# On Global Solution of Nonlinear Hyperbolic Equations

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Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$. Points in $\Omega$ are denoted by $x=(x_1, \ldots, x_n)$ and the time variable is denoted by $x_0$ or $t$. Consider the non-linear hyperbolic boundary-initial value problem

$$u_{tt}-\nabla^2 u+f(x, u)=0, \tag{0.1}$$

$$u=0 \quad \text{on} \quad \partial\Omega, \tag{0.2}$$

$$u(x, 0)=U(x), \quad u_t(x, 0)=V(x). \tag{0.3}$$

In this paper the question of the existence of global weak solutions of (0.1)–(0.3) is investigated.

The method of analysis of (0.1)–(0.3) is based on the energy integrals of (0.1). The "kinetic" and "potential" energies associated with (0.1) are the functionals

$$K(u)=\int_\Omega \tfrac{1}{2}u_t^2\, dx, \quad J(u)=\int_\Omega \{\tfrac{1}{2}|\operatorname{grad} u|^2+F(x, u)\}\, dx. \tag{0.4}$$

respectively, where $F$ is an antiderivative of $f$: $F_u=f$. Suppose that $J$ has a local minimum at $u=u_0(x)$. Then, in analogy with the local minimum of a potential function for a mechanical system with a finite number of degrees of freedom, we may imagine a potential well $W$ situated at $u=u_0$ in function space. If $U$ lies in $W$ and if the total energy of the initial data is less than the depth of $W$ then we expect that (0.1)–(0.3) has a global solution.

In this paper it is shown that under certain conditions on the nonlinear term $f$ these conjectures are correct. The main result is the following theorem.

**Theorem 1.** *Let the functional $J$ have a local minimum at $u=u_0(x)$. By a simple transformation (see section 2) $u_0$ may be translated to the origin, and it can be assumed that $u_0\equiv 0$ and that $f(x,0)=F(x,0)\equiv 0$. Let the non-linear term $f(x,u)$ satisfy the assumptions* (i)–(vii) *of section 1. Then there is a positive number $d$ given by* (3.3) *and a potential well $W$ of depth $d$ given by* (3.4). *Furthermore the boundary-initial value problem* (0.1)–(0.3) *has a global weak solution (in the sense* (2.2), (2.3)*), provided that $U\in W$ and that the total initial energy is less than $d$:*

$$\int_\Omega \left\{\frac{V^2}{2}+\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial U}{\partial x_i}\right)^2+F(x, U)\right\} dx<d.$$

The weak derivatives $u_t$ and $u_i$, $i=1, \ldots, n$ of the solution $u$ satisfy the energy inequality

$$\int_\Omega \left\{\frac{u_t^2}{2}+\sum_{i=1}^{n}\frac{u_i^2}{2}+F(x, u)\right\} dx \leq \int_\Omega \left\{\frac{V^2}{2}+\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial U}{d x_i}\right)^2+F(x, U)\right\} dx. \tag{0.5}$$

The assumptions (i)–(vii) on the non-linear term $f$ are overly restrictive, but they allow us to apply the method of this paper under a single set of conditions. In section 6 it will be seen that in some cases these conditions can be relaxed.

An immediate consequence of the variational formulation of (3.3) of $d$ is that the depth of $W$ decreases as the size of the domain increases. Thus the boundary-initial value problem (0.1)–(0.3) is relatively less stable on larger domains. On an infinite domain equation (0.1) may not have a global solution for any set of initial data.

Theorem 1 is proved by approximating the initial value problem (0.1)–(0.3) by non-linear systems of ordinary differential equations. The global existence of the solutions of these systems is proved by using the energy equality $K+J=\text{const.}$ which is valid for finite dimensional systems. Compactness theorems and the diagonal method are used to select a convergent subsequence from the sequence of approximate solutions. The limit of this subsequence is then shown to satisfy the boundary-initial value problem (0.1)–(0.3) in weak form. The main difficulties of the approximation method are in establishing the existence and compactness of a potential well associated with a local minimum of $J$. A convergence theorem (Lemma 4.2) must also be proved.

Equations of the form (0.1) arise in diverse areas of mathematical physics. See, for example, the references in [7]. Furthermore, the treatment of the initial value problem (0.1)–(0.3) may provide a model for the investigation of other types of non-linear partial differential equations of evolution. For example, the method of approximating a non-linear system of partial differential equations by non-linear systems of ordinary differential equations has been used by Hopf [4] in his investigation of the Navier-Stokes equations. Thus the approach appears to be a powerful and elegant one which should be applicable to a wide class of semi-linear partial differential equations or systems of evolution whose solutions formally satisfy an energy inequality.

In [3] the existence of a strong solution of (0.1)–(0.3) for some finite time interval $0 \leqq t \leqq \delta$ is proved by an iteration method. This method allows one to establish a regular (differentiable) solution for the given time interval, but it does not give any insight into the existence of a global solution of (0.1)–(0.3). On the other hand, Keller [5] has given examples of equations of the type (0.1) for which the solutions diverge to infinity in a finite time interval for certain initial data.

In [7] the non-linear equation (0.1) with the presence of the damping term $2\alpha u_t$ is discussed. The existence of a strong solution is proved there using a perturbation technique. It is assumed in that paper that $f(u)$ is analytic; but no growth assumptions are made on $f$, and $f$ may even have singularities on the real axis. Furthermore it is shown that the solution $u(x,t)$ tends uniformly to the stationary point $u_0(x)$ as $t$ tends to infinity. The initial data is assumed to lie sufficiently close to the stationary point $u_0$; but no precise description of the admissible initial data is given.

One of the features of the approximation method of the present paper is that it is possible to specify precisely that class of initial data for which (0.1)–(0.3) has a global solution. The assumption of analyticity of $f$ can be dropped, but assumptions about the asymptotic behavior of $f$ as $|u| \to \infty$ must be added. On the other hand, the solutions obtained by the approximation method are weak solutions.

The actual construction of a solution of (0.1)−(0.3) is not discussed until section 5. In sections 1−4 the problem is restated in weak form, and the necessary concepts and analysis are developed. In section 6 some specific examples are discussed.

## 1. Preliminaries

In this section we introduce some notation, basic ideas, and important lemmas which will be needed in the course of the paper.

The volume element of integration in $\mathbb{R}^n$ is denoted by $dx = dx_1 \dots dx_n$. In case $n=1$, $\Omega$ is an interval on the real line. It is usually assumed that $n=1$, 2, or 3, although some results hold for higher dimensions as well. The closure of $\Omega$ is denoted by $\bar{\Omega}$. The boundary $\partial\Omega$ is assumed to satisfy some sort of regularity condition, such as the cone condition [3].

For any $T>0$ let $\Omega_T = \Omega \times (0,T)$ and let $\Omega_\infty = \Omega \times (0,\infty)$. The volume element of integration in space-time is denoted by $dx\, dx_0$.

The partial derivatives of the function $u(x,t)$ are denoted by

$$\frac{\partial u}{\partial x_i},\; i=0,\dots,n, \quad \text{where} \quad \frac{\partial u}{\partial x_0} = \frac{\partial u}{\partial t}.$$

For a function $u(x)$ defined on $\Omega$ we introduce the norms:

$$\|u\|_p = \left(\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}}, \quad 1 \leqq p < \infty \tag{1.1}$$

$$|||u|||_2 = \left(\int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 dx\right)^{\frac{1}{2}}, \tag{1.2}$$

$$\|u\|_{1,2} = \left(\int_\Omega \left\{\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 + u^2\right\} dx\right)^{\frac{1}{2}}. \tag{1.3}$$

The class of measurable functions $u$ for which $\|u\|_p$ is finite is the Banach space $L_p(\Omega)$.

We now construct a space of functions with weak derivatives as follows. Let $C_0^\infty(\Omega)$ denote the class of $C^\infty$ functions with compact support in $\Omega$. The completion of the class $C_0^\infty(\Omega)$ under the norm $\|\;\|_{1,2}$ is denoted by $\mathring{H}^1$. The Hilbert space $\mathring{H}^1$ is a subspace of the Sobolev space $W^{1,2}(\Omega)$.

Let $C^k(\bar{\Omega})$ be the class of functions which are $k$ times continuously differentiable on $\bar{\Omega}$. If $\eta \in C^1(\bar{\Omega})$ we have by an integration by parts

$$\int u \frac{\partial \eta}{\partial x_i}\, dx = -\int w_i \eta\, dx \tag{1.4}$$

where $u \in C_0^\infty(\Omega)$ and $w_i = \dfrac{\partial u}{\partial x_i}$. It is easily seen by the customary arguments [1] that for any $u \in \mathring{H}^1$ there exist functions $w_i$, $i=1,\dots,n$ such that (1.4) holds for any $\eta \in C^1(\bar{\Omega})$. The functions $w_i$ are called the *weak derivatives* of $u$. The notation

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$$

will always be interpreted in the weak sense (1.4) where necessary. For $u \in \overset{\circ}{H}{}^1$ the derivatives appearing in the norms $||| \; |||_2$ or $\| \; \|_{1,2}$ are to be taken in the weak sense.

The function space $\overset{\circ}{H}{}^1$ consists of the class of functions $u \in L_2(\Omega)$ which vanish on $\partial\Omega$ (in the weak sense) and which have weak $L_2$ first derivatives.

We have the following inequality of S.L. SOBOLEV [8] for functions in $\overset{\circ}{H}{}^1$:

**Lemma 1.1.** *For all* $u \in \overset{\circ}{H}{}^1$

$$\|u\|_q \leqq C \, |||u|||_2 \tag{1.5}$$

*where* $1 \leqq q \leqq 2n/(n-2)$ *if* $n \geqq 3$ *and* $1 \leqq q < \infty$ *if* $n=2$. *The constant* $C$ *depends only on* $\Omega$, $q$, *and* $n$.

