A Partial Differential Equation with Infinitely many Periodic Orbits: Chaotic Oscillations of a Forced Beam

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Communicated by D. D. JOSEPH

Abstract

This paper delineates a class of time-periodically perturbed evolution equations in a Banach space whose associated Poincaré map contains a Smale horseshoe. This implies that such systems possess periodic orbits with arbitrarily high period. The method uses techniques originally due to MELNIKOV and applies to systems of the form $\dot{x} = f_0(x) + \varepsilon f_1(x, t)$, where $\dot{x} = f_0(x)$ is Hamiltonian and has a homoclinic orbit. We give an example from structural mechanics: sinusoidally forced vibrations of a buckled beam.

§ 1. Introduction: A Physical Model

In this paper we give sufficient conditions on T-periodically forced evolution equations in a Banach space for the existence of a Smale horseshoe for the time-T map of the dynamics. This implies the existence of infinitely many periodic orbits of arbitrarily high period and suggests the existence of a strange attractor. The results here are an extension to infinite dimensions of some of those in HOLMES [1979a, b, 1980a] and CHOW, HALE & MALLET-PARET [1980].

The techniques used are invariant manifolds, nonlinear semigroups and an extension to infinite dimensions of MELNIKOV's method [1963] for planar ordinary differential equations. The results are applied to the equations of a nonlinear, periodically forced, buckled beam. As the external force is increased, we show that a global bifurcation occurs, resulting in the transversal intersection of stable and unstable manifolds. This leads to all the complex dynamics of a horseshoe (SMALE [1963]).

The study of chaotic motion in dynamical systems is now a burgeoning industry. The mechanism given here is just one of many that can lead to chaotic dynamics. For a different mechanism occuring in reaction-diffusion equations, see GUCKENHEIMER [1979].

A physical model will help motivate the analysis. Consider a beam that is buckled by an external load Γ, so there are two stable and one unstable equilibrium states (see Figure 1). The whole structure is then shaken with a transverse periodic displacement, $f \cos \omega t$. The beam moves due to its inertia. In a related experiment (see TSENG & DUGUNDJI [1971] and MOON & HOLMES [1979], and remarks below), one observes periodic motion about either of the two stable equilibria for small f, but as f is increased, the motion becomes aperiodic or chaotic. The mathematical problem is to provide theorems that help to explain this behavior.

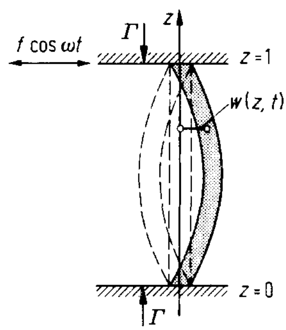

Fig. 1. The forced, buckled beam

There are a number of specific models that can be used to describe the beam in Figure 1. One of these is the following partial differential equation for the transverse deflection $w(z, t)$ of the centerline of the beam:

$$\ddot{w} + w'''' + \Gamma w'' - \varkappa \left(\int_0^1 [w']^2 \, d\zeta \right) w'' = \varepsilon (f \cos \omega t - \delta \dot{w}), \tag{1}$$

where $\dot{} = \partial/\partial t$, $' = \partial/\partial z$, $\Gamma =$ external load, $\varkappa =$ stiffness due to "membrane" effects, $\delta =$ damping, and ε is a parameter used to measure the relative size of f and δ. Amongst many possible boundary conditions we shall choose $w = w'' = 0$ at $z = 0, 1$; *i.e.*, simply supported, or hinged ends. With these boundary conditions, the eigenvalues of the linearized, unforced equations, *i.e.*, complex numbers λ such that

$$\lambda^2 w + w'''' + \Gamma w'' = 0$$

for some non-zero w satisfying $w = w'' = 0$ at $z = 0, 1$, form a countable set

$$\lambda_j = \pm \pi j \sqrt{\Gamma - \pi^2 j^2}, \quad j = 1, 2, \ldots$$

Thus, if $\Gamma < \pi^2$, all eigenvalues are imaginary and the trivial solution $w = 0$ is formally stable; for positive damping it is Liapunov stable. We shall henceforth assume that

$$\pi^2 < \Gamma < 4\pi^2,$$

in which case the solution $w = 0$ is unstable with one positive and one negative eigenvalue and the nonlinear equation (1) with $\varepsilon = 0$, $\varkappa > 0$ has two nontrivial stable buckled equilibrium states.

A simplified model for the dynamics of (1) is obtained by seeking lowest mode solutions of the form

$$w(z, t) = x(t) \sin(\pi z).$$

Substitution into (1) and taking the inner product with the basis function $\sin(\pi z)$, gives a Duffing type equation for the modal displacement $x(t)$:

$$\ddot{x} - \beta x + \alpha x^3 = \varepsilon(\gamma \cos \omega t - \delta \dot{x}), \tag{2}$$

where $\beta = \pi^2(\Gamma - \pi^2) > 0$, $\alpha = \varkappa\pi^4/2$ and $\gamma = 4f/\pi$. Equation (2) was studied at length in earlier papers (see HOLMES [1979a, 1979b] and HOLMES & MARSDEN [1979]). This work uses MELNIKOV's method; see MELNIKOV [1963], ARNOLD [1964], and HOLMES [1980a]. Closely related results are obtained by CHOW, HALE & MALLET-PARET [1980]. This method allows one to estimate the separation between stable and unstable manifolds and to determine when they intersect transversally. The method given in the above references applies to periodically perturbed two-dimensional flows such as the dynamics of equation (2). In this paper we extend these ideas to infinite dimensional evolution equations on Banach spaces and apply the method to the evolution equation (1).

TSENG & DUGUNDJI [1971] studied the one and two mode Galerkin approximations of (1) and found "chaotic snap-through" motions in numerical integrations. Such motions were also found experimentally but were not studied in detail. Subsequently, MOON & HOLMES [1979] found similar motions in experiments with an elastic, ferromagnetic beam and showed that a single-mode Galerkin approximation could indeed admit infinite sets of periodic motions of arbitrarily high period (HOLMES [1979b]).

It is known that the time t-maps of the Euler and Navier-Stokes equations written in Lagrangian coordinates are smooth. Thus the methods of this paper apply to these equations, in principle. On regions with no boundary, one can regard the Navier-Stokes equations with forcing as a perturbation of a Hamiltonian system (the Euler equations); see EBIN & MARSDEN [1970]. Thus, if one knew that a homoclinic orbit existed for the Euler equations, then the methods of this paper would produce infinitely many periodic orbits with arbitrarily high period, indicative of turbulence. No specific examples of this are known to us. One possibility, however, is periodically forced surface waves. See GOLLUB [1980].

An unforced sine-Gordon equation possesses heteroclinic orbits, as was shown by LEVI, HOPPENSTEADT & MIRANKER [1978]. Methods of this paper were used by HOLMES [1980b] to show that this system, with weak periodic forcing and damping and defined on a finite spatial domain, contains horseshoes. The methods should also be useful for travelling wave problems on infinite domains, such as the Korteweg-de Vries equation.

Acknowledgements. We are grateful to the following people for their interest in this work and their comments: HAIM BREZIS, JACK CARR, PAUL CHERNOFF, SHUI-NEE CHOW, JOHN GUCKENHEIMER, JACK HALE, MARK LEVI, RICHARD MCGEHEE, SHELDON NEWHOUSE, DAVID RAND and JEFFREY RAUCH.

The work of PHILIP HOLMES was supported in part by the U.S. National Science Foundation Grant ENG 78-02891 and that of JERROLD MARSDEN by the U.S. National Science Foundation Grant MCS 78-06718, U.S. Army Research Office Grant DAAG 29-79C-0086, and a Killam visiting fellowship at the University of Calgary.

§ 2. Abstract Hypotheses

We consider an evolution equation in a Banach space X of the form

$$\dot{x} = f_0(x) + \varepsilon f_1(x, t) \tag{3}$$

where f_1 is periodic of period T in t. Our hypotheses on (3) are as follows:

(H1). (a) *Assume $f_0(x) = Ax + B(x)$ where A is an (unbounded) linear operator which generates a C^0 one parameter group of transformations on X and where $B: X \to X$ is C^∞. Assume that $B(0) = 0$ and $DB(0) = 0$.*

(b) *Assume $f_1: X \times S^1 \to X$ is C^∞ where $S^1 = \mathbb{R}/(T)$, the circle of length T.*

Assumption 1 implies that the associated suspended autonomous system on $X \times S^1$,

$$\begin{aligned} \dot{x} &= f_0(x) + \varepsilon f_1(x, \theta), \\ \dot{\theta} &= 1, \end{aligned} \tag{4}$$

has a smooth local flow, F_t^ε. This means that $F_t^\varepsilon: X \times S^1 \to X \times S^1$ is a smooth map defined for small $|t|$ which is jointly continuous in all variables ε, t, $x \in X$, $\theta \in S^1$ and for x_0 in the domain of A, $t \mapsto F_t^\varepsilon(x_0, \theta_0)$ is the unique solution of (4) with initial condition x_0, θ_0.

This implication results from a theorem of SEGAL [1962]. For a simplified proof, see HOLMES & MARSDEN [1978, Prop. 2.5] and for generalizations, see MARSDEN & MCCRACKEN [1976].

The final part of assumption 1 follows:

(c) *Assume that F_t^ε is defined for all $t \in \mathbb{R}$ for $\varepsilon > 0$ sufficiently small.*

To verify this in examples, one must obtain an *a priori* bound on the X-norm of solutions of (4) to ensure they do not escape to infinity in a finite time. This is sufficient by the local existence theory alluded to above. In examples of concern to us, (c) will be verified using straightforward energy estimates. See HOLMES & MARSDEN [1978] for related examples.

Our second assumption is that the unperturbed system is Hamiltonian. This means that X carries a skew symmetric continuous bilinear map $\Omega: X \times X \to \mathbb{R}$ which is weakly non-degenerate (*i.e.*, $\Omega(u, v) = 0$ for all v implies $u = 0$) called the *symplectic form* and there is a smooth function $H_0: X \to \mathbb{R}$ such that

$$\Omega(f_0(x), u) = dH_0(x) \cdot u$$

for all x in D_A, the domain of A. Consult ABRAHAM & MARSDEN [1978] and CHERNOFF & MARSDEN [1974] for details about Hamiltonian systems. For example, these assumptions imply that the unperturbed system conserves energy:

$$H_0(F_t^0(x)) = H_0(x).$$

(For $\varepsilon = 0$ we drop the dependence on θ.) We summarize this condition and further restrictions as follows:

(H2). (a) *Assume that the unperturbed system* $\dot{x} = f_0(x)$ *is Hamiltonian with energy* $H_0 : X \to \mathbb{R}$.

(b) *Assume these is a symplectic 2-manifold* $\Sigma \subset X$ *invariant under the flow* F_t^0 *and that on* Σ *the fixed point* $p_0 = 0$ *has a homoclinic orbit* $x_0(t)$, i.e.,

$$\dot{x}_0(t) = f_0(x_0(t))$$

and

$$\operatorname*{limit}_{t \to +\infty} x_0(t) = \operatorname*{limit}_{t \to -\infty} x_0(t) = 0.$$

Remarks on Assumption 2.

(a) For a non-Hamiltonian two-dimensional version, see Holmes [1980a] and Chow, Hale & Mallet-Paret [1980]. Non-Hamiltonian infinite dimensional analogues could probably be developed by using the methods of this paper.

(b) The condition that Σ be symplectic means that Ω restricted to vectors tangent to Σ defines a non-degenerate bilinear form. We also note that by a general theorem of Chernoff & Marsden [1974], the restriction of F_t^0 to Σ is generated by a smooth vector field on Σ; *i.e.*, the dynamics within Σ is governed by *ordinary* differential equations. The situation described in assumption 2 is illustrated in Figure 2(a).

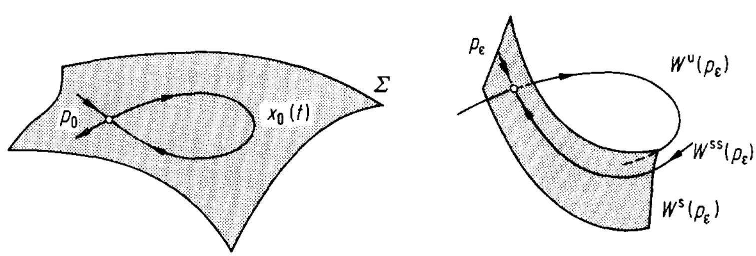

Fig. 2a. Phase portrait on Σ for $\varepsilon = 0$; b. Perturbation of invariant manifolds; $\varepsilon > 0$.

