On the foundations of Landau's theory of superfluid helium

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Dedicated to J. L. Ericksen on the occasion of his 60th birthday

1. Introduction

Superfluid helium exhibits some of the most fascinating effects known to continuum physics. Standard texts such as WILKS [1] and PUTTERMAN [2] may be consulted for comprehensive surveys. The generally accepted theory describing the most fundamental aspects of the behaviour of this liquid is that due to LANDAU [3]. At the foundation of this theory is a notional mixture of two ingredients referred to as normal fluid and superfluid. As LANDAU has emphasized, this mixture is not of the conventional kind. Since all the atoms present are helium atoms, distinct constituents cannot exist independently of each other. Some motivation for the mixture idea may be found from quantum statistical mechanics in terms of the Bose-Einstein condensate, but we consider here only those approaches which may be directly related to the principles of continuum mechanics.

A macroscopic thermodynamic framework for the LANDAU theory has been presented by KHALATNIKOV [4]. In this work the densities ϱ_n, ϱ_s and the velocities $\boldsymbol{v}^n$, $\boldsymbol{v}^s$ of the normal fluid and superfluid are introduced in a plausible manner, but the plausibility stems in some measure from conventional mixture ideas. Aware of the dangers of unwarranted assumptions arising from this source, PUTTERMAN [2] gives KHALATNIKOV'S treatment a different emphasis. He shows how the postulates of the theory may be stated using conventional single-fluid variables with the addition of just one vector field which he takes to be $\boldsymbol{v}^s$. Having developed the theory he is then able to *define* ϱ_n, ϱ_s, $\boldsymbol{v}^n$ using thermodynamic functions and the relations

$$\varrho = \varrho_n + \varrho_s, \quad \varrho\boldsymbol{v} = \varrho_n\boldsymbol{v}^n + \varrho_s\boldsymbol{v}^s, \tag{1.1}$$

where ϱ, $\varrho\boldsymbol{v}$ are the density and momentum of the whole fluid. By this means he shows that the LANDAU theory is the *only* theory of a certain class, and the mixture model upon which it is based is not essential to the derivation of the equations; the model may be viewed only as a convenient interpretation of some of the variables. Although this result is reassuring, some investigation of the class of materials studied by PUTTERMAN is desirable.

Now in modern theories of continuum mechanics the classification of a material is based upon the nature of its constitutive equations (*e.g.* functional or rate-type) together with the variables appearing (*e.g.* deformation gradients or velocity gradients). Our objective here is to examine a general class of materials, defined in this sense, of which superfluid helium is a special case. We follow PUTTERMAN to the extent that we use conventional single continuum variables together with one generalised velocity $\boldsymbol{v}^s$, and we postulate a rate-type equation for this velocity, relating the superfluid acceleration to a certain 'driving force'. However, we do not restrict this driving force to be the gradient of a scalar potential as PUTTERMAN does. The use of the scalar potential leads, subject to the conditions of Kelvin's theorem, to the condition

$$\operatorname{curl} \boldsymbol{v}^s = \boldsymbol{0}, \tag{1.2}$$

and gives the LANDAU theory a deceptively simple appearance. When the whole set of independent variables required for such a driving force is examined a rather complex situation emerges. Although LANDAU regarded (1.2) as essential, we cannot raise this condition to the status of a constraint because it is not invariant.

We present here a set of constitutive equations, of sufficient complexity, and containing a sufficient number of independent variables, to encompass the non-dissipative LANDAU theory in a logical and systematic manner. We consider the restrictions imposed upon these equations by the Clausius-Duhem inequality using techniques similar to those of COLEMAN & NOLL [5]. These restrictions turn out to be severe, and yet leave the theory a little more general than LANDAU'S.

This work extends that of ATKIN & FOX [6, 7]. In these papers we took the constitutive equations for the conventional variables to be in the form directly suggested by the LANDAU equations, and we then proceeded to examine the most general superfluid equation of motion consistent with entropy balance. The resulting non-dissipative theory was shown to reduce to LANDAU'S if certain relations between parameters were assumed. Further discussion of related continuum approaches to superfluid helium will also be found in these papers.