We also need to discuss functions $u(x,t)$ with weak space and time derivatives. A class of test functions is defined as follows. We let $\mathscr{S}$ be the class of all $\eta \in C^2(\bar{\Omega}_\infty)$ which vanish on $\partial\Omega$ and which vanish identically for sufficiently large $t$. The class $\mathscr{S}$ is called the class of test functions. We define $\mathscr{S}_0$ to be the subclass of $\mathscr{S}$ consisting of all $\eta$ for which $\eta = \eta_t = 0$ on the hyperplane $t=0$.

The function $u(x,t)$ is said to have weak space and time derivatives $u_i$, $i = 0, \ldots, n$, if

$$\int_0^\infty \int_\Omega u \frac{\partial \eta}{\partial x_i} \, dx \, dx_0 = -\int_0^\infty \int_\Omega u_i \eta \, dx \, dx_0 \tag{1.6}$$

for all $\eta \in \mathscr{S}_0$. The Sobolev space $W^{1,2}(\Omega_T)$ is the class of all such functions for which the norm

$$|u|_{\Omega_T}^{1,2} = \left[ \int_0^T \int_\Omega \left( \sum_{i=0}^n u_i^2 + u^2 \right) dx \, dx_0 \right]^{\frac{1}{2}}$$

is finite.

Weak solutions of the initial value problem (0.1)–(0.3) lie in the closed subspace of $W^{1,2}(\Omega_T)$ which consists of functions which vanish on $\partial\Omega$. This subspace is constructed as follows. Let $\hat{\mathscr{B}}_{1,2}(\Omega_T)$ denote functions $u(x,t) \in C^1(\bar{\Omega}_T)$ with the property that $u(x,t)=0$ on $\partial\Omega$ for $0 \leqq t \leqq T$. Let $\mathscr{B}_{1,2}(\Omega_T)$ denote the closure of $\hat{\mathscr{B}}_{1,2}(\Omega_T)$ under the norm $|u|_{\Omega_T}^{1,2}$. For every $u \in \hat{\mathscr{B}}_{1,2}(\Omega_T)$ (1.6) holds for every $\eta(x,t) \in \mathscr{S}_T$. In fact, if $u \in \mathscr{B}_{1,2}(\Omega_T)$ we may approximate $u$ by a sequence $\{u_m\}$ in $\hat{\mathscr{B}}_{1,2}(\Omega_T)$. Equation (1.6) holds for every $u_m \in \hat{\mathscr{B}}_{1,2}(\Omega_T)$ and, passing to the limit, we see that (1.6) holds for $u$ as well.

The following imbedding theorem by Sobolev will be an important tool in constructing weak solutions of (0.1)–(0.3).

**Lemma 1.2.** *Let* $\{w_k\}$ *be a sequence of functions in* $W^{1,2}(\Omega_T)$ *such that the norms* $|w_k|_{\Omega_T}^{1,2}$ *are uniformly bounded as k tends to infinity. Then there exists a subsequence* $\{w_{k'}\}$ *which is a Cauchy sequence in* $L_2(\Omega_T)$.

Thus bounded sets in $W^{1,2}(\Omega_T)$ are compact in $L_2(\Omega_T)$. A proof of Lemma 1.2 may be found in [1].

A sequence of functions $\{w_k\}$ is said to have uniformly absolutely continuous integrals on $\Omega$ if given any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_E |w_k| \, dx < \epsilon, \quad k = 1, 2, \ldots$$

whenever $\operatorname{meas}(E) < \delta$ and $E \subset \Omega$. The following lemma can be found in [6].

**Lemma 1.3.** *Let $\Phi(u)$ be a non-negative function which increases monotonically as u tends to infinity and which satisfies the condition*

$$\lim_{u\to\infty} \frac{\Phi(u)}{u} = +\infty.$$

*Let $\{w_k\}$ be a sequence of functions on $\Omega_T$ for which the integrals*

$$\int_0^T \int_\Omega \Phi(|w_k|)\,dx\,dt$$

*are uniformly bounded as k tends to infinity. Then the sequence $\{w_k\}$ has uniformly absolutely continuous integrals on $\Omega_T$.*

Lemma 1.3 will be used in conjunction with the following lemma.

**Lemma 1.4.** *Let* $\lim w_k(x,t) = w(x,t)$ a.e. *in $\Omega_T$ and suppose that the sequence $\{w_k\}$ has uniformly absolutely continuous integrals. Then*

$$\lim_{k\to\infty} \int_0^T \int_\Omega w_k(x,t)\,\varphi\,dx\,dx_0 = \int_0^T \int_\Omega w(x,t)\,\varphi\,dx\,dx_0$$

*for all bounded measurable functions $\varphi$ on $\Omega_T$.*

The proof of Lemma 1.4 uses standard measure-theoretic arguments and will be omitted. Finally, we have

**Lemma 1.5.** *Let $u_k$ converge to u in $L_2(\Omega)$ and suppose that $\|u_k\|_p \leqq A$, $k=1,2,\ldots$ for $p>2$. Then $u_k$ converges to u in $L_q(\Omega)$ for $2\leqq q<p$.*

**Proof.** By Fatou's lemma we see that $\|u\|_p \leqq A$. For $2<q<p$ we have by Hölder's inequality

$$\int_\Omega |u_k-u|^q\,dx \leqq \int_\Omega |u_k-u|^\alpha |u_k-u|^\beta\,dx \leqq \Big[\int_\Omega |u_k-u|^{\alpha s}\,dx\Big]^{1/s} \Big[\int_\Omega |u_k-u|^{\beta t}\Big]^{1/t} \qquad (1.7)$$

where $\alpha$, $\beta$, $s$, and $t$ must satisfy

$$\alpha+\beta=q, \qquad \frac{1}{s}+\frac{1}{t}=1, \qquad s>1,\ t>1. \qquad (1.8)$$

We take

$$\alpha = 2\left(\frac{p-q}{p-1}\right), \quad \beta = q-2\left(\frac{p-q}{p-1}\right), \quad s=\frac{p-1}{p-q}, \quad t=\frac{p-1}{q-1}.$$

Then the relations (1.8) are satisfied; and, furthermore, $\alpha s=2$ and $\beta t \leqq p$. Therefore the first factor on the right side of (1.7) converges to zero while the second factor remains bounded. Q.E.D.

The solution of (0.1)–(0.3) will be constructed by expanding the solution in "normal modes"

$$u(x,t) = \sum_{i=1}^{\infty} q_i(t)\,\psi_i(x).$$

The functions $\{\psi_i(x)\}$ are chosen to be the eigenfunctions of the Laplacian on $\Omega$:

$$\nabla^2 \psi_i + \mu_i \psi_i = 0, \qquad \psi_i = 0 \quad \text{on} \quad \partial\Omega. \qquad (1.8)$$

By Gauss's theorem we have

$$\begin{aligned}\int_{\Omega}(\operatorname{grad}\psi_i)\cdot(\operatorname{grad}\psi_j)\,dx&=\int_{\Omega}\{\operatorname{div}(\psi_i\operatorname{grad}\psi_j)-\psi_i\nabla^2\psi_j\}\,dx\\&=\int_{\partial\Omega}\psi_i\frac{\partial\psi_j}{\partial\nu}\,ds+\mu_j\int\psi_i\psi_j\,dx=\mu_i\delta_{ij}\end{aligned}\tag{1.9}$$

by the orthogonality properties of the eigenfunctions. The functions $\psi_i$ are in the class $C^1(\bar{\Omega})$ and vanish on $\partial\Omega$.

Listed below is a general set of conditions on the non-linear term $f(x,u)$ under which the approximation method will work to give an existence theorem for (0.1)–(0.3) in the neighborhood of a stable equilibrium of the potential. Most of the lemmas in sections 3 and 4 can be proved under more general assumptions or under assumptions of a quite different nature. These more general assumptions, however, vary from one lemma to the next, while the conditions below are in the spirit of a lowest common denominator.

We assume that

(i) $f(x,u)$ is twice continuously differentiable in $u$ for each fixed $x\in\Omega$.

(ii) either $F(x,u)$ is uniformly bounded below for $x\in\Omega$ and $-\infty<u<\infty$, or there is a constant $N$ and a constant $c$, $0<c<\frac{1}{2}$, such that

$$c\,u\,f(x,u)-F(x,u)\leqq N$$

for all $x\in\Omega$ and $-\infty<u<\infty$.

(iii) the operator

$$A=-\nabla^2+f_u(x,0)$$

with the boundary conditions (0.2) is positive definite.

(iv) $f_u(x,0)$ is bounded and measurable on $\Omega$.

(v) If $n=1$ there is a constant $M_\rho$ such that $|f_{uu}(x,u)|\leqq M_\rho$ if $|u|\leqq\rho$. It follows that $|f(x,u)|\leqq\frac{1}{2}\rho^2M_\rho$ if $|u|\leqq\rho$.