(c) Assumption 2 can be replaced by a similar assumption on the existence of heteroclinic orbits connecting two saddle points and the existence of transverse heteroclinic orbits can then be proven using the methods below. For details in the two-dimensional case, see Holmes [1980a]. Theorems guaranteeing the existence of saddle connections may be found in Conley & Smoller [1974] and Kopell & Howard [1979].

(d) To apply the techniques that follow, one must be able to calculate $x_0(t)$ either explicitly or numerically. In our examples, we find it analytically; for numerical methods, see Hassard [1980].

Next we introduce a non-resonance hypothesis.

(H3) (a) *Assume that the forcing term* $f_1(x, t)$ *in* (3) *has the form*

$$f_1(x, t) = A_1 x + f(t) + g(x, t) \tag{5}$$

where $A_1 : X \to X$ *is a bounded linear operator,* f *is periodic with period* T, $g(x, t)$ *is t-periodic with period* T *and satisfies* $g(0, t) = 0$, $D_x g(0, t) = 0$, *so* g *admits the estimate*

$$\|g(x, t)\| \leqq (\text{Const}) \|x\|^2 \tag{6}$$

for x *in a neighborhood of* 0.

(b) *Suppose that the "linearized" system*

$$\dot{x}_L = A x_L + \varepsilon A_1 x_L + \varepsilon f(t) \tag{7}$$

has a T-periodic solution $x_L(t, \varepsilon)$ *such that* $x_L(t, \varepsilon) = O(\varepsilon)$.

Remarks on (H3)

1. For finite dimensional systems, (H3) can be replaced by the assumption that 1 does not lie in the spectrum of e^{TA}; *i.e.* none of the eigenvalues of A resonate with the forcing frequency.

2. For the beam problem, with $f(t) = f(z) \cos \omega t$, (b) means that $\omega \neq \pm\lambda_n$, $n = 1, 2, \ldots$, where $i\lambda_n$ are the purely imaginary eigenvalues of A. This is seen by solving the component forced linear oscillator equations. As we shall see, more delicate non-resonance requirements would be necessary for general (smooth) T-periodic perturbations, not of the form (5).

3. For the beam problem we can take $g = 0$. We have included it in the abstract theory for use in other examples such as the sine-Gordon equation.

Next, we need an assumption that A_1 contributes positive damping and that $p_0 = 0$ is a saddle.

(H4) (a) *For* $\varepsilon = 0$, e^{TA} *has a spectrum consisting in two simple real eigenvalues* $e^{\pm\lambda T}$, $\lambda \neq 0$, *with the rest of the spectrum on the unit circle.*

(b) *For* $\varepsilon > 0$, $e^{T(A+\varepsilon A_1)}$ *has a spectrum consisting in two simple real eigenvalues* $e^{T\lambda_\varepsilon^{\pm}}$ *(varying continuously in* ε *from perturbation theory*; *cf.* KATO [1977]) *with the rest of the spectrum,* σ_R^ε, *inside the unit circle* $|z| = 1$ *and obeying the estimates*

$$C_2 \varepsilon \leqq \text{distance } (\sigma_R^\varepsilon, |z| = 1) \leqq C_1 \varepsilon \tag{8}$$

for C_1, C_2 *positive constants.*

Remarks on (H4). 1. In general it can be awkward to estimate the spectrum of e^{TA} in terms of the spectrum of A. Some information is contained in HILLE & PHILLIPS [1957] and CARR [1980]. See also CARR & MALHARDEEN [1980], VIDAV [1970], SHIZUTA [1979] and RAUCH [1979]. For the beam problem with $\varepsilon = 0$ it is sufficient to use these two facts or a direct calculation:

(a) if A is skew adjoint, then $\sigma(e^{tA}) =$ closure of $e^{t\sigma(A)}$;

(b) if $X = X_1 \oplus X_2$, where X_2 is finite dimensional (the eigenspace of the real eigenvalues in the beam problem) and B_1 is skew adjoint on X_1 and $B_2 : X_2 \to X_2$

is a (bounded) linear operator, then

$$\sigma(e^{t(B_1 \oplus B_2)}) = \text{closure}\,(e^{\sigma t B_1}) \cup e^{t\sigma(B_2)}.$$

For $\varepsilon > 0$ the abstract theorems are not very helpful. In the beam example the eigenfunctions of $A + \varepsilon A_1$ can be computed explicitly and form a basis for X, so the estimates (8) can be done directly; in fact σ_R^ε consists of a circle a distance $O(\varepsilon)$ inside the unit circle; see Appendix A.

2. The estimate dist $(\sigma_R^\varepsilon, |z| = 1) \geqq C_2\varepsilon$ guarantees that

$$L_\varepsilon = Id - e^{T(A+\varepsilon A_1)}$$

is invertible and

$$\|L_\varepsilon^{-1}\| \leqq \text{const}/\varepsilon. \tag{9}$$

3. The estimate dist $(\sigma_R^\varepsilon, |z| = 1) \leqq C_1\varepsilon$ guarantees that the eigenvalue $\exp(T\lambda_\varepsilon^-)$ will be the closest to the origin for ε small. This is needed below for the existence of an invariant manifold corresponding to λ_ε^-.

Finally, we need an extra hypothesis on the nonlinear term. We have already assumed B vanishes at least quadratically, as does g. Now we assume B vanishes cubically.

(H5) $B(0) = 0$, $DB(0) = 0$ *and* $D^2B(0) = 0$.

This means that in a neighborhood of 0,

$$\|B(x)\| \leqq \text{Const}\,\|x\|^3.$$

(Actually $B(x) = o(\|x\|^2)$ would do).

Remarks on (H5). 1. The necessity of having B vanish cubically is due to the possibility of the spectrum of A accumulating at zero. If this can be excluded for other reasons, (H5) can be dropped and (H4) simplified. There is a similar phenomenon for ordinary differential equations noted by JACK HALE. Namely, if the linear system

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ x \\ -\varepsilon y \end{bmatrix}$$

is perturbed by nonlinear terms plus forcing, to guarantee that the trivial solution (0, 0, 0) perturbs to a periodic solution as in lemma 1 below, one needs the nonlinear terms to be $o(|x| + |\dot{x}| + |y|)^3$.

2. For nonlinear wave equations, positivity of energy may force $D^2B(0) = 0$.

Consider the suspended system (4) with its flow $F_t^\varepsilon : X \times S^1 \to X \times S^1$. Let $P^\varepsilon : X \to X$ be defined by

$$P^\varepsilon(x) = \pi_1 \cdot (F_T^\varepsilon(x, 0))$$

where $\pi_1 : X \times S^1 \to X$ is the projection onto the first factor. The map P^ε is just the Poincaré map for the flow F_t^ε. Note that $P^0(p_0) = p_0$, and that fixed points of P^ε correspond to periodic orbits of F_t^ε.

Lemma 1. *For $\varepsilon > 0$ small, there is a unique fixed point p_ε of P^ε near $p_0 = 0$; moreover $p_\varepsilon - p_0 = O(\varepsilon)$, i.e. there is a constant K such that $\|p_\varepsilon\| \leqq K\varepsilon$ for all (small) ε.*

For ordinary differential equations, lemma 1 is a standard fact about persistence of fixed points, assuming 1 does not lie in the spectrum of e^{TA} (*i.e.*, p_0 is hyperbolic). For general partial differential equations, the validity of lemma 1 can be a delicate matter. In our context of smooth perturbations of linear systems with assumptions (H1)–(H5), the result is proved in Appendix A, along with the following.

Lemma 2. *For $\varepsilon > 0$ sufficiently small, the spectrum of $DP^\varepsilon(p_\varepsilon)$ lies strictly inside the unit circle with the exception of the single real eigenvalue $e^{T\lambda_\varepsilon^+} > 1$.*

In lemma 1 we saw the fixed point p_0 perturbs to another fixed point p_ε for the perturbed system. The same is true for the invariant manifolds; see Figure 2(b):

Lemma 3. *Corresponding to the eigenvalues $e^{T\lambda_\varepsilon^\pm}$ there are unique invariant manifolds $W^{ss}(p_\varepsilon)$ (the strong stable manifold) and $W^u(p_\varepsilon)$ (the unstable manifold) of p_ε for the map P^ε such that*

i. *$W^{ss}(p_\varepsilon)$ and $W^u(p_\varepsilon)$ are tangent to the eigenspaces of $e^{T\lambda_\varepsilon^\pm}$ respectively at p_ε;*
ii. *they are invariant under P^ε;*
iii. *if $x \in W^{ss}(p_\varepsilon)$, then*

$$\operatorname*{limit}_{n\to\infty} (P^\varepsilon)^n (x) = p_\varepsilon,$$

and if $x \in W^u(p_\varepsilon)$ then

$$\operatorname*{limit}_{n\to-\infty} (P^\varepsilon)^n (n) = p_\varepsilon.$$

iv. *For any finite t^*, $W^{ss}(p_\varepsilon)$ is C^r close as $\varepsilon \to 0$ to the homoclinic orbit $x_0(t)$, $t^* \leqq t < \infty$ and for any finite t_*, $W^u(p_\varepsilon)$ is C^r close to $x_0(t)$, $-\infty < t \leqq t_*$ as $\varepsilon \to 0$. Here, r is any fixed integer, $0 \leqq r < \infty$.*

This lemma follows from the invariant manifold theorems (HIRSCH, PUGH & SHUB [1977] and the smoothness of the flow of equations (4), discussed in Appendix A.

The Poincaré map P^ε was associated with the section $X\times\{0\}$ in $X\times S^1$. Equally well, we can take the section $X\times\{t_0\}$ to get Poincaré maps $P_{t_0}^\varepsilon$. By definition,

$$P_{t_0}^\varepsilon(x) = \pi_1(F_T^\varepsilon(x, t_0)).$$

[The Poincaré maps on different sections are related as follows: let $U_{t,s}^\varepsilon\colon X \to X$ be the evolution operators defined by $U_{t,s}^\varepsilon(x) = \pi_1(F_{t-s}^\varepsilon(x, s))$. Then $U_{t,s}^\varepsilon = U_{t,r}^\varepsilon \circ U_{r,s}^\varepsilon$ and $P_{t_0}^\varepsilon = U_{T+t_0,T+s_0} \circ P_{s_0}^\varepsilon \circ U_{t_0,s_0}^{-1}$.]

There is an analogue of Lemmas 1, 2 and 3 for $P_{t_0}^\varepsilon$. Let $p_\varepsilon(t_0)$ denote its unique fixed point and $W_\varepsilon^{ss}(p_\varepsilon(t_0))$ and $W_\varepsilon^u(p_\varepsilon(t_0))$ be its strong stable and unstable manifolds. Lemma 2 implies that the stable manifold $W^s(p_\varepsilon)$ of p_ε has codimension 1 in X. The same is then true of $W^s(p_\varepsilon(t_0))$ as well.

Let $\gamma_\varepsilon(t)$ denote the periodic orbit of the (suspended) system (4) with $\gamma_\varepsilon(0) = (p_\varepsilon, 0)$. We have

$$\gamma_\varepsilon(t) = (p_\varepsilon(t), t). \tag{10}$$

The invariant manifolds for the periodic orbit γ_ε are denoted $W_\varepsilon^{ss}(\gamma_\varepsilon)$, $W_\varepsilon^s(\gamma_\varepsilon)$ and $W_\varepsilon^u(\gamma_\varepsilon)$. We have

$$W_\varepsilon^s(p_\varepsilon(t_0)) = W_\varepsilon^s(\gamma_\varepsilon) \cap (X \times \{t_0\}),$$

$$W_\varepsilon^{ss}(p_\varepsilon(t_0)) = W_\varepsilon^{ss}(\gamma_\varepsilon) \cap (X \times \{t_0\}),$$

and

$$W_\varepsilon^u(p_\varepsilon(t_0)) = W_\varepsilon^u(\gamma_\varepsilon) \cap (X \times \{t_0\}).$$

See Figure 3.

We wish to study the structure of $W_\varepsilon^u(p_\varepsilon(t_0))$ and $W_\varepsilon^s(p_\varepsilon(t_0))$ and their intersections. To do this, we first study the perturbation of solution curves in $W_\varepsilon^{ss}(\gamma_\varepsilon)$, $W_\varepsilon^s(\gamma_\varepsilon)$ and $W_\varepsilon^u(\gamma_\varepsilon)$.

