Quasilinear Parabolic Systems Under Nonlinear Boundary Conditions

HERBERT AMANN

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Introduction

In this paper we prove existence and regularity for solutions of abstract quasilinear parabolic systems

$$\frac{\partial u}{\partial t} + \mathscr{A}(t, u)\, u = f(t, u),$$

$$\mathscr{B}(t, u)\, u = g(t, u),$$

$$u(s) = u_0,$$

$s < t \leqq T$, which can be interpreted as quasilinear evolution equations of the form

$$\dot{u} + A(t, u)\, u = F(t, u), \qquad s < t \leqq T, \qquad u(s) = u_0 \tag{1}$$

in an appropriate Banach space W. The results are applicable to rather general parabolic systems of the form

$$\frac{\partial u}{\partial t} + \mathscr{A}(x, t, u, \ldots, D^k u)\, u = f(x, t, u, \ldots, D^{2m-1} u) \quad \text{in } \Omega \times (s, T],$$

$$\mathscr{B}(x, t, u, \ldots, D^k u)\, u = g(x, t, u, \ldots, D^k u) \quad \text{on } \partial\Omega \times (s, T],$$

$$u(\cdot, s) = u_0 \quad \text{on } \Omega,$$

where $2m$ is the order of the system and $k < 2m - 1$.

To be more specific we describe now the main results for some simple second-order systems. We denote by Ω a smooth bounded domain in $\mathbb{R}^n$ and consider a family of second-order differential operators of the form

$$\mathscr{A}(t, u)\, u := -\sum_{j,k=1}^{n} D_j(a_{jk}(\cdot, t, u)\, D_k u) + \sum_{j=1}^{n} a_j(\cdot, t, u)\, D_j u, \qquad 0 \leqq t \leqq T < \infty,$$

where $D_j := \partial/\partial x^j$, acting on N-vector valued real functions $u : \Omega \to \mathbb{R}^N$. We assume that the coefficient functions are smooth, that is,

$$a_{jk} = a_{kj}, \quad a_j \in C^\infty(\bar{\Omega} \times [0, T] \times \mathbb{R}^N, \mathscr{L}(\mathbb{R}^N)),$$

where $\mathscr{L}(\mathbb{R}^N)$ is the space of all endomorphisms ($\equiv (N \times N)$-matrices) of $\mathbb{R}^N$. Moreover we assume that each $\mathscr{A}(t, u)$ is a strongly elliptic system, that is,

$$\sum_{r,s=1}^{N} \sum_{j,k=1}^{n} a_{jk}^{rs}(x, t, \eta)\, \xi^j \xi^k \zeta_r \zeta_s > 0$$

for all $(x, t, \eta) \in \bar{\Omega} \times [0, T] \times \mathbb{R}^N$, all $\xi := (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n \setminus \{0\}$, and all $\zeta := (\zeta_1, \ldots, \zeta_N) \in \mathbb{R}^N \setminus \{0\}$, where, of course, a_{jk}^{rs} are the elements of the matrices a_{jk}. We denote by $\nu := (\nu^1, \ldots, \nu^n)$ the outer normal on $\partial\Omega$ and put

$$\mathscr{B}(t, u)\, u := \sum_{j,k=1}^{n} a_{jk}(\cdot, t, u)\, \nu^j D_k u.$$

Finally we suppose that

$$f \in C^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{nN}, \mathbb{R}^N)$$

and

$$g \in C^\infty([0, T] \times \partial\Omega^N \times \mathbb{R}, \mathbb{R}^N) \quad \text{with } g(\cdot, \cdot, 0) = 0.$$

Of course, the above assumptions of differentiability are only imposed for simplicity. They can be considerably relaxed.

We consider the initial-boundary value problem for the quasilinear parabolic system

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathscr{A}(t, u)\, u &= f(x, t, u, Du) \quad \text{in } \Omega \times (s, T], \\ \mathscr{B}(t, u)\, u &= g(x, t, u) \quad \text{on } \partial\Omega \times (s, T], \\ u(\cdot, s) &= u_0 \quad \text{on } \Omega, \end{aligned} \tag{2}$$

where $0 \leqq s < T$. By a classical solution of (2) on an interval $J \subset [s, T]$, with $s \in J$ and $\dot{J} := J \setminus \{s\} \neq \emptyset$, we mean a function

$$u \in C(\bar{\Omega} \times J, \mathbb{R}^N) \cap C^1(\bar{\Omega} \times \dot{J}, \mathbb{R}^N) \cap C^{2,0}(\Omega \times \dot{J}, \mathbb{R}^N),$$

which satisfies (2) pointwise. A noncontinuable classical solution is said to be maximal.

We suppose now that $n < p < \infty$ and we denote by $W_p^\tau := W_p^\tau(\Omega, \mathbb{R}^N)$, $\tau \in [0, \infty)$, the standard Sobolev-Slobodeckii spaces.

Theorem. *Suppose that $0 \leqq s < T$, that $1 + n/p < \tau \leqq 2$, and that $u_0 \in W_p^\tau$ satisfies the compatibility condition*

$$\mathscr{B}(s, u_0)\, u_0 = g(\cdot, s, u_0) \quad \text{on } \partial\Omega. \tag{3}$$

Then problem (2) *has a maximal classical solution u, defined on a perfect subinterval J of* $[s, T]$, *which is open in* $[s, T]$. *If*

$$\sup_{t \in J} \|u(t)\|_{W_p^\tau} < \infty, \tag{4}$$

then $J = [s, T]$, *that is, u is a global solution. Finally, if f is independent of Du, then u is the only solution of* (2).

Suppose now that f is independent of Du and denote by $u(\cdot, s, u_0)$ the unique maximal solution of (2) for each $u_0 \in W_p^\tau$ satisfying (3). Then it will be shown that $u(\cdot, s, u_0)$ depends continuously upon u_0 in an appropriate neighborhood of s and with respect to appropriate topologies.

In the important autonomous case, that is, if $\mathscr{A}$, $\mathscr{B}$, f and g are independent of t, we shall also establish global continuity results. In particular we shall give conditions guaranteeing that the autonomous system (2) generates a local semiflow on appropriate "spaces of initial values".

The above theorem is a special case of the more general results in Section 6 below. There we admit also boundary conditions which correspond to Dirichlet boundary conditions for some components of u on some components of $\partial\Omega$. Moreover the system does not need to be strongly parabolic. A typical example of a case to which the results apply also is given by the following two-component system

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{j=1}^{n} D_j(A(x,u)\, D_j u) &= f(x, u, Du) \quad \text{in } \Omega \times (s, \infty), \\ A(x,u) \frac{\partial u}{\partial \nu} + B(x,u)\, u &= 0 \quad \text{on } \partial\Omega \times (s, \infty) \\ u(\cdot, s) &= u_0 \quad \text{on } \Omega, \end{aligned} \tag{5}$$

where $N = 2$ and

$$A(x,u) = \begin{bmatrix} a^{11}(x,u) & a^{12}(x,u) \\ 0 & a^{22}(x,u) \end{bmatrix}$$

with smooth functions $a^{jk}: \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$, such that $a^{11}(x, \eta)$ and $a^{22}(x, \eta)$ are positive for each $(x, \eta) \in \bar{\Omega} \times \mathbb{R}^2$ and where $B(\cdot) \in C^\infty(\bar{\Omega} \times \mathbb{R}^2, \mathscr{L}(\mathbb{R}^2))$ is also upper triangular, that is, $B(x, u)$ has the same form as $A(x, u)$. Observe that (5) is not strongly parabolic, in general. Systems of this form occur in many applications, for example in the mathematical theory of chemical reactions, in certain biological models, or in problems of fluid dynamics (*e.g.* [2, 10, 13] and have, in fact, motivated this investigation.

This paper concerns the case in which the boundary conditions depend upon the unknown solution. The case in which the boundary operator is independent of u and t — which covers in particular the case of Dirichlet boundary conditions — has been thoroughly studied in an earlier paper [7].

The existence of classical solutions of a single second-order quasilinear parabolic equation under (nonlinear) Neumann boundary conditions has been extensively studied by LADYŽENSKAJA, SOLONNIKOV & URAL'CEVA [17]. Their method is based upon *a priori* estimates for Hölder norms of the first derivatives of the solutions and upon the Leray-Schauder fixed point theorem. The necessary *a priori* estimates are ultimately derived from *a priori* estimates for the maximum norm of the solutions, which are obtained by maximum principle arguments. It is well known that these techniques require growth restrictions as well as further structural conditions for the non-linearities.

Unfortunately the methods of LADYŽENSKAJA, SOLONNIKOV & URAL'CEVA do not carry over to general parabolic systems (except, of course, to some systems whose principal parts are in diagonal form). There are two reasons. Firstly for general systems, there are no maximum principles which could be used to derive *a priori* bounds for the maximum norms of solutions. Secondly, even if an *a priori* bound could be obtained for the maximum norm (for example due to geometrical restrictions), bounded weak solutions of system are not generally Hölder continuous (*cf.* [11, 12, 20, 21]).

These facts explain why nothing seems to be known about classical solutions of general quasilinear parabolic systems under general nonlinear boundary conditions.

The approach here is completely different and uses functional analysis essentially. Because of the negations above we cannot expect to get global classical solutions in general. For this reason we investigate the local solvability, for which we need not impose any growth condition or other structural condition upon the nonlinearities. However it should be noted that we also get global solutions if we can bound *a priori* a sufficiently strong norm of the solution (*cf.* (4)). Thus we can recover essentially the results of LADYŽENSKAJA, SOLONNIKOV & URAL'CEVA for a single equation.).

The basic idea of this paper is rather simple. We interpret the system as an abstract equation evolution (1) in an appropriate Banach space W, an L_p-space. We associate with (1) the unique solution $u(v)$ of the linear equation of evolution

$$\dot{u} + A(t, v)\, u = F(t, v), \qquad s < t \leqq T, \qquad u(s) = u_0, \tag{6}$$

where v belongs to an appropriate subset of W. Then we try to establish a fixed point of the map $v \mapsto u(v)$, which gives obviously a solution to our problem.

Unfortunately for the following reasons there are serious problems in carrying out this simple idea. Formally the solution of the linear problem (6) is given by the "variation-of-constants" formula

$$u(v)\,(t) = U_v(t, s)\, u_0 + \int_s^t U_v(t, \tau)\, F(\tau, v(\tau))\, d\tau, \qquad s \leqq t \leqq T, \tag{7}$$

where U_v is the fundamental solution for the family $\{A(t, v) \mid 0 \leqq t \leqq T\}$. For each fixed argument $-A(t, v)$ is the infinitesimal generator of a strongly continuous analytic semigroup on W. This is essentially a consequence of the L_p-estimates for elliptic systems (*cf.* [1]). The difficulty lies, however, in the fact that the domains of the unbounded operators $A(t, v)$ depend upon t and v.

There are well known results due to KATO & TANABE [15] (*cf.* also [22] and YAGI [24, 25]), which guarantee the existence of a unique fundamental solution for time-dependent parabolic evolution equations. However if one tries to satisfy the hypotheses of these theorems in our concrete situation (we consider now the case of second-order systems, for simplicity) one finds that one has to assume that $v \in C^1([s, T], C^1(\bar{\Omega}, \mathbb{R}^N))$. (This is due to the fact that the L_p-estimates involve C^1-norms of the coefficients of the boundary operators, which contain v.) Unfortunately this good regularity is not preserved by $u(v)$, and so the fixed-point operator $v \mapsto u(v)$ is not well defined, in the sense that it does not preserve the regularity of v.

In a previous paper [8] I have shown that a fundamental solution can be constructed under weaker assumptions of regularity provided certain additional hypotheses are satisfied. These results apply to the present situation. The basic idea is to extend the whole problem to a larger space such that the domains of the extensions of $A(t, v)$ become independent of the argument (t, v). In terms of differential equations this corresponds to the study of a weak formulation of the problem.

For constant domains it is possible to obtain sharp estimates for the dependence of the (extended) evolution operator upon v. These estimates have been derived in [7] and are crucial for the present paper. It is most important that in the extended, or weak, formulation the first-order boundary terms disappear. As a consequence we can obtain continuous dependence of U_v upon v with respect to the topology of $C(\bar{\Omega})$ (and not with respect to the C^1-topology). It is then possible to lift back the problem within the abstract framework to the space W, so that the weak formulation does not appear explicitly in this paper.

The existence proof is rather delicate since we have to work simultaneously with two topologies. Namely, the weak formulation gives us continuity of the fixed-point operator $v \mapsto u(v)$ with respect to the C-norm. On the other hand the whole theory depends heavily upon the L_p-estimates which are uniform only on bounded subsets of C^1. Hence one has to ensure that the function $v \mapsto u(v)$ is bounded in C^1 although it is only continuous in C.

Finally we have to use a third set of topologies, namely the topology of Hölder spaces, to get classical solutions. I prefer to get the existence theorems in the setting of Sobolev spaces, since in these spaces the associated semigroups are strongly continuous, which is not true for Hölder spaces. Having established existence in Sobolev spaces, we use the Hölder spaces only for proving regularity. Although the semigroups are not strongly continuous in the Hölder spaces, they behave well enough for this purpose. A similar technique has been used in [7].

All these considerations are carried through in an abstract framework. This has the advantage that the results are also applicable to a variety of parabolic systems of higher order. It remains only to verify that the hypotheses of the general results are satisfied.

In Section 1 we introduce the basic assumptions and the abstract formulation of the problem. Section 2 contains a detailed study of the regularities of the linear Cauchy problem. These regularities are crucial for the study of the fixed point map $v \mapsto u(v)$, which is carried out in Section 3. The main abstract results of this paper are contained in Section 4. Besides of general existence and uniqueness theorems we derive also results concerning the continuous dependence upon the initial values. Section 5 contains the proof of the abstract regularity theorem, and in Section 6 we apply the results to rather general quasilinear second-order parabolic systems. In the last section we indicate briefly how the general theorems can be applied to higher-order parabolic systems.

Notations

Throughout this paper we use standard notation. All vector spaces are over $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K} = \mathbb{R}$ and we use complex numbers (for example in the context of spectral theory) it is always understood that we work with the natural complexification (of spaces and operators). Thus, by $\varrho(A)$, the resolvent set of the linear operator A with domain $D(A)$, kernel $\ker(A)$ and range $R(A)$, we mean always the resolvent set of its complexification, if $\mathbb{K} = \mathbb{R}$.

We denote by $\mathscr{L}(X, Y)$ the Banach space of all continuous linear operators from the Banach space X to the Banach space Y, and $\mathscr{L}(X) := \mathscr{L}(X, X)$. Moreover, Isom (X, Y) is the (open) set of all isomorphisms in $\mathscr{L}(X, Y)$. We write $X \hookrightarrow Y$ if X is continuously injected in Y, that is, X is a linear subspace of Y and the natural injection i, given by $i(x) = x$, is continuous. If i is compact, then we write $X \subset\subset\hookrightarrow Y$, and $X \doteq Y$ means that X and Y coincide as vector spaces and carry equivalent norms, that is, $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

Let M be a metric space and X a Banach space. Then we denote by $B(M, X)$ the Banach space of all bounded functions from M to X with the supremum norm. $BC(M, X)$ and $BUC(M, X)$ are the closed subspaces of all bounded and continuous functions and all bounded and uniformly continuous functions, respectively. If $0 < \alpha < 1$, we denote by $C^{\alpha}(M, X)$ the space of all functions $f: M \to X$ which are α-Hölder continuous. This means that each point in M possesses a neighborhood U such that f is uniformly α-Hölder continuous on U, that is,

$$[f]_{\alpha}^{U} := \sup_{x,y \in U} \frac{\|f(x) - f(y)\|}{[d(x, y)]^{\alpha}} < \infty, \tag{8}$$

where d denotes the metric in M. The space $C^{\alpha}(M, X)$ is given the locally convex topology induced by the seminorms

$$\sup_{x \in U} \|f(x)\| + [f]_{\alpha}^{U}, \qquad U \subset M, \qquad U \text{ compact}.$$

This topology is well defined since every α-Hölder continuous functions is uniformly α-Hölder continuous on compact sets. If $\alpha = 1$, we denote the above space by $C^{-1}(M, X)$, the space of all Lipschitz continuous functions. Finally, $UC^{\alpha}(M, X)$, $\alpha \in (0, 1) \cup \{-1\}$, denotes the space of all uniformly α-Hölder continuous (respectively Lipschitz continuous) functions from X to M, that is, the subspace of all $f \in C^{\alpha}(M, X)$ for which the seminorm (8) is finite if U is replaced by M.

