoyed by the response functions of elastic materials,## and entropy has an extremum at constant histories. Theorems 12.3 and 12.4 below show that similar results hold here.

If we let ${}_r\boldsymbol{F}^\dagger$, *etc.*, be the restriction to $\mathbb{R}^{++}$ of $\boldsymbol{F}^\dagger$, *etc.*, defined in (12.63), and if we put ${}_r\sigma^\dagger = ({}_r\boldsymbol{F}^\dagger, {}_r e^\dagger, {}_r\boldsymbol{\beta}^\dagger)$, then (12.64) can be written

$$\begin{aligned} \boldsymbol{T}^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma^\dagger), \\ \theta^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma^\dagger), \\ \boldsymbol{q}^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma^\dagger). \end{aligned} \tag{12.65}$$

It follows from (12.51) and this that, for each triple $(\boldsymbol{F}, e, \boldsymbol{\beta})$ in Σ_i,

$$\begin{aligned} \boldsymbol{T}^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{0}, {}_r\sigma^\dagger), \\ \theta^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) &= \theta(\boldsymbol{F}, e, \boldsymbol{0}, {}_r\sigma^\dagger). \end{aligned} \tag{12.66}$$

For each $t > 0$ we define $P^{[t]}: [0, t) \to U$ by

$$P^{[t]}(\tau) = (\boldsymbol{L}(\tau), h(\tau), \boldsymbol{\gamma}(\tau)) \equiv (\boldsymbol{0}, 0, \boldsymbol{0}), \tag{12.67}$$

for all τ in $[0, t)$. Every such function $P^{[t]}$ is an element of Π, and as each state $\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)$ in Σ is an initial state for the constant solution

$$\boldsymbol{f}(\tau) \equiv (\boldsymbol{F}, e, \boldsymbol{\beta}), \quad \tau \in [0, t],$$

of the system (12.19) with $\boldsymbol{L}$, h, and $\boldsymbol{\beta}$ as in (12.67), we have $\mathscr{D}(P^{[t]}) = \Sigma$ for each $t > 0$. Clearly,

$$P^{[t_1]} P^{[t_2]} = P^{[t_1 + t_2]}, \quad \text{for } t_1, t_2 \in \mathbb{R}^{++}. \tag{12.68}$$

[1964, 1], particularly Thm. 1.

[1964, 1], § 8 and § 12.

Furthermore, it is obvious that if

$$\sigma^\dagger = (F^\dagger, e^\dagger, \beta^\dagger) \tag{12.69}$$

with $F^\dagger$, $e^\dagger$, and $\beta^\dagger$ as in (12.63), then

$$P^{[t]}\sigma^\dagger = \sigma^\dagger, \qquad \text{for all } t>0. \tag{12.70}$$

In the terminology of Definitions 6.1, 7.1, and 7.2 we can assert

Remark 12.6. The set $\mathsf{P} = \{P^{[t]} \mid t \in \mathbb{R}^{++}\}$, with each $P^{[t]}$ as in (12.67), is a stagnating family of processes with $\Sigma_{\mathsf{P}} = \Sigma$ and with each state $\sigma^\dagger$ of the form (12.69) in $\mathsf{P}\Sigma_{\mathsf{P}}$, *i.e.* a stagnant state for P. Moreover P is a null family of processes for the action $\mathfrak{s}$ of (9.7) at each state σ' of the form

$$\sigma' = (F', e', \mathbf{0}, {}_r\sigma') \tag{12.71}$$

(with ${}_r\sigma'$ arbitrary), and P relaxes each such state (12.71) to a corresponding state $\sigma'_\circ = (F'^\dagger, e'^\dagger, \mathbf{0}^\dagger)$. Every state $\sigma^\dagger$ of the form $(F^\dagger, e^\dagger, \mathbf{0}^\dagger)$ is a relaxed state for $\mathfrak{s}$.

Proof. The first sentence of the remark follows immediately from Theorem 6.2 and the discussion containing equations (12.68)–(12.70). Now, for a state σ' of the type (12.71), we have, for $P^{[t]}$ in P, by (12.16) and (12.21),

$$\begin{aligned} \|P^{[t]}\sigma' - \sigma'_\circ\| &= |{}_0P^{[t]}\sigma' - {}_0\sigma'_\circ| + \|{}_rP^{[t]}\sigma' - {}_r\sigma'_\circ\|_r \\ &= 0 + \|\mathrm{T}^{(t)}{}_r\sigma' + \mathfrak{d}f^t - {}_r\sigma'_\circ\|_r \end{aligned} \tag{12.72}$$

where $\sigma'_\circ = (F'^\dagger, e'^\dagger, \mathbf{0}'^\dagger)$ and $f(\tau) \equiv (F', e', \mathbf{0})$ is the solution of the system (12.19) with $(L, h, \gamma) \equiv (\mathbf{0}, 0, \mathbf{0})$ and with σ' the initial state. By (12.7) and (12.17),

$${}_r\sigma'_\circ = \mathrm{T}^{(t)}{}_r\sigma'_\circ + \mathfrak{d}f^t;$$

hence, by (12.72),

$$\|P^{[t]}\sigma' - \sigma'_\circ\| = \|\mathrm{T}^{(t)}({}_r\sigma' - {}_r\sigma'_\circ)\|_r,$$

and (12.12) yields

$$\lim_{t\to\infty} \|P^{[t]}\sigma' - \sigma'_\circ\| = 0,$$

i.e.

$$\lim_{t\to\infty} P^{[t]}\sigma' = \sigma'_\circ \qquad \text{or } \mathsf{P}\sigma' = \sigma'_\circ. \tag{12.73}$$

Furthermore, when $P^{[t]}$ is in P, we have, for τ in $(0, t)$, not only $h(\tau) \equiv 0$ but also, by $(12.19)_3$, $\dot{\beta}(\tau) \equiv \mathbf{0}$, and when the initial state for (12.19) is $\sigma' = (F', e', \mathbf{0}, {}_r\sigma')$, we have $\beta(\tau) \equiv 0$. Thus, (9.7) here yields

$$\mathfrak{s}(P^{[t]}, \sigma') = 0, \qquad \text{for all } t \in \mathbb{R}^{++}. \tag{12.74}$$

In view of (12.73), (12.74), and Definitions 7.1 and 7.2, it is clear that P is a null family for $\mathfrak{s}$, at each state σ' of the form (12.71), and P relaxes σ' to $\sigma'_\circ$. As this argument is independent of the values of F' and e', each state of the form $(F^\dagger, e^\dagger, \mathbf{0}^\dagger)$ is a relaxed state for $\mathfrak{s}$; q.e.d.

Theorem 12.2. If (F, e, β) in U is such that the corresponding constant function $(F^\dagger, e^\dagger, \beta^\dagger)$ in U characterizes a state at which $\mathfrak{s}$ has Clausius property, then

$$q^\circ(F, e, \beta) \cdot \beta \leq 0. \tag{12.75}$$

Proof. Consider the process $P^{[1]}=(\mathbf{0}, 0, \mathbf{0})$ defined by (12.67) with $t=1$. Clearly the state $\sigma^\dagger=(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger)$ is in $\mathscr{D}(P^{[1]})$, and, by (12.70), for each τ in (0, 1) we have $P^{[\tau]}\sigma^\dagger=\sigma^\dagger$, where $P^{[\tau]}=(\mathbf{0}, 0, \mathbf{0})$ is the restriction of $P^{[1]}$ to $[0, \tau)$. Hence the hypothesis of Theorem 9.1 holds here, and (12.75) follows from (9.12); q.e.d.

Let

$$\Theta_i=\{(\boldsymbol{F}, e) \mid (\boldsymbol{F}, e, \boldsymbol{\beta})\in\Sigma_i \text{ for some } \boldsymbol{\beta}\in\mathscr{V}\}. \tag{12.76}$$

Theorem 12.3.# Let $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ be an element with fading memory which obeys the Second Law of Thermodynamics, and let S: $\Sigma^\circ\to\mathbb{R}$ be the entropy function of equation (12.54) in the proof of Theorem 12.1. If $(\boldsymbol{F}_*, e_*)$ is an element of Θ_i, then the state σ_* given by

$$\sigma_*=(\boldsymbol{F}_*^\dagger, e_*^\dagger, \mathbf{0}^\dagger) \tag{12.77}$$

is in Σ° and has the following "maximizing property" for S: for each state σ in Σ° with

$${}_0\sigma=(\boldsymbol{F}_*, e_*, \boldsymbol{\beta}) \quad \text{for some } \boldsymbol{\beta} \text{ in } \mathscr{V}, \tag{12.78}$$

there holds

$$S(\sigma)\leqq S(\sigma_*). \tag{12.79}$$

Proof. By item (1) of Theorem 12.1, Σ° is not empty, and if σ' is in Σ° then so also is every state σ'' with ${}_r\sigma''={}_r\sigma'$. Thus, if $(\boldsymbol{F}_*, e_*)$ is in Θ_i, then there is in Σ° a state of the form $(\boldsymbol{F}_*, e_*, \boldsymbol{\beta}_*, {}_r\sigma)$. Let σ be any state in Σ° with this form. [σ will then obey (12.78).] Again by item (1) of Theorem 12.1, because σ is in Σ° so also is the state $\sigma_\circ=(\boldsymbol{F}_*, e_*, \mathbf{0}, {}_r\sigma)$. By Remark 12.6, the set $\mathsf{P}=\{P^{[t]}: t\in\mathbb{R}^{++}\}$, with each $P^{[t]}$ as in (12.67), is a null family of processes for $\mathfrak{s}$ at $\sigma_\circ$, and $\mathsf{P}\sigma_\circ=\sigma_*$ with σ_* as in (12.77). It therefore follows from Theorem 7.6 that σ_* is in Σ° and $S(\sigma_*)\geqq S(\sigma_\circ)$. But, by (12.50), $S(\sigma_\circ)=S(\sigma)$, and (12.79) follows forthwith; q.e.d.