(vi) If $n=2$ there are positive constants $C_1$, $C_2$ and $q$, $1\leqq q<\infty$, such that

$$|f_{uu}(x,u)|\leqq C_1+C_2|u|^q,$$
$$|f(x,u)|\leqq C_1+C_2|u|^q$$
$$|u\,f(x,u)|\leqq C_1+C_2|u|^q.$$

(vii) If $n=3$ there are constants $C_1$ and $C_2$ and $q<3$ such that

$$|f_{uu}(x,u)|\leqq C_1+C_2|u|^q$$
$$|f(x,u)|\leqq C_1+C_2|u|^{q+2}$$
$$|u\,f(x,u)|\leqq C_1+C_2|u|^{q+3}.$$

Note that the growth conditions on $f(x,u)$ and $u\,f(x,u)$ in assumptions (vi) and (vii) follow from the growth condition on $f_{uu}$. The significance of the condition

$$c\,u\,f(x,u)-F(x,u)\leqq N$$

is discussed in section 4.

## 2. Weak Formulation of the Problem

In this section the boundary-initial value problem (0.1)–(0.3) is stated in weak form. The boundary condition (0.2) is met by requiring that the solution $u$ belong to the class $\mathscr{B}_{1,2}(\Omega_T)$ for all $T>0$. As discussed in section 1, $\mathscr{B}_{1,2}(\Omega_T)$ is the closure under the norm $|\ |_{\Omega_T}^{1,2}$ of the class of $C^1$ functions which vanish on the boundary.

The weak form of the initial condition $u(x,0)=U(x)$ is stated as follows: if $u$ is the weak solution with weak partial derivative $u_0$ with respect to time, then

$$\int_0^\infty \int_\Omega u_0 \eta + u\,\eta_0\, dx\, dx_0 = -\int U(x)\,\eta\, dx \tag{2.1}$$

for all test functions $\eta \in \mathscr{S}$. It is easily seen that (2.1) is always satisfied if $u$ is differentiable in the usual sense and satisfies the initial condition $u(x,0)=U(x)$.

The weak form of the partial differential equation (0.1) and the initial condition $u_t(x,0)=V(x)$ is expressed as follows. Multiply (0.1) by any test function $\eta$, integrate over $\Omega_\infty$, and integrate by parts. We get, using the condition $u_t(x,0)=V(x)$,

$$\int_\Omega \eta\, V(x)\, dx + \int_0^\infty \int_\Omega \left\{ u_t\,\eta_t - \sum_{i=1}^n u_i \frac{\partial \eta}{\partial x_i} - f(x,u)\,\eta \right\} dx\, dt = 0. \tag{2.2}$$

We require that $u$ and its weak derivatives $u_i$, $i=0, \dots, n$, satisfy (2.1) and (2.2) for all $\eta \in \mathscr{S}$. Weak solutions of (2.2) are constructed provided the initial data is sufficiently close to a point of stable equilibrium in function space. The function $u_0(x)$ is a stationary solution of (0.1) if it satisfies the equation

$$-\nabla^2 u_0 + f(x,u_0) = 0, \qquad u_0 = 0 \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

Equation (2.3) is Euler's equation for the extremals of the potential functional $J$. The weak form of (2.3),

$$\int_\Omega \left\{ \sum_{i=1}^n u_i \frac{\partial \eta}{\partial x_i} + f(x,u_0)\,\eta \right\} dx = 0, \tag{2.4}$$

is simply the statement that the first variation of $J$ vanishes at $u_0$. The variation $\eta$ in (2.4) is assumed to be any function of class $C^1(\bar{\Omega})$ which vanishes on $\partial\Omega$.

We assume that $u_0$ is a stationary solution of (0.1) in either the strong sense (2.3) or the weak sense (2.4). Of course, if $u_0$ satisfies (2.4) and has continuous second partial derivatives, then $u_0$ satisfies (2.3). Let us show that by a simple transformation it can always be assumed that the stable equilibrium under consideration is at the origin. In fact, let

$$u(x,t) = u_0(x) + v(x,t)$$

where $u$ is a solution of (0.1). We then get the following boundary-initial value problem for $v$:

$$v_{tt} - \nabla^2 v + g(x,v) = 0 \tag{2.5}$$

$$v = 0 \quad \text{on} \quad \partial\Omega \tag{2.6}$$

$$v(x,0) = U(x) - u_0(x), \quad v_t(x,0) = V(x), \tag{2.7}$$

where

$$g(x, v) = f(x, u_0 + v) - f(x, u_0).$$

The problem (2.5)–(2.7) is formally the same as (0.1)–(0.3), except that now the stationary solution is the identically zero function. Note that $g(x,0)=0$.

From now on we always assume that the stationary solution $u_0$ of (0.1)–(0.3) is at the origin; hence it is always assumed that $f(x,0)=0$. Furthermore, we always take

$$F(x, u) = \int_0^u f(x, s)\, ds$$

so that $F(x,0)=0$ also.

From assumption (i) it follows by Taylor's theorem that

$$F(x, u) = f_u(x, 0)\frac{u^2}{2} + R(x, u) \tag{2.8}$$

where

$$R(x, u) = \tfrac{1}{2}\int_0^u f_{uu}(x, t)(u-t)^2\, dt; \tag{2.9}$$

and

$$f(x, u) = f_u(x, 0)\, u + Q(x, u) \tag{2.10}$$

where

$$Q(x, u) = \int_0^u f_{uu}(x, t)(u-t)\, dt. \tag{2.11}$$

### 3. Local Minima and Potential Wells

Our aim is to construct global solutions of (0.1)–(0.3) when the function $U$ lies sufficiently close to a local minimum of the potential functional $J$. The fact that the first variation (2.4) of $J$ vanishes at $u_0$ is not sufficient to guarantee that $J$ attains a local minimum at $u_0$. To determine whether $u_0$ is a local minimum the second variation of $J$ must be examined. As discussed in section 2, it is assumed that the local minimum in question is the function $u_0 \equiv 0$. In that case the second variation of $J$ is

$$\delta^2 J(\eta) = \int_\Omega \left\{\frac{1}{2}\sum_{i=1}^n \left(\frac{\partial \eta}{\partial x_i}\right)^2 + f_u(x, 0)\,\eta^2\right\} dx,$$

where $\eta$ is any function in $\mathring{H}^1$.

It is easily seen that $\delta^2 J(\eta)$ is the quadratic form for the operator

$$A = -\nabla^2 + f_u(x, 0) \tag{3.2}$$

with the boundary conditions (0.2). Thus a necessary condition that $J$ have a local minimum at the origin is that the operator $A$ be positive definite. (This is assumption (iii) of section 1.) It follows that there is a constant $r>0$ such that

$$r\,\delta^2 J(\eta) \geqq (\|\eta\|_2)^2$$

for all $\eta \in \mathring{H}^1$.

We now proceed to construct a "potential well" associated with the minimum of $J$ at the origin. We proceed formally at first; then the construction is put on a firm mathematical basis. Assume that the functional $J$ is defined for all $u\in \mathring{H}^1$. Since $J$ has a minimum at the origin the function $J(\lambda u)$ is an increasing function of $\lambda$ in a neighborhood of the origin (for $\lambda>0$). Let $\lambda_1>0$ be the first value of $\lambda$ at which $J(\lambda u)$ begins to decrease. We write $\lambda_1=\lambda_1(u)$ to indicate the dependence of $\lambda_1$ on $u$. The depth $d$ of the potential well is given by

$$d=\inf_{u\in \mathring{H}^1} J(\lambda_1 u). \tag{3.3}$$

The number $d$ has the interpretation as the level of the lowest "pass" leading out of the valley situated at $u_0=0$. It is easily seen that $0\leqq d\leqq\infty$. If $d>0$, the potential well $W$ may be defined as follows:

$$W=\{u: u\in \mathring{H}^1, \quad 0\leqq J(\lambda u)<d \quad \text{for } 0\leqq\lambda\leqq 1\}. \tag{3.4}$$

Thus, in order that $u\in W$, we require that $J(u)<d$ and moreover that all points between 0 and $u$ – that is, all points of the form $\lambda u$ where $0\leqq\lambda\leqq 1$ – lie below the potential level $d$. These geometric ideas are immediate in the case of a potential function $V(q_1, \ldots, q_N)$ which is positive in a neighborhood of the origin (in $N$-dimensional space) and has a local minimum there.

In order to put the preceding construction of $W$ on a rigorous basis we must show that

(a) the existence of the positive number $\lambda_1(u)$ is guaranteed for each $u\in \mathring{H}^1$.

(b) the depth $d$ given by the variational principle (3.3) is strictly positive.

Conditions under which (a) and (b) hold are given in Lemmas 3.1 and 3.2 below.

**Lemma 3.1.** *Let $u\in \mathring{H}^1$ and suppose that $f(x,u)$ satisfies the assumptions* (v)–(vii). *Then $J(\lambda u)$ is continuously differentiable and*

$$J'(\lambda u)=\int_\Omega \{\lambda\,|\operatorname{grad} u|^2+f(x,\lambda u)\}\,u\,dx.$$

**Proof.** It is clear that the first term in $J(\lambda u)$ is continuously differentiable. Therefore it is sufficient to show that the term

$$j(\lambda)=\int_\Omega F(x,\lambda u)\,dx$$

is continuously differentiable.