More generally, if M is a subspace of some product space $A \times B$, we write $f \in C^{\alpha,\beta}(M, X)$, $\alpha, \beta \in (0, 1) \cup \{1-\}$, $\alpha \neq \beta$, if each point in M has a neighborhood $U \times V$ such that there is a constant c with

$$\|f(x, y) - f(x', y')\| \leqq c\{[d_A(x, x')]^\alpha + [d_B(y, y')]^\beta\}$$

for all (x, y), $(x', y') \in U \times V$ (where $t^{1-} := t$ for $t \geqq 0$). Clearly $UC^{\alpha,\beta}(M, X)$ has the now obvious meaning.

Suppose now that M and N are smooth manifolds. Then we write $f \in C^{0,l-}(M \times N, X)$, $l \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, if f is continuous (that is, $f \in C(M \times N, X)$), $f(x, \cdot): N \to X$ is $l - 1$ times continuously differentiable for each $x \in M$, and the derivatives of order $l - 1$ are Lipschitz continuous (with respect to the natural metric, present in all cases which will occur). Moreover, we put

$$C^{k,0} \cap C^{0,l-}(M \times N, X) := C^{k,0}(M \times N, X) \cap C^{0,l-}(M \times N, X),$$

$k \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}^*$, where $C^{k,0}(M \times N, X)$ has the obvious meaning.

After having finished this paper I learned of the work by K. FURUYA [26]. That author proves, among other things, the existence of a local solution of the abstract initial-value problem (1) under the assumption that an appropriate fractional power of A has a constant domain. His proof is based on KATO's result in [27]. However, the abstract existence theorems of FURUYA do not seem to be applicable to quasilinear parabolic equations and systems under nonlinear boundary conditions.

1. The Abstract Setting

Throughout this paper T denotes a fixed positive number, $\dot{T}_\Delta := \{(t, s) \in \mathbb{R}^2 \mid 0 \leqq s < t \leqq T\}$, and T_Δ is the closure of $\dot{T}_\Delta$ in $\mathbb{R}^2$. Moreover $\Sigma_\vartheta := \{z \in \mathbb{C} \mid |\arg z| \leqq \vartheta + \pi/2\}$ for $0 \leqq \vartheta < \pi/2$.

We fix a number $p \in [1, \infty)$ and choose, for each $\theta \in (0, 1)$, either the real interpolation functor $(\cdot, \cdot)_{\theta,p}$ or the complex interpolation functor $[\cdot, \cdot]_\theta$, and denote this fixed choice by $(\cdot, \cdot)_\theta$. (We refer to [9, 23] for the basic facts about interpolation theory.)

We denote by $W := (W, \|\cdot\|)$, $(W^1, \|\cdot\|_1)$ and $(\partial W, \|\ \|_{\partial W})$ three fixed Banach spaces, and we assume that

(A1) *W is reflexive and $W^1 \overset{d}{\hookrightarrow} W$.*

Then we let

$$W^\sigma := (W^\sigma, \|\cdot\|_\sigma) := (W, W^1)_\sigma, \quad 0 < \sigma < 1,$$

and observe that (A1) implies)

$$W^\sigma \overset{d}{\hookrightarrow} W^\tau, \quad 0 \leqq \tau < \sigma \leqq 1, \tag{1}$$

where $W^0 := W$ (*cf.* [9, Corollary 3.8.2 and Theorem 4.7.1]). Moreover we impose the following compatibility condition:

(A2) *For each pair of numbers* $0 \leqq \tau \leqq \sigma \leqq 1$ *there is a constant* $\gamma(\tau, \sigma)$ *such that*

$$\|x\|_\tau \leqq \gamma(\tau, \sigma) \|x\|^{1-\tau/\sigma} \|x\|_\sigma^{\tau/\sigma} \quad \forall x \in W^\sigma.$$

We fix four numbers such that

$$0 < \zeta \leqq \eta < \beta < 1, \quad \eta \leqq \xi < 1, \tag{A3}$$

and assume that

(A4) V *is an open subset of* W^ξ, *and*

$$(\mathscr{A}, \mathscr{B}) \in C([0, T] \times V, \mathscr{L}(W^1, W \times \partial W)).$$

Hence

$$W^1_{\mathscr{B}(t,y)} := \ker \mathscr{B}(t, y)$$

is, for each $(t, y) \in [0, T] \times V$, a closed linear subspace of W^1, and we let

$$W^\sigma_{\mathscr{B}(t,y)} := (W, W^1_{\mathscr{B}(t,y)})_\sigma, \quad 0 < \sigma < 1.$$

Observe that

$$W^\sigma_{\mathscr{B}(t,y)} \hookrightarrow W^\sigma, \quad 0 < \sigma < 1, \quad (t, y) \in [0, T] \times V,$$

and, in general, the topology of $W^\sigma_{\mathscr{B}(t,y)}$ is strictly finer than the one of W^σ. Hence $\sigma \in [0, 1]$ is called *regular*, if $W^\sigma_{\mathscr{B}(t,y)}$ is a closed linear subspace of W^σ for each $(t, y) \in [0, T] \times V$, where $W^0_{\mathscr{B}(t,y)} := W$.

Now we assume that

(A5) β, ξ, η *and* ζ *are regular*,

$$(\cdot, \cdot)_\beta = [\cdot, \cdot]_\beta,$$

and there is a point $\hat{y} \in V$ *with*

$$W^\beta_{\mathscr{B}(t,y)} \doteq W^\beta_{\mathscr{B}(0,\hat{y})} =: W^\beta_{\mathscr{B}}$$

for all $(t, y) \in [0, T] \times V$.

We let

$$A(t, y) := \mathscr{A}(t, y) \mid W^1_{\mathscr{B}(t,y)}$$

and consider $A(t, y)$ as a linear operator in W. Then we assume that

(A6) *for each nonempty bounded subset* $S \subset V$ *there are positive constants* γ_0, γ_1, ϑ *and* M *and a number* $\omega \in \mathbb{R}$, *such that*

$$R((\lambda + \mathscr{A}(t, y), \mathscr{B}(t, y)) = W \times \partial W,$$

$$\gamma_0 \|x\|_1 \leqq \|(\lambda + \mathscr{A}(t, y)) x\| + \|\mathscr{B}(t, y) x\|_{\partial W} \leqq \gamma_1 \|x\|_1 \quad \forall x \in W^1$$

and

$$\|(\lambda + A(t, y))^{-1}\|_{\mathscr{L}(W)} \leqq M/(1 + |\lambda - \omega|)$$

for all $\lambda \in \omega + \Sigma_\vartheta$ *and* $(t, y) \in [0, T] \times S$.

Observe that the assumptions upon $(\mathscr{A}, \mathscr{B})$ imply, in particular, that $\omega + \Sigma_\vartheta \subset \varrho(-A(t, y))$. Hence the last estimate is meaningful. Moreover it follows that $A(t, y)$ is a closed linear operator which is densely defined in W on account of the above resolvent estimate and the fact that W is reflexive (*cf.* [14]). Finally it follows that $-A(t, y)$ is the infinitesimal generator of a strongly continuous analytic semigroup $\{e^{-sA(t,y)} \mid s \geqq 0\}$ on W (that is, in $\mathscr{L}(W)$).

For each $y \in V$ we choose now a constant $\omega(y)$ such that the semigroup $e^{-s(\omega(y)+A(t,y))} = e^{-s\omega(y)}e^{-sA(t,y)}$, $s \geqq 0$, shall satisfy an estimate of the form

$$\| e^{-s(\omega(y)+A(t,y))}\|_{L(W)} \leqq \hat{M} e^{-\hat{\omega} s}, \qquad s \geqq 0,$$

for some $\hat{M} \geqq 1$ and $\hat{\omega} > 0$. It is well known that this is always possible. Then the fractional powers $[\omega(y) + A(t, y)]^z$ are well defined for each $z \in \mathbb{C}$ and $(t, y) \in [0, T] \times V$ (*e.g.* [16, 23]).

We put

$$D^\alpha(t, y) := (D([\omega(y) + A(t, y)]^\alpha, \| [\omega(y) + A(t, y)]^\alpha \cdot \|) \quad \text{if } \alpha \geqq 0,$$

whereas

$D^\alpha(t, y)$ is the completion of W in the norm

$$\| [\omega(y) + A(t, y)]^\alpha \cdot \| \quad \text{if } \alpha < 0.$$

Observe that $D^0(t, y) = W$ and that $\| [\omega(y) + A(t, y)]^\alpha \cdot \|$ is equivalent to the graph norm of $D([\omega(y) + A(t, y)]^\alpha)$, if $\alpha > 0$. Hence each $D^\alpha(t, y)$ is a Banach space, and

$$D^\alpha(t, y), \qquad \alpha \in \mathbb{R},$$

is called the *scale of fractional power spaces of* $A(t, y)$. This notation is justified since another admissible choice of $\omega(y)$ gives the same scales, up to equivalent norms.

We assume now that

(A7) *for each* $(t, y) \in [0, T] \times V$ *there are positive constants* ε *and* a *such that* $[\omega(y) + A(t, y)]^{i\tau} \in \mathscr{L}(W)$ *and* $\| [\omega(y) + A(t, y)]^{i\tau} \|_{\mathscr{L}(W)} \leqq a$ *for* $-\varepsilon \leqq \tau \leqq \varepsilon$.

Then it follows from [8, Theorem 3.3] that

$$[D^{\alpha_0}(t, y), D^{\alpha_1}(t, y)]_\theta \doteq D^{\alpha_0(1-\theta)+\alpha_1\theta}(t, y) \tag{2}$$
$$\text{for } 0 < \theta < 1, \ -\infty < \alpha_0 < \alpha_1 < \infty, \ \text{and } (t, y) \in [0, T] \times V.$$

Hence we deduce from (A5) that

$$W^\beta_{\mathscr{B}} \doteq D^\beta(t, y) \qquad \text{for all } (t, y) \in [0, T] \times V, \tag{3}$$

which means that the domain of $[\omega(y) + A(t, y)]^\beta$ is independent of (t, y).

It is a consequence of (2), the reiteration theorem for the complex interpolation functor (*cf.* [9, Theorem 4.6.1]) and the commutativities of the real and complex interpolation functors (*cf.* [23, Theorem 1.10.2] and [9, Theorem 4.7.2]), that

$$W^\theta_{\mathscr{B}(t,y)} \doteq (W, [W, W^1_{\mathscr{B}(t)}]_\beta)_{\theta/\beta} \doteq (W, W^\beta_{\mathscr{B}})_{\theta/\beta} =: W^\theta_{\mathscr{B}} \tag{4}$$

for $0 < \theta < \beta$ and all $(t, y) \in [0, T] \times V$. Thus $W^\theta_{\mathscr{B}(t,y)}$ is independent of (t, y), up to equivalent norms, provided $0 < \theta \leqq \beta$.

We denote by $A'(t, y)$ the dual of $A(t, y)$ and define the *dual scale* of fractional power spaces

$$[D']^\alpha (t, y), \quad \alpha \in \mathbb{R},$$

with the help of the linear operators $\omega(y) + A'(t, y)$ in the dual space W' of W. Then we assume that

$$\text{(A8)} \qquad [D']^{1-\beta}(t, y) \doteq [D']^{1-\beta}(0, \hat{y}) \quad \textit{for all } (t, y) \in [0, T] \times V.$$

In [8, Theorem 1.3] it has been shown that $[D']^\alpha (t, y) = [D^{-\alpha} (t, y)]'$ for every $\alpha \in \mathbb{R}$ (with respect to the natural extension of the duality pairing $\langle W', W\rangle$). Hence (A8) implies that

$$\text{(5)} \qquad D^{\beta-1}(t, y) \doteq D^{\beta-1}(0, \hat{y}) =: W^{\beta-1}_{\mathscr{B}} \quad \text{for all } (t, y) \in [0, T] \times V.$$

We denote by $A_{\beta-1}(t, y)$ the closure of $A(t, y)$ in $W^{\beta-1}_{\mathscr{B}}$. It follows from [8, Theorem 2.1 and Corollary 2.2] that $A_{\beta-1}(t, y)$ is well defined and that

$$D(A_{\beta-1}(t, y)) = W^\beta_{\mathscr{B}}.$$

Hence we can assume that

(A9) *there are a number $\varrho \in (1 - \beta, 1)$ and, for each nonempty bounded subset $S \subset V$, constants L_0 and M_0 such that*

$$\|A_{\beta-1}(0, y)\|_{\mathscr{L}(W^\beta_{\mathscr{B}}, W^{\beta-1}_{\mathscr{B}})} \leqq M_0$$

and

$$\|A_{\beta-1}(s, y) - A_{\beta-1}(t, z)\|_{\mathscr{L}(W^\beta_{\mathscr{B}}, W^{\beta-1}_{\mathscr{B}})} \leqq L_0(|s - t|^\varrho + \|y - z\|_\eta)$$

for all (s, y), $(t, z) \in [0, T] \times S$.

Finally we assume that

(A10) *$f \in C([0, T] \times V, W^\zeta_{\mathscr{B}})$, and there are a number $\sigma \in [\eta, \xi]$ and, for each nonempty bounded subset $S \subset V$, constants L_1 and M_1 such that*

$$\|f(t, y)\|_\zeta \leqq M_1$$

and

$$\|f(t, y) - f(t, z)\|_\zeta \leqq L_1 \|y - z\|_\sigma$$

for all $t \in [0, T]$ and $y, z \in S$.

Under these assumptions we consider the *quasilinear Cauchy problem*

$$(\text{QCP})_{(s,x)} \qquad \dot{u} + A(t, u)\, u = f(t, u), \quad s < t \leqq T, \; u(s) = x$$

for $(s, x) \in [0, T) \times V$. By a *solution u* of $(\text{QCP})_{(s,x)}$ *on J* we mean a function

$$u \in C(J, V) \cap C^1(\dot{J}, W),$$

such that J is a perfect subinterval of $[s, T]$ containing s and $\dot{J} := J \setminus \{s\}$, and such that $u(s) = x$, $u(t) \in W^1_{\mathscr{B}(t,u(t))}$ and $\dot{u}(t) + A(t, u(t))\, u(t) = f(t, u(t))$ for $t \in \dot{J}$, where $\dot{u}$ denotes the derivative of u. A solution u is *maximal* if there is no solution which is a proper extension of u. In this case J is a *maximal interval of existence for* $(\mathrm{QCP})_{(s,x)}$. If $J = [s, T]$, then u is a *global solution.*

2. The Linear Cauchy Problem

Throughout this section we assume that $\mathscr{A}$, $\mathscr{B}$ and f are independent of $y \in V$, and that $\omega = 0$. Then $\Sigma_\vartheta \subset \varrho(-A(t))$,

$$\|(\lambda + A(t))^{-1}\|_{\mathscr{L}(W)} \leqq M/(1 + |\lambda|) \qquad \forall (\lambda, t) \in \Sigma_\vartheta \times [0, T] \tag{1}$$

and

$$\|A_{\beta-1}(s) - A_{\beta-1}(t)\|_{\mathscr{L}(W^\beta_{\mathscr{B}}, W^{\beta-1}_{\mathscr{B}})} \leqq L_0\, |s - t|^\varrho \qquad \forall s, t \in \Sigma_\vartheta \times [0, T], \tag{2}$$

as well as

$$\|A_{\beta-1}(0)\|_{\mathscr{L}(W^\beta_{\mathscr{B}}, W^{\beta-1}_{\mathscr{B}})} \leqq M_0 \tag{3}$$

and

$$\|f(t)\|_\zeta \leqq M_1 \qquad \forall t \in [0, T]. \tag{4}$$

Moreover we conclude from [8, Theorem 2.1 and Lemma 2.3] that $\varrho(-A_{\beta-1}(t)) \subset \Sigma_\vartheta$

$$\|(\lambda + A_{\beta-1}(t))^{-1}\|_{\mathscr{L}(W^{\beta-1}_{\mathscr{B}})} \leqq M/(1 + |\lambda|) \qquad \forall (\lambda, t) \in \Sigma_\vartheta \times [0, T], \tag{5}$$

and that

$$W^\beta_{\mathscr{B}} \overset{d}{\hookrightarrow} W \overset{d}{\hookrightarrow} W^{\beta-1}_{\mathscr{B}}, \tag{6}$$

where the letter d denotes dense imbedding.

Throughout this section we denote by c various positive constants, which may depend upon L_0, M, M_0, M_1, T, β, γ_0, γ_1, ϱ, ζ and ϑ, but are always independent of the specific independent variables occuring in a given formula. Moreover the constants c may be different from formula to formula, that is, we use c much in the same way as the Landau symbol O.