It follows from (12.76) and item (1) of Definition 12.1 that the set Θ_i is an open connected subset of $\mathrm{Lin}(\mathscr{V})\oplus\mathbb{R}$. If c is an oriented polygonal curve in Θ_i, we write $J(c)$ for the line integral

$$J(c)=\int_c \left[-\frac{\boldsymbol{T}^\circ(\boldsymbol{F}, e, \mathbf{0})\,\boldsymbol{F}^{T^{-1}}}{\theta^\circ(\boldsymbol{F}, e, \mathbf{0})}\cdot d\boldsymbol{F}+\frac{1}{\theta^\circ(\boldsymbol{F}, e, \mathbf{0})}\,de\right], \tag{12.80}$$

where $\boldsymbol{T}^\circ$ and θ° are the equilibrium response functions defined in $(12.64)_{1\&2}$.

Lemma 12.2. Let $(\boldsymbol{F}_a, e_a)$ and $(\boldsymbol{F}_b, e_b)$ be two points in Θ_i, let σ_a and σ_b be the states given by

$$\sigma_a=(\boldsymbol{F}_a^\dagger, e_a^\dagger, \mathbf{0}^\dagger), \qquad \sigma_b=(\boldsymbol{F}_b^\dagger, e_b^\dagger, \mathbf{0}^\dagger), \tag{12.81}$$

and let c be a closed oriented polygonal curve joining $(\boldsymbol{F}_a, e_a)$ to $(\boldsymbol{F}_b, e_b)$ and lying in Θ_i. For each pair of positive numbers ν, ε, there is a process P in Π such that σ_a is in $\mathscr{D}(P)$,

$$\|P\sigma_a-\sigma_b\|<\nu, \tag{12.82}$$

and

$$|J(c)-\mathfrak{s}(P, \sigma_a)|<\varepsilon. \tag{12.83}$$

Cf. COLEMAN [1964, 1], Remark 31, p. 38.

Proof. The norm $|(\boldsymbol{F}, e)|$ of an element $(\boldsymbol{F}, e)$ of $\mathrm{Lin}(\mathscr{V}) \oplus \mathbb{R}$ equals the norm of the triple $(\boldsymbol{F}, e, \boldsymbol{0})$ regarded as an element of $\mathrm{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}$, *i.e.* $|(\boldsymbol{F}, e)| = |(\boldsymbol{F}, e, \boldsymbol{0})|$. If $(\boldsymbol{F}, e)$ is in Θ_i, then $(\boldsymbol{F}, e, \boldsymbol{0})$ is in Σ_i, the constant function $(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger)$, defined through (12.63), determines an element of Σ, and we have

$$\begin{aligned} \|(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger)\| &= |{}_0(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger)| + \|{}_r(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger)\|_r \\ &= |(\boldsymbol{F}, e, \boldsymbol{0})| + \left[\int_0^\infty k(s)\,|(\boldsymbol{F}, e, \boldsymbol{0})|^p\, ds\right]^{\frac{1}{p}} \\ &= |(\boldsymbol{F}, e)|\, N, \end{aligned} \tag{12.84}$$

with

$$N = 1 + \left[\int_0^\infty k(s)\, ds\right]^{\frac{1}{p}}. \tag{12.85}$$

Let $(\boldsymbol{F}_a, e_a)$, $(\boldsymbol{F}_b, e_b)$, and $\mathscr{c}$ be as in the hypothesis. As $\mathscr{c}$ is compact, the number,

$$M \overset{\text{def}}{=} \{\sup |(\boldsymbol{F}, e)| \,\big|\, (\boldsymbol{F}, e) \in \mathscr{c}\}, \tag{12.86}$$

is finite. For each $\delta > 0$, let

$$\mathfrak{T}_\delta = \{\sigma = (\boldsymbol{F}, e, \boldsymbol{0}, {}_r\sigma) \mid \sigma \in \Sigma,\ (\boldsymbol{F}, e) \in \mathscr{c},\ \|{}_r\sigma - {}_r(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger)\|_r < \delta\}. \tag{12.87}$$

The set

$$C = \{(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger) \mid (\boldsymbol{F}, e) \in \mathscr{c}\}$$

is a connected compact subset of Σ, because it is the image in Σ, via a continuous function, of the curve $\mathscr{c}$ which is, of course, a connected compact subset of Θ_i. Hence, by Remark 12.3 and item (2) of Definition 12.1, there are positive numbers L and δ such that if g equals $\boldsymbol{T}$ or $1/\theta$, then

$$|g(\sigma_i) - g(\sigma_{ii})| \leq L \|\sigma_i - \sigma_{ii}\|$$

whenever σ_i and σ_{ii} are within distance δ of C. As each element of $\mathfrak{T}_\delta$ is within distance δ of C, we may conclude that for each $\varepsilon > 0$ there exists η in $(0, \delta]$ such that if σ_i and σ_{ii} are in $\mathfrak{T}_\delta$ and satisfy $\|\sigma_i - \sigma_{ii}\| < \eta$, then (12.39) holds (with l again the length of $\mathscr{c}$).

Now, let $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau))$ be a piecewise linear parameterization on $[0, 1]$ of the given curve $\mathscr{c}$. Since this parameterization is a uniformly continuous function on $[0, 1]$, there is, for each $\mu > 0$, a positive integer $n = n(\mu)$ such that

$$|(\boldsymbol{F}(\tau'), e(\tau')) - (\boldsymbol{F}(\tau''), e(\tau''))| < \mu \tag{12.88}$$

for all τ', τ'' in $[0, 1]$ with $|\tau' - \tau''| < 1/n$. For each k in $\{1, 2, \ldots, n\}$, let $\mathscr{c}^{(k)}$ be the segment of $\mathscr{c}$ obtained by restricting the parameterization $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau))$ to the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$. We now construct a second parameterization $\tau \mapsto (\boldsymbol{F}_*(\tau), e_*(\tau))$ of $\mathscr{c}$ which traces out the segments $\mathscr{c}^{(1)}, \mathscr{c}^{(2)}, \ldots, \mathscr{c}^{(n)}$, and "pauses" at the endpoint $\left(\boldsymbol{F}\left(\frac{k}{n}\right), e\left(\frac{k}{n}\right)\right)$ of each segment $\mathscr{c}^{(k)}$ for an interval of length $\lambda_k > 0$. As in the proof of Lemma 12.1, we can use item (3) of Definition 12.1 to obtain from such a parameterization of $\mathscr{c}$ a process P; our goal here is to show that this process P

satisfies (12.82) and (12.83) when the numbers $\lambda_1, \ldots, \lambda_n$ are chosen to be sufficiently large. To construct the function $\tau \mapsto (\boldsymbol{F}_*(\tau), e_*(\tau))$, we shall first choose a parameterization of each segment c and so obtain processes $P^{(k)}$, whose properties are easily analyzed; we shall then employ the formula $P = P^{(n)} P^{(n-1)} \ldots P^{(1)}$ to show that P has the properties desired of it. To this end, let λ_1 be a positive number, let $t_1 = \frac{1}{n} + \lambda_1$, and let $(\boldsymbol{F}^{(1)}, e^{(1)}, \mathbf{0})$: $[0, t_1] \to \operatorname{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}$ be defined by

$$(\boldsymbol{F}^{(1)}(\tau), e^{(1)}(\tau), \mathbf{0}) = \begin{cases} (\boldsymbol{F}(\tau), e(\tau), \mathbf{0}), & \text{for } \tau \in \left[0, \frac{1}{n}\right) \\ \left(\boldsymbol{F}\left(\frac{1}{n}\right), e\left(\frac{1}{n}\right), \mathbf{0}\right), & \text{for } \tau \in \left[\frac{1}{n}, t_1\right]. \end{cases} \tag{12.89}$$

Clearly, $(\boldsymbol{F}^{(1)}, e^{(1)})$ is a parameterization of $c^{(1)}$ and is piecewise differentiable on $[0, t_1]$. By item (3) of Definition 12.1, the function $(\boldsymbol{F}^{(1)}, e^{(1)}, \mathbf{0})$ determines a process P_{t_1} in Π such that $\sigma^{(1)}$ is in $\mathscr{D}(P_{t_1})$ and such that, for each τ in $(0, t_1]$, the corresponding process P_τ obeys

$$\begin{aligned} {}_0P_\tau \sigma_a &= (\boldsymbol{F}^{(1)}(\tau), e^{(1)}(\tau), \mathbf{0}), \\ {}_rP_\tau \sigma_a &= \mathsf{T}^{(\tau)}{}_r(\boldsymbol{F}_a^\dagger, e_a^\dagger, \mathbf{0}^\dagger) + \mathfrak{d}\boldsymbol{f}_1^\tau, \end{aligned} \tag{12.90}$$

with

$$\mathfrak{d}\boldsymbol{f}_1^{(\tau)}(s) = \begin{cases} (\boldsymbol{F}^{(1)}(\tau - s), e^{(1)}(\tau - s), \mathbf{0}), & \text{for } s \in (0, \tau], \\ \mathbf{0}, \quad \text{for } s \in (\tau, \infty). \end{cases} \tag{12.91}$$

We now fix τ in $(0, t_1]$ and observe that, for the state $(\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)$, we have

$$\begin{aligned} {}_0(\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger) &= (\boldsymbol{F}^{(1)}(\tau), e^{(1)}(\tau), \mathbf{0}) \\ {}_r(\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger) &= \mathsf{T}^{(\tau)}{}_r(\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger) + \mathfrak{d}\boldsymbol{g}_1^\tau, \end{aligned}$$

with

$$\mathfrak{d}\boldsymbol{g}_1^\tau(s) = \begin{cases} (\boldsymbol{F}^{(1)}(\tau), e^{(1)}(\tau), \mathbf{0}), & \text{for } s \in (0, \tau], \\ \mathbf{0}, \quad \text{for } s \in (\tau, \infty), \end{cases}$$

and in view of (12.90), (12.91), and (12.9), these last formulae tell us that

$$\begin{aligned} &\|P_\tau \sigma_a - (\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)\| = \|{}_rP_\tau \sigma_a - {}_r(\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)\|_r \\ &\quad = \|\mathsf{T}^{(\tau)}[{}_r(\boldsymbol{F}_a^\dagger, e_a^\dagger, \mathbf{0}^\dagger) - {}_r(\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)] + \mathfrak{d}\boldsymbol{f}_1^\tau - \mathfrak{d}\boldsymbol{g}_1^\tau\|_r \\ &\quad \leqq \|\mathsf{T}^{(\tau)}[{}_r(\boldsymbol{F}_a^\dagger, e_a^\dagger, \mathbf{0}^\dagger) - {}_r(\boldsymbol{F}^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)]\|_r \\ &\qquad + \left[\int_0^\tau |(\boldsymbol{F}^{(1)}(\tau - s), e^{(1)}(\tau - s), \mathbf{0}) - (\boldsymbol{F}^{(1)}(\tau), e^{(1)}(\tau), \mathbf{0})|^p k(s)\, ds\right]^{\frac{1}{p}}. \end{aligned} \tag{12.92}$$