2. Balance laws and entropy inequality

Let ϱ be the mass density of the liquid, v_i, b_i the components of velocity and body force per unit mass, and σ_{ij} the cartesian components of the stress tensor. Then the equations of mass and momentum balance have their usual forms

$$\frac{D\varrho}{Dt} + \varrho v_{k,k} = 0, \tag{2.1}$$

$$\varrho \frac{Dv_i}{Dt} = \sigma_{ji,j} + \varrho b_i, \qquad \sigma_{ij} = \sigma_{ji}, \tag{2.2}$$

where D/Dt denotes the material time derivative, a comma followed by a suffix denotes partial differentiation with respect to the current cartesian coordinates x_i, repeated suffixes implying summation.

We postulate an energy balance equation which differs slightly from that used for conventional fluids, and the various terms appearing are most easily explained if the equation is first presented in integral form. Suppose that an arbitrary set of material particles of the fluid occupy a region V bounded by a surface Σ in the current configuration at time t. Then we assume

$$\frac{d}{dt}\int_V (\tfrac{1}{2}\varrho v_i v_i + \varrho U)\, dV + \int_\Sigma (q_i + l_i - \sigma_{ij} v_j)\, n_i\, d\Sigma = \int_V \varrho(b_i v_i + r)\, dV. \tag{2.3}$$

In this equation $\frac{1}{2}\varrho v_i v_i + \varrho U$ denotes the total energy density per unit volume. (We avoid the use of the term 'internal energy' here because of our later assumptions that U depends on a velocity field.) The flux of energy across Σ arises from the rate of working of surface traction together with the flux of $\boldsymbol{q} + \boldsymbol{l}$. Here $\boldsymbol{q}$ denotes the heat flux and the vector $\boldsymbol{l}$ allows for an additional mechanical energy flux. The introduction of $\boldsymbol{l}$ is mathematically equivalent to the assumption, developed by Müller [8], that the entropy flux may differ from $\boldsymbol{q}/T$, where T is the absolute temperature. This equivalence has been pointed out by Gurtin & Vargas [9]. Finally, r denotes the rate of heat supply from the external world.

We use the Clausius-Duhem inequality in the conventional form

$$\frac{d}{dt}\int_V \varrho S\, dV + \int_\Sigma (q_i/T)\, n_i\, d\Sigma - \int_V \varrho r/T\, dV \geqq 0, \tag{2.4}$$

where S denotes the entropy per unit mass. The point forms of (2.3) and (2.4) may be written

$$\varrho \frac{DU}{Dt} + l_{i,i} + q_{i,i} - \sigma_{ij} D_{ij} = \varrho r, \tag{2.5}$$

$$\varrho T \frac{DS}{Dt} + q_{i,i} - \varrho r - \frac{g_i q_i}{T} \geqq 0, \tag{2.6}$$

where

$$g_i = T_{,i}, \qquad D_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}). \tag{2.7}$$

By use of the Helmholtz free energy $F = U - TS$, (2.5) and (2.6) may be combined to give

$$-\varrho \frac{DF}{Dt} - \varrho S \frac{DT}{Dt} - l_{i,i} + \sigma_{ij} D_{ij} - \frac{g_i q_i}{T} \geqq 0. \tag{2.8}$$