It follows from (2), (3) and (5) that

$$\|A_{\beta-1}(s)\, [A_{\beta-1}(t)]^{-1}\|_{\mathscr{L}(W^{\beta-1}_{\mathscr{B}})} \leqq c \qquad \forall s, t \in [0, T]. \tag{7}$$

The quasilinear Cauchy problem reduces now to the *linear Cauchy problem*

$$\dot{u} + A(t)\, u = f(t), \quad s < t \leqq T, \quad u(s) = x, \tag{$\mathrm{LCP}_{(s,x)}$}$$

where we can now assume that $(s, x) \in [0, T) \times W$. By a *solution* of $(\mathrm{LCP})_{(s,x)}$ we mean always a global solution belonging to

$$C([s, T], W) \cap C^1((s, T], W).$$

Let X and Y be Banach spaces. Then we denote by $\mathscr{K}(X, Y, \alpha)$, $\alpha \in \mathbb{R}$, the Banach space of all functions $k \in C(\dot{T}_\Delta, \mathscr{L}(X, Y))$ satisfying

$$\|k\|_{(\alpha)} := \sup_{(t,s)\in\dot{T}_\Delta} (t-s)^\alpha \|k(t,s)\| < \infty,$$

endowed with the norm $\|\cdot\|_{(\alpha)}$, and $\mathscr{K}(X, \alpha) := \mathscr{K}(X, X, \alpha)$. It is easily seen that

$$\mathscr{K}(X, Y, \alpha_0) \hookrightarrow \mathscr{K}(X, Y, \alpha_1) \quad \text{if } \alpha_0 < \alpha_1, \tag{8}$$

and that

$$\mathscr{K}(X, Y, \alpha) \hookrightarrow C(T_\Delta, \mathscr{L}(X, Y)) \quad \text{if } \alpha < 0, \tag{9}$$

provided $k \in \mathscr{K}(X, Y, \alpha)$ is extended over T_Δ by letting $k(t, t) := 0$ for $0 \leqq t \leqq T$. For $k \in \mathscr{K}(Y, Z, \alpha_0)$ and $h \in \mathscr{K}(Y, Z, \alpha_1)$ with $\alpha_0, \alpha_1 < 1$, we let

$$h*k(t,s) := \int_s^t h(t,\tau)\, k(\tau, s)\, d\tau, \qquad (t,s) \in T_\Delta.$$

Then it is not difficult to see that

$$h*k \in \mathscr{K}(X, Z, \alpha_0 + \alpha_1 - 1) \tag{10}$$

and that

$$\|h*k\|_{(\alpha_0+\alpha_1-1)} \leqq B(1-\alpha_0, 1-\alpha_1)\, \|h\|_{(\alpha_1)}\, \|k\|_{(\alpha_0)}, \tag{11}$$

where $B(\cdot, \cdot)$ is the beta function (*cf.* [7, Lemma 1.1]).

If $X = \mathbb{K}$, we identify $\mathscr{L}(\mathbb{K}, Y)$ naturally with Y via $\mathscr{L}(\mathbb{K}, Y) \ni B \leftrightarrow B \cdot 1 \in Y$. Hence $k \in \mathscr{K}(\mathbb{K}, Y, \alpha)$ if and only if $k \in C(\dot{T}_\Delta, Y)$ and $\|k(t,s)\| \leqq c(t-s)^{-\alpha}$ for all $(t,s) \in \dot{T}_\Delta$. In particular,

$$C([0,T], Y) \hookrightarrow \mathscr{K}(\mathbb{K}, Y, 0) = BC(\dot{T}_\Delta, Y) \tag{12}$$

by the obvious identification

$$C([0,T], Y) \ni u \leftrightarrow [(t,s) \mapsto u(t)] \in BC(\dot{T}_\Delta, Y). \tag{13}$$

If $\{X(t) \mid 0 \leqq t \leqq T\}$ is a family of Banach spaces, then we write

$$k \in \mathscr{K}(X(s), Y, \alpha)$$

provided $k(t,s) \in \mathscr{L}(X(s), Y)$ for $(t,s) \in \dot{T}_\Delta$,

$$\sup_{(t,s)\in\dot{T}_\Delta} (t-s)^\alpha \|k(t,s)\|_{\mathscr{L}(X(s),Y)} < \infty$$

and $k(\cdot, s) \in C((s, T], \mathscr{L}(X(s), Y))$ for every $s \in [0, T)$.

Below we use the following simplifying notation: whenever φ is a function of two real variables and ψ is a function of one real variable, we let

$$\psi\varphi(t,s) := \psi(t)\,\varphi(t,s), \qquad \varphi\psi(t,s) := \varphi(t,s)\,\psi(s),$$

provided the right-hand sides be meaningful.

Finally we denote by $\mathscr{L}_s(X, Y)$ the vector space $\mathscr{L}(X, Y)$ endowed with the strong topology, that is, the topology of pointwise convergence, and $\mathscr{L}_s(X) := \mathscr{L}_s(X, X)$.

It follows from [8, Theorem 4.4] that a unique parabolic fundamental solution U exists for $\{A(t) \mid 0 \leqq t \leqq T\}$ and has the following properties:

(U1) $$U \in C(T_\Delta, \mathscr{L}_s(W)) \cap C(\dot{T}_\Delta, \mathscr{L}(W)),$$

(U2) $$U(t, t) = \mathrm{id}_W \quad \text{and} \quad U(t, s) = U(t, \tau)\, U(\tau, s) \quad \text{for } 0 \leqq s \leqq \tau \leqq t \leqq T,$$

(U3) $$R(U(t, s)) \subset W^1_{\mathscr{B}(t)}, \quad AU \in C(\dot{T}_\Delta, \mathscr{L}(W)), \quad U(\cdot, s) \in C^1((s, T], \mathscr{L}(W))$$
$$\text{for } 0 \leqq s < T, \quad \text{and} \quad D_1 U = -AU,$$

(U4) $$(U \mid W^\beta_{\mathscr{B}})(t, \cdot) \in C^1([0, t], \mathscr{L}(W^\beta_{\mathscr{B}}, W)) \quad \text{for } 0 < t \leqq T \quad \text{and}$$
$$D_2 U(t, s)\, x = UA(t, s)\, x \quad \text{for } x \in W^1_{\mathscr{B}(s)} \quad \text{and } (t, s) \in \dot{T}_\Delta,$$

(U5) $$AUA^{-1} \in C(T_\Delta, \mathscr{L}_s(W)),$$

(U6) $$\| U(t, s) \|_{\mathscr{L}(W)}, \ \| AUA^{-1}(t, s) \|_{\mathscr{L}(W)} \leqq c \quad \forall (t, s) \in T_\Delta$$

and

$$\| AU(t, s) \| \leqq c(t - s)^{-1} \quad \forall (t, s) \in \dot{T}_\Delta,$$

where the constants c depend only upon L, M, M_0, T, β and ϱ, but not upon the individual operators $A(t)$, $0 \leqq t \leqq T$.

Moreover it follows from [8, Theorem 6.2] that $(\mathrm{LCP})_{(s,x)}$ has, for each $(s, x) \in [0, T) \times W$, a unique solution $u(\cdot, s, x)$, which is given by

(14) $$u(\cdot, \cdot, x) = Ux + U * f.$$

The following theorem shows that $u(\cdot, s, x)$ is more regular if $x \in W^\theta_{\mathscr{B}(s)}$.

Theorem 2.1. *Suppose that* $0 \leqq \tau \leqq \theta \leqq 1$ *and that* θ *is regular. Then*

(15) $$u(\cdot, s, x) \in C((s, T], W^1) \cap C^{\theta - \tau}([s, T], W^\tau),$$

provided $x \in W^\theta_{\mathscr{B}(s)}$. *Moreover*

(16) $$\| u(\cdot, s, x) \|_{C^{\theta-\tau}([s,T], W^\tau)} \leqq c(\tau, \theta, N)$$

for all $s \in [0, T)$ *and all* $x \in W^\theta_{\mathscr{B}(s)}$ *satisfying* $\| x \|_\theta \leqq N$.

Proof. Let $v := u(\cdot, s, x)$ and observe that

(17) $$v(t) \in W^1_{\mathscr{B}(t)} \quad \text{and} \quad \dot{v}(t) = -A(t)\, v(t) + f(t) \quad \text{for } s < t \leqq T.$$

Hence

$$\mathscr{A}(t)\, (v(t) - v(r)) = f(t) - f(r) - (\dot{v}(t) - \dot{v}(r)) - (\mathscr{A}(t) - \mathscr{A}(r))\, v(r)$$

and

$$\mathscr{B}(t)\, (v(t) - v(r)) = -(\mathscr{B}(t) - \mathscr{B}(r))\, v(r)$$

for $s < r, t \leqq T$. Thus, by (A6),

$$\gamma_0 \|v(t) - v(r)\|_1 \leqq \|f(t) - f(r)\| + \|\dot{v}(t) - \dot{v}(r)\| + (\|\mathscr{A}(t) - \mathscr{A}(r)\|_{\mathscr{L}(W^1,W)} + \|\mathscr{B}(t) - \mathscr{B}(r)\|_{\mathscr{L}(W^1,\partial W)}) \|v(r)\|_1 .$$

Now (A4), (A10) and the fact that v is a solution of $(\mathrm{LCP})_{(s,x)}$ imply

$$v \in C((s, T], W^1). \tag{18}$$

From (U1), (U3), (U6) and (A6) we deduce that $U \in \mathscr{K}(W, 0) \cap \mathscr{K}(W, W^1, 1)$. Hence, by interpolation,

$$U \in \mathscr{K}(W, W^\theta, \theta). \tag{19}$$

Since (U5), (U6) and (A6) imply $AU \in \mathscr{K}(W^1_{\mathscr{B}(s)}, W, 0)$, interpolation gives

$$AU \in \mathscr{K}(W^\theta_{\mathscr{B}(s)}, W, 1 - \theta). \tag{20}$$

Observe that

$$\gamma_0 \|x\|_1 \leqq \|A(t)\, x\| \leqq \gamma_1 \|x\|_1 \qquad \forall x \in W^1_{\mathscr{B}(t)},\ t \in [0, T], \tag{21}$$

and that

$$W^1_{\mathscr{B}(s)} \overset{d}{\hookrightarrow} W^\theta_{\mathscr{B}(s)} \qquad \forall s \in [0, T] \tag{22}$$

by [9, Theorem 3.4.2 and 4.4.2]. Moreover

$$\mathscr{A}(t)\,[U(t, s)\, x - U(r, s)\, x] = AU(t, s)\, x - AU(r, s)\, x - [\mathscr{A}(t) - \mathscr{A}(r)]\, U(r, s)\, x$$

and

$$\mathscr{B}(t)\,[U(t, s)\, x - U(r, s)\, x] = -[\mathscr{B}(t) - \mathscr{B}(r)]\, U(r, s)\, x,$$

so that

$$\gamma_0 \|U(t, s)\, x - U(r, s)\, x\|_1 \leqq \|AU(t, s)\, x - AU(r,s)\, x\| + (\|\mathscr{A}(t) - \mathscr{A}(r)\|_{\mathscr{L}(W^1,W)} + \|\mathscr{B}(t) - \mathscr{B}(r)\|_{\mathscr{L}(W^1,\partial W)}) \|U(r, s)\, x\|_1$$

for all $x \in W^1_{\mathscr{B}(s)}$ and $s \leqq t, r \leqq T$, by (A6). By using these facts, together with (U1), (U5), (U6) and (A2), we conclude easily that

$$U \in \mathscr{K}(W^\theta_{\mathscr{B}(s)}, W^1, 1 - \theta), \tag{23}$$

$$U(\cdot, s) \in C([s, T], \mathscr{L}_s(W^\theta_{\mathscr{B}(s)}, W^\theta)), \qquad s \in [0, T), \tag{24}$$

and

$$U(t, s)\,(W^\theta_{\mathscr{B}(s)}) \subset W^1_{\mathscr{B}(t)} \hookrightarrow W^\theta_{\mathscr{B}(t)}, \qquad (t, s) \in \dot{T}_\Lambda,$$

Suppose now that $x \in W^\theta_{\mathscr{B}(s)}$. Then $U(\cdot, s)\, x \in C([s, T], W^\theta)$ by (24). Moreover we deduce from (19), from

$$f \in C([s, T], W^\zeta_{\mathscr{B}}) \hookrightarrow \mathscr{K}(\mathbb{K}, W^\zeta_{\mathscr{B}}, 0) \hookrightarrow \mathscr{K}(\mathbb{K}, W, 0), \tag{25}$$

and from (8) and (9) that $U*f \in C(T_\Delta, W^\theta)$, if $\theta < 1$. If $\theta = 1$, we obtain $U*f \in C(T_\Delta, W^1)$ from (25), (23) (with θ replaced by ζ), (8) and (9). Thus

(26) $$v \in C([s, T], W^\theta) \quad \text{and} \quad \|v\|_{C([s,T],W^\theta)} \leqq c(N).$$

Now let $0 < \theta \leqq 1$. Then

$$(t-s)^{1-\theta} \|AU(t,s)x\| \leqq cN \quad \forall (t,s) \in \dot{T}_\Delta$$

by (20), and (4), (11)–(13) and (20) imply

$$\|(AU)*f(t,s)\| \leqq c \quad \forall (t,s) \in T_\Delta.$$

Hence, since $(AU)*f = A(U*f)$,

$$(t-s)^{1-\theta} \|A(Ux + U*f)(t,s)\| \leqq c(N), \quad s < t \leqq T,$$

which, due to $v = [Ux + U*f](\cdot, s)$ and (17), gives

$$(t-s)^{1-\theta} \|\dot{v}(t)\| \leqq c(N), \quad s \leqq t \leqq T.$$

Now we conclude from

$$\|v(t) - v(r)\| \leqq \int_r^t \|\dot{v}(\tau)\| \, d\tau \leqq \int_r^t (\tau - s)^{1-\theta} \|\dot{v}(\tau)\| (\tau - r)^{\theta-1} \, d\tau$$

that

$$\|v(t) - v(r)\| \leqq c(\theta, N)(t-r)^\theta, \quad s \leqq r \leqq t \leqq T.$$

Thus (26) and (A2) imply

(27) $$\|v(t) - v(r)\| \leqq c(\tau, \theta, N)\,|t-r|^{\theta-\tau}, \quad s \leqq t, r \leqq T.$$

Hence (18) and (27) prove the assertion, provided $\theta - \tau < 1$. However, if $\theta = 1$, we deduce from (26), (A4) and (A10) that

(28) $$\dot{v} = -\mathscr{A}(\cdot)v + f \in C([s, T], W).$$

that is, $v \in C^1([s, T], W)$. Finally we obtain

$$\|v\|_{C^1([s,T],W)} \leqq c(N)$$

from (26), (28) and (A6). □

Remark 2.2. It should be noted that for the above proof the compactness of the injection of W^1 into W has not been used. □

3. A Local Existence Theorem

Let S be a nonempty bounded subset of V. Then we fix constants γ_0, γ_1, ϑ, ω, L_0, L_1, M_0 and M_1 such that the inequalities (A6), (A7) and (A10) are satisfied.

For $0 \leqq t_0 \leqq t_1 \leqq T$ and $v \in C([t_0, t_1], S)$ we denote by $\bar{v}$ the continuous extension of v over $[0, T]$ which equals $v(t_0)$ on $[0, t_0]$ and $v(t_1)$ on $[t_1, T]$. Moreover we put

$$\mathscr{A}_v := \omega + \mathscr{A}(\cdot, \bar{v}(\cdot)), \qquad \mathscr{B}_v := \mathscr{B}(\cdot, \bar{v}(\cdot))$$

and

$$f_v := \omega\bar{v} + f(\cdot, \bar{v}(\cdot)).$$

Suppose that

$$\|v(s) - v(t)\|_\eta \leqq L' \, |s - t|^\varrho \qquad \forall s, t \in [t_0, t_1].$$

Then it follows from (A6), (A9) and (A10) that A_v and f_v satisfy the estimates (2.1)–(2.4), where L_0 is replaced by $L_0 + L'$, where M_0 depends also upon ω and β, and where M_1 depends now also upon ω, ξ, ζ and diam (S). Hence a unique parabolic fundamental solution U_v exists for $\{A_v(t) \mid 0 \leqq t \leqq T\}$, and the function

$$\psi_s(v, x)(t) := U_v(t, s)\, x + U_v * f_v(t, s), \qquad s \leqq t \leqq T,$$

is, for each $(s, x) \in [0, T) \times W$, the unique solution of the linear Cauchy problem

$$\dot{u} + A_v(t)\, u = f_v(t), \qquad s < t \leqq T, \qquad u(s) = x. \tag{1}$$

Observe that this means that

$$u := \psi_s(v, x) \in C([s, T], W) \cap C^1((s, T], W), \tag{2}$$

that

$$u(t) \in W^1_{\mathscr{B}(t, \bar{v}(t))}, \qquad s < t \leqq T, \tag{3}$$

and that

$$\dot{u} + A(t, \bar{v}(t))\, u(t) = f(t, \bar{v}(t)), \qquad s < t \leqq T, \qquad u(s) = x. \tag{4}$$

Moreover u satisfies the assertions of Theorem 2.1, where the constant $c(\tau, \theta, N)$ depends now also upon ω, β, ξ and diam (S).