By (12.89) and (12.88),

$$\left[\int_0^\tau |(\boldsymbol{F}^{(1)}(\tau - s), e^{(1)}(\tau - s), \mathbf{0}) - (\boldsymbol{F}^{(1)}(\tau), e^{(1)}(\tau), \mathbf{0})|^p k(s)\, ds\right]^{\frac{1}{p}} < (N-1)\mu \tag{12.93}$$

with N given by (12.85). COLEMAN & MIZEL* have shown that there exists a positive number K such that, for each $\phi \in \mathfrak{U}_r$ and $\sigma \geqq 0$,

$$\|\mathsf{T}^{(\sigma)}\phi\|_r \leqq K\|\phi\|_r.$$

Employing this and (12.84), we find that

$$\begin{aligned}&\|\mathsf{T}^{(\tau)}[{}_r(F_a^\dagger, e_a^\dagger, \mathbf{0}^\dagger) - {}_r(F^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)]\|_r \\ &\quad \leqq KN\,|(F_a, e_a) - (F^{(1)}(\tau), e^{(1)}(\tau))|,\end{aligned} \tag{12.94}$$

and (12.89) and (12.88) yield

$$|(F_a, e_a) - (F^{(1)}(\tau), e^{(1)}(\tau))| = |(F(0), e(0)) - (F(\tau), e(\tau))| < \mu.$$

In view of (12.92)–(12.94) and this last inequality we have, for each τ in $(0, t_1]$,

$$\|P_\tau \sigma_a - (F^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)\| < [N(K+1) - 1]\,\mu. \tag{12.95}$$

It is easy to see that our derivation of (12.95) places no restriction on the positive number λ_1. The relation (12.12) implies that for λ_1 sufficiently large there holds

$$\left\|\mathsf{T}^{(\lambda_1)}\left[{}_r P_{\frac{1}{n}}\sigma_a - {}_r\!\left(F\left(\frac{1}{n}\right)^\dagger, e\left(\frac{1}{n}\right)^\dagger, \mathbf{0}^\dagger\right)\right]\right\|_r < \mu. \tag{12.96}$$

We choose λ_1 and P_{t_1} so that both (12.95) and (12.96) are valid. In order to make use of (12.96), we note that, by (12.89)–(12.91), $P_{t_1}\sigma_a$ obeys

$${}_0 P_{t_1}\sigma_a = \left(F\left(\frac{1}{n}\right), e\left(\frac{1}{n}\right), \mathbf{0}\right),$$

$${}_r P_{t_1}\sigma_a = \mathsf{T}^{(\lambda_1)}{}_r(P_{\frac{1}{n}}\sigma^{(1)}) + \mathfrak{d}\mathbf{h}_1^{\lambda_1},$$

with

$$\mathfrak{d}\mathbf{h}_1^{\lambda_1}(s) = \begin{cases}\left(F\left(\frac{1}{n}\right), e\left(\frac{1}{n}\right), \mathbf{0}\right), & \text{for } s \in (0, \lambda_1], \\ 0, \quad \text{for } s \in (\lambda_1, \infty),\end{cases}$$

and from this it follows immediately that

$$\begin{aligned}\|P_{t_1}\sigma_a - (F^{(1)}(t_1)^\dagger, e^{(1)}(t_1)^\dagger, \mathbf{0}^\dagger)\| &= \left\|P_{t_1}\sigma_a - \left(F\left(\frac{1}{n}\right)^\dagger, e\left(\frac{1}{n}\right)^\dagger, \mathbf{0}^\dagger\right)\right\| \\ &= \left\|{}_r P_{t_1}\sigma_a - {}_r\!\left(F\left(\frac{1}{n}\right)^\dagger, e\left(\frac{1}{n}\right)^\dagger, \mathbf{0}^\dagger\right)\right\|_r \\ &= \left\|\mathsf{T}^{(\lambda_1)}\left[{}_r P_{\frac{1}{n}}\sigma_a - {}_r\!\left(F\left(\frac{1}{n}\right)^\dagger, e\left(\frac{1}{n}\right)^\dagger, \mathbf{0}^\dagger\right)\right]\right\|_r.\end{aligned}$$

Thus, (12.96) tells us that

$$\|P_{t_1}\sigma_a - (F^{(1)}(t_1)^\dagger, e^{(1)}(t_1)^\dagger, \mathbf{0}^\dagger)\| < \mu. \tag{12.97}$$

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* [1966, 2], Remark 6.4, p. 111.

In summary, we have shown that there exists a positive number t_1, a process P_{t_1} in Π, and a parameterization $\tau \mapsto (F^{(1)}(\tau), e^{(1)}(\tau))$ of $c^{(1)}$ on $[0, t_1]$ such that, for each τ in $(0, t_1]$,

$$_0P_\tau\sigma_a = (F^{(1)}(\tau), e^{(1)}(\tau), \mathbf{0}),$$

and

$$\|P_\tau\sigma_a - (F^{(1)}(\tau)^\dagger, e^{(1)}(\tau)^\dagger, \mathbf{0}^\dagger)\| < [N(K+1)-1]\,\mu,$$

and such that (12.97) holds. Repetition of the argument given above, with one additional estimate, shows that there exist positive numbers $t_2, t_3, \ldots, t_n$, processes $P_{t_2}, P_{t_3}, \ldots, P_{t_n}$ in Π, and parameterizations $\tau \mapsto (F^{(k)}(\tau), e^{(k)}(\tau))$ of $c^{(k)}$ on $[0, t_k]$ such that, for each $k = 2, 3, \ldots, n$ and τ in $(0, t_k]$,

$$_0P_\tau P_{t_{k-1}} \cdots P_{t_2} P_{t_1}\sigma_a = (F^{(k)}(\tau), e^{(k)}(\tau), \mathbf{0}),$$

$$\|P_\tau P_{t_{k-1}} \cdots P_{t_2} P_{t_1}\sigma_a - (F^{(k)}(\tau)^\dagger, e^{(k)}(\tau)^\dagger, \mathbf{0}^\dagger)\| < [N(K+1)-1+K]\,\mu,$$

and such that

$$\|P_{t_k} P_{t_{k-1}} \cdots P_{t_2} P_{t_1}\sigma_a - (F^{(k)}(t_k)^\dagger, e^{(k)}(t_k)^\dagger, \mathbf{0}^\dagger)\| < \mu.$$

Let us now put

$$\mu = \min\{\nu, \eta/[N(K+1)+(K-1)]\},$$

$$t = t_1 + \cdots + t_n, \qquad t_0 = 0,$$

and, for each k in $\{1, \ldots, n\}$ and all τ in $\left[\sum_{j=0}^{k-1} t_j, \sum_{j=0}^{k} t_j\right]$,

$$(F_*(\tau), e_*(\tau)) = \left(F^{(k)}\left(\tau - \sum_{j=0}^{k-1} t_j\right), e^{(k)}\left(\tau - \sum_{j=0}^{k-1} t_j\right)\right).$$

If we let $P_t\colon [0, t) \to U$ be determined by the function $\tau \mapsto (F_*(\tau), e_*(\tau))$ as in the proof of Lemma 12.1, then it follows that P_t is a process,

$$P_t = P_{t_n} \cdots P_{t_1},$$

$$\|P_t\sigma_a - \sigma_b\| < \nu,$$

and

$$\begin{aligned} &_0P_\tau = (F_*(\tau), e_*(\tau), \mathbf{0}),\\ &\|P_\tau\sigma_a - (F_*(\tau)^\dagger, e_*(\tau)^\dagger, \mathbf{0}^\dagger)\| < \eta. \end{aligned} \tag{12.98}$$

In particular, P_t obeys (12.82) and, by $(12.98)_{1,2}$, (12.87), and (12.16), we have

$$P_\tau\sigma_a \in \mathfrak{T}_\delta, \tag{12.99}$$

for each τ in $(0, t]$. Clearly, each state $(F_*(\tau)^\dagger, e_*(\tau)^\dagger, \mathbf{0}^\dagger)$ also is in $\mathfrak{T}_\delta$, and, by $(12.98)_{1,2}$ and (12.99), we can set $\sigma_i = P_\tau\sigma_a$ and $\sigma_{ii} = (F_*(\tau)^\dagger, e_*(\tau)^\dagger, \mathbf{0}^\dagger)$ in (12.39) to obtain

$$\left|\frac{T(P_\tau\sigma_a)F_*(\tau)^{T^{-1}}}{\theta(P_\tau\sigma_a)} - \frac{T(F_*(\tau)^\dagger, e_*(\tau)^\dagger, \mathbf{0}^\dagger)F_*(\tau)^{T^{-1}}}{\theta(F_*(\tau)^\dagger, e_*(\tau)^\dagger, \mathbf{0}^\dagger)}\right|^2 + \left|\frac{1}{\theta(P_\tau\sigma_a)} - \frac{1}{\theta(F_*(\tau)^\dagger, e_*(\tau)^\dagger, \mathbf{0}^\dagger)}\right|^2 < \frac{\varepsilon^2}{4l^2}$$

which, by (12.64), yields

$$\left|\frac{T(P_\tau \sigma_a) F_*(\tau)^{T^{-1}}}{\theta(P_\tau \sigma_a)} - \frac{T^\circ(F_*(\tau), e_*(\tau), 0) F_*(\tau)^{T^{-1}}}{\theta^\circ(F_*(\tau), e_*(\tau), 0)}\right|^2 + \left|\frac{1}{\theta(P_\tau \sigma_a)} - \frac{1}{\theta^\circ(F_*(\tau), e_*(\tau), 0)}\right|^2 < \frac{\varepsilon^2}{4l^2}, \tag{12.100}$$

for each τ in $(0, t]$. By (12.80), (9.7), (9.6), (9.3), $(12.98)_1$, and (12.100) we have

$$\begin{aligned} |J(c) - s(P_t, \sigma_a)| &= \left| -\int_0^t \frac{T^\circ(F_*(\tau), e_*(\tau), 0) F_*(\tau)^{T^{-1}} \cdot \dot{F}_*(\tau) - \dot{e}_*(\tau)}{\theta^\circ(F(\tau), e(\tau), 0)} d\tau \right. \\ &\quad \left. + \int_0^t \frac{T(P_\tau \sigma_a) F_*(\tau)^{T^{-1}} \cdot \dot{F}_*(\tau) - \dot{e}_*(\tau)}{\theta(P_\tau \sigma_a)} d\tau \right| \\ &\leqq \int_0^t \sqrt{\frac{\varepsilon^2}{4l^2}} \sqrt{|\dot{F}_*(\tau)|^2 + |\dot{e}_*(\tau)|^2}\, d\tau = \frac{\varepsilon}{2l} l = \frac{\varepsilon}{2} < \varepsilon; \end{aligned}$$

q.e.d.