3. Constitutive theory

Many of the novel phenomena observed in superfluid helium are believed to arise from the motion through the fluid of certain microscopic excitations known as rotons and phonons. In phenomenological theories account is taken of this motion by the introduction of an additional vector field. Here we postulate the existence of a velocity field $\boldsymbol{v}^s$ which we later identify with the variable commonly known as the superfluid velocity. We suppose that under a change of frame

$$x_i^* = Q_{ij}(t)\, x_j + c_i(t), \tag{3.1}$$

where $Q(t)$ is an orthogonal tensor, the vector $\boldsymbol{v}^s$ transforms as an absolute velocity:

$$v_i^{s*} = Q_{ij} v_j^s + \Omega_{ij}(x_j^* - c_j) + \dot{c}_i, \tag{3.2}$$

where

$$\Omega_{ij} = \dot{Q}_{ik} Q_{jk}. \tag{3.3}$$

We further postulate that $\boldsymbol{v}^s$ satisfies an equation of motion of the form

$$\frac{D_s v_i^s}{Dt} = f_i + b_i, \tag{3.4}$$

where D_s/Dt denotes the convected time derivative based on the velocity field $\boldsymbol{v}^s$:

$$D_s/Dt = \partial/\partial t + v_i^s\, \partial/\partial x_i. \tag{3.5}$$

The vector $\boldsymbol{f}$ is assumed to be objective, transforming as

$$f_i^* = Q_{ij} f_j \tag{3.6}$$

and is to be determined by a constitutive equation. The appearance of the body force $\boldsymbol{b}$ in (3.4) ensures the correct invariance of the energy equation when the constitutive equation for F contains a dependence on $\boldsymbol{v}^s$. This starting point is similar to, but more general than, that of PUTTERMAN [2], where the driving force $\boldsymbol{f}$ is assumed at the outset to be the gradient of a scalar potential.

Under change of frame (3.1), the velocity $\boldsymbol{v}$ can be shown to have the same transformation property as that postulated for $\boldsymbol{v}^s$ in (3.2). Hence, the vector

$$\boldsymbol{s} = \boldsymbol{v} - \boldsymbol{v}^s \tag{3.7}$$

is objective, transforming according to

$$s_i^* = Q_{ij} s_j. \tag{3.8}$$

We now postulate constitutive equations in which F, S, q_i, l_i, f_i, σ_{ij} depend on ϱ, T, s_i, h_i, g_i, M_{ij}, N_{ij}, where

$$h_i = \varrho_{,i}, \qquad M_{ij} = \tfrac{1}{2}(s_{i,j} + s_{j,i}), \qquad N_{ij} = \tfrac{1}{2}(s_{i,j} - s_{j,i}). \tag{3.9}$$

The inequality (2.8) may now be written

$$-\varrho\left\{\frac{\partial F}{\partial \varrho}\frac{D\varrho}{Dt} + \frac{\partial F}{\partial s_i}\frac{Ds_i}{Dt} + \frac{\partial F}{\partial h_i}\frac{Dh_i}{Dt} + \frac{\partial F}{\partial g_i}\frac{Dg_i}{Dt} + \frac{\partial F}{\partial M_{ij}}\frac{DM_{ij}}{Dt} + \frac{\partial F}{\partial N_{ij}}\frac{DN_{ij}}{Dt}\right\}$$

$$-\varrho\left\{\frac{\partial F}{\partial T} + S\right\}\frac{DT}{Dt} - l_{i,i} + \sigma_{ij} D_{ij} - \frac{q_i g_i}{T} \geqq 0. \tag{3.10}$$

Using (3.5) and (3.7) we can show that

$$\frac{Ds_i}{Dt} = \frac{Dv_i}{Dt} - \frac{D_s v_i^s}{Dt} - s_j v_{i,j} + s_j s_{i,j}, \tag{3.11}$$

and using (2.2) and (3.4) we deduce

$$\frac{Ds_i}{Dt} = -f_i + \varrho^{-1}\sigma_{ji,j} - s_j v_{i,j} + s_j s_{i,j}. \tag{3.12}$$