We now put

$$CV := CV(S) := \{v \in C([0, T], S) \mid \|v(s) - v(t)\|_\eta \leqq |s - t|^\varrho \quad \text{for } 0 \leqq s, t \leqq T\}$$

and prove the following crucial

Lemma 3.1. *For each $b > 0$ there is a constant c such that*

$$\|\psi_s(v, x)(t) - \psi_s(w, y)(t)\|_\eta \leqq c\{(t - s)^{\beta - \eta} \|v - w\|_{C([s,t], W^\sigma)} + \|x - y\|_\eta\}$$

for all $v, w \in CV$, all $(t, s) \in T_\Delta$, and all $x, y \in W^\beta_{\mathscr{B}}$ satisfying $\|x\|_\beta, \|y\|_\beta \leqq b$.

Proof. The results in [8] imply that $\psi_s(v, x)$ is the unique solution of the linear Cauchy problem

$$\dot{u} + A_{v, \beta - 1}(t)\, u = f_v(t), \qquad s < t \leqq T, \qquad u(s) = x$$

in $W_{\mathscr{B}}^{\beta-1}$. By the reiteration and commutativity properties of the real and complex interpolation functors we deduce from (1.2) and (1.4) that

$$W_{\mathscr{B}}^{\gamma} = (W, W_{\mathscr{B}}^{\beta})_{\gamma/\beta} \doteq ([W_{\mathscr{B}}^{\beta-1}, W_{\mathscr{B}}^{\beta}]_{1-\beta}, W_{\mathscr{B}}^{\beta})_{\gamma/\beta} \doteq (W_{\mathscr{B}}^{\beta-1}, W_{\mathscr{B}}^{\beta})_{1-\beta+\gamma} \tag{5}$$

for $0 < \gamma < \beta$. Hence we can apply [7, Theorem 5.4], with (X, X^1) replaced by $(W_{\mathscr{B}}^{\beta-1}, W_{\mathscr{B}}^{\beta})$ (where it is easily verified that the last term, namely $(t-s)^{\lambda} \|x-y\|_{\theta}$ can be replaced by $\|x-y\|_{\eta}$), to obtain the assertion. □

In the following we let

$$J_{s,\delta} := [s, s+\delta] \cap [0, T]$$

for $\delta > 0$ and $s \in [0, T)$. Then Lemma 3.1 implies the

Corollary 3.2. *Let $b > 0$ be fixed. Then there are positive constants c and δ such that*

$$\|\psi_s(v, x) - \psi_s(w, y)\|_{C(J_{s,\delta}, W^{\eta})} \leqq \tfrac{1}{2} \|v - w\|_{C(J_{s,\delta}, W^{\sigma})} + c\, \|x - y\|_{\eta}$$

for all $v, w \in CV$, all $x, y \in W_{\mathscr{B}}^{\beta}$ satisfying $\|x\|_{\beta}, \|y\|_{\beta} \leqq b$, and all $s \in [0, T)$.

Let K be a nonempty subset of W^{τ} for some $\tau \in [0, 1]$. Then we denote by $\overline{\mathrm{B}}_{\tau}(K, \varepsilon)$ the closed ε-neighborhood of K in W^{τ}.

We suppose now that K is a nonempty subset of S such that $\overline{\mathrm{B}}_{\xi}(K, 2\varepsilon) \subset S$ for some $\varepsilon > 0$. For each $(s, x) \in [0, T) \times \overline{\mathrm{B}}_{\xi}(K, \varepsilon)$ we let

$$\mathscr{K}_{\delta}(s, x) := \{v \in C(J_{s,\delta}, W^{\xi}) \mid \bar{v} \in CV,\ v(s) = x,\ \|v(t) - x\|_{\xi} \leqq \varepsilon \quad \text{for } t \in J_{s,\delta}\},$$

and observe that $\mathscr{K}_{\delta}(s, x)$ is a nonempty closed and convex subset of the Banach space $C(J_{s,\delta}, W^{\xi})$ such that $v(J_{s,\delta}) \subset \overline{\mathrm{B}}_{\xi}(K, 2\varepsilon)$ for every $v \in \mathscr{K}_{\delta}(s, x)$.

Moreover we put

$$g_{s,\delta}(v, x) := \psi_s(v, x) \mid J_{s,\delta}$$

for all $v \in C(J_{s,\delta}, W^{\xi})$ with $\bar{v} \in CV$ and prove the following

Lemma 3.3. *Let $\tau, \theta \in (\xi, 1]$ be regular such that $\theta - \eta > \varrho$ and $\tau \leqq \theta$. Moreover suppose that N and b are positive constants. Then there are positive numbers δ, c_0 and c_1 such that:*

(i) $$g_{s,\delta}(v, x) \in \mathscr{K}_{\delta}(s, x),$$

(ii) $$\|g_{s,\delta}(v, x)\|_{C^{\theta-\tau}(J_{s,\delta}, W^{\tau})} \leqq c_0,$$

(iii) $$\|g_{s,\delta}(v, y) - g_{s,\delta}(w, z)\|_{C(J_{s,\delta}, W^{\eta})} \leqq \tfrac{1}{2} \|v - w\|_{C(J_{s,\delta}, W^{\sigma})} + c_1 \|y - z\|_{\eta}$$

for all $s \in [0, T)$, $x \in W_{\mathscr{B}(s,x)}^{\theta} \cap \overline{\mathrm{B}}_{\xi}(K, \varepsilon) \cap \overline{\mathrm{B}}_{\theta}(0, N)$ and $v \in \mathscr{K}_{\delta}(s, x)$, and for all $y, z \in W_{\mathscr{B}}^{\beta} \cap \overline{\mathrm{B}}_{\beta}(0, b)$ and $u, w \in C(J_{s,\delta}, W^{\xi})$ with $\bar{u}, \bar{w} \in CV$.

Proof. We deduce from Theorem 2.1 that

$$\|\psi_s(v, x)\|_{C^{\theta-\lambda}([s,T],W^\lambda)} \leqq c, \qquad \lambda \in \{\tau, \xi, \eta\},$$

for all admissible s, v and x. Consequently

$$\|\psi_s(v, x)(t) - x\|_\xi \leqq c\,\delta^{\theta-\xi}$$

and

$$\|\psi_s(v, x)(t_0) - \psi_s(v, x)(t_1)\|_\eta \leqq c\,\delta^{\theta-\eta-\varrho}\,|t_0 - t_1|^\varrho$$

for all $t, t_0, t_1 \in J_{s,\delta}$. Now the assertion follows from Corollary 3.2. □

It is convenient to exhibit the following simple general facts, which we shall use below.

Lemma 3.4. *Let Ω be a metric space and suppose that X, Y and Z are Banach spaces such that $X \subset\subset\hookrightarrow Y \hookrightarrow Z$.*

(i) *If $g \in C(\Omega, Z)$ and $g(\Omega)$ is bounded in X, then $g \in C(\Omega, Y)$.*

(ii) *If, in addition, Ω is a nonempty closed and convex subset of Y, and $g(\Omega) \subset \Omega$, then g has a fixed point, which is unique, provided g be a strict contraction with respect to the norm of Z.*

Proof. (i) Let (x_j) be a sequence in Ω converging to some $x \in \Omega$. Then $(g(x_j))$ is a bounded sequence in X, hence a relatively compact sequence in Y. Thus there are a subsequence (x_{j_k}) and a point $y \in Y$ such that $g(x_{j_k}) \to y$ in Y. Since $Y \hookrightarrow Z$ and $g \in C(\Omega, Z)$, it follows that $y = g(x)$. Now the assertion follows since, by this argument, $g(x)$ is the only cluster point of the sequence $(g(x_j))$ in Y.

(ii) The existence of a fixed point follows from (i) and Schauder's fixed point theorem. The asserted uniqueness is a trivial consequence of $Y \hookrightarrow Z$. □

Let D be a nonempty subset of W and let $0 \leqq \tau \leqq 1$. Then we put

(6) $D_\tau := D \cap W^\tau$, *endowed with the topology induced by* W^τ.

After these preparations we can easily prove the following *local existence, uniqueness and continuity*

Theorem 3.5. *Let N be a positive constant and let K be a nonempty subset of S such that $\overline{\mathbb{B}}_\xi(K, 2\varepsilon) \subset S$ for some $\varepsilon > 0$. Suppose that $\theta \in (\xi, 1]$ is regular, that $\theta - \eta > \varrho$, and that there is a regular $\tau \in (\xi, \theta)$.*

Then there is a positive number δ such that $(QCP)_{(s,x)}$ has, for each $s \in [0, T)$ and

$$x \in W^\theta_{\mathscr{B}(s,x)} \cap \overline{\mathbb{B}}_\xi(K, \varepsilon) \cap \overline{\mathbb{B}}_\theta(0, N) =: D$$

a solution $u_{s,x}$ on $J_{s,\delta}$, such that $u_{s,x}(J_{s,\delta}) \subset \overline{\mathbb{B}}_\xi(K, 2\varepsilon)$. If $\sigma = \eta$, then $u_{s,x}$ is the only solution of $(QCP)_{(s,x)}$ on $J_{s,\delta}$, and

$$(x \mapsto u_{s,x}) \in C(D_\eta, C(J_{s,\delta}, W^\xi)) \cap UC^{1-}(D_\eta, C(J_{s,x}, W^\eta)).$$

Proof. Let b be a positive constant such that $\overline{\mathbb{B}}_\xi(K, \varepsilon) \subset \overline{\mathbb{B}}_\beta(0, b)$, and let δ be the positive number of Lemma 3.3.

Now let $(s, x) \in [0, T) \times D$ be fixed, and let $\Omega := \mathscr{K}_\delta(s, x)$. Moreover, let $X := C^{\theta-\tau}(J_{s,\delta}, W^\tau)$, $Y := C(J_{s,\delta}, W^\xi)$ and $Z := C(J_{s,\delta}, W^\eta)$. Then (1.1) and the Arzelà-Ascoli theorem imply $X \subset\subset\hookrightarrow Y \hookrightarrow Z$. Furthermore it follows from Lemma 3.3 and the fact that $Y \hookrightarrow C(J_{s,\delta}, W^\sigma)$, that $g := g_{s,\delta}(\cdot, x)$ belongs to $C(\Omega, Z)$ and that $g(\Omega)$ is bounded in X and contained in Ω. Hence Lemma 3.4 implies the existence of a fixed point $u(x)$ of g, and it follows from (1)–(4) that $u(x)$ is a solution of $(\mathrm{QCP})_{(s,x)}$ on $J_{s,\delta}$.

Suppose now that $\sigma = \eta$. Then we see from Lemma 3.3 (iii) that g is a strict contraction with respect to the norm of Z. Hence $u(x)$ is the unique fixed point of g and, consequently, the unique solution of $(\mathrm{QCP})_{(s,x)}$ on $J_{s,\delta}$.

For each $y \in D$ let $u(y)$ be the unique fixed point of $g_{s,\delta}(\cdot, y)$ in $\mathscr{K}_\delta(s, y)$. Then we deduce from Lemma 3.3 (iii) that

$$\|u(x) - u(y)\|_z \leqq 2c_1 \|x - y\|_\eta .$$

Hence $u \in UC^{1-}(D_\eta, Z)$. Since $u(D)$ is bounded in X, by Lemma 3.3 (ii), the remaining continuity assertion is a consequence of Lemma 3.4 (i). □

4. Global Existence and Continuity

Throughout this section we assume that

(A11) $\theta \in (\xi, 1]$ *is regular,* $\theta - \eta > \varrho$, *and there is a regular* $\tau \in (\xi, \theta)$.

Then we can prove our main abstract result, namely

Theorem 4.1. *Suppose that* $(s, x) \in [0, T) \times V$ *such that*

$$x \in W^\theta_{\mathscr{B}(s,x)} .$$

Then the quasilinear Cauchy problem $(QCP)_{(s,x)}$ *has a maximal solution* $u(\cdot, s, x)$ *and*

$$u(\cdot, s, x) \in C^{\theta-\lambda}(J, W^\lambda) \cap C(\dot{J}, W^1) \tag{1}$$

for every $\lambda \in [0, \theta]$. *Each maximal interval of existence of* $(QCP)_{(s,x)}$ *is open in* $[s, T]$.

If $\sigma = \eta$, *then* $u(\cdot, s, x)$ *is the only maximal solution of* $(QCP)_{(s,x)}$, *and there are positive constants* ε *and* δ *such that* $u(\cdot, s, y)$ *exists on* $J_{s,\delta}$ *for each* $y \in W^\theta_{\mathscr{B}(s,y)} \cap \overline{\mathbb{B}}_\theta(x, \varepsilon) =: D$. *Moreover*

$$[x \mapsto u(\cdot, s, x)] \in C(D_\eta, C(J_{s,\delta}, W^\xi)) \cap UC^{1-}(D_\eta, C(J_{s,\delta}, W^\eta)) .$$

Proof. Let $K := \{x\}$ and choose $\varepsilon_0 > 0$ such that $S := \overline{\mathbb{B}}_\xi(x, 2\varepsilon_0) \subset V$. Moreover choose $\varepsilon > 0$ such that $\overline{\mathbb{B}}_\theta(x, \varepsilon) \subset \overline{\mathbb{B}}_\xi(x, \varepsilon_0)$. Then it follows from Theorems 3.5 and 2.1 that there are a $\delta > 0$ and a solution $u_{s,y}$ of $(\mathrm{QCP})_{(s,y)}$ on $J_{s,\delta}$, for every $y \in D$, possessing all the properties of the assertion (suitably restricted to $J_{s,\delta}$, of course).

Now let $(s_1, x_1) := (s + \delta, u_{s,x}(s + \delta))$, provided $s + \delta < T$. Then, by the above, there is a $\delta_1 > 0$ such that $(QCP)_{(s_1,x_1)}$ has a solution v on J_{s_1,δ_1}, which is unique if $\sigma = \eta$. Since $x_1 \in W^1_{\mathscr{B}(s_1,x_1)}$, it follows from Theorem 2.1 that $v \in C^1(J_{s_1,\delta_1}, W)$. Now it is easily verified that the function u, defined to be equal to $u_{s,x}$ on $J_{s,\delta}$ and to v on J_{s_1,δ_1}, is a solution of $(QCP)_{(s,x)}$ on $J_{s,\delta+\delta_1}$. This implies that every maximal interval of existence is open in $[s, T]$, and that there is at most one maximal solution if $\sigma = \eta$. Finally the existence of a maximal solution follows now by a standard application of Zorn's lemma. □

In the next theorem we give a sufficient condition for global existence. In this connection we define the distance to the empty set to be ∞.

Theorem 4.2. *Suppose that* $(s, x) \in [0, T) \times V$ *such that* $x \in W^\theta_{\mathscr{B}(s,x)}$, *and let* $u(\cdot, s, x)$ *be a maximal solution of* $(QCP)_{(s,x)}$. *Let* J *be the corresponding maximal interval of existence and suppose that*

(2) $$u(J, s, x) \quad \text{is bounded in } W^\theta$$

and

(3) $$u(J, s, x) \quad \text{has a positive distance to } \partial V.$$

Then $u(\cdot, s, x)$ *is a global solution.*

Proof. Let $K := u(J, s, x)$ and observe that $y \in W^\theta_{\mathscr{B}(t,y)}$ whenever $t \in J$ and $y := u(t, s, x)$. By (3) we can find a number $\varepsilon > 0$ such that $S := \overline{\mathrm{B}}_\xi(K, 2\varepsilon) \subset V$. Hence (2) and Theorem 3.5 imply the existence of a positive number δ such that $(QCP)_{(t,u(t,s,x))}$ has, for every $t \in J$, a solution on $J_{t,\delta}$. Thus we could continue $u(\cdot, s, x)$ beyond J, if $J \neq [s, T]$, which contradicts the maximality of J. □

Let X be a metric space and suppose that

$$t^+ : X \to (0, \infty]$$

and

$$\varphi : \mathscr{D} := \{(t, x) \in [0, \infty) \times X \mid 0 \leqq t < t^+(x)\} \to X.$$

Then we put $\varphi^t := \varphi(t, \cdot)$ and call φ *a preflow* (more precisely, a local semi-preflow) *on* X if

(i) $$\varphi(\cdot, x) \in C([0, t^+(x)), X) \quad \text{for every } x \in X.$$

(ii) $$\varphi^0 = \mathrm{id}_X.$$

(iii) *If* $0 \leqq t < t^+(x)$ *and* $0 \leqq s < t^+(\varphi^t(x))$, *then* $s + t < t^+(x)$ *and* $\varphi^{s+t}(x) = \varphi^s(\varphi^t(x))$.