Theorem 12.4. Let $\hat{S}: \mathscr{S} \to \mathbb{R}$ be an entropy function for an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with fading memory, and let $\mathscr{S}^\circ$ be the set of all triples $(\boldsymbol{F}, e, \boldsymbol{\beta})$ in Σ_i with $(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger)$ in $\mathscr{S}$. If the domain $\mathscr{S}$ of $\hat{S}$ contains the domain Σ° of the entropy function S of Theorem 12.1, then the "equilibrium response function" $\hat{S}^\circ$, *i.e.* the function on $\mathscr{S}^\circ$ defined by

$$\hat{S}^\circ(\boldsymbol{F}, e, \boldsymbol{\beta}) = \hat{S}(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{\beta}^\dagger), \tag{12.101}$$

has the following properties:

(1) if $(\boldsymbol{F}_{(1)}, e_{(1)})$ and $(\boldsymbol{F}_{(2)}, e_{(2)})$ are in Θ_i, then $(\boldsymbol{F}_{(1)}, e_{(1)}, \boldsymbol{0})$ and $(\boldsymbol{F}_{(2)}, e_{(2)}, \boldsymbol{0})$ are in $\mathscr{S}^\circ$, and, for every oriented rectifiable curve in Θ joining $(\boldsymbol{F}_{(1)}, e_{(1)})$ to $(\boldsymbol{F}_{(2)}, e_{(2)})$, there holds

$$\hat{S}^\circ(\boldsymbol{F}_{(2)}, e_{(2)}, \boldsymbol{0}) - \hat{S}^\circ(\boldsymbol{F}_{(1)}, e_{(1)}, \boldsymbol{0}) = J(c^+), \tag{12.102}$$

where J is as in (12.80);

(2) the function $\hat{S}^\circ(\cdot, \cdot, \boldsymbol{0})$ is continuously differentiable on its domain Θ_i and obeys the familiar "equilibrium stress and temperature relations",

$$\begin{aligned} \boldsymbol{T}^\circ(\boldsymbol{F}, e, \boldsymbol{0}) &= -\theta^\circ(\boldsymbol{F}, e, \boldsymbol{0})[\partial_{\boldsymbol{F}} \hat{S}^\circ(\boldsymbol{F}, e, \boldsymbol{0})]\, \boldsymbol{F}^T, \\ \theta^\circ(\boldsymbol{F}, e, \boldsymbol{0}) &= [\partial_e \hat{S}^\circ(\boldsymbol{F}, e, \boldsymbol{0})]^{-1}. \end{aligned} \tag{12.103}$$

Proof. As each state σ of the form $\sigma = (\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger)$, with $(\boldsymbol{F}, e)$ in Θ_i, is, by Theorem 12.3, an element of Σ°, such states are also in $\mathscr{S}$. Hence $(\boldsymbol{F}_{(j)}, e_{(j)})$ in Θ_i implies that $(\boldsymbol{F}_{(j)}, e_{(j)}, \boldsymbol{0})$ is in the domain $\mathscr{S}^\circ$ of $\hat{S}^\circ$. Now, let there be given $(\boldsymbol{F}_{(1)}, e_{(1)})$ and $(\boldsymbol{F}_{(2)}, e_{(2)})$ in Θ_i, an oriented polygonal curve c^+ lying in Θ_i and joining $(\boldsymbol{F}_{(1)}, e_{(1)})$ to $(\boldsymbol{F}_{(2)}, e_{(2)})$, and a positive number ε. Since $\hat{S}$ is an entropy function, *i.e.* an upper potential for s, and since the states $\sigma^{(1)} = (\boldsymbol{F}^\dagger_{(1)}, e^\dagger_{(1)}, \boldsymbol{0}^\dagger)$ and $\sigma^{(2)} = (\boldsymbol{F}^\dagger_{(2)}, e^\dagger_{(2)}, \boldsymbol{0}^\dagger)$ are in $\mathscr{S}$, there exists $\nu > 0$ such that

$$s(P, \sigma^{(1)}) < \hat{S}(\sigma^{(2)}) - \hat{S}(\sigma^{(1)}) + \frac{\varepsilon}{2}, \tag{12.104}$$

for each process P in Π with

$$\|P\sigma^{(1)}-\sigma^{(2)}\|<\nu. \tag{12.105}$$

Lemma 12.2 tells us not only that there exists a process P obeying (12.105) and hence (12.104), but also that P may be chosen so that

$$J(c^+)-s(P,\sigma^{(1)})<\frac{\varepsilon}{2},$$

and, for such a process, this inequality when added to (12.104) yields

$$J(c^+)<\hat{S}(\sigma^{(2)})-\hat{S}(\sigma^{(1)})+\varepsilon.$$

Because $\varepsilon>0$ is arbitrary and $\hat{S}^\circ$ obeys (12.101), we have

$$J(c^+)\leqq\hat{S}^\circ(\boldsymbol{F}_{(2)},e_{(2)},\boldsymbol{0})-\hat{S}^\circ(\boldsymbol{F}_{(1)},e_{(1)},\boldsymbol{0}). \tag{12.106}$$

If we now interchange $(\boldsymbol{F}_{(1)},e_{(1)})$ and $(\boldsymbol{F}_{(2)},e_{(2)})$ and replace c^+ by the curve c^- which differs from c^+ only in orientation, then the argument which gave (12.106) gives

$$-J(c^+)=J(c^-)\leqq\hat{S}^\circ(\boldsymbol{F}_{(1)},e_{(1)},\boldsymbol{0})-\hat{S}^\circ(\boldsymbol{F}_{(2)},e_{(2)},\boldsymbol{0}),$$

which, in view of (12.106), yields (12.102) when c is polygonal (and hence when c is rectifiable). In view of (12.80), it follows from (12.102) that $\hat{S}^\circ(\cdot,\cdot,\boldsymbol{0})$ is a potential (in the usual sense) for the following vector field on $\Theta_i\subset\mathrm{Lin}(\mathscr{V})\oplus\mathbb{R}$:

$$(\boldsymbol{F},e)\mapsto\left(\frac{-\boldsymbol{T}^\circ(\boldsymbol{F},e,\boldsymbol{0})\boldsymbol{F}^{T^{-1}}}{\theta^\circ(\boldsymbol{F},e,\boldsymbol{0})},\frac{1}{\theta^\circ(\boldsymbol{F},e,\boldsymbol{0})}\right);$$

thus $\hat{S}^\circ(\cdot,\cdot,\boldsymbol{0})$ is differentiable on Θ_i and has this vector field for its gradient; in other words, $\hat{S}^\circ$ obeys (12.103). Because the function $(\boldsymbol{F},e,\boldsymbol{\beta})\mapsto(\boldsymbol{F}^\dagger,e^\dagger,\boldsymbol{\beta}^\dagger)$ is continuous with its domain Σ_i in U and its range in $\mathfrak{U}$, and the functions $\boldsymbol{T}$ and θ are continuous on $\Sigma\subset\mathfrak{U}$, it follows that T°, θ° are continuous functions on Θ_i, and hence (12.103) implies that the gradient of $\hat{S}(\cdot,\cdot,\boldsymbol{0})$ is continuous on Θ_i. Thus $\hat{S}^\circ$ has properties (1) and (2); q.e.d.

The following remark is an immediate corollary of the theorem just proven.

Remark 12.7. Consider an element with fading memory that obeys the Second Law of Thermodynamics, let Σ° be the domain of the entropy function S of Theorem 12.1, and let Σ^* be the set of all states in Σ° which have the form $(\boldsymbol{F}^\dagger,e^\dagger,\boldsymbol{0}^\dagger)$. If $\hat{S}$ is any entropy function for the same element whose domain contains Σ° as a subset, then the restriction of $\hat{S}$ to Σ^* must equal, to within a constant, the restriction of S to Σ^*, and both restrictions must obey (12.103).

An argument employing Theorem 12.1 and Lemma 12.1, instead of Theorem 12.3 and Lemma 12.2, but otherwise entirely analogous to that used to prove Theorem 12.4, yields

Theorem 12.5. Let $\hat{S}:\mathscr{S}\to\mathbb{R}$ be an entropy function for an element $(\Sigma,\Pi,\theta,\boldsymbol{T},\boldsymbol{q})$ with fading memory. If Σ°, the domain of the entropy function S of Theorem 12.1, is a subset of $\mathscr{S}$, then $\hat{S}$ has the following properties:

(1) if $(\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma')$ is in Σ° and $(\boldsymbol{F}'', e'', \boldsymbol{\beta}'')$ is in Σ_i, then, for every oriented rectifiable curve c^+ in Σ_i joining $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ to $(\boldsymbol{F}'', e'', \boldsymbol{\beta}'')$, there holds

$$\hat{S}(\boldsymbol{F}'', e'', \boldsymbol{\beta}'', {}_r\sigma') - \hat{S}(\boldsymbol{F}', e', \boldsymbol{\beta}', {}_r\sigma') = I(c^+),$$

where I is the line integral defined in (12.33);

(2) at each state $\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)$ in Σ°, $\hat{S}$ has instantaneous derivatives $D_{\boldsymbol{F}}\hat{S}$, $D_e\hat{S}$, $D_{\boldsymbol{\beta}}\hat{S}$ obeying the relations

$$\begin{aligned} \boldsymbol{T}(\sigma) &= -\theta(\sigma)\, D_{\boldsymbol{F}}\hat{S}(\sigma)\, \boldsymbol{F}^T, \\ \theta(\sigma) &= [D_e\hat{S}(\sigma)]^{-1}, \\ \boldsymbol{0} &= D_{\boldsymbol{\beta}}\hat{S}(\sigma); \end{aligned} \tag{12.107}$$

(3) if ${}_r\sigma$ is the past history of a state $\sigma = ({}_0\sigma, {}_r\sigma)$ in Σ°, then the function $\hat{S}(\cdot, {}_r\sigma): \Sigma_i \to \mathbb{R}$ and the function $S(\cdot, {}_r\sigma)$, with S given by Theorem 12.1, obey a relation of the form

$$\hat{S}(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma) = S(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma) + f({}_r\sigma)$$

for all triples $(\boldsymbol{F}, e, \boldsymbol{\beta})$ in Σ_i.