Consider now thermodynamic processes in which, at an arbitrary point and instant of time, ϱ, T, s_i, v_i and the gradients of these quantities take prescribed values. Then $D\varrho/Dt$ is determined through (2.1), and Ds_i/Dt is determined through (3.12) and the constitutive equations. However, the material time derivatives of h_i, g_i, M_{ij}, N_{ij}, T, appearing in (3.10) may be chosen arbitrarily and independently of ϱ, T, s_i, v_i and their gradients. Since (3.10) must hold for all admissible thermodynamic processes, we deduce

$$\frac{\partial F}{\partial h_i} = \frac{\partial F}{\partial g_i} = 0, \qquad \frac{\partial F}{\partial M_{ij}} = \frac{\partial F}{\partial N_{ij}} = 0, \qquad S = -\frac{\partial F}{\partial T}. \tag{3.13}$$

Since F must be invariant under change of frame, we must have therefore

$$F = F(\varrho, T, s^2), \qquad s^2 = s_i s_i. \tag{3.14}$$

To proceed further with the analysis of (3.10) some simplification of the constitutive equations is desirable. Here we study a particular class of constitutive equations which keeps the algebraic manipulations down to a reasonable level while retaining sufficient generality to include the traditional LANDAU theory as a special case. We consider constitutive equations for q_i, l_i, f_i, σ_{ij} which are linear in the gradients h_i, g_i, M_{ij}, N_{ij}. We restrict all equations to be at most of order two in s_i and its gradients with the exception of that for l_i which is allowed to be of order three. The motivation for the latter exception may be seen from (3.10) where l_i is the only variable not occurring in a product. Bearing in mind these restrictions and the requirements of frame indifference we find that F takes the form

$$F = \phi_0(\varrho, T) + \tfrac{1}{2}\phi_1(\varrho, T)\, s^2, \tag{3.15}$$

and

$$f_i = \alpha_1 s_i + \alpha_2 h_i + \alpha_3 g_i + \alpha_4 s_j M_{ij} + \alpha_5 s_j N_{ij}, \tag{3.16}$$

$$l_i = \beta_1 s_i + \beta_2 h_i + \beta_3 g_i + \beta_4 s_j M_{ij} + \beta_5 s_j N_{ij}, \tag{3.17}$$

$$q_i = \gamma_1 s_i + \gamma_2 h_i + \gamma_3 g_i + \gamma_4 s_j M_{ij} + \gamma_5 s_j N_{ij}, \tag{3.18}$$

$$\sigma_{ij} = \sigma_1 s_i s_j + \sigma_2(h_i s_j + h_j s_i) + \sigma_3(g_i s_j + g_j s_i) + \sigma_4 M_{ij} - p\delta_{ij}. \tag{3.19}$$

The coefficients in (3.16)–(3.19) depend on ϱ, T, s^2, $s_i g_i$, $s_i h_i$, M_{ii}, although in some cases the dependence on these variables is limited by the restrictions mentioned above. Explicit forms for the coefficients are given below. Substituting (3.15)–(3.19) into (3.10) and making use of (2.1), (3.12) and (3.13) we find

$$\begin{aligned}
&\varrho\phi_1 s_i\{\alpha_1 s_i + \alpha_2 h_i + \alpha_3 g_i + \alpha_4 s_j M_{ij}\} - \varrho\phi_1 s_i s_j M_{ij} \\
&\quad - \{\beta_1 s_i + \beta_2 h_i + \beta_3 g_i + \beta_4 s_j M_{ij} + \beta_5 s_j N_{ij}\}_{,i} \\
&\quad - (g_i/T)\{\gamma_1 s_i + \gamma_2 h_i + \gamma_3 g_i + \gamma_4 s_j M_{ij} + \gamma_5 s_j N_{ij}\} \\
&\quad - \phi_1 s_i\{\sigma_1 s_i s_j + \sigma_2(h_i s_j + h_j s_i) + \sigma_3(g_i s_j + g_j s_i) + \sigma_4 M_{ij} - p\delta_{ij}\}_{,j} \\
&\quad + \{(\varrho\phi_1 + \sigma_1)\, s_i s_j + \sigma_2(h_i s_j + h_j s_i) + \sigma_3(g_i s_j + g_j s_i) + \sigma_4 M_{ij} \\
&\quad + \{\varrho^2(\partial F/\partial\varrho) - p)\,\delta_{ij}\}\, D_{ij} \geqq 0. \qquad (3.20)
\end{aligned}$$