Choose a point, say $x_0(0)$ on the homoclinic orbit for the unperturbed system. Choose a codimension 1 hyperplane H transverse to the homoclinic orbit at $x_0(0)$. Since $W_\varepsilon^{ss}(p_\varepsilon(t_0))$ is C^r close to $x_0(0)$, it intersects H in a unique point, say $x_\varepsilon^s(t_0, t_0)$. Define $(x_\varepsilon^s(t, t_0), t)$ to be the unique integral curve of the suspended system (4) with initial condition $x_\varepsilon^s(t_0, t_0)$. Define $x_\varepsilon^u(t, t_0)$ in a similar way.

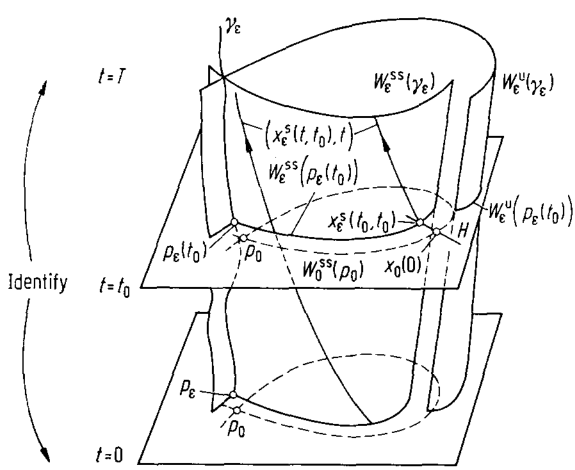

Fig. 3. The perturbed manifolds

The initial conditions $x_\varepsilon^s(t_0, t_0)$ and $x_\varepsilon^u(t_0, t_0)$ are not conveniently computable. This difficulty is, however, unimportant and is taken care of by the boundary conditions at $t \to \pm\infty$. We have

$$x_\varepsilon^s(t_0, t_0) = x_0(0) + \varepsilon v^s + O(\varepsilon^2)$$

and (11)

$$x_\varepsilon^u(t_0, t_0) = x_0(0) + \varepsilon v^u + O(\varepsilon^2)$$

by construction, where $\|O(\varepsilon^2)\| \leq \text{Constant} \cdot \varepsilon^2$ and v^s and v^u are fixed vectors. Notice that

$$(P_{t_0}^\varepsilon)^n \, x_\varepsilon^s(t_0, t_0) = x_\varepsilon^s(t_0 + nT, t_0) \to p_\varepsilon(t_0) \text{ as } n \to \infty.$$

Since $x_\varepsilon^s(t, t_0)$ is an integral curve of a perturbation, we can write

$$x_\varepsilon^s(t, t_0) = x_0(t - t_0) + \varepsilon x_1^s(t, t_0) + O(\varepsilon^2), \tag{12}$$

where $x_1^s(t, t_0)$ is the solution of the first variation equation

$$\frac{d}{dt} x_1^s(t, t_0) = Df_0(x_0(t - t_0)) \cdot x_1^s(t, t_0) + f_1(x_0(t - t_0), t), \tag{13}$$

with

$$x_1^s(t_0, t_0) = v^s.$$

This linearization procedure is justified by the proof of smoothness of the time t-maps; see Appendix A. There is a similar formula for $x_\varepsilon^u(t, t_0)$. Notice that when $\varepsilon \to 0$, the curve $x_\varepsilon^s(t, t_0)$ approaches the homoclinic orbit $x(t - t_0)$ with a phase shift t_0. In (12), $O(\varepsilon^2)$ means a term bounded by a constant$\times\varepsilon^2$, on each finite time interval. For $x_\varepsilon^s(t, t_0)$, the error $O(\varepsilon^2)$ is uniform as $t \to +\infty$ since $(x_\varepsilon^s(t, t_0), t)$ converges to the periodic orbit $\gamma_\varepsilon(t)$, by construction. Similarly the error $O(\varepsilon^2)$ in the corresponding equation for $x_\varepsilon^u(t, t_0)$ is uniform as $t \to -\infty$.

§ 3. The Melnikov Function

Recall that $\Omega: X \times X \to \mathbb{R}$ denotes the symplectic form on X relative to which f_0 is Hamiltonian. Define the *Melnikov function* by

$$\Delta_\varepsilon(t, t_0) = \Omega(f_0(x_0(t - t_0)), x_\varepsilon^s(t, t_0) - x_\varepsilon^u(t, t_0)) \tag{14}$$

and set

$$\Delta_\varepsilon(t_0) = \Delta_\varepsilon(t_0, t_0).$$

Lemma 4. *If ε is sufficiently small and $\Delta_\varepsilon(t_0)$ has a simple zero at some t_0 and maxima and minima that are at least $O(\varepsilon)$, then $W_\varepsilon^u(p_\varepsilon(t_0))$ and $W_\varepsilon^s(p_\varepsilon(t_0))$ intersect transversally near $x_0(0)$.*

Proof. First note that, by lemma 2, $W_\varepsilon^s(p_\varepsilon)$ has codimension 1. As $\varepsilon \to 0$, the perturbation theory of invariant manifolds shows that $W_\varepsilon^s(p_\varepsilon) \xrightarrow{C^r} W^{sc}(p_0)$, where $W^{sc}(p_0)$ is the center-stable manifold for F_T, the time T map for the unperturbed system.

Let $d_\varepsilon(t, t_0) = x_\varepsilon^s(t, t_0) - x_\varepsilon^u(t, t_0)$ and $d_\varepsilon(t_0) = d_\varepsilon(t_0, t_0)$.

Let $T_{x_0(0)}\Sigma$ be the tangent space to Σ at $x_0(0)$ and let

$$(T_{x_0(0)}\Sigma)^\Omega = \{v \in X \mid \Omega(v, w) = 0 \text{ for all } w \in T_{x_0(0)}\Sigma\}$$

be its Ω-orthogonal complement. Because Σ is finite (2)-dimensional and symplectic,

$$X = T_{x_0(0)}\Sigma \oplus (T_{x_0(0)}\Sigma)^\Omega. \tag{15}$$

Let $\mathbb{P}: X \to T_{x_0(0)}\Sigma$ be the projection onto the first factor, associated with the decomposition (15). Set

$$d_\varepsilon^\Sigma(t_0) = \mathbb{P}(d_\varepsilon(t_0)) \tag{16}$$

(see Fig. 4).

By definition,

$$\Delta_\varepsilon(t_0) = \Omega(f_0(x_0(0)), d_\varepsilon(t_0)) = \Omega(f_0(x_0(0)), d_\varepsilon^\Sigma(r_0)).$$

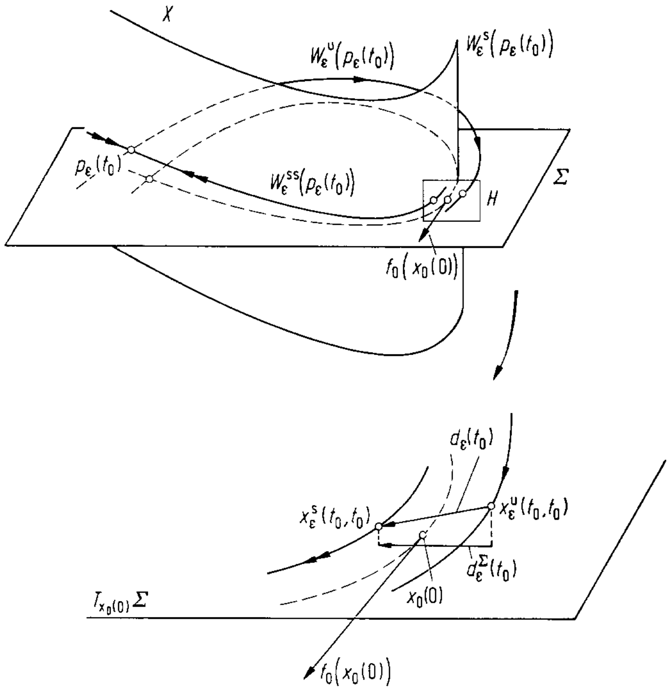

Fig. 4. Geometry for the Melnikov function

If $\Delta_\varepsilon(t_0)$ has a simple zero, then as a function of t_0, $d_\varepsilon^\Sigma(t_0)$ changes orientation relative to $f_0(x_0(0))$ within Σ as t_0 changes. Thus, as $W_0^{sc}(p_0)$ is codimension 1, $d_\varepsilon^\Sigma(t_0)$ also changes orientation relative to it.

The tangent space of $W_\varepsilon^s(p_\varepsilon(t_0))$ near $x_0(t)$ is ε-close to that of $W_0^{sc}(p_0)$; and the tangent space of $W_\varepsilon^u(p_\varepsilon(t_0))$ near $x_0(0)$ is ε-close to the vector $f_0(x_0(0))$, uniformly for $0 \leqq t_0 \leqq T$. This follows from the perturbation theory of invariant manifolds. Let $v_\varepsilon(t_0)$ denote the vector from $x_\varepsilon^u(t_0, t_0)$ to the nearest point on $W_\varepsilon^s(p_\varepsilon(t_0))$ (it is easy to see that there is a unique such point in an ε-neighborhood of $x_0(0)$). By (11) and the tangent space estimates just discussed, it follows that

$$v_\varepsilon(t_0) = d_\varepsilon^\Sigma(t_0) + O(\varepsilon^2). \tag{17}$$

Thus if $d_\varepsilon^\Sigma(t_0)$ passes through zero, changing orientation relative to $f_0(x_0(0))$ with an amplitude $O(\varepsilon)$, then by (17), $v_\varepsilon(t_0)$ must do the same. It follows that $W_\varepsilon^u(p_\varepsilon(t_0))$ and $W_\varepsilon^s(p_\varepsilon(t_0))$ then intersect transversally near the t_0 at which $\Delta_\varepsilon(t_0)$ has its zero. ∎

The next lemma gives a formula for $\Delta_\varepsilon(t_0)$ that uses the Hamiltonian nature of f_0. This formula is needed in examples to check effectively that $\Delta_\varepsilon(t_0)$ has simple zeros.

Lemma 5. *The following formula holds*:

$$\Delta_\varepsilon(t_0) = -\varepsilon \int_{-\infty}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0)), t)\, dt + O(\varepsilon^2). \tag{18}$$

Proof. By (12) we can write $\Delta_\varepsilon(t, t_0) = \Delta_\varepsilon^+(t, t_0) - \Delta_\varepsilon^-(t, t_0) + O(\varepsilon^2)$, where

$$\Delta_\varepsilon^+(t, t_0) = \Omega(f_0(x_0(t - t_0)), \varepsilon x_1^s(t, t_0))$$

and

$$\Delta_\varepsilon^-(t, t_0) = \Omega(f_0(x_0(t - t_0)), \varepsilon x_1^u(t, t_0)).$$

Using (13), we get

$$\frac{d}{dt}\Delta_\varepsilon^+(t, t_0) = \Omega(Df_0(x_0(t, t_0)) \cdot f_0(x_0(t - t_0)), \varepsilon x_1^s(t, t_0))$$
$$+ \Omega(f_0(x_0(t - t_0)), \varepsilon\{Df_0(x_0(t - t_0)) \cdot x_1^s(t, t_0) + f_1(x_0(t - t_0), t)\}).$$

Since f_0 is Hamiltonian, Df_0 is Ω-skew. Therefore the terms involving x_1^s drop out, leaving

$$\frac{d}{dt}\Delta_\varepsilon^+(t, t_0) = \Omega(f_0(x_0(t - t_0)), \varepsilon f_1(x_0(t - t_0), t)).$$

Integrating, we have

$$-\Delta_\varepsilon^+(t_0, t_0) = \varepsilon \int_{t_0}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt, \tag{19}$$

since

$$\Delta_\varepsilon^+(\infty, t_0) = \Omega(f_0(p_0), \varepsilon f_1(p_0, \infty)) = 0, \text{ because } f_0(p_0) = 0.$$

Similarly, we obtain

$$\Delta_\varepsilon^-(t_0, t_0) = \varepsilon \int_{-\infty}^{t_0} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt$$

and adding gives the stated formula. ▮

The expression $\int_{-\infty}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt$ is an "averaged bracket" over the orbit $x_0(t)$; if f_1 is Hamiltonian (time dependent), this is just an integrated Poisson bracket over the orbit $x_0(t)$; *cf.* ARNOLD [1964]. The power of MELNIKOV's method rests in the fact that this formula renders the leading term of $\Delta_\varepsilon(t_0)$ computable.