The following theorem shows that a condition analogous to a "work axiom" proposed by DAY [1968, 3] implies the Second Law for an element with fading memory.

Theorem 12.6. Let σ° be a state of an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with fading memory, and let $\mathfrak{s}$ be as in (9.7). If the set

$$\mathfrak{C} \overset{\text{def}}{=} \{\mathfrak{s}(P, \sigma^\circ) \mid P \in \Pi,\ {}_0(P\sigma^\circ) = {}_0\sigma^\circ\} \tag{12.108}$$

is bounded above, then $\mathfrak{s}$ has the Clausius property at σ°.

Proof. We shall show that if $\mathfrak{s}$ does *not* have the Clausius property at σ°, then the set $\mathfrak{C}$ is *not* bounded above. To this end we observe that there is an $M > 0$ and a neighborhood $\mathcal{O}$ of σ° such that, for each state $\sigma = (\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)$ in $\mathcal{O}$,

$$\left| \frac{\boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)\, \boldsymbol{F}^{T^{-1}}}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)} \right|^2 + \left| \frac{1}{\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, {}_r\sigma)} \right|^2 < M^2, \tag{12.109}$$

and the line segment c_σ in U joining $(\boldsymbol{F}, e, \boldsymbol{\beta})$ to ${}_0\sigma^\circ$ lies in Σ_i and has length less than 1. The existence of M and $\mathcal{O}$ follows from items (1) and (2) of Definition 12.1. By (12.109), we have, for each σ in $\mathcal{O}$,

$$|I(c_\sigma)| < \int_{c_\sigma} M\, dl = M\, l < M, \tag{12.110}$$

where $I(c_\sigma)$ is defined by (12.33) with ${}_r\sigma$ playing the role of ${}_r\sigma'$, and l is the length of c_σ. Now, suppose that $\mathfrak{s}$ does *not* have the Clausius property at σ°, and let b be an arbitrary number. Lemma 3.1 tells us that, for each neighborhood $\mathcal{O}'$ of σ°, the set $\mathfrak{s}\{\sigma^\circ \to \mathcal{O}'\}$ is *not* bounded above. In particular, there is a process P_1 such that the state $\sigma' \overset{\text{def}}{=} P_1 \sigma^\circ$ is in $\mathcal{O}$ and

$$\mathfrak{s}(P_1, \sigma^\circ) > M + 1 + b. \tag{12.111}$$

If in Lemma 12.1 we put $\sigma'' = ({}_0\sigma^\circ, {}_r\sigma')$ and $c = c_{\sigma'}$, we may conclude from that lemma and its proof that there exists a process P_2 such that

$$_0(P_2 \sigma') = {}_0\sigma^\circ \tag{12.112}$$

and

$$\mathfrak{s}(P_2, \sigma') > I(c_{\sigma'}) - 1.$$

From this, the additivity of $\mathfrak{s}$, and (12.111), we obtain

$$\mathfrak{s}(P_2 P_1, \sigma^\circ) = \mathfrak{s}(P_2, \sigma') + \mathfrak{s}(P_1, \sigma^\circ) > I(c_{\sigma'}) + M + b,$$

and, since (12.110) yields $I(c_{\sigma'}) > -M$, we have

$$\mathfrak{s}(P_2 P_1, \sigma^\circ) > b. \tag{12.113}$$

Moreover, since $\sigma' = P_1 \sigma^\circ$, we have $P_2 P_1 \sigma^\circ = P_2 \sigma'$, and it follows from (12.112) that $_0(P_2 P_1 \sigma^\circ) = {}_0\sigma^\circ$. Hence the number $\mathfrak{s}(P_2 P_1, \sigma^\circ)$ is in the set $\mathfrak{C}$, and, by (12.113), the number b is not an upper bound for $\mathfrak{C}$. Since b is arbitrary, $\mathfrak{C}$ is *not* bounded above; q.e.d.

Theorem 12.7. Let σ° be a state of an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with fading memory, suppose that σ° has the form

$$\sigma^\circ = (\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger), \tag{12.114}$$

and let $\mathfrak{C}$ be the following set of numbers:

$$\mathfrak{C} = \{\mathfrak{s}(P, \sigma^\circ) \mid P \in \Pi, \ _0(P\sigma^\circ) = {}_0\sigma^\circ\},$$

with $\mathfrak{s}$ as in (9.7). The action $\mathfrak{s}$ has the Clausius property at σ° if and only if

$$\sup \mathfrak{C} = 0. \tag{12.115}$$

Proof. It follows from Theorem 12.6 that if (12.115) holds, $\mathfrak{s}$ has the Clausius property at σ°. Thus, the only thing to prove here is the assertion that if $\mathfrak{s}$ has the Clausius property at σ° and if, in addition, σ° has the form (12.114), then (12.115) holds. To prove this, we let $\sigma^\circ = (\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger)$, we let P in Π be such that $_0(P\sigma^\circ) = {}_0\sigma^\circ$, and we let S be the entropy function of (12.54). By Theorem 12.3 and item (1) of Theorem 12.1, both σ° and $P\sigma^\circ$ are in Σ°, and there holds

$$S(P\sigma^\circ) \leqq S(\sigma^\circ). \tag{12.116}$$

As S is an upper potential for $\mathfrak{s}$, there also holds

$$\mathfrak{s}(P, \sigma^\circ) \leqq S(P\sigma^\circ) - S(\sigma^\circ). \tag{12.117}$$

It follows from (12.116) and (12.117) that

$$\mathfrak{s}(P, \sigma^\circ) \leqq 0,$$

and, as P is an arbitrary process with $_0(P\sigma^\circ)={}_0\sigma$, we conclude that 0 is an upper bound for $\mathfrak{C}$. Because, for $t>0$, the process $P^{[t]}$ of (12.67) obeys $_0(P\sigma^\circ)={}_0\sigma^\circ$ and

$$\mathfrak{s}(P^{[t]}, \sigma^\circ)=0,$$

the number 0 is in $\mathfrak{C}$ and hence is the least upper bound for $\mathfrak{C}$; q.e.d.

As an immediate consequence of Theorems 12.6 and 12.7, we have

Remark 12.8. Let σ° be a state of an element with fading memory and have the form (12.114). If the set $\mathfrak{C}$ defined in (12.108) is bounded above, then $\sup \mathfrak{C}=0$.

The main results of this section are contained in Theorems 12.1, 12.3, and 12.4 and are summarized below.

Remark 12.9. For an element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ with fading memory that obeys the Second Law of Thermodynamics, there exists an upper semicontinuous entropy function S, given by (5.5), which has the properties listed below:

(i) the domain Σ° of S is given by (5.8), contains σ°, is closed under $\mathfrak{s}$-approach, and is such that if σ' is in Σ°, then so also is every σ in Σ with $_r\sigma={}_r\sigma'$;

(ii) at each state in Σ°, the function S obeys (12.50) and has instantaneous derivatives which obey the "generalized stress relations" (12.47) and (12.48);

(iii) for each choice of $(\boldsymbol{F}_*, e_*)$ in Θ_i, of all states in Σ° whose present values have the form $_0\sigma=(\boldsymbol{F}_*, e_*, \boldsymbol{\beta})$, the state $(\boldsymbol{F}_*^\dagger, e_*^\dagger, \boldsymbol{0}^\dagger)$ maximizes S;

(iv) the "equilibrium entropy function" $S^\circ(\cdot,\cdot,\boldsymbol{0})$, defined by

$$S^\circ(\boldsymbol{F}, e, \boldsymbol{0}) \stackrel{\text{def}}{=} S(\boldsymbol{F}^\dagger, e^\dagger, \boldsymbol{0}^\dagger),$$

is differentiable on Θ_i and obeys the "equilibrium stress relations" (12.103).

13. Viscous Elements

Definition 13.1. A simple material element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ is called a **viscous element** if

(1) Σ equals $\Sigma_i \times \operatorname{Lin}(\mathscr{V})$, where Σ_i is an open connected subset of

$$\operatorname{Lin}(\mathscr{V})^{++} \times \mathbb{R} \times \mathscr{V}$$

such that, for each triple $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ in Σ_i, the set $\{(\boldsymbol{F}', e', \boldsymbol{\beta}) \mid \boldsymbol{\beta} \in \mathscr{V}\}$ is contained in Σ_i; the topology of Σ is that induced by the natural topology of

$$\operatorname{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V} \oplus \operatorname{Lin}(\mathscr{V}) = U \oplus \operatorname{Lin}(\mathscr{V});$$

the mapping $\sigma \mapsto {}_0\sigma$ takes Σ into Σ_i and is defined to be

$$\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{M}) \quad \Rightarrow \quad {}_0\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta}); \tag{13.1}$$

(2) the functions θ and $\boldsymbol{q}$ of (9.5) are continuous, and $\boldsymbol{T}$ is continuously differentiable;