The function t^+ is the (positive) *exit time* of φ, and φ is *global* if $t^+(x) = \infty$ for all $x \in X$. The set $\gamma^+(x) := \varphi([0, t^+(x)), x)$ is the *orbit* (more precisely, the positive semi-orbit) *of* x. A subset $Y \subset X$ is *positively invariant* if $\gamma^+(Y) \subset Y$, and Y is *invariant* if, for each $y \in Y$, there is a solution u through y with $u(\mathrm{dom}(u)) \subset Y$.

Here we mean by a *solution u of the flow φ* through $x \in X$ a continuous function $u: J \to X$ such that J is an open interval in $\mathbb{R}$ containing $[0, t^+(x))$, such that $u(0) = x$ and $u(t+\tau) = \varphi^t(u(\tau))$ for all $t, \tau \in \mathbb{R}$ with $(t, u(\tau)) \in \mathscr{D}$ and $t + \tau \in J$, and such that there is no proper continuous extension of u with these properties. Clearly every critical point is invariant, where x is a *critical point of* φ if $\gamma^+(x) = \{x\}$. Finally,

$$\omega(x) := \bigcap_{t>0} \overline{\gamma^+(x)}$$

is the *ω-limit set of* x. Observe that $\omega(x) = \emptyset$ if $t^+(x) < \infty$.

For each $\Phi \in C(X, \mathbb{R})$ we define $\dot{\Phi}(x)$, the *orbital derivative* of Φ at x, to be the lower right Dini derivative of the continuous function $\Phi \circ \varphi(\cdot, x)$ at $t = 0$. Then Φ is said to be a *Liapounov* function for φ on the subset $Y \subset X$ if $\dot{\Phi}(y) \leqq 0$ for all $y \in Y$.

Lastly, a preflow φ is a *flow* (more precisely, a local semiflow) *on* X if $\mathscr{D}$ is open in $\mathbb{R}^+ \times X$ and $\varphi \in C(\mathscr{D}, X)$.

Suppose now that $\sigma = \eta$ and that $\mathscr{A}$, $\mathscr{B}$ and f are independent of t. Then Theorems 4.1 and 4.2 are valid for every $T > 0$. This implies, in particular, that the *autonomous* quasilinear Cauchy problem

$$(\mathrm{AQCP})_{(s,x)} \qquad \dot{u} + A(u)\, u = f(u), \quad s < t < \infty, \quad u(s) = x$$

has, for each $(s, x) \in \mathbb{R}^+ \times V$ with $x \in W^{\theta}_{\mathscr{B}(x)}$, a unique maximal solution $u(\cdot, s, x)$, which is now defined on some open subinterval $J(s, x)$ of $[s, \infty)$, containing s, such that

$$(4) \qquad u(\cdot, s, x) \in C^{\theta-\lambda}(J, W^{\lambda}) \cap C(\dot{J}, W^1) \cap C^1(\dot{J}, W)$$

for every $\lambda \in [0, \theta]$.

We put

$$\mathscr{X} := \{x \in V \mid x \in W^{\theta}_{\mathscr{B}(x)}\}$$

and

$$\varphi(\cdot, x) := u(\cdot, 0, x), \qquad t^+(x) := \sup J(0, x) \qquad \forall x \in \mathscr{X},$$

as well as

$$\mathscr{D} := \{(t, x) \in \mathbb{R}^+ \times \mathscr{X} \mid 0 \leqq t < t^+(x)\}.$$

Then the unique solvability of $(\mathrm{AQCP})_{(s,x)}$ implies that φ is a preflow on $\mathscr{X}_\lambda$ for every $\lambda \in [0, \theta]$ (*cf.* (3.6)), possessing the additional regularity properties of (4). Moreover φ has the "local continuity" given in Theorem 4.1. The following theorem shows that the restriction of φ to certain positively invariant subsets of $\mathscr{X}$ has the corresponding global continuity.

Theorem 4.3. *Let $\mathscr{P}$ be a positively invariant bounded subset of $\mathscr{X}_\theta$, which is closed in V. Then*

$$\mathscr{P}\mathscr{D} := \{(t, x) \in \mathscr{D} \mid x \in \mathscr{P}\}$$

is open in $\mathbb{R}^+ \times \mathcal{P}_\xi$ *and*

$$\varphi \in C(\mathcal{PD}_\eta, \mathcal{P}_\xi) \cap C^{\theta-\eta,1-}(\mathcal{PD}_\eta, \mathcal{P}_\eta),$$

where $\mathcal{PD}_\eta$ *is* $\mathcal{PD}$, *endowed with the topology induced by* $\mathbb{R} \times W^\eta$.

Proof. Let $(t_0, x_0) \in \mathcal{PD}$ be fixed and choose a constant N such that $\mathcal{P} \subset \overline{\mathbb{B}}_\theta(0, N)$. Since $K := \varphi([0, t_0], x_0)$ is compact in V, by (4), there is a number $\varepsilon > 0$ such that $S := \overline{\mathbb{B}}_\xi(K, 2\varepsilon) \subset V$. Thus, by Theorem 3.5, we find positive constants c and δ such that $t^+(x) > \delta$ for every $x \in \mathcal{P} \cap \overline{\mathbb{B}}_\xi(K, \varepsilon) =: D$, such that

$$(5) \qquad [x \mapsto \varphi(\cdot, x)] \in C(D_\eta, C([0, \delta], W^\xi)),$$

and such that

$$(6) \qquad \|\varphi^t(x) - \varphi^t(y)\|_\eta \leqq c\,\|x - y\|_\eta \quad \forall t \in [0, \delta],\ x, y \in D.$$

Since D is closed in V and $\operatorname{dist}(D, \partial V) \geqq \varepsilon$, the boundedness of $\mathcal{P}$ in W^θ and $W^\theta \subset\subset\hookrightarrow W^\xi$ imply that D_ξ is compact. Thus D_η is also compact, due to $W^\xi \hookrightarrow W^\eta$. Hence the map (5) is uniformly continuous.

We now fix points $s_0 := 0 < s_1 < \ldots < s_m := t_0 < s_{m+1}$ with $s_{j+1} - s_j \leqq \delta$, and let $\alpha \in (0, \varepsilon)$ be arbitrary. By the uniform continuity of the map (5), and by $W^\xi \hookrightarrow W^\eta$, we find numbers $0 < \alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{m+1} := \alpha$ such that

$$(7) \qquad \|\varphi^t(x) - \varphi^t(y)\|_\xi \leqq \alpha_{j+1}$$

for all $t \in [0, \delta]$ and all $x, y \in D$ with $\|x - y\|_\xi \leqq \alpha_j$, $j = 0, \ldots, m$. This shows that $\varphi^t(x) \in \overline{\mathbb{B}}_\xi(\varphi^t(x_0), \alpha_1)$ for all $x \in \overline{\mathbb{B}}_\xi(x_0, \alpha_0)$. Thus $t^+(\varphi^{s_1}(x)) > \delta$ and $\varphi^{s_2}(x) = \varphi^{s_2 - s_1}(\varphi^{s_1}(x)) \subset \overline{\mathbb{B}}_\xi(\varphi^{s_2}(x_0), \alpha_2)$ for all $x \in \overline{\mathbb{B}}_\xi(x_0, \alpha_0)$, by (7). By repeating this argument m times, we see that $t^+(x) > t_0 + \delta$ for every $x \in \overline{\mathbb{B}}_\xi(x_0, \alpha_0)$ which proves that $\mathcal{PD}$ is open in $\mathbb{R}^+ \times \mathcal{P}_\xi$. Moreover, it follows from (5) that there is a number $\gamma > 0$ such that $\varphi^{s_1}(x) \in \overline{\mathbb{B}}_\xi(\varphi^{s_1}(x_0), \alpha_1)$ for all $x \in \overline{\mathbb{B}}_\eta(x_0, \gamma)$, which implies that $\varphi^t(\overline{\mathbb{B}}_\eta(x_0, \gamma)) \subset \overline{\mathbb{B}}_\xi(\varphi^t(x_0), \alpha)$ for all $t \in [0, t_0 + \delta]$. From this we deduce easily that $\varphi \in C(\mathcal{PD}_\eta, \mathcal{P}_\xi)$. The second continuity assertion is now an immediate consequence of (4) and (6). □

Corollary 4.4 *Let* $\mathcal{P}$ *be a positively invariant bounded subset of* $\mathcal{X}_\theta$, *which is closed in* V. *Then* φ *is a flow on* $\mathcal{P}_\xi$. *If* $\gamma^+(x)$ *is an orbit in* $\mathcal{P}_\xi$, *which has a positive distance from* ∂V, *then* $\gamma^+(x)$ *is relatively compact. Hence* $\omega(x)$ *is a nonempty, invariant, compact and connected subset of* $\mathcal{P}_\xi$, *and* $\varphi^t(x) \to \omega(x)$ *as* $t \to \infty$. *If* Φ *is a Liapounov function on* $\mathcal{P}_\xi$, *then there is a number* $\alpha \in \mathbb{R}$ *such that*

$$\omega(x) \subset \{y \in \mathcal{P} \mid \dot{\Phi}(y) = 0\} \cap \Phi^{-1}(\alpha).$$

Proof. The fact that φ is a flow on $\mathcal{P}_\xi$ follows from Theorem 4.3 and $W^\xi \hookrightarrow W^\eta$. The compactness of $\overline{\gamma^+(x)}$ in $\mathcal{P}_\xi$ is a consequence of the fact that $\overline{\gamma^+(x)}$ is a closed subset of V, and of $W^\theta \subset\subset\hookrightarrow W^\xi$. The remaining assertions are well known facts from the general theory of semiflows (*e.g.* [3, Theorems 17.2 and 18.3]). □

Remarks 4.4. (a) *Let* $V := W^{\xi}$ *and let* $\mathscr{P}$ *be a positively invariant bounded subset of* $\mathscr{X}_{\theta}$, *which is closed in* W^{ξ}. *Then* φ *is a global flow on* $\mathscr{P}_{\xi}$ *such that every orbit is relatively compact.*

(b) *If* $y \in \mathscr{X}$ *is a critical point for the preflow* φ, *then* $y \in W^1_{\mathscr{B}(y)}$ *and* $A(y)\,y = f(y)$.

This follows from (4).

(c) *In the special case* $\eta = \xi$, *it follows that* φ *is a flow on* $\mathscr{X}$ *such that* $\varphi \in C^{\theta-\eta,1-}(\mathscr{D}, \mathscr{X})$. In fact, denoting by $J(s, x)$ the maximal interval of existence of the unique maximal solution $u(\cdot, s, x)$ of $(\mathrm{QCP})_{(s,x)}$ and letting

$$\mathscr{X}(s) := \{x \in V \mid x \in W^{\theta}_{\mathscr{B}(s,x)}\}$$

and

$$\mathscr{D}(s) := \{(t, x) \in [s, T] \times \mathscr{X}(s) \mid t \in J(s, x)\},$$

we conclude that $\mathscr{D}(s)$ is open in $[s, T] \times (\mathscr{X}(s))_{\eta}$ and that $u(\cdot, s, \cdot) \in C^{\theta-\eta,1-}([\mathscr{D}(s)]_{\eta}, [\mathscr{X}(s)]_{\eta})$. Moreover, in this case it suffices to assume that $W^1 \hookrightarrow W$.

This follows from the above proofs since Theorem 3.5 can be obtained by the contraction mapping principle in this case. □

5. Higher Regularity

In addition to the assumptions (A1)–(A11) we presuppose throughout this section the assumptions (HR1)–(HR3) below.

There are Banach spaces H^1, H *and* ∂H *and a number* $\gamma \in (0, \eta]$ *such that*

(HR1) $$H^1 \hookrightarrow W^1 \hookrightarrow W^{\gamma} \hookrightarrow H \hookrightarrow W$$

and $\partial H \hookrightarrow \partial W$, *and such that*

$$(\mathscr{A}, \mathscr{B}) \in C([0, T] \times V, \mathscr{L}(H^1, H \times \partial H)).$$

We put

$$H^1_{\mathscr{B}(t,y)} := \ker \mathscr{B}(t, y) \mid H^1 \qquad \forall (t, y) \in [0, T] \times V$$

and observe that $H^1_{\mathscr{B}(t,y)}$ is a closed linear subspace of H^1 such that

$$H^1_{\mathscr{B}(t,y)} \hookrightarrow W^1_{\mathscr{B}(t,y)}.$$

Let X and Y be Banach spaces with $Y \hookrightarrow X$, and let A be a linear operator in X. Then we define the *Y-realization* A_Y of A to be the maximal restriction of A to Y, that is, A_Y is the linear operator *in* Y, defined by

$$A_Y y := Ay \quad \text{for all } y \in \{y \in D(A) \cap Y \mid Ay \in Y\} =: D(A_Y).$$

Clearly A_Y is closed if A is closed.

It is obvious that the H-realization $A_H(t, y)$ of $A(t, y)$ satisfies

$$A_H(t, y) = \mathscr{A}(t, y) \mid H^1_{\mathscr{B}(t,y)}.$$

Moreover it is an easy consequence of (HR1) that $\varrho(-A_H(t,y)) \supset \varrho(-A(t,y))$ and that

$$(\lambda + A_H(t,y))^{-1} = (\lambda + A(t,y))^{-1} \mid H \quad \forall \lambda \in \varrho(-A(t,y)). \tag{1}$$

Hence the following assumption is meaningful, where ω and ϑ are the constants of (A6).

(HR2) *There are a number $\varkappa \in [0, 1)$ and, for each bounded subset $S \subset V$, positive constants $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{M}$ such that*

$$\hat{\gamma}_0 \|x\|_{H^1} \leqq \|(\lambda + \mathscr{A}(t,y))\,x\|_H + \|\mathscr{B}(t,y)\,x\|_{\partial H} \leqq \hat{\gamma}_1 \|x\|_{H^1} \qquad \forall x \in H^1$$

and

$$\|(\lambda + A_H(t,y))^{-1}\|_{\mathscr{L}(H)} \leqq \hat{M}/|\lambda - \omega|^{1-\varkappa}$$

for all $\lambda \in \omega + \Sigma_\vartheta$ and $(t,y) \in [0,T] \times V$.

Finally we strengthen the requirements for the nonlinearity f.

(HR3) *There are numbers $\nu \in (\varkappa, 1)$ and $\lambda \in (\gamma, \beta)$ such that*

$$f \in C^{\nu,1-}([0,T] \times V, H)$$

and

$$f(\cdot, u(\cdot)) \in B([0,T], W^{\lambda}_{\mathscr{B}})$$

for every $u \in C([0,T], V)$ satisfying $u(t) \in W^{\xi}_{\mathscr{B}(t,u(t))}$ for $0 \leqq t \leqq T$.