Again the inequality must be satisfied by all admissible thermodynamic processes. Considering first those in which D_{ij}, M_{ij}, h_i, g_i, s_i, N_{ij}, $N_{ij,k}$ may be varied arbitrarily and independently, we may deduce

$$\begin{aligned}
\varrho\phi_1 + \sigma_1 = 0, \quad & p = \varrho^2\, \partial F/\partial\varrho, \\
\sigma_2 = \sigma_3 = \sigma_4 = 0, \quad & \beta_5 = 0, \quad \gamma_5 = 0.
\end{aligned} \qquad (3.21)$$

Bearing in mind the restrictions that we have imposed on the constitutive equations, we may write

$$\begin{aligned}
\alpha_1 &= \alpha_{10} + \alpha_{11} s_i g_i + \alpha_{12} s_i h_i + \alpha_{13} M_{ii}, \\
\alpha_2 &= \alpha_{20} + \alpha_{21} s^2, \\
\alpha_3 &= \alpha_{30} + \alpha_{31} s^2,
\end{aligned} \qquad (3.22)$$

$$\begin{aligned}
\beta_1 &= \beta_{10} + \beta_{11} s^2 + \beta_{12} s_i g_i + \beta_{13} s_i h_i + \beta_{14} M_{ii}, \\
\beta_2 &= \beta_{20} + \beta_{21} s^2, \\
\beta_3 &= \beta_{30} + \beta_{31} s^2,
\end{aligned} \qquad (3.23)$$

$$\begin{aligned}
\gamma_1 &= \gamma_{10} + \gamma_{11} s_i g_i + \gamma_{12} s_i h_i + \gamma_{13} M_{ii}, \\
\gamma_2 &= \gamma_{20} + \gamma_{21} s^2, \\
\gamma_3 &= \gamma_{30} + \gamma_{31} s^2,
\end{aligned} \qquad (3.24)$$

where all coefficients of the form α_{ij}, β_{ij}, γ_{ij}, together with α_4, α_5, β_4, β_5, γ_4, γ_5 depend only on ϱ and T. Using (3.15) and $(3.21)_2$ we find

$$p = p_0 + p_1 s^2, \qquad (3.25)$$

where

$$p_0 = \varrho^2\, \partial\phi_0/\partial\varrho, \quad p_1 = \tfrac{1}{2}\varrho^2\, \partial\phi_1/\partial\varrho. \qquad (3.26)$$

Using (3.23) we see that the only terms on the left-hand side of (3.20) involving the second-order gradients of ϱ, T, s_i are

$$\begin{gathered}-\beta_{12}s_is_jg_{j,i}-\beta_{13}s_is_jh_{j,i}-\beta_{14}s_iM_{jj,i}\\ -\beta_2h_{i,i}-\beta_3g_{i,i}-\beta_4s_jM_{ij,i}.\end{gathered}\tag{3.27}$$