We have for $h>0$,

$$\frac{F(x,(\lambda+h)\,u)-F(x,\lambda u)}{h}=\left[\int_0^1 f(x,(\lambda+\tau h)\,u)\,u\,d\tau\right].$$

Therefore

$$\frac{j(\lambda+h)-j(\lambda)}{h}=\int_\Omega\int_0^1 f(x,\lambda u+\tau h u)\,u\,d\tau\,dx.$$

In case $n=1$, $u\in \mathring{H}^1$ implies that $u$ is uniformly bounded on $\Omega$. Therefore the integrand $f(x,[\lambda+\tau h]u)$ is uniformly bounded for fixed $\lambda$, $0\leqq\tau\leqq 1$, and $h$ tending to zero. Let $\{h_m\}$ be any sequence which tends to zero. By the Lebesgue dominated convergence theorem we get

$$\lim_{m\to\infty} \frac{j(\lambda+h_m)-j(\lambda)}{h_m} = \int_\Omega \int_0^1 f(x,\lambda u)\, u\, d\tau\, dx = \int_\Omega f(x,\lambda u)\, dx.$$

Since this result holds for any sequence $\{h_m\}$ we must kave

$$j'(\lambda) = \int_\Omega f(x,\lambda u)\, dx.$$

To show that $j'(\lambda)$ is continuous at an arbitrary point $\lambda_0$, let $\{\lambda_m\}$ be any sequence tending to $\lambda_0$. Again by the dominated convergence theorem it follows that $j'(\lambda_m)\to j'(\lambda_0)$; hence $j'(\lambda)$ is continuous at $\lambda_0$.

In case $n=2$ it follows from condition (vi) that

$$|f(x,[\lambda+\tau h]\, u)\, u| \leqq C_1 + C_2 |\lambda+\tau h|^q \cdot |u|^q.$$

By Lemma 1.1 the function $u$ belongs to the class $L_q(\Omega)$ for $1\leqq q<\infty$. Therefore, for fixed $\lambda$, $h$ tending to zero, and $0\leqq\tau\leqq 1$, the integrand in the difference quotient for $j(\lambda)$ is dominated in absolute value by the fixed integrable function

$$C_1 + C_2' |u|^q.$$

Here $C_2'$ is some constant that dominates $C_2 |\lambda+\tau h|^q$ for $\lambda$, $h$, and $0\leqq\tau\leqq 1$. The dominated convergence theorem is again applied to all sequences $h_m$ tending to zero to show that $j'(\lambda)$ exists. Similarly it can be shown that $j'(\lambda)$ is continuous.

The case $n=3$ is similar to the case $n=2$. The integrand in the difference quotient for $j(\lambda)$ is bounded in absolute value by the fixed function

$$C_3 + C_4' |u|^q$$

which is integrable by Lemma 1.1 provided that $1\leqq q\leqq 6$. Since condition (vii) is actually more restrictive than this, the proof of Lemma 3.1 is established.

**Lemma 3.2.** *Let the functional $J$ have a local minimum at $u_0\equiv 0$, and suppose that conditions* (i) *and* (iii)–(vii) *are satisfied by $f$. Then the depth $d$ given by* (3.3) *is positive.*

**Proof.** To prove lemma 3.2 it is sufficient to show that there are positive numbers $\alpha$ and $\beta$ such that

(a) $J'(\lambda u)>0$ for $0<\lambda<1$ whenever $0<|||u|||_2\leqq\alpha$.

(b) $J(u)\geqq\beta$ whenever $|||u|||_2=\alpha$.

It then follows that $d\geqq\beta$. In fact, let $u\neq 0$, $u\in\mathring{H}^1$, and assume that $u$ is normalized so that $|||u|||_2=\alpha$. (This renormalization of $u$ simply changes the scale along the ray $\{\lambda u\colon \lambda>0\}$ and involves no loss of generality.) Let $\lambda_1$ be the first value of $\lambda$ at which $J(\lambda u)$ begins to decrease; by (a) we have $\lambda_1\geqq 1$. If $\lambda_1=\infty$ then $u$ is not an admissible trial function for the variational problem (3.3). On the other hand, if $\lambda_1$ is finite then by (b) we have

$$J(\lambda_1 u)\geqq J(u)\geqq\beta,$$

since $J(\lambda u)$ increases over $0\leqq\lambda\leqq\lambda_1$. Since $d$ is the infimum of the numbers $J(\lambda_1 u)$ over $u\in\mathring{H}^1$, we have immediately that $d\geqq\beta$.

We now establish the existence of positive numbers $\alpha$ and $\beta$ for which properties (a) and (b) are valid.

Let $\theta_1$, $\theta_2$, and $\theta_3$ be three positive numbers such that $\theta_1+\theta_2+\theta_3=1$. $\theta_2$ and $\theta_3$ are arbitrary, but $\theta_1$ will be restricted later. We may write, by virtue of equations (2.8) and (2.10),

$$\begin{aligned} J(u) &= \tfrac{1}{2}\int_\Omega \{\theta_2 |\operatorname{grad} u|^2 + f_u(x,0)\,u^2\}\,dx \\ &\quad + \int_\Omega \{\tfrac{1}{2}\theta_1 |\operatorname{grad} u|^2 + R(x,u)\}\,dx + \tfrac{1}{2}\theta_3 \int_\Omega |\operatorname{grad} u|^2\,dx \end{aligned} \tag{3.5}$$

$$\begin{aligned} J'(\lambda u) &= \lambda \int_\Omega \theta_2 |\operatorname{grad} u|^2 + f_u(x,0)\,u^2\,dx \\ &\quad + \int_\Omega \lambda(1-\theta_1)|\operatorname{grad} u|^2 + u\,Q(x,\lambda u)\,dx. \end{aligned} \tag{3.6}$$

By assumption the operator $A$ in (3.2) is positive definite. By perturbation theory, the operator

$$A(\theta_1) = -\theta_1 \nabla^2 + f_u(x,0)$$

will be positive definite for $\theta_1$ sufficiently close to 1. Thus $\theta_1$ is restricted to be sufficiently close to 1 so that

$$\int_\Omega \{\theta_1 |\operatorname{grad} u|^2 + f_u(x,0)\,u^2\}\,dx \geqq 0 \tag{3.7}$$

for all $u \in \mathring{H}^1$.

We show below that there is a number $\alpha>0$ such that

$$\int_\Omega \{\tfrac{1}{2}\theta_2 |\operatorname{grad} u|^2 + R(x,u)\}\,dx \geqq 0 \tag{3.8}$$

and

$$\int_\Omega \{\lambda(1-\theta_1)|\operatorname{grad} u|^2 + u\,Q(x,\lambda u)\}\,dx \geqq 0 \tag{3.9}$$

for all $0\leqq\lambda\leqq 1$ whenever $|||u|||_2 \leqq \alpha$. Property (a) then holds for this choice of $\alpha$ by virtue of (3.6), (3.7) and (3.9). Moreover, for $|||u|||_2=\alpha$ we have

$$J(u) \geqq \frac{\theta_3}{2}\int_\Omega |\operatorname{grad} u|^2\,dx = \frac{\alpha^2\theta_3}{2}$$

by virtue of (3.5), (3.7), and (3.8). Thus we may take $\beta = \dfrac{\alpha^2}{2}\theta_3$.

To show that (3.8) holds whenever $|||u|||_2$ is sufficiently small we first estimate the term $R(x,u)$. From (2.9)

$$|R(x,u)| \leqq \tfrac{1}{2}|u| \max_{0\leqq|t|\leqq|u|} |f_{uu}(x,t)(u-t)^2|.$$

In case $n=1$, $u\in\mathring{H}^1$ implies that $u$ is uniformly bounded. Therefore for $|||u|||_2$ bounded by some fixed constant, there is a constant $M_\rho$ such that $|f_{uu}(x,t)| \leqq M_\rho$ if $|t|\leqq\rho$ (assumption (v)). In that case $|R(x,u)| \leqq \frac{1}{2}M_\rho |u|^3$. In case $n=2$, 3 it follows from assumption (vi) or (vii) that

$$|R(x,u)| \leqq \tfrac{1}{2}|u|^3 (C_1 + C_2|u|^q)$$

where $1\leqq q<\infty$ if $n=2$ or $q<3$ if $n=3$.

The above estimate also holds in case $n=1$ by taking $C_1=M_p$ and $C_2=0$. Therefore if $n=1, 2, 3$,

$$\left|\int_\Omega R(x,u)\,dx\right| \leqq \tfrac{1}{2}\{C_1(|||u|||_3)^3 + C_2(|||u|||_{q+3})^{q+3}\}.$$

From the Sobolev inequality (1.5) it follows that the above expression is dominated by

$$C_1'(|||u|||_2)^3 + C_2'(|||u|||_2) \tag{3.10}$$

for some constants $C_1'$ and $C_2'$. The quantity (3.10) vanishes to third order in the variable $|||u|||_2$ at zero. Therefore there is a constant $\alpha_1$ such that $|||u|||_2 \leqq \alpha_1$ implies that (3.10) is dominated by $\frac{1}{2}\theta_2(|||u|||_2)^2$, and (3.8) holds for $|||u|||_2 \leqq \alpha_1$.

Similarly, since the term $uQ(x,\lambda u)$ behaves as $u^3$ as $u\to 0$, there is a constant $\alpha_2>0$ such that (3.9) holds whenever $|||u|||_2 \leqq \alpha_2$. The proof of Lemma 3.2 is completed by taking $\alpha=\min\{\alpha_1,\alpha_2\}$.

An immediate consequence of the variational expression (3.3) for $d$ is that the depth of $W$ decreases as the size of the domain increases. In fact, let $\Omega$ be contained in a larger domain $\Omega'$ and $d$ and $d'$ denote the depths of the respective potential wells. Assuming that the boundary of $\Omega$ is sufficiently regular, the function space $\mathring{H}^1(\Omega)$ can be embedded as a subspace of $\mathring{H}^1(\Omega')$. Thus $d'$ is determined by taking the infimum over a larger class of functions, and therefore $d'<d$.

The following lemma on the continuity of the functional $J$ will be needed in the sequel.

**Lemma 3.3.** *Let the conditions* (v)–(vii) *be satisfied. If* $\{u_k\}$ *is a sequence in* $\mathring{H}^1$ *tending strongly to* $u$ *then* $\lim J(u_k)=J(u)$.