We summarize the situation as follows:

Theorem 1. *Let hypotheses* 1–5 *hold. Let*

$$M(t_0) = \int_{-\infty}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))\, dt. \tag{20}$$

Suppose that $M(t_0)$ *has a simple zero as a function of* t_0. *Then for* $\varepsilon > 0$ *sufficiently small, the stable manifold* $W_\varepsilon^s(p_\varepsilon(t_0))$ *of* p_ε *for* $P_{t_0}^\varepsilon$ *and the unstable manifold* $W_\varepsilon^u(p_\varepsilon(t_0))$ *intersect transversally. (We shall also call* M *the Melnikov function).*

In Section 5 we discuss consequences of this result. Before doing so we discuss some examples related to the physical model of the beam.

§ 4. Examples

1. (See HOLMES [1979 b]): Consider the forced Duffing equation

$$\ddot{x} - \beta x + \alpha x^3 = \varepsilon(\gamma \cos \omega t - \delta \dot{x}).$$

Here the unperturbed ($\varepsilon = 0$) system is $\ddot{x} - \beta x + \alpha x^3 = 0$, *i.e.*,

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \beta x - \alpha x^3 \end{pmatrix} \tag{21}$$

which is Hamiltonian in $X = \mathbb{R}^2 = \Sigma$ with

$$H(x, \dot{x}) = \frac{\dot{x}^2}{2} - \frac{\beta x^2}{2} + \frac{\alpha x^4}{4}. \tag{22}$$

The flow of this system is the "figure eight" pattern shown in Figure 5(a). The homoclinic orbit is given by

$$(x_0(t), \dot{x}_0(t)) = \left(\sqrt{\frac{2\beta}{\alpha}} \operatorname{sech}(\sqrt{\beta}\, t),\ -\beta \sqrt{\frac{2}{\alpha}} \operatorname{sech} \sqrt{\beta}\, t \tanh \sqrt{\beta}\, t\right). \tag{23}$$

We have based the solution at $(x_0(0), \dot{x}_0(0)) = (\sqrt{2\beta/\alpha}, 0)$. The symplectic form is $\Omega((x, \dot{x}), (y, \dot{y})) = x\dot{y} - \dot{x}y$, so by (20) the Melnikov function is

$$M(t_0) = \int_{\infty}^{\infty} \Omega\left(\begin{pmatrix} \dot{x} \\ \beta x - \alpha x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \cos \omega t - \delta \dot{x} \end{pmatrix}\right) dt = \int_{-\infty}^{\infty} \dot{x}(\gamma \cos \omega t - \delta \dot{x})\, dt, \tag{24}$$

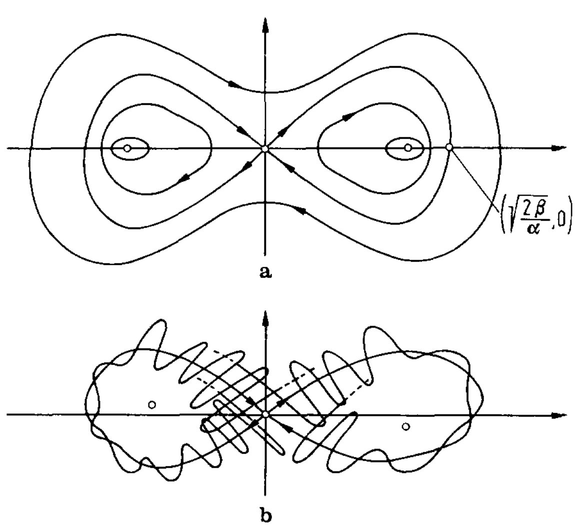

Fig. 5a and b. The single mode model.
a Unperturbed case; $\varepsilon = 0$; **b** Perturbed Case; $\varepsilon >$ small, $\gamma > \gamma_c$

where x and $\dot{x}$ are given in (23) (with t replaced by $t - t_0$). The integral (24) may be evaluated by standard methods (HOLMES [1979b]) to yield

$$M(t_0) = -\frac{4\delta\beta^{\frac{3}{2}}}{3\alpha} + 2\gamma\omega\sqrt{\frac{2}{\alpha}}\,\frac{\sin(\omega t_0)}{\cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right)}. \tag{25}$$

Thus, if

$$\gamma > \gamma_c = \frac{2\delta\beta^{\frac{3}{2}}}{3\omega\sqrt{2\alpha}}\cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right), \tag{26}$$

then M has simple zeros and so by Theorem 1 the stable and unstable manifolds intersect transversely for ε sufficiently small. See Fig. 5(b).

2. The two mode Galerkin approximation of the beam equation (1) is given as follows (*cf.* TSENG & DUGUNDJI [1971], HOLMES [1979a]):

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \beta_1 x - \alpha(x^2 + 4y^2)\,x \\ \dot{y} \\ -\beta_2 x - 4\alpha(x^2 + 4y^2)\,y \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \gamma_1 \cos\omega t - \delta_1 \dot{x} \\ 0 \\ \gamma_2 \cos\omega t - \delta_2 \dot{y} \end{pmatrix} \tag{27}$$

where $\beta_1 = \pi\sqrt{\Gamma - \pi^2} > 0$, $\beta_2 = \pi\sqrt{4\pi^2 - \Gamma} > 0$ and $\alpha = \frac{\varkappa\pi^4}{2}$.

Here the plane Σ, spanned by the vectors $(1, 0, 0, 0)^T$ and $(0, 1, 0, 0)^T$, is a symplectic 2 manifold and the unperturbed homoclinic orbit is given by

$$(x_0(t_0), \dot{x}_0(t_0), y_0(t_0), \dot{y}_0(t_0))$$
$$= \left(\sqrt{\frac{2\beta}{\alpha}}\,\mathrm{sech}\,(\sqrt{\beta}\,t), -\beta\sqrt{\frac{2}{\alpha}}\,\mathrm{sech}\,\sqrt{\beta}\,t\tanh\sqrt{\beta}\,t, 0, 0\right). \tag{28}$$

The Melnikov function is found to be

$$M(t_0) = \int_{-\infty}^{\infty} \dot{x}[(\gamma_1\cos\omega t - \delta_1\dot{x}) + \dot{y}\,(\gamma_2\cos\omega t - \delta_2\dot{y})]\,dt, \tag{29}$$

and since $\dot{y}_0 = 0$ by (28), the computation of (29) reduces to that of Example 1. The non-resonance condition (H3) for this example becomes $\omega^2 \neq \beta_2$.

3. The partial differential equation of the beam:

$$\ddot{w} + w'''' + \Gamma w'' - \varkappa\left(\int_0^1 [w']^2\,d\zeta\right) w'' = \varepsilon(f\cos\omega t - \delta\dot{w})$$

with boundary conditions

$$w = w'' = 0 \text{ at } z = 0, 1.$$

The basic space is $X = H_0^2 \times L^2$ where H_0^2 denotes the set of all H^2 functions on $[0, 1]$ satisfying the boundary conditions $w = 0$ at $z = 0, 1$. For $x = (w, \dot{w}) \in X$, the X-norm is the "energy" norm $\|x\|^2 = |w''|^2 + |\dot{w}|^2$ where $|\cdot|$ denotes the

L_2 norm. Write the equation as

$$\frac{dx}{dt} = f_0(x) + \varepsilon f_1(x, t),$$

where

$$f_0(x) = Ax + B(x) \text{ and } f_1(x, t) = \begin{pmatrix} 0 \\ f\cos\omega t - \delta\dot{w} \end{pmatrix} = A_1 x + f(t).$$

Here A and A_1 are the linear operators

$$A\begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \dot{w} \\ -w'''' - \Gamma w' \end{pmatrix}, \; A_1\begin{bmatrix} w \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 \\ -\delta\dot{w} \end{bmatrix},$$

with domains $D(A_1) = X$ (A_1 is bounded) and

$$D(A) = \{(w, \dot{w}) \in H^4 \times H^2 \mid w = 0, w'' = 0 \text{ and } \dot{w} = 0 \text{ at } z = 0, 1\},$$

and B is the nonlinear mapping of X to X given by

$$B(x) = \begin{pmatrix} 0 \\ \varkappa\left(\int_0^1 [w']^2\, d\zeta\, w''\right) \end{pmatrix}.$$

In the forcing term $f(t) = f\cos\omega t$, f can be a function of z, *i.e.*, a spatially distributed load. Let $\bar{f}$ denote the mean of f. We expand f in a Fourier series with period twice the beam length:

$$f(z) = \bar{f} + \sum_{n=1}^{\infty} \{\alpha_n \sin(n\pi z) + \beta_n \cos(n\pi z)\}.$$

The coefficients $\bar{f}$ and α_1 will be important in calculations that follow.

The theorems of HOLMES & MARSDEN [1978] show that A is a generator and that B and f_1 are smooth maps. This, together with the energy estimates, shows that the equations generate a global flow $F_t^\varepsilon\colon X \times S^1 \to X \times S^1$ consisting of C^∞ maps for each ε and t. See appendix A for details. If x_0 lies in the domain of the (unbounded) operator A, then $F_t^\varepsilon(x_0, s)$ is t-differentiable and the equation is literally satisfied. Thus Hypothesis 1 holds.

For $\varepsilon = 0$ the equation is readily verified to be Hamiltonian using the symplectic form

$$\Omega((w_1, \dot{w}_1), (w_2, \dot{w}_2)) = \int_0^1 \{\dot{w}_2(z)\, w_1(z) - \dot{w}_1(z)\, w_2(z)\}\, dz$$

and

$$H(w, \dot{w}) = \frac{1}{2}|\dot{w}|^2 - \frac{\Gamma}{2}|w'|^2 + \frac{1}{2}|w''|^2 + \frac{\varkappa}{4}|w'|^4.$$

The invariant symplectic 2-manifold Σ is the plane in X spanned by the functions $(a\sin\pi z, b\sin\pi z)$ and the homoclinic loop is given by

$$w_0(z, t) = \frac{2}{\pi}\sqrt{\frac{\Gamma - \pi^2}{\varkappa}}\sin(\pi z)\operatorname{sech}\left(t\pi\sqrt{\Gamma - \pi^2}\right).$$

Hypothesis 2 therefore holds. For $\varepsilon = 0$ one finds by direct calculation that the spectrum of $Df_0(p_0)$, where $p_0 = (0, 0)$, is discrete with two real eigenvalues

$$\pm\lambda = \pm\pi\sqrt{\Gamma - \pi^2}$$

and the remainder pure imaginary (since $\Gamma < 4\pi^2$) at

$$\lambda_n = \pm n\pi\sqrt{\Gamma - n^2\pi^2}, \quad n = 2, 3, \ldots$$

(*cf.* HOLMES [1979a]). Hypothesis (H3a) clearly holds with $g \equiv 0$. Condition (H3b) holds by direct calculation using Fourier series and the following nonresonance assumption:

$$j^2\pi^2(j^2\pi^2 - \Gamma) \neq \omega^2, \quad j = 2, 3, 4, \ldots \tag{30}$$

Expanding $w(z, t)$ in the eigenfunctions $\sin j\pi z$ of the linearized problem and using Galerkin's method we obtain an infinite set of second order ordinary differential equations for the modal coefficient $a_j(t)$:

$$\ddot{a}_j + \varepsilon\,\delta\dot{a}_j + j^2\pi^2(j^2\pi^2 - \Gamma)\,a_j = \varepsilon\gamma_j\cos\omega t, \quad j = 1, 2, \ldots, \tag{31}$$

where

$$w(z, t) = \sum_{j=1}^{\infty} a_j(t)\sin(j\pi z) \text{ and } \gamma_j = \int_0^1 f(z)\sin(j\pi z)\,dz.$$

It is easy to check that if (30) holds, then (31) has a unique periodic solution $w_L(z, t) = w_L(z, t + 2\pi/w)$ of $O(\varepsilon)$. Moreover, the eigenvalues of the perturbed operator $e^{T(A+\varepsilon A_1)}$ and the unperturbed operator e^{TA} can be calculated directly from (31), with the γ_j set to zero (*cf.* HOLMES [1979a]). One obtains a countable set of eigenvalues for the flow given by

$$\lambda_n^\varepsilon = \frac{1}{2}\left[-\varepsilon\,\delta \pm \sqrt{\varepsilon^2\,\delta^2 + n^2\pi^2(\Gamma - n^2\pi^2)}\right], \quad n = 1, 2, \ldots \tag{32}$$

Exponentiation of (32) reveals that hypothesis (H4) holds (here $T = 2\pi/\omega$). Finally, it is clear that (H5) also holds for this example.