Then we prove the following *abstract regularity*

Theorem 5.1. *Let $(s,x) \in [0,T) \times V$ satisfy $x \in W^{\theta}_{\mathscr{B}(s,x)}$, and suppose that $\theta - \xi \geqq \nu$ and $\varrho > 1 - \beta + \gamma$. Then every maximal solution $u(\cdot, s, x)$ of $(\mathrm{QCP})_{(s,x)}$ satisfies*

$$u(\cdot, s, x) \in C(\dot{J}, H^1) \cap C^1(\dot{J}, H).$$

Proof. Let $t_1 \in \dot{J}$ be arbitrarily fixed and put $v : u(\cdot, s, x) \mid [s, t_1]$. Then $S := v([s, t_1])$ is a compact subset of V and $v \in C^{\varrho}([s, t_1], W^{\eta})$ by (A11) and Theorem 4.1. Hence we can define $\mathscr{A}_v$, $\mathscr{B}_v$, f_v and U_v as in the beginning of Section 3. Moreover, if

$$u := U_v(\cdot, s)\,x + U_v * f_v(\cdot, s), \tag{2}$$

it follows that

$$u \mid [s, t_1] = u(\cdot, s, x) \mid [s, t_1].$$

Hence it suffices to show that

$$u \in C((s,T], H^1) \cap C^1((s,T], H). \tag{3}$$

Since $\theta - \gamma \geqq \theta - \xi \geqq \nu$, we deduce from (HR1), (HR3) and Theorem 4.1 that

$$f_v \in B([0,T], W^{\lambda}_{\mathscr{B}}) \cap C^{\nu}([0,T], H). \tag{4}$$

In the remainder of this proof we shall be concerned with the functions u, $\mathscr{A}_v$, $\mathscr{B}_v$, f_v and U_v only, and so we will drop the subscript v from now on. Moreover we let

$$B(t) := A_{\beta-1}(t), \quad 0 \leqq t \leqq T.$$

It follows from (2.5) and (2.6) that $-B(t)$ is the infinitesimal generator of a strongly continuous analytic semigroup $\{e^{-sB(t)} \mid s \geqq 0\}$ on $W_{\mathscr{B}}^{\beta-1}$. If

$$a(t,s) := e^{-(t-s)B(s)}, \quad 0 \leqq s, t \leqq T,$$

it is an easy consequence of (2.7) and of the analyticity of the above semigroup that $a \in \mathscr{K}(W_{\mathscr{B}}^{\beta-1}, 0) \cap \mathscr{K}(W_{\mathscr{B}}^{\beta}, W_{\mathscr{B}}^{\beta-1}, 1)$. Hence we deduce from (3.5) by interpolation that

(5) $$a \in \mathscr{K}(W_{\mathscr{B}}^{\beta-1}, W_{\mathscr{B}}^{\gamma}, 1+\gamma-\beta).$$

Let

$$k(t,s) := -[B(t) - B(s)]\, a(t,s), \quad 0 \leqq s, t \leqq T,$$

and denote by w the unique solution in $\mathscr{K}(W_{\mathscr{B}}^{\beta-1}, 1-\varrho)$ of the equation $w = k + k*w$, whose existence is guaranteed by [7, Theorem 1.2]. Then we know from [8, formula (5.14)] that $w \in \mathscr{K}(W, W_{\mathscr{B}}^{\beta-1}, \beta-\varrho)$. Hence we obtain from (5) that $aw \in \mathscr{K}(W, W_{\mathscr{B}}^{\gamma}, 1+\gamma-\varrho)$, which implies, due to (HR1), that

(6) $$aw \in \mathscr{K}(H, 1+\gamma-\varrho)$$

(where, of course, $aw(t,s) := a(t,s)\, w(t,s)$).

In [8] it has been shown that $U(\cdot, s) \in C^1((s,T], \mathscr{L}(W_{\mathscr{B}}^{\beta-1}))$, that

(7) $$D_1 U = D_1 a + D_1(a*w),$$

and that

(8) $$D_1(a*w) = aw + b_1 + b_2,$$

where

$$b_1(t,s) := \int_s^t [B(t)\, e^{-(t-\tau)B(t)} - B(\tau)\, e^{-(t-\tau)B(\tau)}]\, w(\tau,s)\, d\tau$$

and

$$b_2(t,s) := \int_s^t B(t)\, e^{-(t-\tau)B(t)}[w(t,s) - w(\tau,s)]\, d\tau$$

for $(t,s) \in T_\Delta$ (*cf.* [8, formulas (5.14) and (5.15)]).

From (2.5), (2.7) and (3.5) we obtain by interpolation

$$\|(\lambda + B(t))^{-1}\|_{\mathscr{L}(W_{\mathscr{B}}^{\beta-1}, W_{\beta}^{\gamma})} \leqq c\,|\lambda|^{\gamma-\beta}$$

for all $(\lambda, t) \in \Sigma_\vartheta \times [0,T]$. Hence, by modifying in an obvious way the arguments following the proof of [8, Lemma 5.3], we find that $b_1 \in \mathscr{K}(W, W_{\mathscr{B}}^{\gamma}, 1+\gamma-\varrho)$.

Thus, using again (HR1), we conclude that

$$(9)\qquad b_1 \in \mathscr{K}(H, 1+\gamma-\varrho).$$

From [8, Lemma 5.2] and (3.5) we deduce by interpolation that

$$\|B(t)\, e^{-(t-\tau)B(t)}\|_{\mathscr{L}(W_{\mathscr{B}}^{\beta-1}, W_{\mathscr{B}}^{\gamma})} \leqq c(t-\tau)^{\beta-\gamma-2}$$

for all $(t,\tau) \in \dot{T}_\Delta$. Hence [8, Lemma 5.3] implies easily that $b_2 \in \mathscr{K}(W, W_{\mathscr{B}}^{\gamma}, 1+\gamma-\varrho)$, whence

$$(10)\qquad b_2 \in \mathscr{K}(H, 1+\gamma-\varrho),$$

provided $\varrho > 1-\beta+\gamma$, which we have assumed. Thus, by collecting (6), (9) and (10), we conclude from (8) that

$$(11)\qquad D_1(a*w) \in \mathscr{K}(H, 1+\gamma-\varrho).$$

Denote by Γ any piecewise smooth curve in Σ_ϑ running from $\infty e^{-i(\vartheta+\pi/2)}$ to $\infty e^{i(\vartheta+\pi/2)}$, and let $e^{-oA_H(s)} := \mathrm{id}_H$ and

$$(12)\qquad e^{-tA_H(s)} := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}(\lambda + A_H(s))^{-1}\, d\lambda, \qquad t > 0.$$

Standard arguments show that $\{e^{-tA_H(s)} \mid t \geqq 0\}$ is a semigroup on H, which is differentiable for $t>0$ (but not strongly continuous at $t=0$, in general) and satisfies

$$(13)\qquad \|A_H^j(s)\, e^{-tA_H(s)}\|_{\mathscr{L}(H)} \leqq ct^{-j-\varkappa}$$

for $t>0$, $s \in [0,T]$ and $j=0,1$. It is an easy consequence of (1) and (12) that

$$(14)\qquad e^{-tA_H(s)} = e^{-tA(s)} \mid H.$$

Since $A(s)$ is the W-realization of $B(s)$ by [8, Proposition 3.1], and since $D_1 a(t,s) = -B(s)\, e^{-(t-s)B(s)}$, it follows from (14) that

$$(15)\qquad D_1 a(t,s) \mid H = -A_H(s)\, e^{-(t-s)A_H(s)} \quad \text{for } (t,s) \in \dot{T}_\Delta.$$

Let $z \in H$ and $\lambda \in \Sigma_\vartheta$ be fixed and let $z(t) := (\lambda + A_H(t))^{-1} z$. Then $z(t) \in H^1$ and $\mathscr{B}(t)\, z(t) = 0$, so that

$$(\lambda + \mathscr{A}(t))\,(z(t)-z(s)) = -(\mathscr{A}(t)-\mathscr{A}(s))\, z(s)$$

$$\mathscr{B}(t)\,(z(t)-z(s)) = -(\mathscr{B}(t)-\mathscr{B}(s))\, z(s)$$

for $0 \leqq s,\ t \leqq T$. Thus (HR2) implies the estimate

$$\hat{\gamma}_0 \|z(t)-z(s)\|_{H^1} \leqq (\|\mathscr{A}(t)-\mathscr{A}(s)\|_{\mathscr{L}(H^1,H)} + \|\mathscr{B}(t)-\mathscr{B}(s)\|_{\mathscr{L}(H^1,\partial H)})\, \|z(s)\|_1$$

for all $s, t \in [0, T]$. Since $\|z(s)\|_1 \leqq \|(\lambda + A_H(s))^{-1}\|_{\mathscr{L}(H,H^1)} \|z\|_H$, it follows that

(16) $$(\lambda + A_H(\cdot))^{-1} \in C([0, T], \mathscr{L}(H, H^1))$$

for each fixed $\lambda \in \Sigma_\vartheta$. Thus, since $H^1 \hookrightarrow H$ and

$$A_H(s)\, e^{-tA_H(s)} = -\frac{1}{2\pi i} \int_\Gamma \lambda e^{\lambda t}(\lambda + A_H(s))^{-1}\, d\lambda,$$

we deduce from (13), (15) and (16) that

(17) $$D_1 a(t, s) \mid H \in \mathscr{K}(H, 1 + \varkappa).$$

Observe now that $W^1 \hookrightarrow H$ and Theorem 2.1 imply

(18) $$u \in C((s, T], H).$$

Moreover,

(19) $$\dot{u}(t) = -A(Ux + U{*}f)(t, s) + f(t), \quad s < t \leqq T$$

in W. By replacing s by any $s_1 \in (s, T]$ and by using (18), we can assume that $x \in H$. Thus, since

(20) $$-AU|H = D_1 U|H = D_1 a|H + D_1(a{*}w)|H \in \mathscr{K}(H, 1 + \varkappa)$$

by (11), (7) and $\gamma - \varrho < \beta - 1 < 0 < \varkappa$, we obtain

(21) $$u \in C^1((s, T], H)$$

from (4), (18) and (19), provided we can show that

$$A(U{*}f)(\cdot, s) \in C((s, T], H).$$

Since $A(U{*}f) = (AU){*}f$ in W, we see from (20), (11) and (2.10) that it suffices to show that $(D_1 a){*}f(\cdot, s) \in C((s, T], H)$. But

$$(D_1 a){*}f(t, s) = -\int_s^t A_H(\tau)\, e^{-(t-\tau)A_H(\tau)} [f(\tau) - f(t)]\, d\tau - \int_s^t B(\tau)\, e^{-(t-\tau)B(\tau)} f(t)\, d\tau,$$

and the first integrand can be estimated, because of (4), (15) and (17), by $c(t - \tau)^{\nu - \varkappa - 1}$. From (4), (3.5), [7, Lemma 2.1] and $W^\gamma_{\mathscr{B}} \hookrightarrow W^\gamma \hookrightarrow H$ we deduce that the second integrand can be estimated by $c(t - \tau)^{\lambda - \gamma - 1}$ in the norm of H. Now [5, Lemma 1.1] implies $(D_1 a){*}f(\cdot, s) \in C([s, T], H)$ and (21) has been proven.

From (4), (19) and (21) we see that $u(t) \in H$ and $A(t)\, u(t) = f(t) - \dot{u}(t) \in H$ for $s < t \leqq T$. Hence $u(t) \in D(A_H(t)) = H^1_{\mathscr{B}(t)}$ for $s < t \leqq T$. Now we conclude, on the basis of (HR2), that $u \in C((s, T], H^1)$, much as we derived (2.18) from (2.17). □

6. Second Order Parabolic Systems

In the following we identify $\mathscr{L}(\mathbb{R}^N)$ with the space of all real $(N\times N)$-matrices by identifying $a\in\mathscr{L}(\mathbb{R}^N)$ with its matrix representation $[a^{rs}]_{1\leqq r,s\leqq N}$ with respect to the standard basis of $\mathbb{R}^N$. Moreover a family $a_{jk}\in\mathscr{L}(\mathbb{R}^N), j, k=1,\ldots,n$, is said to be *strongly elliptic* if $a_{jk}=a_{kj}$ and

$$\sum_{r,s=1}^{N}\sum_{j,k=1}^{n} a_{jk}^{rs}\xi^j\xi^k\zeta_r\zeta_s>0$$

for all $\xi:=(\xi^1,\ldots,\xi^n)\in\mathbb{R}^n\setminus\{0\}$ and $\zeta:=(\zeta_1,\ldots,\zeta_N)\in\mathbb{R}^N\setminus\{0\}$.

A matrix $a\in\mathscr{L}(\mathbb{R}^N)$ is said to be *block-upper-triangular* if there are positive integers l and N_λ, $1\leqq\lambda\leqq l$, with $N_1+N_2+\ldots+N_l=N$ such that

$$a=[\alpha^{\lambda\varkappa}]_{1\leqq\lambda,\varkappa\leqq l}\quad\text{with } \alpha^{\lambda\varkappa}\in\mathscr{L}(\mathbb{R}^{N_\varkappa},\mathbb{R}^{N_\lambda})$$

$$\text{for } 1\leqq\lambda,\varkappa\leqq l\quad\text{with } \alpha^{\lambda\varkappa}=0 \text{ for } \varkappa<\lambda.$$

Then the family $a_{jk}\in\mathscr{L}(\mathbb{R}^N)$, $1\leqq j, k\leqq n$ is said to be *block-upper-triangular strongly elliptic* if each a_{jk} is block-upper-triangular, with l and $N_1,\ldots,N_l$ being independent of $j,k\in\{1,\ldots,n\}$, such that the diagonal families $a_{jk}^{\lambda\lambda}$, $1\leqq j,k\leqq n$, are strongly elliptic for each $\lambda\in\{1,\ldots,l\}$.

In the remainder of this paper Ω denotes a bounded smooth domain in $\mathbb{R}^n$, that is, $\bar\Omega$ is a compact connected n-dimensional C^∞-submanifold of $\mathbb{R}^n$ with boundary $\partial\Omega$. We denote by $\nu:=(\nu^1,\ldots,\nu^n)$ the outer unit normal vector field on $\partial\Omega$ and we put

$$Q_T:=\bar\Omega\times[0,T]\quad\text{and}\quad\Sigma_T:=\partial\Omega\times[0,T].$$

Throughout the remainder of this section N is a fixed positive integer and

(P1) $\qquad a_{jk}=a_{kj}\in C^{\infty,0}\cap C^{0,2-}(Q_T\times\mathbb{R}^N,\mathscr{L}(\mathbb{R}^N)),\quad 1\leqq j,k\leqq n.$

Moreover we suppose that

(P2) *$a_{jk}(x,t,\eta)$, $1\leqq j$, $k\leqq n$, is block-upper-triangular strongly elliptic for every $(x,t)\in Q_T$, $\eta\in\mathbb{R}^N$, where the block structure is independent of (x,t,η).*

In the following we employ the summation convention with the indices j, k running from 1 to n. Then we define a family of linear differential operators, acting on N-vector valued functions $u:\Omega\to\mathbb{R}^N$, by

$$\mathscr{A}(x,t,\eta)\,u:=-D_j(a_{jk}(\cdot,t,\eta)\,D_ku),\quad (t,\eta)\in[0,T]\times\mathbb{R}^N$$

(where the letter x indicates that the coefficients are functions of $x\in\bar\Omega$).

We denote by I the identity matrix in $\mathscr{L}(\mathbb{R}^N)$, consider functions

$$\delta_r\in C(\partial\Omega,\{0,1\}),\quad 1\leqq r\leqq N,$$

and put

$$\delta := \operatorname{diag}(\delta_1, \ldots, \delta_N) \in C(\partial\Omega, \mathscr{L}(\mathbb{R}^N)).$$

Observe that each δ_r is constant on each component of $\partial\Omega$. Then we define a family of boundary operators on $\partial\Omega$ by

$$\mathscr{B}(x, t, \eta)\, u := \delta a_{jk}(\cdot, t, \eta)\, \nu^j D_k u + (I - \delta)\, u, \qquad (t, \eta) \in [0, T] \times \mathbb{R}^N.$$

Thus $[\mathscr{B}(x, t, \eta)\, u]^r = u^r$ if $\delta_r = 0$, that is, the r^{th} component of $\mathscr{B}(x, t, \eta)\, u$ reduces to the Dirichlet boundary operator for the r^{th} component of u on each component of $\partial\Omega$ on which $\delta_r = 0$.

We suppose that

(P3) $$f \in C^{\infty,0} \cap C^{0,2-}(Q_T \times \mathbb{R}^{N+nN}, \mathbb{R}^N),$$

that

(P4) $$g_0 \in C^\infty(\Sigma_T, \mathbb{R}^N)$$

and that

(P5) $$g_1 \in C^{\infty,0} \cap C^{0,3-}(\Sigma_T \times \mathbb{R}^N, \mathbb{R}^N),$$

and we put

$$g := \delta g_1 + (I - \delta)\, g_0.$$

Then we consider the second-order block-upper-triangular quasilinear *parabolic initial boundary value problem*

$$\frac{\partial u}{\partial t} + \mathscr{A}(x, t, u)\, u = f(x, t, u, Du) \quad \text{in } \Omega \times (s, T],$$

$(\mathrm{IBV})_{(s,u_0)}$ $$\mathscr{B}(x, t, u)\, u = g(x, t, u) \quad \text{on } \partial\Omega \times (s, T],$$

$$u(\cdot, s) = u_0 \quad \text{on } \Omega.$$

By a *classical solution* u of $(\mathrm{IBV})_{(s,u_0)}$ on K we mean a function

(1) $$u \in C(\bar{\Omega} \times J, \mathbb{R}^N) \cap C^1(\bar{\Omega} \times \dot{J}, \mathbb{R}^N) \cap C^{0,2}(\Omega \times \dot{J}, \mathbb{R}^N),$$

where J is a perfect subinterval of $[s, T]$ containing s and $\dot{J} := J \setminus \{s\}$, such that u satisfies $(\mathrm{IBV})_{(s,u_0)}$ pointwise (in the obvious sense).