(3) let $P_t=(\boldsymbol{L}, h, \gamma)$ be a piecewise continuous function mapping an interval $[0, t)$, $t \in \mathbb{R}^{++}$, into the set U of (9.1), and let $\mathscr{D}(P_t)$ be the set of states σ such

that the differential equation,

$$\begin{aligned} \dot{\boldsymbol{F}} &= \boldsymbol{L}\boldsymbol{F}, \\ \dot{e} &= \boldsymbol{T}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{L}) \cdot \boldsymbol{L} + h, \\ \dot{\boldsymbol{\beta}} &= \gamma, \end{aligned} \tag{13.2}$$

with initial condition

$$(\boldsymbol{F}(0), e(0), \boldsymbol{\beta}(0)) = {}_0\sigma, \tag{13.3}$$

has a solution $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ whose values lie in Σ_i for $\tau \in [0, t]$. If $\mathscr{D}(P_t)$ is not empty, then P_t is a member of Π, and for each σ in $\mathscr{D}(P_t)$, the state $P_t\sigma$ is the element of $\Sigma_i \times \operatorname{Lin}(\mathscr{V})$ given by

$$P_t\sigma \stackrel{\text{def}}{=} (\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t), \boldsymbol{L}(t-)), \tag{13.4}$$

where

$$\boldsymbol{L}(t-) = \lim_{\tau \uparrow t} \boldsymbol{L}(\tau). \tag{13.5}$$

Arguments given after Definition 10.1 can be applied here to show that (a) for each P_t in Π the mapping $\sigma \to P_t\sigma$ of (13.4) is well defined (*i.e.* single-valued) and is continuous, (b) the set $\mathscr{D}(P_t)$ is open in Σ, and (*c*) for each initial condition (13.3) the mapping $\tau \mapsto \sigma_\tau$ of (9.3) has the properties mentioned in item (2) of Definition 9.1. It is also easily verified that item (3) of Definition 9.1 holds here. Thus, items (1)–(3) of Definition 9.1 hold for viscous elements, and, as Definition 2.1 requires, for such an element each P_t in Π is a continuous Σ-valued function on a non-empty open subset of Σ. Since the successive application $P_{t_2}P_{t_1} = P_{t_2+t_1}$ is defined as in item (1) of Definition 9.1, axiom **II** of Definition 2.1 holds here, and it is a corollary of Theorem 13.1 below that axiom **I** of Definition 2.1 also holds. Thus we can assert

Remark 13.1. Definition 13.1 is consistent with Definitions 9.1 and 2.1.

Theorem 13.1. For each pair of states σ' and σ'' of a viscous element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$ there is a sequence $n \mapsto P_n$ of processes such that $\sigma'' = \lim_{n\to\infty} P_n\sigma'$ and $\lim_{n\to\infty} \mathfrak{s}(P_n, \sigma')$ exists, where $\mathfrak{s}$ is as in (9.7). That is, every state of viscous element is $\mathfrak{s}$-approachable from every other state.

Proof. Let $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{M}')$ and $\sigma'' = (\boldsymbol{F}'', e'', \boldsymbol{\beta}'', \boldsymbol{M}'')$ be in Σ. By item (1) of Definition 13.1, the triples ${}_0\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}')$ and ${}_0\sigma'' = (\boldsymbol{F}'', e'', \boldsymbol{\beta}'')$ are in Σ_i and, as Σ_i is an open connected subset of U, there is an oriented polygonal curve $\mathfrak{c}$ which lies in Σ_i and joins ${}_0\sigma'$ to ${}_0\sigma''$. Let $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ be a piecewise linear parameterization of $\mathfrak{c}$ on an interval $[0, t]$, and consider the function $(\boldsymbol{L}^{(1)}, h^{(1)}, \gamma^{(1)})\colon [0, t) \to U$ defined by

$$\begin{aligned} \boldsymbol{L}^{(1)}(\tau) &= \dot{\boldsymbol{F}}(\tau)\boldsymbol{F}(\tau)^{-1}, \\ h^{(1)}(\tau) &= \dot{e}(\tau) - \boldsymbol{T}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau), \boldsymbol{L}^{(1)}(\tau)) \cdot \boldsymbol{L}^{(1)}(\tau), \\ \gamma^{(1)}(\tau) &= \dot{\boldsymbol{\beta}}(\tau). \end{aligned} \tag{13.6}$$

If we put $(\boldsymbol{L}, h, \gamma)=(\boldsymbol{L}^{(1)}, h^{(1)}, \gamma^{(1)})$, then the function $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ is a solution of (13.2) whose values lie in Σ_i and which obeys the initial condition (13.3) with ${}_0\sigma = {}_0\sigma'$. It follows from item (3) of Definition 13.1 that $P^{(1)} = (\boldsymbol{L}^{(1)}, h^{(1)}, \gamma^{(1)})$ is a member of Π, and, by (13.4) and the fact that $\tau \mapsto (\boldsymbol{F}(\tau), e(\tau), \boldsymbol{\beta}(\tau))$ parameterizes c, we have

$$\begin{aligned} P^{(1)}\sigma' &= (\boldsymbol{F}(t), e(t), \boldsymbol{\beta}(t), \boldsymbol{L}^{(1)}(t-)) \\ &= (\boldsymbol{F}'', e'', \boldsymbol{\beta}'', \boldsymbol{L}^{(1)}(t-)). \end{aligned} \tag{13.7}$$

For each positive integer n we choose α_n in $(0,1/n]$ so that

$$(e^{|\boldsymbol{M}''|\alpha_n} - 1)\,|\boldsymbol{F}''| < \frac{1}{n}, \tag{13.8}$$

and we let $\boldsymbol{F}_n^{(2)}: [0, \alpha_n] \to \mathrm{Lin}\,(\mathscr{V})^{++}$ be the solution on $[0, \alpha_n]$ of

$$\dot{\boldsymbol{F}}_n^{(2)}(\tau) = \boldsymbol{M}''\boldsymbol{F}_n^{(2)}(\tau), \qquad \tau \in [0, \alpha_n],$$

with the initial value

$$\boldsymbol{F}_n^{(2)}(0) = \boldsymbol{F}'';$$

i.e.

$$\boldsymbol{F}_n^{(2)}(\tau) = e^{\boldsymbol{M}''\tau}\boldsymbol{F}''.$$

It follows from (13.8) that, for each τ in $[0, \alpha_n]$,

$$|\boldsymbol{F}_n^{(2)}(\tau) - \boldsymbol{F}''| < \frac{1}{n},$$

and hence

$$|(\boldsymbol{F}_n^{(2)}(\tau), e'', \boldsymbol{\beta}'', \boldsymbol{M}'') - (\boldsymbol{F}'', e'', \boldsymbol{\beta}'', \boldsymbol{M}'')| < \frac{1}{n}. \tag{13.9}$$

Therefore, there is an $N>0$ such that if, for each $n>N$, we put

$$\begin{aligned} &\boldsymbol{L}^{(2)}(\tau) = \dot{\boldsymbol{F}}_n^{(2)}(\tau)\,\boldsymbol{F}_n^{(2)}(\tau)^{-1} = \boldsymbol{M}'', \\ &h_n^{(2)}(\tau) = -\boldsymbol{T}(\boldsymbol{F}_n^{(2)}(\tau), e'', \boldsymbol{\beta}'', \boldsymbol{M}'')\cdot \boldsymbol{M}'', \\ &\gamma_n^{(2)}(\tau) = \boldsymbol{0}, \end{aligned}$$

then $P_n^{(2)} = (\boldsymbol{L}_n^{(2)}, h_n^{(2)}, \gamma_n^{(2)}): [0, \alpha_n) \to U$ is a process, and

$$P^{(1)}\sigma' = (\boldsymbol{F}'', e'', \boldsymbol{\beta}'', \boldsymbol{L}^{(1)}(t-))$$

is in $\mathscr{D}(P_n^{(2)})$. In view of (13.2), the process $P_n^{(2)}$ so constructed leaves e, $\boldsymbol{\beta}$, and $\boldsymbol{M}$ unchanged, and we have

$$P_n^{(2)}P^{(1)}\sigma' = (\boldsymbol{F}_n^{(2)}(\alpha_n), e'', \boldsymbol{\beta}'', \boldsymbol{M}''). \tag{13.10}$$

Because (13.9) holds for each τ in $[0, \alpha_n]$, it follows from (13.10) that, if we put (for $n>N$)

$$P_n \overset{\text{def}}{=} P_n^{(2)}P^{(1)}, \tag{13.11}$$

then

$$|P_n \sigma' - \sigma''| < \frac{1}{n}.$$

Since this last inequality holds for each $n > N$, (13.11) gives us a sequence $n \mapsto P_n$ such that

$$\sigma'' = \lim_{n \to \infty} P_n \sigma'.$$

By the additivity of $\mathfrak{s}$, *i.e.* by (9.7) or (2.3), to show that this same sequence $n \mapsto P_n$ is such that $\lim_{n \to \infty} \mathfrak{s}(P_n, \sigma')$ exists, it suffices to show that $\lim_{n \to \infty} \mathfrak{s}(P_n^{(2)}, P^{(1)}\sigma')$ exists. Now, it follows from (9.7) and the definition of $P_n^{(2)}$, that

$$\mathfrak{s}(P_n^{(2)}, P^{(1)}\sigma')$$
$$= \int_0^{\alpha_n} \left[-\frac{\boldsymbol{T}(\boldsymbol{F}_n^{(2)}(\tau), e'', \boldsymbol{\beta}'', \boldsymbol{M}'') \cdot \boldsymbol{M}''}{\theta(\boldsymbol{F}_n^{(2)}(\tau), e'', \boldsymbol{\beta}'', \boldsymbol{M}'')} + \boldsymbol{q}(\boldsymbol{F}_n^{(2)}(\tau), e'', \boldsymbol{\beta}'', \boldsymbol{M}'') \cdot \boldsymbol{\beta}'' \right] d\tau. \tag{13.12}$$

Because $\boldsymbol{T}$, $\boldsymbol{q}$, and θ are continuous on Σ, it follows from (13.9) that there is a $Z > 0$ such that, for all $n > Z$, the integrand in (13.12) is within unit distance of

$$\frac{-\boldsymbol{T}(\sigma'') \cdot \boldsymbol{M}''}{\theta(\sigma'')} + \boldsymbol{q}(\sigma'') \cdot \boldsymbol{\beta}'',$$

and hence

$$\left| \mathfrak{s}(P_n^2, P^1 \sigma') - \left[-\frac{\boldsymbol{T}(\sigma'') \cdot \boldsymbol{M}''}{\theta(\sigma'')} + \boldsymbol{q}(\sigma'') \cdot \boldsymbol{\beta}'' \right] \alpha_n \right| < \alpha_n \tag{13.13}$$

for $n > Z$. As α_n is in $(0, 1/n]$, (13.13) yields, forthwith,

$$\lim_{n \to \infty} \mathfrak{s}(P_n^{(2)}, P^{(1)}\sigma') = 0;$$

q.e.d.