The Melnikov function (20) is given by

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} \Omega\begin{pmatrix} \dot{w} & 0 \\ -w'''' + \varkappa\,|w'|^2\,w'' - \Gamma w'' & f\cos\omega t - \delta\dot{w}\end{pmatrix} dt \\ &= \int_{-\infty}^{\infty}\left(\int_0^1 f\cos\omega t\,\dot{w}(z, t - t_0) - \delta\dot{w}(z, t - t_0)\,\dot{w}(z, t - t_0)\,dz\right) dt. \end{aligned}$$

Substituting the expressions for w, $\dot{w}$ along the homoclinic orbit, we evaluate the integral as in Example 1. One gets

$$-M(t_0) = \frac{2\omega}{\pi}\sqrt{\frac{\Gamma - \pi^2}{\varkappa}}\left(\frac{\alpha_1}{2} + \frac{2\bar{f}}{\pi}\right)\frac{\sin(\omega t_0)}{\cosh\left(\dfrac{\omega}{2\sqrt{\Gamma - \pi^2}}\right)} + \frac{4\delta(\Gamma - \pi^2)^{\frac{3}{2}}}{3\pi\varkappa}$$

Thus, if

$$\left|\frac{\alpha_1}{2} + \frac{2\bar{f}}{\pi}\right| > \frac{2\delta(\Gamma - \pi^2)}{3\omega\sqrt{\varkappa}}\left[\cosh\left(\frac{\omega}{2\sqrt{\Gamma - \pi^2}}\right)\right],$$

then the hypotheses of Theorem 1 hold and so the stable and unstable manifolds intersect transversally. Note that in the spatial integral evaluated, in the expression for Ω, only the components $\bar{f}$ and α_1 of $f(z)$ survive, due to orthogonality of the other Fourier components with the solution

$$\dot{w}_0(t) = \frac{2}{\sqrt{\varkappa}} (\Gamma - \pi^2) \sin(\pi z) \operatorname{sech}(t\pi\sqrt{\Gamma - \pi^2}) \tanh(t\pi\sqrt{\Gamma - \pi^2}).$$

It should be realized that, while the formal calculations of $M(t_0)$ in the second and third examples are similar to that of $M(t_0)$ in the two dimensional example given first, the full power of Theorem 1 is necessary since in the four and infinite dimensional cases, the perturbed manifolds $W_\varepsilon^{ss}(p_\varepsilon(t_0))$ and $W_\varepsilon^u(p_\varepsilon(t_0))$ do not lie in Σ.

We close this section with a comment on the bifurcations in which the transversal intersections are created. Since the Melnikov function $M(t_0)$ has non-degenerate maxima and minima in all three examples, it can be shown that, near the parameter values at which $M(t_0) = M'(t_0) = 0$, but $M''(t_0) \neq 0$, the stable and unstable manifolds $W_\varepsilon^s(p_\varepsilon(t_0))$, $W_\varepsilon^u(p_\varepsilon(t_0))$ have quadratic tangencies. The mechanism, described by Newhouse [1974, 1979], then implies that $P_{t_0}^\varepsilon$ can have infinitely many stable periodic orbits of arbitrarily high periods near the bifurcation point, at least in the first two (finite dimensional) examples. In practice it may be difficult to distinguish these long period stable periodic points with their small basins, from transient chaos, noise and from "true" chaos itself. We note that transient chaos has been observed in experimental work (Moon & Holmes [1979]).

§ 5. Consequences of Transversal Intersection

If the hypotheses of Theorem 1 hold, we obtain a Poincaré map $P_{t_0}^\varepsilon: X \to X$ having the following property: there is a hyperbolic saddle point p_ε which has a 1 dimensional unstable manifold intersecting a codimension 1 stable manifold transversally. For $X = \mathbb{R}^2$, this situation implies that the dynamics contains a horseshoe (see Smale [1967]). For instance, one can conclude the existence of infinitely many periodic points with arbitrarily high period. See Figure 6. Together with global attractivity due to positive damping, this suggests the presence of a strange attractor (*cf.* Holmes [1979a, b]).

A particularly noteworthy method for analysis in such cases has been given by Conley & Moser; see Moser [1973], Chapter III. The attractive feature of their method is that it reduces the proof to one of finding *explicit estimates* on what $P_{t_0}^\varepsilon$ does to horizontal and vertical strips near the saddle point. This enables one to generalize the argument to dimension $\geqq 2$ and to Banach spaces X. In particular, it applies to the beam example. Specifically, we prove the following result in Appendix B.

Theorem 2. *If the diffeomorphism $P_{t_0}^\varepsilon: X \to X$ possesses a hyperbolic saddle point p_ε and an associated transverse homoclinic point $q \in W_\varepsilon^u(p_\varepsilon) \cap W_\varepsilon^s(p_\varepsilon)$, with $W_\varepsilon^u(p_\varepsilon)$*

of dimension 1 *and* $W^s_\varepsilon(p_\varepsilon)$ *of codimension* 1, *then some power of* $P^\varepsilon_{t_0}$ *possesses an invariant zero dimensional hyperbolic set* Λ *homeomorphic to a Cantor set on which a power of* $P^\varepsilon_{t_0}$ *is conjugate to a shift on two symbols.*

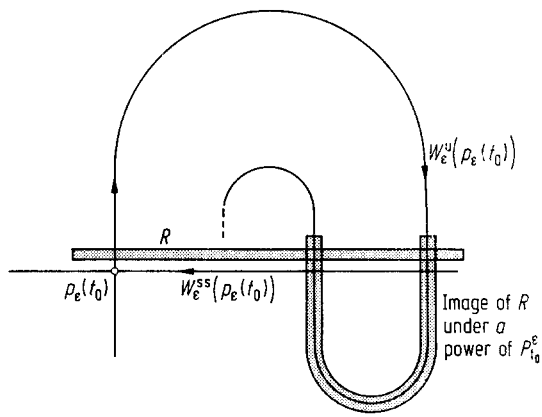

Fig. 6

As in the finite dimensional case, this implies

Corollary 1. *A power of* $P^\varepsilon_{t_0}$ *restricted to* Λ *possesses a dense set of periodic points there are points of arbitrarily high period and there is a non-periodic orbit dense in* Λ.

The hyperbolicity of Λ under a power of $P^\varepsilon_{t_0}$ and the theorem on structural stability of ROBBIN [1971] implies that the situation of Theorem 2 persists under perturbations:

Corollary 2. *If* $\bar{P}: X \to X$ *is a diffeomorphism that is sufficiently close to* $P^\varepsilon_{t_0}$ *in* C^1 *norm, then a power of* $\bar{P}$ *has an invariant set* $\bar{\Lambda}$ *and there is a homeomorphism* $h: \bar{\Lambda} \to \Lambda$ *such that* $(P^\varepsilon_{t_0})^N \circ h = h \circ \bar{P}^N$ *for a suitable power N.*

Thus, the complex dynamics of $(P^\varepsilon_{t_0})^N$ near Λ cannot be removed by making small changes in lower order (bounded) terms in the governing equations (3).

Although the dynamics near Λ is complex, we do not assert that Λ is a strange attractor. In fact, Λ is unstable in the sense that its generalized unstable manifold (or outset), $W^u(\Lambda)$ is non-empty (it is one dimensional) and thus points starting near Λ may wander, remaining near Λ for a relatively long time, but eventually leaving a neighborhood of Λ and approaching an attractor. This kind of behavior has been referred to as *transient chaos* (or pre-turbulence). In two dimensions (Example 1 of § 4), Λ can coexist with two simple sinks of period one or with a strange attractor, depending on the parameter values (see HOLMES [1979b]). As noted earlier, there is experimental evidence for transient chaos in the magnetic cantilever problem, in addition to the evidence for sustained non-periodic notions (HOLMES & MOON [1979]).

Appendix A. Perturbations of Fixed Points and Invariant Manifolds for Partial Differential Equations

We begin by recalling the local existence result.

Proposition A.1. *Let X be a Banach space and U_t a linear semigroup on X with generator A and domain $D(A)$. Let $B: X \to X$ be C^k, $k \geqq 1$. Let $G = A + B$ on $D(A)$. Then*

$$\frac{dx}{dt} = G(x); \quad x_0 = x(0) \in X \tag{A-1}$$

defines a unique local semiflow $F_t(x_0)$: If $x_0 \in D(A)$, then $F_t(x_0) \in D(A) = D(G)$, is X-differentiable and satisfies (A-1) *with initial condition x_0, $F_t(x_0)$ is the unique such solution and moreover, F_t extends to a C^k map of an open set in X to X for each $t \geqq 0$.*

The basic idea is to use Picard iteration on the corresponding integral equation

$$x(t) = U_t x_0 + \int_0^t U_{t-s} B(x(s))\, ds, \tag{A-2}$$

where $U_t = e^{tA}$. One sets $F_t(x_0) = x(t)$. For details, see HOLMES & MARSDEN [1978], Prop. 2.5.

Suppose $G = A + B$ and $G = A + \bar{B}$ both satisfy the conditions of proposition A.1. and F_t and $\bar{F}_t$ are the corresponding semiflows.

Proposition A.2. *We have the following estimates on $F_t - \bar{F}_t$: fix $T > 0$ and suppose F_t and $\bar{F}_t$ map the bounded set $S \subset X$ into the ball B_R of radius R and that on B_R we have*

(i) $$\sup_{x \in B_R} \|B(x) - \bar{B}(x)\| < C,$$

(ii) $$\|B(x) - B(y)\| \leqq K \|x - y\|$$

and assume

(iii) $$\|e^{tA}\| \leqq Me^{|t|\beta} \text{ for } M > 0, \beta > 0.$$

Let $\bar{M} = Me^{T\beta}$. Then

$$\|F_t(t) - \bar{F}_t(x)\| \leqq \bar{M}CTe^{K|t|\bar{M}}. \tag{A-3}$$

Furthermore, assume that for $x, y \in B_R$,

(iv) $$\|[DB(x) - DB(y)] \cdot v\| \leqq K_1 \|v\| \|x - y\|,$$

(v) $$\|[DB(x) - D\bar{B}(x)] \cdot v\| \leqq C_1 \|v\|,$$

(vi) $$\|DB(x) \cdot v\| \leqq M_2 \|v\|$$

and (vii) *for $x \in S$, $\|D\bar{F}_t(x) \cdot u\| \leqq M_3 \|u\|$ for $|t| \leqq T$. Then*

$$\|[DF_t(x) - D\bar{F}_t(x)] \cdot u\| \leqq TM_3\bar{M}(C_1 + K_1\bar{M}CTe^{kT\bar{M}})\, e^{M_2|t|\bar{M}} \|u\|. \tag{A-4}$$

In particular, if B and $\overline{B}$ are close in C^1 norm on bounded sets then $DF_t(x)$ and $D\overline{F}_t(x)$ are close in the *operator* norm; *cf.* Prop. 2.13 of HOLMES & MARSDEN [1978].

Proof. Let $x(t) = F_t(x)$ and $\overline{x}(t) = \overline{F}_t(x)$. From (A-2) we have

$$\begin{aligned} x(t) - \overline{x}(t) &= \int_0^t U_{t-s}[B(x(s)) - \overline{B}(\overline{x}(s))]\, ds \\ &= \int_0^t U_{t-s}[B(x(s)) - B(\overline{x}(s))]\, ds \\ &\quad + \int_0^t U_{t-s}[B(\overline{x}(s)) - \overline{B}(\overline{x}(s))]\, ds. \end{aligned}$$

Thus from (i), (ii) and (iii),

$$\|x(t) - \overline{x}(t)\| \leqq MK \int_0^t \|x(s) - \overline{x}(s)\|\, ds + \overline{M}CT.$$

Estimate (A-3) thus follows. To prove (A-4) we recall (HOLMES & MARSDEN [1978, Eq. 14]) that $DF_t(x)$ satisfies the first variation equation:

$$DF_t(x)\, u = U_t \cdot u + \int_0^t U_{t-s} DB(F_s(x)) \cdot [DF_s(x) \cdot u]\, ds.$$

(Note that from this one can choose $M_3 = \overline{M}e^{\overline{M}M_2T}$.) Therefore

$$\begin{aligned} &DF_t(x) \cdot u - D\overline{F}_t(x) \cdot u \\ &\quad = \int_0^t U_{t-s}[DB(F_s(x)) \cdot (DF_s(x) \cdot u) - D\overline{B}(\overline{F}_s(x)) \cdot (D\overline{F}_s(x) \cdot u)]\, ds \\ &\quad = \int_0^t U_{t-s}[DB(F_s(x)) \cdot \{DF_s(x) \cdot u - D\overline{F}_s(x) \cdot u\}\, ds] \\ &\qquad + \int_0^t U_{t-s}[\{DB(F_s(x)) - DB(\overline{F}_s(x))\} \cdot \{D\overline{F}_s(x) \cdot u\}]\, ds \\ &\qquad + \int_0^t U_{t-s}\{DB(\overline{F}_s(x)) - D\overline{B}(\overline{F}_s(x))\} \cdot \{D\overline{F}_s(x) \cdot u\}\, ds. \end{aligned}$$

Thus $\|DF_t(x) \cdot u - D\overline{F}_t(x) \cdot u\| \leqq \overline{M}M_2 \int_0^t \|DF(x) \cdot u - D\overline{F}_s(x) \cdot u\|\, ds +$
$+ \overline{M}K_1\overline{M}CT^2e^{KT\overline{M}} M_3 \|u\| + \overline{M}C_1TM_3 \|u\|$,
from which (A-4) follows. ∎

Condition (ii) holds if DB is bounded on B_R and (iii) is automatic for any C^0 semigroup and serves only to define the constants. Condition (iv) holds if D^2B is bounded on B_R, (vi) just says DB is bounded on B_R, and we have already noted that one can choose $M_3 = \overline{M}e^{\overline{M}M_2T}$ to obtain (vii).