**Proof.** It is sufficient to show that

$$\int_\Omega F(x,u_k)\,dx \to \int_\Omega F(x,u)\,dx$$

as $k$ tends to infinity, since $|||u_k|||_2$ tends to $|||u|||_2$ as $k\to\infty$. In case $n=1$ the result follows from the Lebesgue dominated convergence theorem and the fact that the functions $\{u_k\}$ and $u$ are uniformly bounded as $k$ tends to infinity.

In case $n=2$ or 3 we have

$$F(x,u_k)-F(x,u)=\int_0^1 f(x,\tau u_k+(1-\tau)u)(u_k-u)\,d\tau.$$

Therefore, by Hölder's inequality,

$$\left|\int_\Omega F(x,u_k)-F(x,u)\,dx\right| \leqq \left[\int_\Omega\int_0^1 |f(x,\tau u_k+(1-\tau)u)^p\,dx\,d\tau\right]^{1/p}\cdot \|u_k-u\|_{p'}, \qquad \frac{1}{p}+\frac{2}{p'}=1.$$

By Lemma 1.5 the second factor above tends to zero as $k\to\infty$ for $1\leqq p'<\infty$ if $n=2$ or for $1\leqq p'<6$ if $n=3$. On the other hand, by condition (vi) or (vii), the Minkowski inequality, and the convexity of the function $|s|^{pq}$ for $pq>1$, the first factor above is dominated by

$$\left[\int_{\Omega}\int_0^1 (C_1+C_2\,|\tau\, u_k+(1-\tau)\, u\,|^q)^p\, dx\, d\tau\right]^{1/p}$$
$$\leqq C_1\,[\mathrm{meas}\,(\Omega)]^{1/p}+C_2\left[\int_{\Omega}\int_0^1 |\tau\, u_k+(1-\tau)\, u\,|^{pq}\, dx\, d\tau\right]^{1/p}$$
$$\leqq C_1\,[\mathrm{meas}\,(\Omega)]^{1/p}+C_2\left[\int_{\Omega}\int_0^1 \tau\,|u_k|^{pq}+(1-\tau)\,|u\,|^{pq}\, dx\, d\tau\right]^{1/p}$$
$$\leqq C_1\,[\mathrm{meas}\,(\Omega)]^{1/p}+C_2\left[\int_{\Omega} |u_k|^{pq}+|u\,|^{pq}\, dx\right]^{1/p}.$$

The constant $q$ satisfies the condition $1\leqq q<\infty$ if $n=2$ or $1\leqq q<5$ if $n=3$. The right side above is finite if $u_k$ and $u$ belong to the class $L_{pq}(\Omega)$. By Sobolev's inequality (1.5) this is the case provided that $1\leqq pq<\infty$ in case $n=2$ or provided that $pq\geqq 6$ in case $n=3$. The lemma is proved for case $n=2$ by taking $p$ and $p'$ subject only to the condition $p>1$, $p'>1$. In case $n=3$ we take $p=6/q$. Since $q<5$, it follows that $p'<6$, and the lemma is proved.

## 4. Compactness and Convergence Theorems

In section 5 the solution of the initial value problem (0.1)–(0.3) is approximated by a sequence of functions $\{u_k(x,t)\}$ which possess the following properties:

$$u_k\in C^1(\bar{\Omega}_\infty) \quad \text{and} \quad u_k=0 \quad \text{on} \quad \partial\Omega \tag{4.1}$$

$$u_k\in W \quad \text{for all } t\geqq 0 \tag{4.2}$$

$$\int_{\Omega}\left\{\sum_{i=0}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2+F(x,u_k)\right\} dx\leqq d \quad \text{for all } t\geqq 0 \tag{4,3}$$

where $d$ is the depth of $W$. In order to prove the existence of a weak solution of (0.1)–(0.3) it must be shown that the sequence $\{u_k\}$ contains a subsequence which converges in an appropriate sense. More precisely, we must show that there exists a subsequence $\{u_{k'}\}$ and a limit function $u$ such that for all $T>0$

$$u\in\mathscr{B}_{1,2}(\Omega_T) \tag{4.4}$$

$$\left.\begin{array}{l}\dfrac{\partial u_{k'}}{\partial x_i} \text{ tends weakly to the weak derivatives } u_i,\ i=0,\dots,n;\\[2ex] u_{k'} \text{ tends to } u \text{ strongly in } L_2(\Omega_T)\end{array}\right. \tag{4.5}$$

$$\lim_{k'\to\infty}\int_0^T\int_{\Omega} f(x,u_{k'})\,\eta\, dx\, dx_0=\int_0^T\int_{\Omega} f(x,u)\,\eta\, dx\, dx_0. \tag{4.6}$$

The function $\eta$ in (4.6) is assumed to be any bounded mesurable function on $\Omega_T$.

In this section it is shown that under the assumptions (i)–(vii) on $f(x,u)$ a subsequence with the properties (4.4)–(4.6) can be selected from any sequence of functions satisfying the conditions (4.1)–(4.3).

First we show that $W$ is a compact subset of $L_2(\Omega)$.

**Lemma 4.1.** *Let* $f(x,u)$ *satisfy condition* (ii) *and suppose that* $f(x,u)$ *is continuously differentiable in* $u$ *for each* $x\in\Omega$. *Suppose also that* $J(\lambda u)$ *is continuously differentiable in* $\lambda$ *for fixed* $u\in\mathring{H}^1$. *Then* $|||u|||_2$ *is uniformly bounded as* $u$ *ranges over* $W$.

**Proof.** If $u\in W$ then

$$0\leqq \int_\Omega \{\tfrac{1}{2}|\operatorname{grad} u|^2+F(x,u)\}\,dx\leqq d,$$

$$0\leqq \tfrac{1}{2}(|||u|||)^2\leqq d-\int_\Omega F(x,u)\,dx. \tag{4.7}$$

First suppose that $F(x,u)$ is bounded below. Then there is a constant $M\geqq 0$ such that $F(x,u)\geqq -M$. From (4.7) we have that

$$0\leqq \tfrac{1}{2}(|||u|||)^2\leqq d+M\cdot \operatorname{meas}(\Omega).$$

To prove Lemma 4.1 in case $F(x,u)$ is not bounded below, note that if $u\in W$ then

$$0\leqq J'(\lambda u)|_{\lambda=1}=\int_\Omega \{|\operatorname{grad} u|^2+f(x,u)\,u\}\,dx. \tag{4.8}$$

From (4.7), (4.8), and condition (ii) we have

$$\tfrac{1}{2}(|||u|||_2)^2\leqq d-\int_\Omega c\,u\,f(x,u)\,dx+\int_\Omega \{c\,u\,f(x,u)-F(x,u)\}\,dx \tag{4.9}$$

$$\leqq d+c(|||u|||_2)^2+N\cdot\operatorname{meas}(\Omega). \tag{4.10}$$

Hence

$$0\leqq(\tfrac{1}{2}-c)(|||u|||_2)^2\leqq d+N\cdot\operatorname{meas}(\Omega)$$

for all $u\in W$. Since $0<c<\frac{1}{2}$, the proof of Lemma 4.1 is complete.

Let us give some examples of functions $f(x,u)$ for which condition (ii) is satisfied.

(1) Suppose that

$$f(x,u)=\sum_{i=1}^{r} a_i\,u^i \tag{4.11}$$

where the functions $a_i=a_i(x)$ are bounded and measurable on $\Omega$. We have

$$F(x,u)=\sum_{i=1}^{r} a_i\,\frac{u^{i+1}}{i+1},$$

and so

$$c\,u\,f(x,u)-F(x,u)=\sum_{i=1}^{r} a_i(x)\left[c-\frac{1}{i+1}\right]u^{i+1}.$$

Suppose that $r$ is even and that $a_{r-1}(x)\geqq\delta$ on $\Omega$, where $\delta$ is some positive number. Taking $c=(r+1)^{-1}$ we get

$$c\,u\,f(x,u)-F(x,u)=-\frac{a_{r-1}\,u^r}{r(r+1)}+\text{lower order terms}.$$

This polynomial in $u$ is bounded above for $x\in\Omega$ and all real $u$.

More generally (ii) holds for functions of the form

$$f(x,u)=\sum_{i=1}^{r} a_i u^i + p(x,u)$$

where $a_{r-1}(x)\geqq\delta$ on $\Omega$ and $|up(x,u)|\leqq C_1+C_2\,|u|^p$ for all $x\in\Omega$, where $C_1$ and $C_2$ are positive constants and $p<r$.

(2) Suppose that $F(x,u)\to-\infty$ as $u\to\pm\infty$ for all $x\in\Omega$. Let $F$ satisfy the condition that there exists a number $u_*>0$ such that $F(x,u)<0$ for all $|u|\geqq u_*$ and

$$F(x,u_2)\leqq F(x,u_1)\left(\frac{u_2}{u_1}\right)^{1/c} \tag{4.12}$$

whenever $0<u_*\leqq u_1<u_2$ or $u_1<u_2\leqq u_*<0$. The number $c$ above satisfies $0<c<\frac{1}{2}$. Then condition (ii) is satisfied.