Since these second-order gradients may be varied independently we deduce

$$\beta_{12}=\beta_{13}=\beta_{14}=\beta_2=\beta_3=\beta_4=0.\tag{3.28}$$

Using (3.21)–(3.28) in (3.20) we find

$$\begin{aligned}&\{-\varrho\phi_1+\varrho\phi_1\alpha_4-\phi_1\sigma_1+2p_1\phi_1-2\beta_{11}\}\,s_is_jM_{ij}-\frac{\gamma_4}{T}s_ig_jM_{ij}\\ &+\{-\beta_{10}-\beta_{11}s^2-\phi_1\sigma_1s^2+\varrho\phi_1\alpha_{13}s^2\}\,M_{ii}-\frac{\gamma_{13}}{T}s_jg_jM_{ii}\\ &+\left\{\varrho\phi_1\alpha_{20}-\frac{\partial\beta_{10}}{\partial\varrho}+\frac{\partial p_0}{\partial\varrho}\phi_1+\left(\varrho\phi_1\alpha_{21}-\phi_1\frac{\partial\sigma_1}{\partial\varrho}+\phi_1\frac{\partial p_1}{\partial\varrho}+\varrho\phi_1\alpha_{12}-\frac{\partial\beta_{11}}{\partial\varrho}\right)s^2\right\}s_ih\\ &+\left\{\varrho\phi_1\alpha_{30}-\frac{\partial\beta_{10}}{\partial T}-\frac{\gamma_{10}}{T}+\frac{\partial p_0}{\partial T}\phi_1\right.\\ &\left.+\left(\varrho\phi_1\alpha_{31}-\phi_1\frac{\partial\sigma_1}{\partial T}+\phi_1\frac{\partial p_1}{\partial T}+\varrho\phi_1\alpha_{11}-\frac{\partial\beta_{11}}{\partial T}\right)s^2\right\}s_ig_i\\ &+\varrho\phi_1\alpha_{10}s^2-(\gamma_{30}+\gamma_{31}s^2)\frac{g^2}{T}-(\gamma_{20}+\gamma_{21}s^2)\frac{g_ih_i}{T}\\ &\qquad-\frac{\gamma_{11}}{T}(s_ig_i)^2-\frac{\gamma_{12}}{T}s_ig_is_jh_j\geqq 0.\end{aligned}\tag{3.29}$$

Considering now the independent variations of s_i, h_i, g_i, M_{ij}, we deduce the following relations:

$$\beta_{10}=\gamma_{12}=\gamma_{13}=\gamma_{20}=\gamma_{21}=\gamma_4=0,\tag{3.30}$$

$$\phi_1\alpha_{10}\geqq 0,\gamma_{30}\leqq 0,\gamma_{31}\leqq 0,|\gamma_{11}|\leqq-\gamma_{31},\tag{3.31}$$

$$-\varrho\phi_1+\varrho\phi_1\alpha_4-\phi_1\sigma_1+2p_1\phi_1-2\beta_{11}=0,\tag{3.32}$$

$$\beta_{11}+\phi_1\sigma_1-\varrho\phi_1\alpha_{13}=0,\tag{3.33}$$

$$\varrho\alpha_{20}+\frac{\partial p_0}{\partial\varrho}=0,\tag{3.34}$$

$$\varrho\phi_1\alpha_{30}+\phi_1\frac{\partial p_0}{\partial T}-\frac{\gamma_{10}}{T}=0,\tag{3.35}$$

$$\varrho\phi_1\alpha_{21}-\phi_1\frac{\partial\sigma_1}{\partial\varrho}+\phi_1\frac{\partial p_1}{\partial\varrho}+\varrho\phi_1\alpha_{12}-\frac{\partial\beta_{11}}{\partial\varrho}=0,\tag{3.36}$$

$$\varrho\phi_1\alpha_{31}-\phi_1\frac{\partial\sigma_1}{\partial T}+\phi_1\frac{\partial p_1}{\partial T}+\varrho\phi_1\alpha_{11}-\frac{\partial\beta_{11}}{\partial T}=0,\tag{3.37}$$

and (3.29) reduces to

$$\varrho\phi_1\alpha_{10}s^2 - (\gamma_{30} + \gamma_{31}s^2)\frac{g^2}{T} - \frac{\gamma_{11}}{T}(s_i g_i)^2 \geqq 0. \tag{3.38}$$

4. Reduction to Landau's theory

The preceding analysis has placed a number of severe restrictions on the proposed constitutive equations (3.16)–(3.19). However, as it stands, the theory is still more general than that of LANDAU. We consider now the further relationships between the coefficients which lead to LANDAU'S non-dissipative equations.