Proposition A.3. *Under assumptions* 1(a), (b) *and* (c) *of* § 2, *the bounded linear operators* $DP^\varepsilon(p_0): X \to X$ *converge in norm as* $\varepsilon \to 0$ *to* $DP^0(p_0)$.

Proof. As $\varepsilon \to 0$, $\varepsilon f_1(x, t) \to 0$ locally in x (uniformly in t) along with its derivative. Thus, by proposition A.2 $DF_t^\varepsilon(x) \to DF_t^0(x)$ as $\varepsilon \to 0$, where the convergence is in norm. Since $P^\varepsilon(x) = \pi_1 \cdot (F_T^\varepsilon(x, 0))$, the result follows. ∎

Remarks. (a) Norm convergence of evolution operators in general is not to be expected, even for linear operators. It is true here because the unbounded part is fixed and the perturbation is bounded; *cf.* the Trotter-Kato theorem, KATO [1977, p. 502]. (b) These estimates generalize to higher derivatives in the obvious way.

We now prove Lemmas 1 and 2 of Section 2.

Proof of Lemma 1. By (H3b) we have an $x_L(t, \varepsilon)$ satisfying

$$x_L(t, \varepsilon) = e^{t(A+\varepsilon A_1)}\, x_L(0, \varepsilon) + \int_0^t \varepsilon e^{(t-s)(A+\varepsilon A_1)} f(s)\, ds \tag{A-5}$$

and

$$x_L(T, \varepsilon) = x_L(0, \varepsilon),\ x_L(t, \varepsilon) = O(\varepsilon).$$

We seek a curve $x(t, \varepsilon)$ such that

$$\begin{aligned} x(t, \varepsilon) &= e^{t(A+\varepsilon A_1)}\, x(0, \varepsilon) + \int_0^t \varepsilon e^{(t-s)(A+\varepsilon A_1)} f(s)\, ds \\ &\quad + \int_0^t e^{(t-s)(A+\varepsilon A_1)} [B(x(s, \varepsilon)) + \varepsilon g\,(x(s, \varepsilon), s)]\, ds \end{aligned} \tag{A-6}$$

and

$$x(T, \varepsilon) = x(0, \varepsilon),\ x(T, \varepsilon) = O(\varepsilon).$$

We first claim that for ε sufficiently small, $\|x(0, \varepsilon)\| \leqq (\text{Const})\, \varepsilon$ implies that $\|x(t, \varepsilon)\| \leqq (\text{Const})\, \varepsilon$ for $0 \leqq t \leqq T$. To obtain this, subtract (A-5) and (A-6):

$$\begin{aligned} x(t, \varepsilon) - x_L(t, \varepsilon) &= e^{t(A+\varepsilon A_1)} [x(0, \varepsilon) - x_L(0, \varepsilon)] \\ &\quad + \int_0^1 e^{(t-s)(A+\varepsilon A_1)} [B(x(s, \varepsilon)) + \varepsilon g(x(s, \varepsilon), s)]\, ds. \end{aligned} \tag{A-7}$$

Thus

$$\|x(t, \varepsilon) - x_L(t, \varepsilon)\| \leqq (\text{Const})\, \varepsilon + (\text{Const}) \int_0^t \{\|x(s, \varepsilon)\|^3 + \varepsilon\|x(s, \varepsilon)\|^2\}\, ds. \tag{A-8}$$

The estimates on B and g are valid since for ε small enough the solutions will remain in a neighborhood of 0 for $0 \leqq t \leqq T$. From (A-8) we obtain an estimate of the form

$$\|x(t, \varepsilon)\| \leqq (\text{Const})\, \varepsilon + \text{Const} \int_0^t \|x(s, \varepsilon)\|\, ds$$

by using $\|x_L(t, \varepsilon)\| \leqq (\text{Const})\ \varepsilon$ and dropping the cube and square. Thus, by Gronwall's inequality,

$$\|x(t, \varepsilon)\| \leqq (\text{Const})\, \varepsilon \tag{A-9}$$

as desired.

Next, let B_ε be the ball of radius ε about $x_L(0, \varepsilon) = x_L(T, \varepsilon)$. Consider the map

$$P^\varepsilon : B_\varepsilon \to X; \quad x(0, \varepsilon) \mapsto x(T, \varepsilon).$$

We seek a fixed point of P^ε.

From (A-7) note that $x(0, \varepsilon)$ is a fixed point of P^{ε} if and only if it is a fixed point of the map

$$\mathscr{F}_{\varepsilon}: B_{\varepsilon} \to X,$$

$$\mathscr{F}_{\varepsilon}(x(0, \varepsilon)) = x_L(0, \varepsilon) + L_{\varepsilon}^{-1} \int_0^T e^{(T-s)(A+\varepsilon A_1)} [B(x(s, \varepsilon)) + \varepsilon g(x(s, \varepsilon), s)]\, ds, \quad \text{(A-10)}$$

where

$$L_{\varepsilon} = (Id - e^{T(a+\varepsilon A_1)}).$$

Claim 1. *For ε small, $\mathscr{F}_{\varepsilon}$ maps B_{ε} to itself.*

Proof. From (9), (A-9) and (H-5),

$$\|\mathscr{F}_{\varepsilon}(x(0, \varepsilon)) - x_L(0, \varepsilon)\| \leqq \frac{\text{Const}}{\varepsilon} \int_0^T (\text{Const}) [\|x(s, \varepsilon)\|^3 + \varepsilon \|x(s, \varepsilon)\|^2]\, ds$$
$$\leqq (\text{Const})\, \varepsilon^2.$$

This is less than ε for ε sufficiently small.

Claim 2. *$\mathscr{F}_{\varepsilon}$ is a contraction*; i.e. *has Lipschitz constant* <1.

Proof. Indeed, the derivative of $\mathscr{F}_{\varepsilon}$ is

$$D\mathscr{F}_{\varepsilon}(x(0, \varepsilon)) = L_{\varepsilon}^{-1} \int_0^1 e^{(T-s)(A+\varepsilon A_1)} [DB(x(s, \varepsilon)) \circ DF_{s,0}^{\varepsilon}(x(0, \varepsilon))$$
$$+ \varepsilon D_x g(x(s, \varepsilon), s) \circ DF_{s,0}^{\varepsilon}(x(0, \varepsilon), s)]\, ds.$$

Estimating as above,

$$\|D\mathscr{F}_{\varepsilon}(x(0, \varepsilon))\| \leqq \frac{(\text{Const})}{\varepsilon} \cdot \int_0^T (\text{const}) \cdot \varepsilon^2\, ds \leqq (\text{Const})\, \varepsilon$$

so if ε is small enough, this is less than 1. ∎

Thus $\mathscr{F}_{\varepsilon}$ has a unique fixed point in B_{ε}, so lemma 1 is proved. ∎

Proof of Lemma 2. The Poincaré map P^{ε} is the map $x(0, \varepsilon) \mapsto x(T, \varepsilon)$ determined by equation (A-6). Thus $DP^{\varepsilon}(x(0, \varepsilon)) = DP^{\varepsilon}(p_{\varepsilon})$ maps $v(0)$ to $v(T)$, determined by

$$v(T) = e^{(A+\varepsilon A_1)T} v(0) + \int_0^T e^{(A+\varepsilon A_1)(T-s)} [DB(x(s, \varepsilon)) \cdot DF_{s,0}^{\varepsilon}(x(0, \varepsilon)) \cdot v(s)$$
$$+ \varepsilon Dg(x(s, \varepsilon)) \cdot DF_{s,0}^{\varepsilon}(x(0, \varepsilon)) \cdot v(s)]\, ds.$$

By lemma 1, $x(s, \varepsilon) = O(\varepsilon)$ and B is cubic, so $DB(x(s, \varepsilon)) = O(\varepsilon^2)$; also $\varepsilon Dg(x(s, \varepsilon)) = O(\varepsilon^2)$. Thus for ε small, $DP^{\varepsilon}(p_{\varepsilon}) - e^{(A+\varepsilon A_1)T} = O(\varepsilon^2)$ where the $O(\varepsilon^2)$ estimate is in norm. Now $e^{(A+\varepsilon A_1)T}$ has spectrum shifted toward the origin by an amount $O(\varepsilon)$ and so it follows by perturbation of spectra that $DP^{\varepsilon}(p_{\varepsilon})$ has its spectrum shifted by $O(\varepsilon)$ in the same direction as well. (See KATO [1977], Chapter 4, § 3.) ∎

Finally we make some remarks on why the flow is global for the beam example. First of all, the fact that A generates a group in X follows the same pattern as the proposition 2.4 of HOLMES & MARSDEN [1978], so is omitted. Secondly, B and f_1 are smooth maps since multiplication $H^1 \times H^1 \to H^1$ is continuous and bilinear. Moreover it is clear that B and f_1 have bounded derivatives on bounded sets. Thus hypotheses 1(a) and (b) hold. To prove (c) we use energy functionals. To begin we consider the unforced case.

Proposition A.4. *Consider equation* (1) *with* $f = 0$ *and* $\delta > 0$. *Then its flow* F_t^ε *on* X *is globally defined. If* $\Gamma < \pi^2$, *and* $\varepsilon > 0$, *then* $(0, 0)$ *is stable: i.e. for any* $x \in X$,

$$\operatorname*{limit}_{t \to +\infty} F_t^\varepsilon(x) = (0, 0).$$

Proof. Consider the energy

$$H(w, \dot{w}) = \frac{1}{2} |\dot{w}|^2 - \frac{\Gamma}{2} |w'|^2 + \frac{1}{2} |w''|^2 + \frac{\varkappa}{4} |w'|^4$$

where $|w|^2 = \int_0^1 |w(\zeta, t)|^2 \, d\zeta$ is the square of the L^2 norm. We compute:

$$\frac{d}{dt} H(w, \dot{w}) = -\varepsilon \, \delta \, |\dot{w}|^2 \leqq 0 .$$

Also, we have the elementary estimate

$$H(w, \dot{w}) \geqq \begin{cases} \frac{1}{2} \|(w, \dot{w})\|_X^2 & \text{if } |w'|^2 \geqq \Gamma/\varkappa, \\ -\Gamma^2/2\varkappa & \text{if } |w'|^2 \leqq \Gamma/\varkappa. \end{cases}$$

In particular, $\|(w, \dot{w})\|_X$ is *a priori* bounded for all t, so the flow is global in time from the local existence theory.

Notice that H strictly decreases along any trajectory if $\varepsilon\delta > 0$. (Thus there can be no closed orbits.) Along any trajectory $H(x(t))$ decreases, so it converges to a limit, say H_∞, as $t \to +\infty$. From

$$H(x(s)) - H(x(t)) = -\int_s^t \frac{d}{dt} H(x(t)) \, dt = \varepsilon\delta \int_s^t |\dot{w}(\tau)|^2 \, d\tau ,$$

we see that $\dot{w}$ satisfies a Cauchy condition, so it converges in L^2. If $\Gamma < \pi^2$, then the estimate $|w''|^2 \geqq \pi^2 \, |w'|^2$ shows that

$$H(w, \dot{w}) \geqq \frac{1}{2} |\dot{w}|^2 + \frac{(\pi^2 - \Gamma)}{2} |w'|^2 \geqq 0 .$$

Now since H is strictly decreasing, and $H \geqq 0$, $\dot{w}$ converges to 0 in L^2. Also H must converge to 0 so from the above inequality, $w' \to 0$ in L^2. From the original expression for H, $w \to 0$ in H^2. ▮

For $\Gamma > \pi^2$, $F_t^\varepsilon(x)$ will generally converge to one of the stable fixed points.

Proposition A.5. *The flow of* (1) *on* $X \times S^1$ *is global in time for any* $\varepsilon > 0$, $\delta > 0$.