Define $b \in C^{\infty,0} \cap C^{0,2-}(\Sigma_T \times \mathbb{R}^N, \mathscr{L}(\mathbb{R}^N))$ by

$$b(\cdot, \cdot, \eta) := \int_0^1 D_3 g_1(\cdot, \cdot, s\eta)\, ds \qquad \forall \eta \in \mathbb{R}^N,$$

so that

$$g_1(\cdot, \cdot, \eta) = b(\cdot, \cdot, \eta)\, \eta + g_1(\cdot, \cdot, 0) \qquad \forall \eta \in \mathbb{R}^N.$$

by the mean-value theorem. Then we assume that

(P6) *the matrix b has the same block-upper-triangular structure as each one of the matrices* a_{jk} *and there is a function* $w_0 \in C^\infty(Q_T, \mathbb{R}^N)$ *such that*

$$(*) \quad \mathcal{B}(x, t, \delta\eta + w_0(x, t))\, w_0(x, t) - \delta b(x, t, \delta\eta + w_0(x, t))\, w_0(x, t) = g(x, t, 0)$$

$$\textit{for all } (x, t, \eta) \in \Sigma_T \times \mathbb{R}^N.$$

There are many conditions sufficient to ensure the validity of (*). For example, (*) *is true* (with $w_0 = 0$) *if* $g(\cdot, \cdot, 0) = 0$. *Condition* (*) *is also satisfied if* $g_1(\cdot, \cdot, 0) = 0$ *and if*

(2)

$$\delta\mathcal{B}(x, t, \delta\eta) - \delta b(x, t, \delta\eta) = \delta\mathcal{B}(x, t, \delta\eta)\,\delta - \delta b(x, t, \delta\eta)\,\delta \qquad \forall (x, t, \eta) \in \Sigma_T \times \mathbb{R}^N,$$

for, in this case, we can take for w_0 any smooth extension of g_0 over Q_T such that w_0^r vanishes in a neighborhood of any component of $\partial\Omega$ on which $\delta_r = 1$. Observe that (2) is trivially true if $\delta = I$, thus, in particular, if $N = 1$. Hence, *if* $N = 1$ *or* $\delta = I$, *then* (*) *can be replaced by the assumption that* $g_1(\cdot, \cdot, 0) = 0$.

We now put

$$\tilde{\mathcal{A}}(x, t, \tilde{\eta}) := \mathcal{A}(x, t, w_0(x, t) + \tilde{\eta})$$

$$\tilde{\mathcal{B}}(x, t, \tilde{\eta}) := \mathcal{B}(x, t, w_0(x, t) + \tilde{\eta}) - \delta b(x, t, w_0(x, t) + \tilde{\eta})$$

and

$$\tilde{f}(x, t, \tilde{\eta}, \tilde{\zeta}) := f(x, t, w_0(x, t) + \tilde{\eta}, Dw_0(x, t) + \tilde{\zeta})$$

$$- \tilde{\mathcal{A}}(x, t, \tilde{\eta})\, w_0(x, t) - \frac{\delta w_0(x, t)}{\partial t}$$

for all $(x, t) \in Q_T, \tilde{\eta} \in \mathbb{R}^N$ and $\tilde{\zeta} \in \mathbb{R}^{nN}$. Then u is a classical solution of $(\mathrm{IBV})_{(s,u_0)}$ on J if and only if $v := u - w_0$ is a classical solution on J of

$$\frac{\partial v}{\partial t} + \tilde{\mathcal{A}}(x, t, v)\, v = \tilde{f}(x, t, v, Dv) \quad \text{in } \Omega \times (s, T],$$

$$(3) \qquad \tilde{\mathcal{B}}(x, t, v)\, v = 0 \quad \text{on } \partial\Omega \times (s, T],$$

$$v(\cdot, s) = v_0 \quad \text{on } \Omega,$$

where

$$(4) \qquad v_0 := u_0 - w_0(\cdot, s).$$

For $1 \leq p \leq \infty$ and $s \in \mathbb{R}^+$ we denote by $W_p^s := (W_p^s, \|\cdot\|_{s,p})$ the Sobolev-Slobodeckii spaces $W_p^s(\Omega, \mathbb{R}^N)$, so that $W_p^0 = L_p := L_p(\Omega, \mathbb{R}^N)$.

We now fix $p \in (n, \infty)$ arbitrarily and put

$$W^1 := W_p^2 \quad \text{and} \quad W := L_p. \tag{5}$$

Then it is well known that (A1) is satisfied. Moreover we let

$$(\cdot, \cdot)_\theta := \begin{cases} (\cdot, \cdot)_{\theta,p} & \text{if } \theta \in (0, 1) \setminus \{1/2\} \\ [\cdot, \cdot]_\theta & \text{if } \theta = 1/2. \end{cases}$$

Thus it is known (*e.g.* [5, Theorem 11.6]) that

$$W^\theta \doteq W_p^{2\theta} \quad \forall \theta \in [0, 1],$$

and [5, Theorem 11.6] implies that (A2) is satisfied.

We denote by $\boldsymbol{\Gamma}$ the set of components of $\partial\Omega$ and by $\delta_r(\Gamma) \in \{0, 1\}$ the value of δ_r on $\Gamma \in \boldsymbol{\Gamma}$. Then we put

$$\partial W := \prod_{\Gamma \in \boldsymbol{\Gamma}} \prod_{r=1}^{N} W_p^{2-\delta_r(\Gamma)-1/p}(\Gamma, \mathbb{R}). \tag{6}$$

We fix $\xi \in ((1 + n/p)/2, 1)$ arbitrarily and put

$$\mathscr{A}(t, u)\, v := \tilde{A}(x, t, u)\, v \tag{7}$$

and

$$\mathscr{B}(t, u)\, v := \tilde{\mathscr{B}}(x, t, u)\, v \tag{8}$$

for all $u \in W^\xi$ and $v \in W^1$. Observe that

$$W^\xi \hookrightarrow C^{1+\tilde{\mu}}(\bar{\Omega}, \mathbb{R}^N) \hookrightarrow C^1(\bar{\Omega}, \mathbb{R}^N) \hookrightarrow C(\bar{\Omega}, \mathbb{R}^N), \quad \tilde{\mu} := 2\xi - 1 - n/p, \tag{9}$$

by a well known Sobolev-type imbedding theorem (*e.g.* [23]). By a using these facts and (P1) and (P5), it is not difficult to see that assumption (A4) is satisfied with $V := W^\xi$.

For each fixed $(t, u) \in [0, T] \times V$ we put

$$W^s_{p,\mathscr{B}(t,u)} := \begin{cases} \{v \in W_p^s \mid \mathscr{B}(t, u)\, v = 0\} & \text{for } 1 + 1/p < s \leqq 2, \\ \{v \in W_p^s \mid (I - \delta)\, v \mid \partial\Omega = 0\} & \text{for } 1/p < s \leqq 1 + 1/p, \\ W_p^s \quad \text{for } 0 \leqq s \leqq 1/p. \end{cases}$$

Then we know from [5, Theorem 13.3] that

$$W^\sigma_{\mathscr{B}(t,u)} \doteq W^{2\sigma}_{p,\mathscr{B}(t,u)} \quad \forall \sigma \in [0, 1] \setminus \{1/2p, (1 + 1/p)/2\}.$$

This implies in particular that every $\sigma \in [0, 1] \setminus (\mathbb{N} + 1/p)/2$ is regular and that $W^{\frac{1}{2}}_{\mathscr{B}(t,u)} \doteq W^{\frac{1}{2}}_{\mathscr{B}(0,0)} =: W^{\frac{1}{2}}_{\mathscr{B}}$ for every $(t, u) \in [0, T) \times V$.

We now fix β, η and ζ subject to

(10)

$$0 < \zeta < 1/2p < \zeta + n/2p < \eta < 1/2 := \beta \quad \text{and} \quad \zeta < \xi - (1 + n/p)/2.$$

Then it follows from the above facts that (A3) and (A5) are satisfied.

It is a consequence of (P2), of [6, Theorem 6.6 and Lemma 6.8] and of (9), that $(\tilde{\mathscr{A}}(x, t, u), \tilde{\mathscr{B}}(x, t, u), \Omega)$, $0 \leqq t \leqq T$, is, for every $u \in V$, a regular parabolic initial boundary value problem of class C^0 in the sense of [6, Section 1]. From this and [5, Theorems 12.1 and 12.2] we deduce easily the validity of (A6).

We can now invoke a result of SEELEY [19, Theorem 1 and Section 3] to verify that (A7) is true. (For this we consider first the case $u \in \mathscr{D}(\Omega, \mathbb{R}^N)$, that is, u is smooth and has compact support. Then we use the density of $\mathscr{D}(\Omega, \mathbb{R}^N)$ in V and the continuous dependence of $[\omega(u) + A(t, u)]^z$, $\operatorname{Re} z < 0$, on $u \in V$, which is an easy consequence of the representation of these operators by a Dunford-type integral, together with a limit argument for $\operatorname{Re} z \to 0$ as in [19, Section 3].)

We define the formally adjoint problem $(\tilde{\mathscr{A}}^{\#}(x, t, u), \tilde{\mathscr{B}}^{\#}(x, t, u), \Omega)$ $0 \leqq t \leqq T$ for each $u \in V$ by

$$\tilde{\mathscr{A}}^{\#}(x, t, u)\, v := -D_j(\tilde{a}^*_{jk}(\cdot, t, u)\, D_k v) \quad \text{on } Q_T$$

and

$$\tilde{B}^{\#}(x, t, u)\, v := \delta \tilde{a}^*_{jk}(\cdot, t, u)\, v^j D_k v - \delta \tilde{b}^*(\cdot, t, u)\, v + (I - \delta)\, v \quad \text{on } \Sigma_T,$$

where

$$\tilde{a}_{jk}(\cdot, t, \tilde{\eta}) := a_{jk}(\cdot, t, w_0 + \tilde{\eta}), \quad \tilde{b}(\cdot, t, \tilde{\eta}) := \tilde{b}(\cdot, t, w_0 + \tilde{\eta}) \quad \forall \tilde{\eta} \in \mathbb{R}^N,$$

and where c^* is the transposed matrix of $c \in \mathscr{L}(\mathbb{R}^N)$. Moreover, we let

$$W^2_{p', \mathscr{B}^{\#}(t,u)} := \{u \in W^2_{p'} \mid \tilde{\mathscr{B}}^{\#}(x, t, u)\, v = 0\},$$

where $p' := p/(p-1)$, and put

$$A^{\#}(t, u)\, v := \tilde{\mathscr{A}}^{\#}(x, t, u)\, v \quad \forall v \in W^2_{p', \mathscr{B}^{\#}(t,u)}.$$

Then it follows from Gauss' theorem that

(11) $$\langle w, A(t, u)\, v \rangle = \langle A^{\#}(t, u)\, v, w \rangle, \quad u \in V, \quad v \in W^1_{\mathscr{B}(t,u)}, \quad w \in W^2_{p', \mathscr{B}^{\#}(t,u)},$$

where $\langle \cdot, \cdot \rangle : W' \times W \to \mathbb{R}$ is the duality pairing and $W' = L_{p'}$. Moreover (P2) and [6, Theorem 6.6 and Remark 6.7] imply that $(\tilde{\mathscr{A}}^{\#}(x, t, u), \tilde{\mathscr{B}}^{\#}(x, t, u), \Omega)$, $0 \leqq t \leqq T$, is also a regular parabolic initial boundary value problem of class C^0 for every $u \in V$. From this and from (11) we deduce by well known arguments (*cf. e.g.* [4, Theorem 7.1]) that $A'(t, u) = A^{\#}(t, u)$. In addition it follows from [5, Theorem 13.3], much as above, that

$$D([A'(t, u)]^{\frac{1}{2}}) = \{v \in W^1_{p'} \,|\, (I - \delta)\, v \,|\, \partial\Omega = 0\} =: [W']^{\frac{1}{2}}_{\mathscr{B}^{\#}}$$

for all $(t, u) \in [0, T] \times V$. Thus assumptions (A8) is satisfied.

For each $(t, u) \in [0, T] \times V$, and for $v \in W^{\beta}_{\mathscr{B}}$ and $w \in [W']^{1-\beta}_{\mathscr{B}\#}$, let

$$a(t, u; v, w) := \int_{\Omega} (D_k w | a_{jk}(\cdot, t, u) D_j v)\, dx - \int_{\partial\Omega} (w | b(\cdot, t, u) v)\, d\sigma,$$

where $(\cdot|\cdot)$ denotes the Euclidean inner product in $\mathbb{R}^N$. Then it is an obvious consequence of (P1) and

$$W^{\eta} = W^{2\eta}_p \hookrightarrow C(\bar{\Omega}, \mathbb{R}^N), \tag{12}$$

which follows from (10) and Sobolev's imbedding theorem, that

$$[(t, u) \mapsto a(t, u; \cdot, \cdot)] \in C^{1-}([0, T] \times W^{\eta}, \mathscr{L}^2(W^{\beta}_{\mathscr{B}}, [W']^{\beta}_{\mathscr{B}\#}; \mathbb{R})), \tag{13}$$

where $\mathscr{L}^2(\cdot, \cdot; \cdot)$ is the Banach space of all continuous real bilinear forms on $W^{\beta}_{\mathscr{B}} \times [W']^{\beta}_{\mathscr{B}\#}$. Observe that, by Gauss' theorem, we obtain for every $(t, u) \in [0, T] \times V$ the relation

$$a(t, u; v, w) := \int_{\Omega} (w | A(t, u) v)\, dx \qquad \forall v \in W^1_{\mathscr{B}(t,u)},\ \forall w \in [W']^{1-\beta}_{\mathscr{B}\#}.$$

Hence we deduce from [8, Theorem 1.3] that

$$a(t, u; v, w) = \langle w, A_{\beta-1}(t, u) v \rangle \tag{14}$$

for all $(t, u) \in [0, T] \times V$, $v \in W^{\beta}_{\mathscr{B}}$ and $w \in [W']^{1-\beta}_{\mathscr{B}\#} = (W^{\beta-1}_{\mathscr{B}})'$. Since the imbedding (12) is in fact compact and since locally Lipschitz-continuous maps are Lipschitz-continuous on compact sets, we easily obtain now from (13) and (14) the validity of (A9) for any ϱ with $1 - \beta = \frac{1}{2} < \varrho < 1$.

Suppose that f is an affine functions of $\tilde{\zeta} \in \mathbb{R}^{nN}$, where $\tilde{\zeta}$ is a dummy variable for Du. Then we can write $f(\cdot, t, u, Du)$ in the form

$$f(\cdot, t, u, Du) = a_j(\cdot, t, u) D_j u + f_0(\cdot, t, u)$$

with well defined functions

$$a_j \in C^{\infty,0} \cap C^{0,2-}(Q_T \times \mathbb{R}^N, \mathscr{L}(\mathbb{R}^N)), \quad j = 1, \ldots, n,$$

and

$$f_0 \in C^{\infty,0} \cap C^{0,2-}(Q_T \times \mathbb{R}^N, \mathbb{R}^N).$$

If we replace $\mathscr{A}(x, t, u) v$ by

$$\mathscr{A}(x, t, u) v - a_j(\cdot, t, u) D_j v,$$

all the above results remain true. Hence we can assume that $\tilde{f}$ is independent of $\tilde{\zeta}$, if f is an affine functions of $\tilde{\zeta}$.

We denote by F the substitution operator induced by $\tilde{f}$, that is,

$$F(t, u)(x) := \tilde{f}(x, t, u(x), Du(x)) \tag{15}$$

for $u: \overline{\Omega} \to \mathbb{R}^N$ and $(t, x) \in Q_T$. Moreover, we put

$$\sigma := \begin{cases} \eta & \text{if } f \text{ is an affine function of } \tilde{\zeta} \in \mathbb{R}^{nN}, \\ \xi & \text{otherwise}. \end{cases}$$

Then it is an easy consequence of (P3), (9), (10) and [5, Proposition 15.6] that assumption (A10) holds for F.

We now fix

$$\mu \in (0, 1 - n/p) \quad \text{such that} \quad \gamma := (\mu + n/p)/2 \leqq \eta \tag{16}$$

and such that $\gamma + \frac{1}{2} < \xi - \eta$, which is possible if we choose $\xi < 1$ large enough. Then we fix any $\varrho \in (\gamma + 1/2, \xi - \eta)$ and observe that (A11) is satisfied for every $\theta \in (\xi, 1]$. Moreover,

$$\varrho > 1 - \beta + \gamma = \gamma + 1/2. \tag{17}$$

Then we put

$$H := C^{\mu}(\overline{\Omega}, \mathbb{R}^N) \quad \text{and} \quad H^1 := C^{2+\mu}(\overline{\Omega}, \mathbb{R}^N), \tag{18}$$

as well as

$$\partial H := \prod_{\Gamma \in \boldsymbol{\Gamma}} \prod_{r=1}^{N} C^{2-\delta_r(\Gamma)+\mu}(\Gamma, \mathbb{R}). \tag{19}$$

Since we can assume that $\mu < \tilde{\mu}$, it is an easy consequence of (9), (P1) and (P5) that condition (HR1) is satisfied (*cf.* the proof of [5, Proposition 15.6]).