To see this, consider the case where $u\geqq u_*$. (The case where $u<-u_*$ is similar.) Since $F<0$ for $u_*\leqq u_1<u_2$ we have by (4.12)

$$\frac{F(x,u_2)}{F(x,u_1)}\geqq\left(\frac{u_2}{u_1}\right)^{1/c},$$

and, since the logarithm is an increasing function,

$$\log\frac{F(x,u_2)}{F(x,u_1)}\geqq\frac{1}{c}\log\frac{u_2}{u_1}.$$

This inequality may be written in the form

$$\int_{u_1}^{u_2}\frac{f(x,s)}{F(x,s)}\,ds\geqq\frac{1}{c}\int_{u_1}^{u_2}\frac{ds}{s}.$$

Since this holds for all $u_1$ and $u_2$ such that $u_*\leqq u_2$ it follows that

$$\frac{f(x,u)}{F(x,u)}\geqq\frac{1}{cu}.$$

Thus $cuf(x,u)-F(x,u)\leqq 0$ for all $u\geqq u_*$. $u\geqq u_*$ and, by a similar argument, for $u\leqq -u_*$ as well. Since $cuf(x,u)-F(x,u)$ is bounded if $|u|\leqq u_*$ condition (ii) follows immediately.

A consequence of Lemma 4.1 is that $W$ is compact in $L_2(\Omega)$. In fact, from the variational inequality

$$\|u\|_2\leqq\frac{1}{\sqrt{\mu_1}}\,|||u|||_2 \tag{4.13}$$

valid for all $u\in\mathring{H}^1$, where $\mu_1$ is the first eigenvalue of the Laplacian (see (1.8)), it follows that $\|u\|_{1,2}$ is uniformly bounded as $u$ varies over $W$. The compactness of $W$ in $L_2(\Omega)$ now follows from the Sobolev imbedding theorem (Lemma 1) for the space $W^{1,2}(\Omega)$.

Now let $\{u_k\}$ be a sequence of functions satisfying conditions (4.1)−(4.3). From (4.2), (4.3), Lemma 4.1, and (4.13) it follows that there is a constant $C$ such that

$$\int_{\Omega}\left\{\sum_{i=0}^{n}\left(\frac{\partial u_k}{\partial x_i}\right)^2+u_k^2\right\}dx\leqq C \tag{4.14}$$

for all $t\geqq 0$. From (4.1) it follows that $u_k\in\hat{\mathscr{B}}_{1,2}(\Omega)$ for all $T>0$.

By the diagonal method a subsequence $\{u_{k'}\}$ is constructed such that $\left\{\frac{\partial u_{k'}}{\partial x_i}\right\}$ converges weakly in $L_2(\Omega_T)$ for $i=0,\dots,n$ and $\{u_{k'}\}$ converges strongly in $L_2(\Omega_T)$ for every $T>0$. Since some subsequence of a strongly $L_2$ convergent sequence converges pointwise a.e. we may assume that $\{u_{k'}\}$ converges a.e. on $\Omega_T$ for all $T>0$. Now let $u(x,t)$ be the strong $L_2$ limit of $\{u_{k'}\}$ on $L_2(\Omega_T)$ for each $T>0$, and let $u_i$, $i=0,\dots,n$ be the weak limits of the $L_2$ weakly convergent sequences $\left\{\frac{\partial u_{k'}}{\partial x_i}\right\}$. For each $T>0$ and for every bounded measurable function $\eta$

$$\lim_{k'\to 0}\int_0^T\int_{\Omega}|u_{k'}-u|^2\,dx\,dx_0=0, \tag{4.16}$$

$$\lim_{k'\to 0}\int_0^T\int_{\Omega}\frac{\partial u_{k'}}{\partial x_i}\eta\,dx\,dx_0=\int_0^T\int_{\Omega}u_i\eta\,dx\,dx_0,\qquad i=0,\dots,n. \tag{4.17}$$

The limit function $u\in\mathscr{B}_{1,2}(\Omega_T)$ for all $T>0$ and its weak derivatives are the functions $u_i$, $i=0,\dots,n$.

Conditions (4.4) and (4.5) are satisfied by the subsequence $\{u_k\}$ and its limit $u$ constructed above. In the following lemma we give sufficient conditions on the non-linear term $f(x,u)$ so that (4.6) will hold also.

**Lemma 4.2.** *Let $\{u_k\}$ be a sequence of functions satisfying conditions* (4.1)–(4.3). *Suppose that $u$ is the $L_2$ limit of $u_k$ in the sense* (4.16) *and let* $\lim u_k=u$ a.e. *in* $\Omega_\infty$. *Let $f(x,u)$ satisfy conditions* (i) *and* (v)–(vii). *Then* (4.6) *holds.*

**Proof.** For almost all $(x,t)\in\Omega_\infty$ we have $\lim f(x,u_k)=f(x,u)$, since $f(x,u)$ is continuous in $u$ for all $x\in\Omega$.

In case $n=1$ we have from (4.14) that

$$|u(x,t)|\leqq\sqrt{l(\Omega)}\left[\int|u'|^2\,dx\right]^{\frac{1}{2}}\leqq\text{const.}$$

for all $t\geqq 0$; here $l(\Omega)$ denotes the length of the interval $\Omega$. Therefore $|f(x,u)|\leqq M_\rho$ where $M_\rho$ is a positive constant, and (4.6) follows by the Lebesgue dominated convergence theorem.

In case $n\geqq 2$ it follows from conditions (vi) or (vii) that for any $p>1$ there are constants $C_1'$ and $C_2'$ such that

$$|f(x,u)|^p\leqq C_1'+C_2'|u|^{pq}.$$

In case $n=2$ we take $p$ to be any number greater than unity. In case $n=3$, $p$ is chosen so that $pq\leqq 6$. This is possible by condition (vii). From Sobolev's inequality (1.5) and (4.14) it then follows that the integrals

$$\int_0^T\int_{\Omega}|f(x,u_k)|^p\,dx\,dt \tag{4.18}$$

are uniformly bounded as $k\to\infty$ for any $T>0$. The limit theorem (4.6) now follows from Lemma 1.4. Q.E.D.

From (4.18) and Fatou's lemma it follows that $f(x,u)\in L_p(\Omega_T)$ for all $T>0$, where $p$ is some constant greater than unity. Since $\operatorname{meas}(\Omega_T)<+\infty$ $L_1(\Omega_T)\subset L_p(\Omega_T)$ and $f(x,u)$ is integrable.

## 5. Approximation by Systems with a Finite Number of Degrees of Freedom

The existence of a weak solution of the boundary-initial value problem (0.1)–(0.3) can now be proved using the analytical machinery developed in sections 1–4. The solution is approximated by functions of the form

$$u_k(x,t)=\sum_{i=1}^{k} q_i(t)\,\psi_i(x) \tag{5.1}$$

where the $\{\psi_i\}$ are the eigenfunctions of the Laplacian. (Cf. (1.8).) The coordinates $q_i(t)$ are solutions of a non-linear system of ordinary differential equations. We denote by $\mathscr{M}_k$ the class of functions $u$ lying in the subspace spanned by $\{\psi_1,\dots,\psi_k\}$. For each $t>0$, $u_k\in\mathscr{M}_k$.

Substituting the linear combination $u_k$ into the functionals $K$ and $J$, we get the expressions for kinetic and potential energies

$$K(\dot q_1,\dots,\dot q_k)=\int_\Omega \tfrac{1}{2}\left|\frac{\partial u_k}{\partial t}\right|^2 dx=\tfrac{1}{2}\sum_{i=1}^{k}(\dot q_i)^2 \tag{5.2}$$

$$\begin{aligned} J(q_1,\dots,q_k)&=\int_\Omega \tfrac{1}{2}|\operatorname{grad} u_k|^2+F(x,u_k)\,dx\\ &=\sum_{i=1}^{k}\mu_i\frac{q_i^2}{2}+\int_\Omega F(x,q_1\psi_1+\cdots+q_k\psi_k)\,dx \end{aligned} \tag{5.3}$$

by the orthogonality relations (1.9). The Lagrangian for this $k$-dimensional system is the function

$$L(\dot q_1,\dots,\dot q_k;q_1,\dots,q_k)=K(\dot q_1,\dots,\dot q_k)-J(q_1,\dots,q_k).$$

By the principle of least action, the ordinary differential system satisfied by $q_1,\dots,q_k$ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}-\frac{\partial L}{\partial q_i}=0,\qquad i=1,\dots,k,$$

hence

$$\ddot q_i+\frac{\partial J}{\partial q_i}=0,\qquad i=1,\dots,k. \tag{5.4}$$

It is easily seen that

$$\frac{\partial J}{\partial q_i}=\mu_i q_i+\int_\Omega f(x,q_1\xi_1+\cdots+q_k\psi_k)\psi_i\,dx.$$

Denote points in $k$-dimensional Euclidean space $E^k$ by $q=(q_1,\dots,q_k)$. From our assumptions concerning the local minimum of the potential functional $J$ it follows that the function $J(q)=J(q_1,\dots,q_k)$ in (5.3) is positive in a neighborhood of the origin, that $J(0)=0$, and that $J$ has a local minimum at $q=0$. A potential well $W_k$ is constructed in $E^k$ with depth $d_k$ in the same way $W$ was constructed in $\overset{\circ}{H}{}^1$. Thus, for $q \neq 0$ we let $\lambda_1 = \lambda_1(q)$ be the first value of $\lambda$ at which $J(\lambda q)$ begins to decrease. Then

$$d_k = \inf_{q \neq 0} J(\lambda_1 q) \tag{5.5}$$

where the infimum in (5.5) is taken over all $q \in E^k$.