Proof. The same energy function has

$$\frac{d}{dt} H(w, \dot{w}) = -\varepsilon\delta \, |\dot{w}|^2 + (\varepsilon \cos \omega t) \left(\int_0^1 f(\zeta) \, \dot{w}(\zeta, b) \, d\zeta \right),$$

which decreases for $|\dot{w}|^2$ large (by the Schwartz inequality) and so $\|(w, \dot{w})\|_X^2$ is again bounded uniformly for all time. ▮

This completes the details of hypothesis 1. (Similar results were obtained for a slightly different equation in HOLMES [1979a].) We have already checked hypotheses 2–5.

Appendix B. The Birkhoff-Smale Homoclinic Theorem in Infinite Dimensions

The goal of this appendix is to outline the proof of theorem 2 for a C^∞ diffeomorphism $P: X \to X$ where X is a Banach space. Let $p \in X$ be a fixed point of P that is hyperbolic; specifically assume that $\sigma(DP(p))$ is the union of two compact sets not meeting the unit circle. In fact, assume the piece exterior to the unit circle is a single point. Let $W^u(p)$ and $W^s(p)$ be the corresponding unstable and stable manifolds, so $W^u(p)$ is one dimensional and $W^s(p)$ has codimension one. Assume $q \in W^u(p) \cap W^s(p)$ is a transverse homoclinic point. We wish to show that there is an integer N such that P^N has an invariant set Λ such that P^N restricted to Λ is conjugate to a shift on two symbols.

We shall only outline the argument since many of the details are similar to the two dimensional case. The plan is that due to CONLEY & MOSER. The reader wishing to reconstruct all the details should consult MOSER [1973], Chapter III. (Some notes of DAVID RAND were also helpful.)

Abstract Shoes

Let Z be a Banach space and let $Q \subset Z \times \mathbb{R}$ denote the unit box:

$$Q = \{(x, y) \in Z \times \mathbb{R} \mid \|x\| \leqq 1, 0 \leqq y \leqq 1\}.$$

Fix a number μ, $0 < \mu < 1$. A map $u: B = \{x \in Z \mid \|x\| \leqq 1\} \to \mathbb{R}$ is called *horizontal* (or μ-horizontal) if $0 \leqq u(x) \leqq 1$ for $x \in B$ and if $|u(x_1) - u(x_2)| \leqq \mu \|x_1 - x_2\|$ for $x_1, x_2 \in B$. Let $\mathscr{H}$ (or $\mathscr{H}_\mu$) denote the set of such maps. For u_1 and u_2 in $\mathscr{H}$ satisfying $0 \leqq u_1(x) < u_2(x) \leqq 1$ let

$$H = \{(x, y) \in Q \mid u_1(x) < y < u_2(x)\}$$

and call such a set a *horizontal strip*. Its *diameter* is

$$d(H) = \sup_{x \in B} (u_2(x) - u_1(x)).$$

A *vertical* (or μ-vertical) curve is a map

$$v: [0, 1] \to B$$

such that $\|v(y_1) - v(y_2)\| \leqq \mu |y_1 - y_2|$ for $y_1, y_2 \in [0, 1]$. The set of all such curves is denoted $\mathscr{V}$ (or $\mathscr{V}_\mu$). A *vertical strip* is the closure of an open set V in Q bounded by open subsets of $B \times \{0\}$ and $B \times \{1\}$ and by vertical curves joining their respective boundaries. We set

$$d(V) = \sup_{\alpha, \beta, y} \|v_\alpha(y) - v_\beta(y)\|,$$

where $\{v_\alpha\}$ denotes the vertical curves comprising the sides of V as described above. See Figure 7.

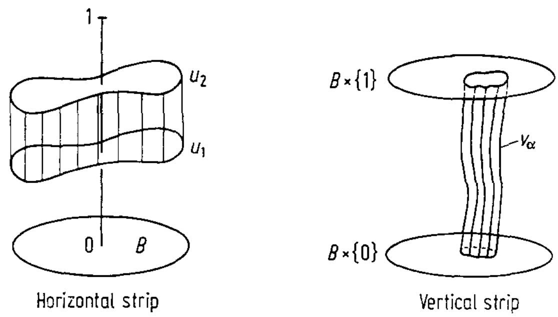

Fig. 7

Lemma B.1. *If $H^1 \supset H^2 \supset \dots$ is a nested sequence of μ-horizontal strips and if $d(H^k) \to 0$ as $k \to \infty$, then $\bigcap_{k=1}^{\infty} H^k$ is the graph of a μ-horizontal map.*

Proof. Let u_1^k and u_2^k be the two functions defining H^k. Then $u_1^k(x) - u_2^k(x) \to 0$. But by the nesting assumption, $u_1^k(x)$ is increasing and $u_2^k(x)$ is decreasing, so they converge to, say, $u(x)$. Letting $k \to \infty$ in $|u_2^k(x_1) - u_2^k(x_2)| \leqq \mu \|x_1 - x_2\|$ gives the μ-horizontality of u. ∎

There is an analogous lemma for vertical strips.

Lemma B.2. *A μ-horizontal graph and a μ-vertical curve intersect in exactly one point.*

Proof. Let u and v describe the μ-horizontal graph and μ-vertical curve respectively. Let (x, y) lie on their graphs: Then $x = v(y)$ and $y = u(x)$, so $x - v(u(x)) = 0$, and $y - u(v(y)) = 0$. Let $g(y) = y - u(v(y))$ so $g: [0, 1] \to \mathbb{R}$. Now if $0 \leqq y_1 < y_2 \leqq a$, then

$$|u(v(y_2)) - u(v(y_1))| \leqq \mu\, |v(y_2) - v(y_1)| \leqq \mu^2\, |y_2 - y_1|$$

and so as $\mu^2 < 1$,

$$g(y_2) - g(y_1) = y_2 - y_1 + [u(v(y_2)) - u(v(y_1))] > 0.$$

Thus g is strictly increasing. However, $g(0) \leqq 0$ and $g(1) \geqq 0$ so g has exactly one zero. ∎

By lemma B.2, there is a well-defined map

$$\chi: \mathscr{H}_\mu \times \mathscr{V}_\mu \to Q.$$

Define norms on $\mathscr{H}_\mu \times \mathscr{V}_\mu$ and Q by

$$\|(u, v)\| = \sup_{x \in B} |u(x)| + \sup_{y \in [0,1]} \|v(y)\|,$$

$$\|(x, y)\| = \|x\| + |y|.$$

Lemma B.3. *χ is Lipschitz continuous with Lipschitz constant* $(1 - \mu)^{-1}$.
This is straightforward (see Moser [1973], p. 71).

Definition B.4. Fix numbers μ and ν satisfying $0 < \mu < 1$ and $0 < \nu < 1$. Let H_1 and H_2 be two disjoint μ-horizontal strips and V_1 and V_2 be two disjoint μ-vertical strips. Let

$$\phi: H_1 \cup H_2 \to V_1 \cup V_2$$

be a homeomorphism satisfying

(i) $\phi(H_i) = V_i$, $i = 1, 2$;

(ii) horizontal (respectively vertical) boundaries of H_i are mapped onto the horizontal (respectively vertical) boundaries of V_i, $i = 1, 2$; and

(iii) if H is a μ-horizontal strip in $H_1 \cup H_2$, then for $i = 1$ or 2

$$\phi^{-1}(H) \cap H_i = \tilde{H}$$

is a non-empty μ-horizontal strip satisfying

$$d(\tilde{H}) \leqq \nu\, d(H).$$

Similarly, if V is a μ-vertical strip in $V_1 \cup V_2$, then for $i = 1$ or 2

$$\phi(V) \cap V_i = \tilde{V}$$

is a non-empty μ-vertical strip satisfying

$$d(\tilde{V}) \leqq \nu\, d(V).$$

Such a homeomorphism ϕ is called a *shoe*. (One can allow i to range over a general set A; we chose $i = 1, 2$ for the present context.)

The Smale horseshoe is a basic example of a shoe; *cf.* SMALE [1967].

Theorem B.5. *Let ϕ be a shoe. Let $\Lambda = \bigcap_{-\infty<n<\infty} \phi^n(V_1 \cup V_2)$. Then Λ is a non-empty invariant set for ϕ and $\phi \mid \Lambda$ is conjugate to the shift automorphism on two symbols.*

In particular, it follows that ϕ has infinitely many periodic points in Λ and there is a point $p \in \Lambda$ whose orbit is dense in Λ.

Proof of B.5. Let $i_0, i_{-1}, \ldots, i_{-n}$ be a sequence of $n + 1$ 0's or 1's. Define $H_{i_0,\ldots,i_{-n}}$ in terms of sequences of length n inductively by

$$H_{i_0,i_{-1},\ldots,i_{-n}} = H_{i_0} \cap \phi^{-1}(H_{i_{-1},\ldots,i_{-n}}).$$

Thus $H_{i_0,i_{-1},\ldots,i_{-n}}$ is μ-horizontal of width $\leqq \nu^n \cdot$ constant. If $i = i_0, i_{-1}, i_{-2}, \ldots$ is a (one sided) sequence of 0's or 1's, then

$$H_{i-} = \bigcap_{n=0}^{\infty} H_{i_0,i_{-1},\ldots,i_{-n}}$$

is the graph of a μ-horizontal map by lemma B.1. Similarly if $i+ = i_1, i_2, \ldots$, define $V_{i_1,\ldots,i_n}$ by

$$V_{i_1,\ldots,i_n} = V_{i_1} \cap \phi(V_{i_2,\ldots,i_n})$$

and obtain the μ-vertical curve

$$V_{i+} = \bigcap_{n=1}^{\infty} V_{i_1,\ldots,i_n}.$$

If $i = \ldots i_{-2}, i_{-1}, i_0, i_1, i_2, \ldots$ is a bi-infinite sequence, then by lemma B.2 V_{i+} and H_{i-} meet in precisely one point, denoted $\tau(i)$. [Lemma B.3 implies τ is continuous from $\{0, 1\}^Z$ to Q.] Thus $\tau:\{0, 1\}^Z \to \Lambda$ and $\tau^{-1} \circ \phi \circ \tau$ is shift automorphism by construction. ∎

Sector Bundles

Assume that ϕ in definition B.4 is C^1. We now give a condition that implies property (iii) in that definition.

Definition B.6. Let $0 < \lambda < 1$ and for $(x, y) \in Q$, let

$$S_\lambda^u(x, y) = \{(\xi, \eta) \in Z \times \mathbb{R} \mid \|\xi\| \leqq \lambda |\eta|\}$$

and

$$S_\lambda^s(x, y) = \{(\xi, \eta) \in Z \times \mathbb{R} \mid |\eta| \leqq \lambda \|\xi\|\}.$$

If R is a closed subset of Q, we call

$$S_\lambda^u = \bigcup_{(x,y)\in R} S^u(x, y) \text{ and } S_\lambda^s = \bigcup_{(x,y)\in R} S_\lambda^s(x, y)$$

the unstable and stable *λ-sector bundles over R.*

Consider the following condition on ϕ:

(iii)′ There exists a μ, $0 < \mu < \dfrac{1}{2}$ such that

(a) the unstable (respectively stable) μ-sector bundle over $H_1 \cup H_2$ (respectively $V_1 \cup V_2$) is mapped to itself by $d\phi$ (respectively $d\phi^{-1}$) and

(b) if $(x_0, y_0) \in H_i$ for $i = 1$ or 2 and $(\xi_0, \eta_0) \in S_\mu^u(x_0, y_0)$, then $|\eta_1| \geqq \mu^{-1} |\eta_0|$ where $(\xi_1, \eta_1) = d\phi(x_0, y_0) \cdot (\xi_0, \eta_0)$, and if $(x_1, y_1) \in V_i$ for $i = 1$ or 2 and $(\xi_1, \eta_1) \in S_\mu^s(x_1, y_1)$, then $\|\xi_0\| \geqq \mu^{-1} \|\xi_1\|$.

Condition (b) says that vectors in S_μ^u are vertically expanded by a factor μ^{-1} by $d\phi$ while vectors in S_μ^s are horizontally contracted by a factor μ^{-1}.

Proposition B.7. *Let ϕ satisfy* (i) *and* (ii) *of definition* B.4 *and be C^1. Then* (iii)′ *implies* (iii).