Viscous elements are simple material elements, and for them the Second Law of Thermodynamics asserts that the action $\mathfrak{s}$ has the Clausius property at some state σ°. Let $\mathfrak{S}$, m, and Σ° be defined by (5.7), (5.6), and $(5.8)_2$. In view of Remark 3.2 (or Lemma 3.3 and Theorem 3.4), it follows from Theorem 13.1 that we here have

Theorem 13.2. For a viscous element which obeys the Second Law, Σ° equals the entire state space Σ, and $\mathfrak{s}$ has the Clausius property at every state.

We now assemble the apparatus needed to prove the main result of this section, Theorem 13.5.

Theorem 13.3. Let $S\colon \mathscr{S} \to \mathbb{R}$ be an upper potential for the action of (9.7). If the system is a viscous element, and if σ' and σ'' are two states in $\mathscr{S}$ with ${}_0\sigma' = {}_0\sigma''$, then

$$S(\sigma') = S(\sigma'').$$

Proof. If σ' and σ'' in $\mathscr{S}$ are such that ${}_0\sigma' = {}_0\sigma''$, then for some element $(\boldsymbol{F}', e', \boldsymbol{\beta}')$ of U, there holds $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{M}')$ and $\sigma'' = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{M}'')$. Let $\varepsilon > 0$ be given.

By Definition 3.2, there is a neighborhood $\mathcal{O}$ of σ'' such that

$$P \in \Pi, \quad P\sigma' \in \mathcal{O} \quad \Rightarrow \quad S(\sigma'') - S(\sigma') > \mathfrak{s}(P, \sigma') - \frac{\varepsilon}{3}. \tag{13.14}$$

As in the proof of Theorem 13.1, we can construct processes $P^{(1)}$, $P_n^{(2)}$, and $P_n = P_n^{(2)} P^{(1)}$, such that

$$\begin{gathered} \sigma'' = \lim_{n\to\infty} P_n \sigma', \\ \lim_{n\to\infty} \mathfrak{s}(P_n^{(2)}, P^{(1)}\sigma') = 0, \end{gathered} \tag{13.15}$$

and hence

$$\lim_{n\to\infty} \mathfrak{s}(P_n, \sigma') = \lim_{n\to\infty} \mathfrak{s}(P_n^{(2)}, P^{(1)}\sigma') + \mathfrak{s}(P^{(1)}, \sigma') = \mathfrak{s}(P^{(1)}, \sigma'). \tag{13.16}$$

Moreover, since ${}_0\sigma' = {}_0\sigma'' = (\boldsymbol{F}', e', \boldsymbol{\beta}')$, instead of employing a curve c and equation (13.6), we may choose $P^{(1)}$ so that, when considered a function on $[0, t)$, $P^{(1)}$ has the form $(\boldsymbol{L}^{(1)}, h^{(1)}, \gamma^{(1)}) = (\mathbf{0}, 0, \mathbf{0})$, and (9.7) yields

$$\mathfrak{s}(P^{(1)}, \sigma') = t\boldsymbol{q}(\boldsymbol{F}', e', \boldsymbol{\beta}', 0) \cdot \boldsymbol{\beta}';$$

As this construction of $P^{(1)}$ is valid for any choice of $t > 0$, we can choose t so that

$$\mathfrak{s}(P^{(1)}, \sigma') > -\frac{\varepsilon}{3}. \tag{13.17}$$

By (13.15) and (13.16), for n sufficiently large we have $P_n \sigma'$ in $\mathcal{O}$ and

$$\mathfrak{s}(P_n, \sigma') > \mathfrak{s}(P^{(1)}, \sigma') - \frac{\varepsilon}{3}; \tag{13.18}$$

when $P_n \sigma'$ is in $\mathcal{O}$, (13.14) yields

$$S(\sigma'') - S(\sigma') > \mathfrak{s}(P_n, \sigma') - \frac{\varepsilon}{3},$$

and hence, in view of (13.17) and (13.18), we have

$$S(\sigma'') - S(\sigma') > -\varepsilon.$$

Because ε is an arbitrary positive number, we may conclude that $S(\sigma'') \geqq S(\sigma')$, and as the argument just given can be applied with σ' and σ'' interchanged, we have $S(\sigma') \geqq S(\sigma'')$; thus $S(\sigma'') = S(\sigma')$; q.e.d.

Let Θ_i be the following subset of $\mathrm{Lin}(\mathscr{V})^{++} \oplus \mathbb{R}$:

$$\Theta_i \overset{\text{def}}{=} \{(\boldsymbol{F}, e) \mid (\boldsymbol{F}, e, \mathbf{0}) \in \Sigma_i\} = \{(\boldsymbol{F}, e) \mid (\boldsymbol{F}, e, \mathbf{0}, \mathbf{0}) \in \Sigma\}. \tag{13.19}$$

For each oriented polygonal curve c lying in Θ_i, we here write $J(c)$ for the line integral

$$J(c) \overset{\text{def}}{=} \int_c \left[-\frac{\boldsymbol{T}(\boldsymbol{F}, e, \mathbf{0}, \mathbf{0})\,\boldsymbol{F}^{T^{-1}}}{\theta(\boldsymbol{F}, e, \mathbf{0}, \mathbf{0})} \cdot d\boldsymbol{F} + \frac{1}{\theta(\boldsymbol{F}, e, \mathbf{0}, \mathbf{0})}\, de \right]. \tag{13.20}$$

Theorem 13.4. Let $\sigma' = (\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{M}')$ and $\sigma'' = (\boldsymbol{F}'', e'', \boldsymbol{\beta}'', \boldsymbol{M}'')$ be two arbitrary states of a viscous element $(\Sigma, \Pi, \theta, \boldsymbol{T}, \boldsymbol{q})$, and let S map Σ into $\mathbb{R}$. If S is an upper potential for the action $\mathscr{s}$, then

$$S(\sigma'')-S(\sigma')=J(\mathscr{c})$$

for every oriented rectifiable curve $\mathscr{c}$ joining $(\boldsymbol{F}', e')$ to $(\boldsymbol{F}'', e'')$ and lying in Θ_i.

Proof. In view of Theorem 13.3, $S(\sigma')$ and $S(\sigma'')$ are independent of the values of $\boldsymbol{M}'$ and $\boldsymbol{M}''$, and since $J(\mathscr{c})$ is obviously independent of $\boldsymbol{M}'$ and $\boldsymbol{M}''$, we may put, without loss of generality, $\boldsymbol{M}'=\boldsymbol{M}''=\boldsymbol{0}$. Let $\mathscr{c}$ be an oriented polygonal curve in Θ_i joining $(\boldsymbol{F}', e')$ to $(\boldsymbol{F}'', e'')$, let $\tau\mapsto(\boldsymbol{F}(\tau), e(\tau))$ be a piecewise linear parameterization of $\mathscr{c}$ on $[0, 1]$, and put

$$\begin{aligned} \boldsymbol{L}(\tau)&=\dot{\boldsymbol{F}}(\tau)\boldsymbol{F}(\tau)^{-1},\\ h(\tau)&=\dot{e}(\tau)-\boldsymbol{T}(\boldsymbol{F}(\tau), e(\tau), \boldsymbol{0}, \boldsymbol{L}(t))\cdot\boldsymbol{L}(\tau). \end{aligned}$$

For each positive integer n, let $(\boldsymbol{L}^n, h^n, \boldsymbol{\gamma}^n)$: $\left[0, n+\frac{2}{n}\right)\to U$ be defined by

$$(\boldsymbol{L}^n(\tau), h^n(\tau), \boldsymbol{\gamma}^n(\tau)) = \begin{cases} (\boldsymbol{0}, 0, -n\boldsymbol{\beta}'), & \text{for } \tau\in\left[0, \frac{1}{n}\right),\\ \left(\frac{1}{n}\boldsymbol{L}\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), \frac{1}{n}h\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), \boldsymbol{0}\right), & \text{for } \tau\in\left[\frac{1}{n}, n+\frac{1}{n}\right),\\ (\boldsymbol{0}, 0, n\boldsymbol{\beta}''), & \text{for } \tau\in\left[n+\frac{1}{n}, n+\frac{2}{n}\right). \end{cases}$$

It is easily verified that $P^{(n)}=(\boldsymbol{L}^n, h^n, \boldsymbol{\gamma}^n)$ is a process in Π with $(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{0})$ in $\mathscr{D}(P^{(n)})$ and

$$P_\tau^{(n)}\sigma'=P_\tau^{(n)}(\boldsymbol{F}', e', \boldsymbol{\beta}', \boldsymbol{0})$$
$$= \begin{cases} \left(\boldsymbol{F}', e', \left(\frac{1}{n}-\tau\right)n\boldsymbol{\beta}', \boldsymbol{0}\right), & \text{for } \tau\in\left(0, \frac{1}{n}\right),\\ \left(\boldsymbol{F}\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), e\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), \boldsymbol{0}, \frac{1}{n}\boldsymbol{L}\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right)\right), & \text{for } \tau\in\left[\frac{1}{n}, n+\frac{1}{n}\right),\\ \left(\boldsymbol{F}'', e'', \left(\tau-n-\frac{1}{n}\right)n\boldsymbol{\beta}'', \boldsymbol{0}\right), & \text{for } \tau\in\left[n+\frac{1}{n}, n+\frac{2}{n}\right). \end{cases}$$