For a non-dissipative theory we must take

$$\alpha_{10} = \gamma_{11} = \gamma_{30} = \gamma_{31} = 0. \tag{4.1}$$

In addition we take

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = 0, \quad \alpha_4 + \alpha_5 = 0, \quad \gamma_{10} = \varrho TS\phi_1. \tag{4.2}$$

With these assumptions and in the absence of heat supply the energy equation reduces to

$$\varrho\frac{DS}{Dt} + (\varrho S\phi_1 s_i)_{,i} = 0. \tag{4.3}$$

By use of the continuity equation, this may be written

$$\frac{\partial}{\partial t}(\varrho S) + \{\varrho S(v_i + \phi_1 s_i)\}_{,i} = 0, \tag{4.4}$$

and so $\varrho S(\boldsymbol{v} + \phi_1\boldsymbol{s})$ is the flux of entropy measured per unit area of elements of surface fixed in space.

Now one of the most basic observations concerning superfluid helium, which has a direct bearing on the two-fluid model, relates to the flow of the fluid through a system of very fine channels known as a superleak. KAPITZA [10] showed that the total entropy content of a reservoir feeding helium into a superleak showed no measurable change.

This motivates the definition of a superleak as a region in which

$$\boldsymbol{v} + \phi_1\boldsymbol{s} = \boldsymbol{0}, \tag{4.5}$$

and suggests a transition to two-fluid formalism. If we wish to view the fluid as a mixture of normal fluid and superfluid, and regard a superleak as a region in which only superfluid flows, then in such a region $\boldsymbol{v}^n = \boldsymbol{0}$, or using $(1.1)_2$,

$$\varrho\boldsymbol{v} - \varrho_s\boldsymbol{v}^s = \boldsymbol{0}. \tag{4.6}$$

Using (3.7), (4.5) becomes

$$(1 + \phi_1)\,\boldsymbol{v} - \phi_1\boldsymbol{v}^s = \boldsymbol{0}. \tag{4.7}$$

In view of $(1.1)_1$, (4.6) and (4.7) coincide if we make the identification

$$\phi_1 = \varrho_s/\varrho_n. \tag{4.8}$$

The whole theory now involves only $\boldsymbol{v}^s$ and ϕ_1 in addition to conventional variables, and the identification (4.8) means that all the terms appearing in the governing equations may be written in terms of the two-fluid variables $\boldsymbol{v}^s$ and ϱ_s/ϱ_n. The relations (1.1) and (4.8) may be used to provide definitions of ϱ_n, ϱ_s, $\boldsymbol{v}^n$ in terms of ϕ_1 if required. We have therefore reduced the theory to the set of field equations (2.1), (2.2), (3.4), and constitutive equations $(3.13)_5$, (3.15), $(3.21)_2$ together with special forms of (3.16)–(3.19) which may now be written

$$f_i = -\mu_{,i}, \tag{4.9}$$

where

$$\mu = F + \frac{p}{\varrho} - \left(\frac{1}{2} + \frac{\varrho_s}{\varrho_n}\right)\frac{\varrho_n^2}{\varrho^2}(v_i^n - v_i^s)(v_i^n - v_i^s), \tag{4.10}$$

$$l_i = \frac{\varrho_n \varrho_s^2}{\varrho^2}(v_j^n - v_j^s)(v_j^n - v_j^s)(v_i^n - v_i^s), \tag{4.11}$$

$$q_i = \varrho_s TS(v_i^n - v_i^s), \tag{4.12}$$

$$\sigma_{ij} = -p\delta_{ij} - \frac{\varrho_s \varrho_n}{\varrho}(v_i^n - v_i^s)(v_j^n - v_j^s). \tag{4.13}$$

This set of equations constitutes the Landau non-dissipative theory.

In conclusion we may say that the Landau non-dissipative theory is a special case of a fluid in which the independent variables of the constitutive equations are density, temperature, and a generalised velocity field (together with its gradients) satisfying a certain rate-type equation. An observed property of a superleak may be formalised to provide a basis for the two-fluid description. Further study may show how other observed properties may be treated similarly to motivate the restrictions (4.2).

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