Proof. Let γ be the graph of a μ-horizontal map in H_i and let $\gamma' = \gamma \cap V_j$ where i, j are 1 or 2. Consider $\phi^{-1}(\gamma')$; we claim it is the graph of a μ-horizontal map. See Figure 8. Because the boundaries of H_j and V_j correspond under ϕ, $\phi^{-1}(\gamma')$ covers all of B. Also, if $v = (\xi, \eta)$ is tangent to γ', then $|\eta| \leqq \mu \|\xi\|$ because γ' is μ-horizontal. Thus by (iii)′ (a), the same is true of vectors tangent to $\phi^{-1}(\gamma')$. Thus, by the mean value theorem, if (x_1, y_1) and (x_2, y_2) lie on $\phi^{-1}(\gamma')$, then $\|y_1 - y_2\| \leqq \mu \|x_1 - x_2\|$. Thus $\phi^{-1}(\gamma')$ is the graph of a μ-horizontal map. This implies the μ-horizontality of $\tilde{H}$ in B.4(iii).

It remains to check the condition on the diameters. Let p_1 and p_2 be points on different horizontal boundaries of $\tilde{H} = \phi^{-1}(H) \cap H_i = \phi^{-1}(H \cap V_i)$ with the same x-component. The image of the μ-vertical segment joining p_1 and p_2 is a μ vertical curve. Let $Z_1 = \phi(p_1)$ and $Z_2 = \phi(p_2)$. By B.3, $\|Z_1 - Z_2\| \leqq (1 - \mu)^{-1} d(H)$.

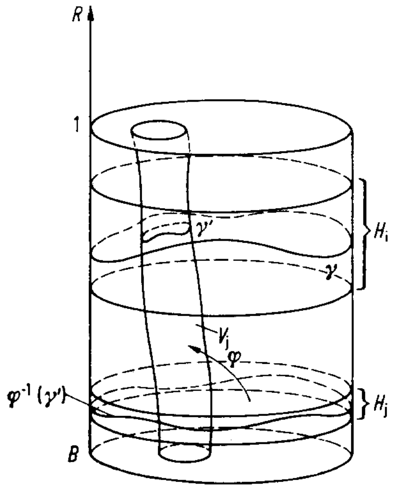

Fig. 8

Let $p(t) = (1 - t)\,p_1 + p_2$ and $Z(t) = \phi(p(t)) = (x(t), y(t))$. Since $p(t)$ is vertical, (iii)′(b) gives $|\dot{y}| \geqq \mu^{-1}\,|\dot{p}| = \|p_1 - p_2\| > 0$. Thus $\dot{y}$ does not change sign, so

$$\|p_1 - p_2\| \leqq \int_0^1 |\dot{y}|\,dt = \mu\,|y(1) - y(0)| \leqq \mu\,\|Z_1 - Z_2\| \leqq \mu(1 - \mu)^{-1}\,d(H).$$

This verifies $d(\tilde{H}) \leqq \nu\,d(H)$ with $\nu = \mu(1 - \mu)^{-1}$; since $0 < \mu < 1/2$ we have $0 < \nu < 1$. The assertions for vertical strips are similar. ▮

We remark that if ϕ is sufficiently close to a volume preserving map (which happens in our example for ε small), then Λ is actually hyperbolic for ϕ.

Homoclinic Points

Now we apply the machinery just developed to prove theorem 2. Introduce local coordinates (x, y) near p, for x in a neighborhood of 0 in a Banach space Z and y in a neighborhood of 0 in $\mathbb{R}$ such that $W^s(p)$ is the codimension 1 submanifold $y = 0$ and $W^u(p)$ is the curve $x = 0$.

Write

$$P(x, y) = (U(x, y), V(x, y)).$$

Then invariance of $W^s(p)$ and $W^u(p)$ implies $U(x, 0) = 0$ and $V(0, y) = 0$. Thus, at $p = (0, 0)$,

$$DP(0, 0) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A: Z \to Z$ and $B \in \mathbb{R}$. The spectral hypotheses imply that we can define a norm on Z such that $\|A\| < 1$ and can assume $B > 1$. (See, for example, MARSDEN & MCCRACKEN [1976], Sec. 2A.) Let $Q_a = \{(x, y) \mid \|x\| \leqq a \text{ and } |y| \leqq a\}$.

Near q points can similarly be coordinatized by pairs of points in $W^s(p)$ and $W^u(p)$ which can in turn be assumed to be linear near q. Relative to these coordinates, let R be defined in a manner like Q_a but on one side of $W^s(p)$ (which makes sense as $W^s(p)$ has codimension 1); *i.e.* relative to coordinates near q,

$$R = \{(x, y) \mid \|x\| < b, \|y\| < b, y \geqq 0\}.$$

We can choose b sufficiently small so that for some integers $l, m > 0$

$$A = P^{-m}(R) \subset Q_a \text{ and } B = P^l(R) \subset Q_a;$$

l and m are large enough so that $P^l(q) \in Q_a$ and $P^{-m}(q) \in Q_a$. We shall locate the desired set A inside B (or, equivalently, inside A). See Figure 9. The rectangle R is chosen small enough and l is chosen large enough so the sides of A and B are C^r close to the coordinate planes.

One now proceeds in several steps; fix a μ, $0 < \mu < \frac{1}{2}$.

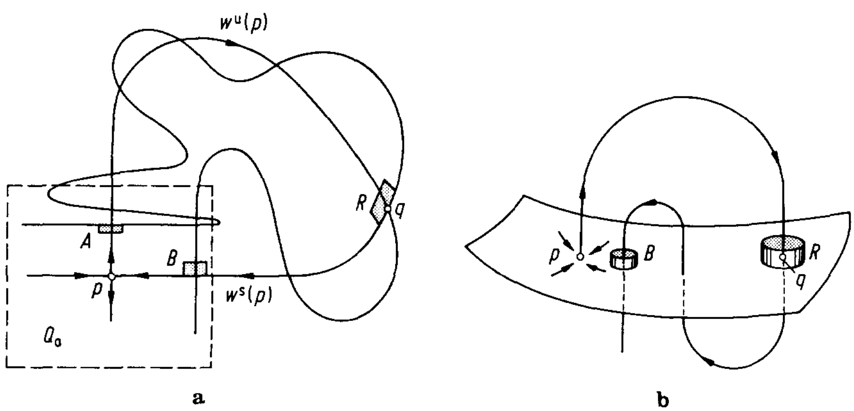

Fig. 9a. $\mathbb{R}^2$ Fig. 9b. $\mathbb{R}^3$

Lemma B.8. *There are disjoint horizontal strips H_1 and H_2 in B and an integer n such that $U_1 = P^n(H_1)$ and $U_2 = P^n(H_2)$ are disjoint vertical strips crossing A from one side to the other.*

The idea is simply to take horizontal strips in B and map them repeatedly forward (keeping track only of things in Q_a). The spectral hypothesis implies that horizontal distances shrink and vertical distances stretch. The details in Moser [1973, p. 186–188] are readily adapted to the present context.

Lemma B.9. *$V_1 = P^{m+l}(U_1 \cap A)$ and $V_2 = P^{m+l}(U_2 \cap A)$ are disjoint vertical strips in B.*

Here we follow our vertical strips in A out to R and back to B. The verticality is maintained if R is small enough. Shrink B horizontally so that P^n maps H_i homeomorphically onto V_i, $i = 1, 2$.

Lemma B.10. *Let $\phi = P^N$ where $N > n + m + l$, is sufficiently large, restricted to $H_1 \cup H_2$. Then ϕ satisfies condition* (i) *and* (ii) *of* B.4 *and* (iii)′ *following* B.6.

This is now straightforward using the given spectral conditions on $DP(p)$. Again, see MOSER [1973] for details. Thus we have set up a map ϕ to which Theorem B.5 applies, producing the desired set Λ.

References

R. ABRAHAM & J. MARSDEN [1978] "Foundations of Mechanics," 2nd Edition, Addison-Wesley.

V. ARNOLD [1964] Instability of dynamical systems with several degrees of freedom. Dokl. Akad. Nauk. SSSR **156**, 9–12.

V. ARNOLD [1966] Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynomique des fluides parfaits. Ann. Inst. Fourier, Grenoble **16**, 319–361.

J. CARR [1980] Application of Center Manifolds, Springer Applied Math. Sciences (to appear).

J. CARR & M. Z. H. MALHARDEEN [1979] Beck's problem, SIAM J. Appl. Math. **37**, 261–262.

P. CHERNOFF & J. MARSDEN [1974] "Properties of Infinite Dimensional Hamiltonian Systems," Springer Lecture Notes in Math., no. 425.

S. N. CHOW, J. HALE & J. MALLET-PARET [1980] An example of bifurcation to homoclinic orbits, J. Diff. Eqn's. **37**, 351–373.

C. CONLEY & J. SMOLLER [1974] On the structure of magneto hydrodynamic shock waves, Comm. Pure Appl. Math. **27**, 367–375, J. Math. of Pures et. Appl. **54** (1975) 429–444.

D. EBIN & J. MARSDEN [1970] Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math. **92**, 102–163.

J. GOLLUB [1980] The onset of turbulence: convection, surface waves, and oscillators, Springer Lecture Notes in Physics.

J. GUCKENHEIMER [1979] On a codimension two bifurcation (preprint).

O. GUREL & O. RÖSSLER (eds) [1979] "Bifurcation theory and applications in scientific disciplines," Ann. of N.Y. Acad. Sciences, vol. **316**.

B. HASSARD [1980] Computation of invariant manifolds, in Proc. "New Approaches to Nonlinear Problems in Dynamics", ed. P. HOLMES, SIAM publications.

E. HILLE & R. PHILLIPS [1957] "Functional Analysis and Semigroups", A.M.S. Colloq. Publ.

M. HIRSCH, C. PUGH & M. SHUB [1977] "Invariant Manifolds", Springer Lecture Notes in Math. no. 583.

P. HOLMES [1979a] Global bifurcations and chaos in the forced oscillations of buckled structures, Proc. 1978 IEEE Conf. on Decision and Control, San Diego, CA, 181–185.

P. HOLMES [1979b] A nonlinear oscillator with a strange attractor, Phil. Trans. Roy. Soc. A **292**, 419–448.

P. HOLMES [1980a] Averaging and chaotic motions in forced oscillations, SIAM J. on Appl. Math. **38**, 65–80.

P. HOLMES [1980b] Space and time-periodic perturbations of the sine-Gordon equation (preprint).

P. HOLMES & J. MARSDEN [1978] Bifurcation to divergence and flutter in flow induced oscillations; an infinite dimensional analysis, Automatica **14**, 367–384.

P. HOLMES & J. MARSDEN [1979] Qualitative techniques for bifurcation analysis of complex systems, in GUREL & RÖSSLER [1979], 608–622.

T. KATO [1977] "Perturbation Theory for Linear Operators" 2nd Ed., Springer.

N. KOPELL & L. N. HOWARD [1976] Bifurcations and trajectories joining critical points, Adv. Math. **18**, 306–358.

M. LEVI, F. C. HOPPENSTADT & W. L. MIRANKER [1978] Dynamics of the Josephson junction, Quart. of Appl. Math. **36**, 167–198.

V. K. MELNIKOV [1963] On the stability of the center for time periodic perturbations, Trans. Moscow Math. Soc. **12**, 1–57.

F. MOON & P. HOLMES [1979] A magneto-elastic strange attractor, J. Sound and Vibrations **65**, 275–296.

J. MOSER [1973] "Stable and Random Motions in Dynamical Systems," Ann. of Math. Studies no. **77**, Princeton Univ. Press.

S. NEWHOUSE [1974] Diffeomorphisms with infinitely many sinks, Topology **12**, 9–18.

S. NEWHOUSE [1979] The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms, Publ. I.H.E.S. **50**, 100–151.

J. RAUCH [1979] Qualitative behavior of dissipative wave equations. Arch. Rational Mech. An. **62**, 77–91.

J. ROBBIN [1971] A structural stability theorem, Ann. of Math. **94**, 447–493.

I. SEGAL [1962] Nonlinear Semigroups, Ann. of Math. **78**, 334–362.

Y. SHIZUTA [1980] On the classical solutions of the Boltzmann equations, Comm. Pure. Appl. Math. (to appear).

S. SMALE [1963] Diffeomorphisms with many periodic points, in "Differential and Combinatorial Topology" (ed. S. S. CAIRNS), Princeton Univ. Press, 63–80.

S. SMALE [1967] Differentiable dynamical systems, Bull. Am. Math. Soc. **73**, 747–817.

W. Y. TSENG & J. DUGUNDJI [1971] Nonlinear vibrations of a buckled beam under harmonic excitation. J. Appl. Mech. **38**, 467–476.

I. VIDAV [1970] Spectra of perturbed semigroups, J. Math. An. Appl. **30**, 264–279.

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(Received August 25, 1980)