We have immediately from (5.5) that $d_k \geqq d$. In fact $d_k$ is obtained by taking the infimum of $J(\lambda_1 u)$ only over $u \in \mathscr{M}_k$. Since $d$ is obtained by taking the infimum over the larger class $\overset{\circ}{H}{}^1$, it follows immediately that $d \leqq d_k$. The potential well $W_k$ is defined by

$$W_k = \{q \colon q \in E^k, 0 \leqq J(\lambda_q) \leqq d_k \text{ for } 0 \leqq \lambda \leqq 1\}.$$

It is clear that $W \in W_k$.

Now let $q(t)$ satisfy the differential system (5.4) and the initial conditions

$$\begin{aligned} q(0) &= q_0 = (q_{10}, q_{20}, \dots, q_{k0}), \\ \dot{q}(0) &= \dot{q}_0 = (\dot{q}_{10}, \dots, \dot{q}_{k0}). \end{aligned} \tag{5.6}$$

Assume that $q_0 \in W$ and that the total energy (kinetic plus potential) is less than $d$; that is,

$$\sum_{i=1}^{k} \frac{(\dot{q}_{i0})^2}{2} + J(q_0) < d \leqq d_k. \tag{5.7}$$

Then the initial value problem (5.4), (5.6) has a global solution which remains in $W$ for all $t \geqq 0$.

The existence of a global solution of (5.4) and (5.6) is a consequence of the energy equality

$$\sum_{i=1}^{k} \frac{\dot{q}_i^2}{2} + J(q_1, \dots, q_k) = \text{const.} \tag{5.8}$$

which follows immediately from the differential equations (5.4). In fact, the set $W$ is a bounded set in $E^k$. Thus there is a constant $b > 0$ such that the initial data (5.6) satisfy $|q_{i0}| \leqq b$; moreover, from (5.7) we have $|\dot{q}_{i0}| \leqq \sqrt{2d}$. It can be shown under the assumption that $f_u(x,u)$ is continuous in $u$ that the functions $\dfrac{\partial J}{\partial q_i}$ are Lipshitz continuous functions of $q_i$, $i = 1, \dots, k$. (See the proof of Lemma 3.1.) Therefore the initial value problem (5.4)–(5.6) has a local solution for some time interval $0 \leqq t \leqq \delta$. The constant $\delta$ depends on the constants $b$, $d$, and the moduli of Lipshitz continuity of the functions $\dfrac{\partial J}{\partial q_i}$. The solution $q(t)$ so constructed is unique and remains in $W$ for $0 \leqq t \leqq \delta$. For, if $q(t)$ passes out of $W$ at time $t = t_1$ then we must have $J(q(t_1)) \geqq d$; but this violates the energy equality (5.8) and the fact that the total initial energy is less than $d$ (5.7). Therefore $q(\delta) \in W$ and (5.7) is satisfied at time $t = \delta$. The above argument may be repeated, and the solution can thus be extended to the time interval $\delta \leqq t \leqq 2\delta$. Continuing in this way, it is seen that (5.4), (5.6) does in fact have a global solution which remains in $W$ for all $t \geqq 0$.

The initial conditions (5.6) appropriate to the $k^{\text{th}}$ approximation of the solution of (0.1)−(0.3) are determined as follows. Let the initial data $U(x)$ and $V(x)$ in (0.3) have the orthogonal expansions

$$U(x)=\sum_{i=1}^{\infty}\alpha_i\psi_i(x) \qquad V(x)=\sum_{i=1}^{\infty}\beta_i\psi_i(x).$$

Let $U_k$ and $V_k$ denote the orthogonal projections of $U$ and $V$ into the space $\mathscr{M}_k$. That is

$$U_k(x)=\sum_{i=1}^{k}\alpha_i\psi_1(x) \qquad V(x)=\sum_{i=1}^{\infty}\beta_i\psi_i(x).$$

We have that $\|V_k-V\|_2\to 0$ as $k\to\infty$; assume that $U\in\overset{\circ}{H}{}^1$ and that $\|U_k-U\|_{1,2}\to 0$ as $k\to\infty$. We take the initial conditions (5.6) to be

$$q_{i\,0}=\alpha_i, \qquad \dot{q}_{i\,0}=\beta_i. \tag{5.9}$$

Now let $U\in W$, $V\in L_2(\Omega)$ and let

$$E(U,V)=\int_{\Omega}\{\tfrac{1}{2}|V|^2+\tfrac{1}{2}|\operatorname{grad}U|^2+f(x,U)\}\,dx.$$

The quantity $E(U,V)$ is the total energy of the initial data. The initial value problem (5.4), (5.7) has a global solution if $U_k\in W$ and $E(U_k,V_k)<d$.

Suppose that $E(U,V)<d$. If $J$ is continuous in the strong topology of $\overset{\circ}{H}{}^1$ then for sufficiently large $k$ $E(U_k,V_k)<d$. In fact, by Lemma 3.3 $\lim J(U_k)=J(U)$ and therefore $\lim E(U_k,V_k)=E(U,V)<d$. Moreover, there must be an infinite subsequence $\{U_{k_1}\}$ such that $U_{k_1}\in W$. For, suppose that $U_k\notin W$ for all but a finite number of integers $k$. Then there is a sequence $\{\lambda_k\}$ such that $J(\lambda_k U_k)\geqq d$. Since $0\leqq\lambda_k\leqq 1$ some subsequence, say $\lambda_{k'}$, is convergent; let $\lambda_0=\lim\lambda_{k'}$. Now $0\leqq\lambda_0\leqq 1$ so that $J(\lambda_0 U)<d$. On the other hand, $\|\lambda_{k'}U_k-\lambda_0 U\|_{1,2}\to 0$ as $\to k'\to\infty$; and this implies that $\lim J(\lambda_{k'}V_{k'})=J(\lambda_0 U)$ by Lemma 3.3. This is a contradiction, and therefore there must be an infinite subsequence $\{U_{k_1}\}$ such that $U_{k_1}\in W$.

We assume therefore that there is a sequence $\{k_1\}$ tending to infinity such that $U_{k_1}\in W$ and $E(U_{k_1},V_{k_1})<d$ for all $k_1$. The corresponding solutions of (5.4), (5.9) exist for all time for each $k_1$. From the energy equality (5.8) we see that the functions $\{u_{k_1}(x,t)\}$ satisfy the conditions (4.1)−(4.3). In section 4 it was shown that there is a subsequence which tends to a limit $u$ in the sense (4.5), (4.6) (see also (4.16) and (4.17)). Let us show that $u$ satisfies (2.1) and (2.2). We denote the subsequence of approximate solutions by $\{u_m\}$; let $N_m$ be the dimension of the corresponding system of ordinary differential equations.

We first show that (2.2) is valid for test functions of the form

$$\eta(x,t)=C_i(t)\,\psi_i(x) \tag{5.10}$$

where $C_i(t)$ vanishes identically for sufficiently large $t$. Let $i$ be fixed and let $m>i$. Multiplying the $i^{\text{th}}$ equation of (5.4) by $C_i(t)$, integrating over $[0,\infty)$, and using the orthogonality properties (1.9) of the functions $\{\psi_i\}$, we get

$$
\begin{aligned}
0 &= \int_0^\infty \{\ddot{q}_i + \mu_i q_i + \int_\Omega f(x,u_m)\psi_i\,dx\}\, C_i(t)\,dt \\
&= \int_0^\infty \int_\Omega \left\{ \sum_{j=1}^{N_m} \ddot{q}_j \psi_j + \sum_{j=1}^{N_m} \mu_j q_j \psi_j + f(x,u_m) \right\} C_i(t)\,\psi_i(x)\,dx\,dt \\
&= \int_0^\infty \int_\Omega \left\{ \frac{\partial^2 u_m}{\partial t} - \nabla^2 u_m + f(x,u_m) \right\} C_i(t)\,\psi_i(x)\,dx\,dt.
\end{aligned}
$$

Now integrating by parts we get

$$
\int_\Omega \frac{\partial u_m}{\partial t}(x,0)\,\eta(x,0)\,dx + \int_0^\infty \int_\Omega \left\{ \frac{\partial u_m}{\partial t}\frac{\partial \eta}{\partial t} - \sum_{i=1}^{N_m} \frac{\partial u_m}{\partial x_i}\frac{\partial \eta}{\partial x_i} - f(x,u_m)\,\eta \right\} dx\,dt = 0. \quad (5.11)
$$

Now let $m$ tend to infinity in (5.11). We take $u_m(x,0) = U_m(x)$ and $\frac{\partial u_m}{\partial t}(x,0) = V_m(x)$. By (4.5), (4.6), and the fact that $\|V_m - V\|_2$ and $\|U_m - U\|_{1,2}$ tend to zero as $m$ tends to infinity, the limit function $u$ and its weak derivatives $u_i$, $i = 0, \ldots, n$ satisfy (2.2) for any $\eta$ of the form (5.10).