Since $(\tilde{\mathscr{A}}(x, t, u), \tilde{\mathscr{B}}(x, t, u), \Omega)$, $0 \leqq t \leqq T$, is, for each $u \in V$, a regular parabolic initial boundary value problem, it satisfies Schauder-type *a priori* estimates (*e.g.* [1, Theorem 9.3]). By use of this fact together with $W^{\xi} \subset\subset\!\!\to C^{1+\mu}(\overline{\Omega}, \mathbb{R}^N)$, the first part of (HR2) is easily established. The second part of (HR2), with $\varkappa := \mu/2$, follows by modifying the proof of [7, Lemma 10.5] in the obvious way.

After these preparations we can prove the main results of this section, where we assume throughout that (P1)–(P6) are satisfied.

Theorem 6.1 *Suppose that* $p > n$ *and that* $1 + n/p < \tau \leqq 2$. *Moreover suppose that* $s \in [0, T)$, *and that* $u_0 \in W_p^{\tau}$ *satisfies the compatibility condition*

$$\mathscr{B}(s, u_0)\, u_0 = g(\cdot, s, u_0) \quad \text{on } \partial\Omega. \tag{20}$$

Then the quasilinear parabolic initial boundary value problem $(\mathrm{IBV})_{(s,u_0)}$ *possesses a maximal classical solution* $u(\cdot, s, u_0)$ *such that*

$$u(\cdot, s, u_0) \in C^{(\tau-\lambda)/2}(J, W_p^{\lambda})$$

for all $\lambda \in [0, \tau]$.

If f is an affine function of Du, then $u(\cdot, s, u_0)$ is the only maximal solution of $(\mathrm{IBV})_{(s,u_0)}$. *Moreover there are positive constants $\hat{\varepsilon}$ and $\hat{\delta}$ such that $u(\cdot, s, v_0)$ exists on $J_{s,\hat{\delta}} := [s, s+\hat{\delta}] \cap [s, T]$ for each*

$$v_0 \in \{v \in W_p^{\tau} \mid (I-\delta)\, v \mid \partial\Omega = g_0(\cdot, s),\ \|v - u_0\|_{s,p} \leqq \hat{\varepsilon}\} =: D$$

and

$$[v_0 \mapsto u(\cdot, s, v_0)] \in C(D_\lambda, C(J_{s,\hat{\delta}}, W_p^{\varkappa})) \cap UC^{1-}(D_\lambda, C(J_{s,\hat{\delta}}, W_p^{\lambda})),$$

where $n/p < \lambda < 1$ and $1 + n/p < \varkappa < \tau$ with $\varkappa - \lambda > 1$, and where D_λ denotes D, endowed with the topology induced by W_p^{λ}.

Proof. We put $\theta := \tau/2$, $\xi := \varkappa/2$ and $\eta := \lambda/2$, and choose ζ such that (10) is satisfied. Then we define W, W^1 and ∂W by (5) and (6), respectively, $(\mathscr{A}, \mathscr{B})$ by (7) and (8), and F by (15), where $V := W^{\xi}$. By the above considerations we know that the assumptions (A1)–(A11) are satisfied and that $(\mathrm{IBV})_{(s,u_0)}$ is equivalent to (3). Observe that $v_0 \in W^{\theta}_{\mathscr{B}(s,v_0)}$ by (4) and (20). Hence we can apply Theorem 4.1 to the abstract quasilinear Cauchy problem

$$\text{(21)} \qquad \dot{v} + A(t, v)\, v = F(t, v), \quad s < t \leqq T,\ v(s) = v_0.$$

Consequently (21) possesses a maximal solution $v(\cdot, s, v_0)$, which is unique, if $\sigma = \eta$, that is, if f is an affine function of Du. If we can show that $v(\cdot, s, v_0)$ is regular in the sense of (1), then $u(\cdot, s, u_0) := v(\cdot, s, v_0) + w_0$ is a maximal classical solution of $(\mathrm{IBV})_{(s,u_0)}$ and the stated continuity follows directly from Theorem 4.1.

Let $v := v(\cdot, s, v_0)$ and let $t_1 \in \dot{J}$ be arbitrary. Denote by $\bar{v}$ the continuous extension of $v \mid [s, t_1]$ over $[0, T]$, which is constant on $[0, s]$ and on $[t_1, T]$, and put $\bar{A}(t) := A(t, \bar{v}(t))$ and $\bar{F}(t) := F(t, \bar{v}(t))$ for $0 \leqq t \leqq T$. Let M be a nonempty compact subset of $\Omega \times (s, t_1]$ and choose a $\varphi \in C^\infty(Q_T, \mathbb{R})$ with $\varphi \mid M = 1$ and $\operatorname{supp}(\varphi) \subset \Omega \times (s, T]$. Moreover let

$$\bar{F}_0 := \varphi \bar{F}(\cdot) + \bar{A}(\cdot)\,(\varphi \bar{v}) - \varphi \bar{A}(\,)\, \bar{v} + \dot{\varphi} \bar{v}$$

and observe that F_0 contains at most first-order x-derivatives of $v(t)$, where $s < t < t_1$.

We now fix μ so as to satisfy (16) and define H, H^1 and ∂H by (18) and (19), respectively. Then the above considerations imply that conditions (HR1) and (HR2) are satisfied for $(\bar{\mathscr{A}}(t), \bar{\mathscr{B}}(t))$, $0 \leqq t \leqq T$, where $\bar{\mathscr{A}}(t) := \mathscr{A}(t, \bar{v}(t))$ and $\bar{\mathscr{B}}(t) := \mathscr{B}(t, \bar{v}(t))$ (*cf.* the beginning of the proof of Theorem 5.1), Since F_0 vanishes in a neighborhood of $(\{s\} \times \bar{\Omega}) \cup (\partial\Omega \times [s, T])$, it is clear that we can assume that (HR3) is satisfied (cf. again the beginning of the proof of Theorem 5.1).

Consider now the linear Cauchy problem

$$\text{(22)} \qquad \dot{w} + \bar{A}(t)\, w = \bar{F}_0(t), \quad s < t \leqq T,\ w(s) = 0.$$

By Theorem 2.1 it has a unique solution $w(\cdot, s, 0)$, and

$$w(\cdot, s, 0) \in C((s, T], H^1) \cap C^1((s, T], H)$$

by Theorem 5.1. On the other hand, by multiplying (21) by φ, we see that the function $\varphi\bar{v}|[s, T]$ is a solution of (22). Hence $\varphi\bar{v}|[s, T] = w(\cdot, s, 0)$. Now the assertion follows from $\varphi | M = 1$, the arbitrariness of M, and from $v(t) \in W_p^2 \hookrightarrow C^1(\bar{\Omega}, \mathbb{R}^N)$ for $t \in \dot{J}$. □

Corollary 6.2. *There exists a constant* $\mu > 0$ *such that*

$$u(\cdot, s, u_0) \in C(\dot{J}, C^{2+\mu}(\Omega, \mathbb{R}^N)) \cap C^1(\dot{J}, C^{\mu}(\Omega, \mathbb{R}^N)).$$

If

$$\sup_{t \in J} \| u(t, s, u_0) \|_{\tau,p} < \infty,$$

then $J = [s, T]$, *that is,* $u(\cdot, s, u_0)$ *is a global solution.*

Proof. The first assertion follows from the above proof and the second one from Theorem 4.2. □

We suppose now that $\mathscr{A}$, $\mathscr{B}$, f and g are independent of t and consider the *autonomous initial boundary value problem*

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathscr{A}(x, u)\, u &= f(x, u, Du) \quad \text{in } \Omega \times (s, \infty), \\ (\mathrm{AIBV})_{(s,u_0)} \qquad \mathscr{B}(x, u)\, u &= g(x, u) \quad \text{on } \partial\Omega \times (s, \infty), \\ u(\cdot, s) &= u_0 \quad \text{on } \Omega. \end{aligned}$$

In addition we assume that f is an affine function of Du. We put

$$\mathscr{X}_\tau := \{u_0 \in W_p^\tau \,|\, \mathscr{B}(x, u_0)\, u_0 = g(\cdot, u_0) \quad \text{on } \partial\Omega\}$$

and

$$\varphi(\cdot, u_0) := u(\cdot, 0, u_0) \qquad \forall u_0 \in \mathscr{X}_\tau$$

Observe that Theorem 6.1 implies that φ is a preflow on $\mathscr{X}_\tau$.

Theorem 6.3. *Suppose that* $\mathscr{A}$, $\mathscr{B}$, f *and* g *are independent of* t *and that* f *is an affine function of* Du. *Let* $n < p < \infty$ *and* $1 + n/p < \sigma < \tau \leqq 2$ *and suppose that* $\mathscr{P}$ *is a positively invariant bounded subset of* $\mathscr{X}_\tau$, *which is closed in* W_p^σ. *Then* φ *is a global flow on* $\mathscr{P}_\sigma$ *such that every orbit is relatively compact, where* $\mathscr{P}_\sigma$ *is* $\mathscr{P}$, *endowed with the topology induced by* W_p^σ. *Moreover every critical point* $u \in \mathscr{X}_\tau$ *of the preflow* φ *belongs to* $C^1(\bar{\Omega}, \mathbb{R}^N) \cap C^2(\Omega, \mathbb{R}^N)$ *and is a solution of the elliptic boundary-value problem*

$$\mathscr{A}(x, u)\, u = f(x, u, Du) \quad \text{in } \Omega,$$

$$\mathscr{B}(x, u)\, u = g(x, u) \quad \text{on } \partial\Omega$$

Proof. This is an easy consequence of Theorem 6.1 and Remark 4.4. □

For simplicity we have restricted our considerations to the case in which $V = W^{\xi}$, that is, when the coefficients of $\mathscr{A}$ and $\mathscr{B}$ and the nonlinearities are defined everywhere, and the dependence upon (x, t) is smooth. I leave it to the reader to obtain more general and more precise results by applying the abstract results of Sections 1–5 to concrete parabolic systems. For the same reason I do not reformulate all the general abstract results of Section 4 in the present concrete situation.

7. Remarks on Parabolic Systems of Higher Order

In this section we indicate briefly how the general abstract results apply to higher-order problems.

We consider a family of quasilinear differential operators of order $2m$

$$\mathscr{A}(x, t, u, \ldots, D^k u)\, u := (-1)^m \sum_{|\alpha| \leqq 2m} a_\alpha(\cdot, t, u, \ldots, D^k u)\, D^\alpha u, \qquad 0 \leqq t \leqq T,$$

acting on N-vector valued functions $u: \Omega \to \mathbb{C}^N$. In addition we suppose that

$$\mathscr{B}(x, t, u, \ldots, D^k u) := \{\mathscr{B}^\sigma(x, t, u, \ldots, D^k u) \mid 1 \leqq \sigma \leqq mN\}$$

is a system of boundary operators of the form

$$\mathscr{B}^\sigma(x, t, u, \ldots, D^k u)\, u := \sum_{|\alpha| \leqq m_\sigma} b_\alpha^\sigma(\cdot, t, u, \ldots, D^k u)\, D^\alpha u.$$

We suppose that $k < 2m - 1$ and that the coefficients of these operators are smooth functions of their arguments. Moreover *we assume that, for each* $(t, u) \in [0, T] \times C^k(\bar{\Omega}, \mathbb{C}^N)$, *the family* $\{\mathscr{A}(x, t, u, \ldots, D^k u), \mathscr{B}(x, t, u, \ldots, D^k u), \Omega\}$, $0 \leqq t \leqq T$, *is a regular parabolic initial-boundary value problem of order* $2m$ in the sense of [5, Section 14].

We fix $p \in (n, \infty)$ and denote by $(\mathscr{A}(t, u), \mathscr{B}(t, u))$, $0 \leqq t \leqq T$, the operators induced by $(\mathscr{A}(x, t, u, \ldots, D^k u), \mathscr{B}(x, t, u, \ldots, D^k u))$, $0 \leqq t \leqq T$, in the natural way on the space $W_p^{2m}(\Omega, \mathbb{C}^N)$. For $0 \leqq s \leqq 2m$ we let

$$W^s_{p,\mathscr{B}(t,u)} := \{v \in W_p^s(\Omega, \mathbb{C}^N) \mid \mathscr{B}^\sigma(t, u)\, v = 0 \quad \text{for } m_\sigma < s - 1/p\},$$

and we put

$$A(t, u) := \mathscr{A}(t, u) \mid W^{2m}_{p,\mathscr{B}(t,u)},$$

considered as an unbounded linear operator in $L_p(\Omega, \mathbb{C}^N)$.

We suppose now that there is a "formally adjoint" system $(\mathscr{A}^\sharp(x, t, u, \ldots, D^k u)$, $\mathscr{B}^\sharp(x, t, u, \ldots, D^k u), \Omega)$, $0 \leqq t \leqq T$, which is also a regular parabolic initial-boundary value problem of order $2m$ for each fixed $(t, u) \in [0, T] \times C^k(\bar{\Omega}, \mathbb{C}^N)$, such that

$$A'(t, u) = A^\#(t, u) \quad \forall (t, u) \in [0, T] \times C^k(\bar{\Omega}, \mathbb{C}^N),$$

where $A^\#(t, u)$ is the $L_{p'}$-realization, given by

$$A^\#(t, u) := \mathscr{A}^\#(x, t, u, \ldots, D^k u) \mid W^{2m}_{p', \mathscr{B}^\#(t,u)},$$

and where $W^{2m}_{p', \mathscr{B}^\#(t,u)}$ has the obvious meaning. Observe that this assumption is always satisfied if $N = 1$ (*cf.* [18, Theorem II.8.4], where the case $p = 2$ is treated).

SEELEY's results [19] imply that

$$D(A^{l/2m}(t, u)) = W^l_{p, \mathscr{B}(t,u)}$$

and

$$D([A']^{l/2m}(t, u)) = W^l_{p', \mathscr{B}^\#(t,u)}$$

for $l = 1, 2, \ldots, 2m$. *We suppose now that the boundary operators* $\mathscr{B}^\sigma(t, u)$ *of order* $m_\sigma \leqq k$ *and the boundary operators* $(\mathscr{B}^\#)^\tau(t, u)$ *of order* $m^\#_\tau < 2m - k - 1$ *are independent of* $(t, u) \in [0, T] \times C^k(\bar{\Omega}, \mathbb{C}^N)$, and we put $\beta := (k + 1)/2m$ and $W^\beta_{\mathscr{B}} := W^{k+1}_{\mathscr{B}(0,0)}$ and $(W^{\beta-1}_{p,\mathscr{B}})' := W^{2m-k-1}_{p', \mathscr{B}^\#(0,0)}$.

Finally *we suppose that there is a function*

$$[(t, u) \mapsto a(t, u; \cdot, \ )] \in C^{1-}([0, T] \times C^k(\bar{\Omega}, \mathbb{C}^N), \mathscr{L}^2(W^\beta_{\mathscr{B}}, (W^{\beta-1}_{\mathscr{B}})', \mathbb{C})),$$

such that

$$a(t, u; v, w) = \langle w, A(t, u)\, v \rangle$$

for all $(t, u) \in [0, T] \times C^k(\bar{\Omega}, \mathbb{C}^N)$, *all* $v \in W^{2m}_{p, \mathscr{B}(t,u)}$ *and all* $w \in (W^{\beta-1}_{\mathscr{B}})'$. Of course, such a function is obtained, in practical cases, by an appropriate "Green's formula".

Given the above assumptions, we can apply the abstract results of this paper to show that the quasilinear parabolic system

$$\frac{\partial u}{\partial t} + \mathscr{A}(x, t, u, \ldots, D^k u)\, u = f(x, t, u, \ldots, D^{2m-1} u) \quad \text{in } \Omega \times (s, T],$$

$$\mathscr{B}(x, t, u, \ldots, D^k u)\, u = 0 \quad \text{on } \partial\Omega \times (s, T],$$

$$u(\cdot, s) = u_0 \quad \text{on } \Omega$$

has a maximal classical solution provided $u_0 \in W^\sigma_{\mathscr{B}(t,u_0)}$ for $2m - 1 + n/p < \sigma \leqq 1$. Moreover this solution is unique if f depends only upon $u, \ldots, D^k u$. In this case one obtains also results concerning the continuity of the solution as a function of its initial value. The proofs and conclusions are similar to those of the preceding Section.

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Mathematisches Institut
Universität Zürich

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