Note that for τ in $\left[\frac{1}{n}, n+\frac{1}{n}\right]$, $P^{(n)}\sigma'$ differs from

$$\left(\boldsymbol{F}\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), e\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), \boldsymbol{0}, \boldsymbol{0}\right),$$

by a term,

$$\left(\boldsymbol{0}, 0, \boldsymbol{0}, \frac{1}{n}\boldsymbol{L}\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right)\right),$$

whose magnitude is $O(1/n)$, uniformly in τ. Since T and θ are uniformly continuous on compact subsets of Σ, for each $\varepsilon>0$, there is an N such that if we put

$$\Delta_n \overset{\text{def}}{=} \int_{\frac{1}{n}}^{\frac{1}{n}+n} \Bigg\{ -\Bigg[\frac{T(P_\tau^{(n)}\sigma')\, F\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right)^{T-1}}{\theta(P_\tau^{(n)}\sigma')} - \frac{T\left(F\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), e\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), 0, 0\right) F\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right)^{T-1}}{\theta\left(F\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), e\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), 0, 0\right)} \Bigg]$$

$$\cdot \frac{d}{d\tau} F\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right) + \Bigg[\frac{1}{\theta(P_\tau^{(n)}\sigma')} - \frac{1}{\theta\left(F\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), e\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right), 0, 0\right)} \Bigg]$$

$$\cdot \frac{d}{d\tau} e\left(\frac{1}{n}\left(\tau-\frac{1}{n}\right)\right) \Bigg\} d\tau,$$

then $|\Delta_n|<\varepsilon/2$ for all $n>N$. Since

$$s(P^{(n)}, \sigma') - J(c) = \int_0^{\frac{1}{n}} \boldsymbol{q}(P_\tau^{(n)}\sigma') \cdot \boldsymbol{\beta}' n \left[\frac{1}{n}-\tau\right] d\tau + \Delta_n + \int_{n+\frac{1}{n}}^{n+\frac{2}{n}} \boldsymbol{q}(P_\tau^{(n)}\sigma') \cdot \boldsymbol{\beta}'' n \left[\tau - n - \frac{1}{n}\right] d\tau,$$

if we put

$$Q' = \sup\{|\boldsymbol{q}(F', e', \boldsymbol{\beta}, 0) \cdot \boldsymbol{\beta}| \,\big|\, |\boldsymbol{\beta}| < |\boldsymbol{\beta}'|\}$$

$$Q'' = \sup\{|\boldsymbol{q}(F'', e'', \boldsymbol{\beta}, 0) \cdot \boldsymbol{\beta}| \,\big|\, |\boldsymbol{\beta}| < |\boldsymbol{\beta}''|\},$$

then

$$|s(P^{(n)}, \sigma') - J(c)| < \frac{1}{n} Q' + \frac{\varepsilon}{2} + \frac{1}{n} Q'', \tag{13.21}$$

for each $n>N$. The continuity of $\boldsymbol{q}$ on Σ implies that Q' and Q'' are finite, and if we choose N sufficiently large that it exceeds $2(Q'+Q'')/\varepsilon$, then, for each $n>N$, we have not only (13.21), but also

$$|s(P^{(n)}, \sigma') - J(c)| < \varepsilon. \tag{13.22}$$

Each process $P^{(n)}$ is so constructed that

$$P^{(n)}\sigma' = \left(F'', e'', \left(\tau - n - \frac{1}{n}\right) n\beta'', 0\right)\Big|_{\tau = n + \frac{2}{n}}$$
$$= (F'', e'', \beta'', 0) = \sigma'',$$

and, hence, if $S: \Sigma \to \mathbb{R}$ is an upper potential for $\mathfrak{s}$, S must obey

$$\mathfrak{s}(P^{(n)}, \sigma') \leqq S(\sigma'') - S(\sigma'),$$

or, by (13.22),

$$J(c) < S(\sigma'') - S(\sigma') + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this last inequality implies that

$$J(c) \leqq S(\sigma'') - S(\sigma'). \tag{13.23}$$

The argument just given may be repeated with σ' and σ'' interchanged and with c replaced by the curve c^- which differs from c only in orientation. On doing this, one obtains the relation

$$-J(c) = J(c^-) \leqq S(\sigma') - S(\sigma''),$$

which is compatible with (13.23) only if

$$J(c) = S(\sigma'') - S(\sigma').$$

This last equation has been derived for an arbitrary oriented *polygonal* curve c which lies in Θ_i and joining (F', e') to (F'', e''), and hence the equation must hold for every such *rectifiable* curve; q.e.d.

It is evident from Theorem 9.3 and item (4) of Remark 3.2 that a simple material element which obeys the Second Law has an entropy function S, *i.e.* an upper potential for $\mathfrak{s}$, and S can be chosen so that it is an upper semicontinuous function with Σ° for its domain. By Theorem 13.2, for a viscous element obeying the Second Law there holds $\Sigma^\circ = \Sigma$, and hence S can be chosen so that the entire state space is its domain. Moreover, it follows from Theorem 13.4 [see (13.20)] that an entropy function S with domain Σ is not only upper semicontinuous, but is continuously differentiable; indeed, it is a potential for the following continuous vector field on Σ,

$$(F, e, \beta, M) \mapsto \left(-\frac{T(F, e, 0, 0)\, F^{T^{-1}}}{\theta(F, e, 0, 0)}, \frac{1}{\theta(F, e, 0, 0)}, 0, 0\right),$$

and hence must have this vector field for its gradient. In particular, the derivatives of S with respect to β and M must vanish, and this, by the structure of Σ prescribed in item (1) of Definition 13.1, implies that S is independent of β and M.[#] Since, by Theorem 13.4, each entropy function S with domain Σ must be such that its differences are determined by the line integral J, which, by (13.20), is specified when θ and T are given, a viscous element has on Σ at most one entropy function which vanishes at a prescribed state. Thus, we have

Theorem 13.5.[##] If a viscous element $(\Sigma, \Pi, \theta, T, q)$ obeys the Second Law of Thermodynamics, then the element has exactly one entropy function S whose

[#] Of course, that S is independent of M follows also from Theorem 13.3.

[##] For derivations of relations equivalent to (13.24)–(13.28) starting from the assumption that there is a differentiable entropy function S on Σ, see COLEMAN & MIZEL [1964, 3] and COLEMAN & NOLL [1963, 2].

domain is Σ and which vanishes at a prescribed point σ_o in Σ; this entropy function is continuously differentiable, obeys on Σ the relations

$$\partial_F S(F, e, \beta, M) = -\frac{T(F, e, 0, 0) F^{T^{-1}}}{\theta(F, e, 0, 0)}, \tag{13.24}$$

$$\partial_e S(F, e, \beta, M) = \frac{1}{\theta(F, e, 0, 0)}, \tag{13.25}$$

$$\partial_\beta S(F, e, \beta, M) = 0, \tag{13.26}$$

$$\partial_M S(F, e, \beta, M) = 0, \tag{13.27}$$

and, for each state $\sigma = (F, e, \beta, M)$,

$$S(\sigma) = S(F, e, 0, 0). \tag{13.28}$$

By (13.28), we can regard S as a function on the set Θ_i defined in (13.19); we write (13.28) in the form $S(\sigma) = S(F, e)$. On Θ_i and Σ, respectively, we define functions $T^{(0)}$ and $T^{(d)}$ by

$$T^{(0)}(F, e) \overset{\text{def}}{=} T(F, e, 0, 0) \tag{13.29}$$

$$T^{(d)}(F, e, \beta, M) = T(F, e, \beta, M) - T^{(0)}(F, e). \tag{13.30}$$

The values of $T^{(0)}$ and $T^{(d)}$ at a state $\sigma = (F, e, \beta, M)$ are called the "equilibrium stress" and the "extra stress" corresponding to σ.

Because the function S in Theorem 13.5 is differentiable, the relations (9.21) and (13.2) imply that there holds, for each state $\sigma = (F, e, \beta, M)$ and each process P_t with (L, h, γ) continuous on $[0, t)$ and with $L(0) = M$,

$$\dot{S} \geqq \frac{h}{\theta(\sigma)} + q(\sigma)\cdot\beta = -\frac{T(F, e, \beta, L(0))\cdot L(0)}{\theta(F, e, \beta, L(0))} + \frac{\dot{e}}{\theta(F, e, \beta, L(0))} + q(F, e, \beta, L(0))\cdot\beta, \tag{13.31}$$

where $\dot{e}$ is the projection on $\mathbb{R}$ of $\left.\frac{d}{d\tau} P_\tau \sigma\right|_{\tau=0}$, regarded as an element of $\operatorname{Lin}(\mathscr{V}) \oplus \mathbb{R} \oplus \mathscr{V}$, and

$$\dot{S} = \left.\frac{d}{d\tau} S(P_\tau \sigma)\right|_{\tau=0}.$$

By (13.28) and $(13.2)_1$,

$$\dot{S} = \partial_F S(\sigma)\cdot L(0) F + \partial_e S(\sigma)\,\dot{e}. \tag{13.32}$$

Employing (13.31), (13.32), and arguments introduced by COLEMAN & NOLL[#] and COLEMAN & MIZEL[##], one may easily show that Theorem 13.5 has the following corollary.

Theorem 13.6. Consider a viscous element $(\Sigma, \Pi, \theta, T, q)$, and let $T^{(d)}$ be defined by (13.29) and (13.30). If the element obeys the Second Law of Thermodynamics,

[#] [1963, 2].
[##] [1964, 3], see § 4.

then at each point $(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{M})$ of Σ there holds

$$\boldsymbol{T}^{(d)}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{M})\cdot\boldsymbol{M}-\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{M})\cdot\boldsymbol{\beta}\geqq 0, \tag{13.33}$$

and

$$\theta(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{M})=\theta(\boldsymbol{F}, e, \boldsymbol{0}, \boldsymbol{0}). \tag{13.34}$$

Remark 13.2. COLEMAN & MIZEL[#] called (13.33) the "general dissipation inequality" (for viscous materials) and explored its consequences in detail for viscous fluids. When $\boldsymbol{M}=\boldsymbol{0}$ the relation (13.33) reduces to the *heat conduction inequality,*

$$\boldsymbol{q}(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{0})\cdot\boldsymbol{\beta}\leqq 0, \tag{13.35}$$

and when $\boldsymbol{\beta}=\boldsymbol{0}$, (13.33) becomes the *mechanical dissipation inequality for viscous materials*:

$$\boldsymbol{T}^{(d)}(\boldsymbol{F}, e, \boldsymbol{0}, \boldsymbol{M})\cdot\boldsymbol{M}\geqq 0. \tag{13.36}$$

For linearly viscous fluids, (13.36) implies the usually assumed inequalities for the viscosity coefficients.

Remark 13.3. It follows from (13.34) that θ can be considered a function on Θ_i; when $\sigma=(\boldsymbol{F}, e, \boldsymbol{\beta}, \boldsymbol{M})$, we write

$$\theta(\sigma)=\theta(\boldsymbol{F}, e).$$

In this notation, the relations (13.24) and (13.25) become

$$\boldsymbol{T}^{(0)}(\boldsymbol{F}, e)=-\theta(\boldsymbol{F}, e)\,\partial_{\boldsymbol{F}} S(\boldsymbol{F}, e), \tag{13.37}$$

$$\theta(\boldsymbol{F}, e)=[\partial_e S(\boldsymbol{F}, e)]^{-1}. \tag{13.38}$$

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