By linearity $u$ satisfies (2.2) for any test function of the form

$$
\eta_k = C_1(t)\,\psi_1(x) + \cdots + C_k(t)\,\psi_k(x). \quad (5.12)
$$

Now let $\eta(x,t)$ be any function in $\mathscr{S}$. Let $\eta$ be approximated by linear combinations (5.13) and suppose that the norms

$$
\int_\Omega \sum_{i=1}^{n} \left( \frac{\partial^2 \eta}{\partial x_i^2} - \frac{\partial^2 \eta_k}{\partial x_i^2} \right)^2 dx
$$

tend to zero uniformly in $t$ on $0 \leqq t < \infty$ as $k$ tends to infinity. Then $\eta_k$ tends uniformly to $\eta$ on $\Omega_\infty$ as $k$ tends to infinity. If $n = 1$ this is obvious. If $n = 2, 3$ the statement follows from the representation

$$
\eta(x,t) = \int_\Omega G(x,y)\,\Delta_y \eta(y,t)\,dy,
$$

where $G(x,y)$ is Green's function on the domain $\Omega$, and the fact that

$$
\int_\Omega |G(x,y)|^2\,dy \leqq \text{const.}, \qquad x \in \Omega.
$$

For, by Schwarz's inequality

$$
\begin{aligned}
|\eta(x,t) - \eta_k(x,t)| &\int_\Omega |G(x,y)| \cdot |\Delta y(\eta - \eta_k)|\,dy \\
&\leqq \text{const.} \left[ \int_\Omega \sum_{i=1}^{n} \left( \frac{\partial^2 \eta}{\partial x_i^2} - \frac{\partial^2 \eta_k}{\partial x_i^2} \right)^2 dx \right]^{\frac{1}{2}}.
\end{aligned}
$$

Now (2.2) holds for each of the approximations $\eta_k$. Letting $k$ tend to infinity and using the fact that the partial derivatives of $\eta_k$ tend strongly in the $L_2$ norm to those of $\eta$, we see that (2.2) is valid for the test function $\eta$ as well.

To show that $u$ and its weak derivative $u_0$ satisfy (2.1) we first note that

$$\int_0^\infty \int_\Omega \left(\frac{\partial u_m}{\partial t}\eta + u_m \frac{\partial \eta}{\partial t}\right) dy\, dt = -\int_\Omega U_m(x)\eta\, dx$$

for any $\eta \in \mathscr{S}$. Letting $m$ tend to infinity, we see that (2.1) is satisfied by $u$, $u_0$ and $U$.

It remains to show that the energy inequality (0.5) is valid. First note that (0.5) holds for the approximations $\{u_m\}$. Since $u_m$ tends to $u$ in $L_2(\Omega_T)$ for any $T>0$, we have that

$$\lim_{m\to\infty} \int_G \int_\Omega |u_m(x,t) - u(x,t)|^2\, dy\, dt = 0$$

for any measurable set $G$ of finite measure in $[0,\infty)$. In the same way in which Lemma 3.3. is proved it can be shown that

$$\lim_{m\to\infty} \int_G \int_\Omega F(x, u_m)\, dx\, dt = \int_G \int_\Omega F(x,u)\, dx\, dt.$$

On the other hand, since the derivatives $\dfrac{\partial u_m}{\partial x_i}$ tend weakly to the weak derivatives $u_i$, $i=0, \ldots, \eta$, we have

$$\int_G \int_\Omega |u_i|^2\, dx\, dx_0 \leqq \int_G \int_\Omega \left|\frac{\partial u_m}{\partial x_i}\right|^2 dx\, dt, \qquad i=0, \ldots, n.$$

Therefore for any set $G$ of finite measure in $[0,\infty)$

$$\begin{aligned}
\int_G \int_\Omega \left\{\sum_{i=0}^n \frac{(u_i)^2}{2} + F(x,u)\right\} dx\, dt &\leqq \lim_{m\to\infty} \int_G \int_\Omega \left\{\sum_{i=0}^n \left(\frac{\partial u_m}{\partial x_i}\right)^2 + F(x,u_m)\right\} dx\, dt \\
&= \int_G \int_\Omega \left\{\frac{V^2}{2} + \sum_{i=1}^n \left(\frac{\partial U}{\partial x_i}\right)^2 + F(x,U)\right\} dx\, dt \\
&= \lim_{m\to\infty} \int_G \int_\Omega \left\{\frac{(V_m)^2}{2} + \sum_{i=1}^n \left(\frac{\partial U_m}{\partial x_i}\right)^2 + F(x,U_m)\right\} dx\, dt \\
&= E(U,V)\cdot \operatorname{meas}(G).
\end{aligned}$$

Hence

$$\int_G \left\{E(U,V) - \int_\Omega \sum_{i=0}^n \frac{u_i^2}{3} + F(x,u)\, dx\right\} dt \geqq 0.$$

Since this holds for arbitrary sets $G$ of finite measure the integrand must be positive for all $t$ except possibly a set of measure zero. If necessary, the weak solution and its derivatives may be modified on a set of measure zero in $[0,\infty)$; (0.5) then holds everywhere.

## 6. Examples

Let us consider as a special case the equation

$$u_{tt} - \nabla^2 u + \gamma u^p = 0 \tag{6.1}$$

where $p$ is an integer and $\gamma$ is a constant. In case $n=1, 2$ no other restrictions are placed on $p$; but if $n=3$ it is assumed that $1 \leqq p \leqq 5$. In the last part of section 6 it is shown that (6.1) has a global solution for any initial data with finite energy if $\gamma>0$ and $p$ is odd. However, if $p$ is even or if $p$ is odd and $\gamma<0$, then the energy integral associated with (6.1) is no longer positive definite, and so the question of the existence of global solutions of (6.1) is non-trivial.

The equation for the stationary solutions of (6.1) is

$$\nabla^2 u=\gamma u^p. \tag{6.2}$$

It is easily seen that the functional $J$ has a local minimum at $u_0\equiv 0$. In case $n=1,2$ the assumptions (i)–(vi) are satisfied and there is a potential well $W$ of positive depth at the origin.

Let us show that $d>0$ in case $n=3$. We first show that for $|||u|||_2$ sufficiently small $J'(\lambda u)>0$ for $0<\lambda\leqq 1$. In fact,

$$J'(\lambda u)=\int_\Omega (\lambda|\operatorname{grad} u|^2+\lambda^p\gamma u^{p+1})\,dx=\lambda\{(|||u|||_2)^2+\gamma\lambda^{p-1}\int_\Omega u^{p+1}\,dx\}.$$

By Sobolev's inequality (1.5)

$$|\gamma\lambda^{p-1}\int_\Omega u^{p+1}\,dx|\leqq C(|||u|||_2)^{p+1}$$

for some positive constant $C$. Since $p+1\geqq 3$, $J'(\lambda u)>0$ for all $0<\lambda\leqq 1$, provided that $|||u|||_2$ is sufficiently small, say $|||u|||_2\leqq\alpha$. Now if $J'(\lambda u)>0$ for $0<\lambda\leqq 1$ we have

$$\int_\Omega \gamma u^{p+1}\,dx\geqq -\int_\Omega |\operatorname{grad} u|^2\,dx;$$

hence

$$\begin{aligned}J(u)&=\int_\Omega\left(\frac{|\operatorname{grad} u|^2}{2}+\gamma\frac{u^{p+1}}{p+1}\right)dx\\&\geqq\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega|\operatorname{grad} u|^2\,dx\geqq\frac{p-1}{2(p+1)}\alpha^2.\end{aligned}$$

Thus $d$ is positive.

Using the fact that

$$F(u)=\gamma\frac{u^{p+1}}{p+1}=\frac{1}{p+1}f(u) \tag{6.3}$$

it can be shown that $|||u|||_2$ remains bounded as $u$ varies over $W$. (See the proof of Lemma 3.2.) A global solution of (6.1) can therefore be constructed using the approximation method discussed in sections 1–5, provided that the initial data are sufficiently close to the origin in the energy norm. Note that the relationship (6.3) has allowed us to weaken assumption (vii) slightly.

The equation

$$u_{tt}-\nabla^2 u+u^2=0 \tag{6.4}$$

is an interesting special case. Suppose that $n=1$. Then the domain $\Omega$ is an interval $(0,l)$, and the non-linear boundary value for the stationary solutions of (6.4) in this case is

$$u''=u^2,\qquad u(0)=u(l)=0. \tag{6.5}$$

In addition to the trivial solution $u_0 \equiv 0$ the boundary value problem (6.5) also has a solution which is convex and negative on the interval $(0,l)$ [7]. Let this solution be denoted by $w$.

The stationary point $w$ is not a stable equilibrium of the functional $J$. In fact, the second variation of $J$ at $w$ is

$$\delta^2 J(\eta) = \int_\Omega \left( \frac{|\eta'|^2}{2} + w\eta^2 \right) dx .$$

Since $w=0$ on $\partial\Omega$ it is an admissible variation; and furthermore

$$0 = J'(\lambda w)|_{\lambda=1} = \int_\Omega w^3 \, dx < 0$$

since $w<0$ on $\Omega$.

Thus the potential energy functional $J$ associated with (6.4) has two stationary points: a local minimum at $u_0 \equiv 0$ and a saddle point at $u_0 \equiv w$. The point $w$ is at the top of the "pass" leading out of the potential well at the origin. We have, by (6.6), that

$$d = J(w) = \tfrac{1}{6} \int_\Omega |w'|^2 \, dx .$$

Equation (6.4) has a global weak solution provided that

$$0 \leqq \int_\Omega \left\{ \frac{|\lambda U'|^2}{2} + \frac{(\lambda U)^3}{3} \right\} dx < d$$

for $0 \leqq \lambda \leqq 1$ and

$$0 \leqq \int_\Omega \left( \frac{V^2}{2} + \frac{|U'|^2}{2} + \frac{U^3}{3} \right) dx < d .$$

Finally we consider the case where $f$ is a function of $u$ only, and $F(u)$ is positive, convex, and even. The function $f(u)$ is therefore odd, and $F(0)=f(0)=0$. References for the following remarks may be found in [6].

A function $F$ satisfying the above assumptions is called an $N$-function. It is said to satisfy the $\Delta_2$-condition if there exist constants $k>0$ and $u_0>0$ such that $F(2u) \leqq k F(u)$ for all $u \geqq u_0$. A necessary and sufficient condition that $F$ satisfy the $\Delta_2$ condition is that there exist constants $c>0$ and $u_0>0$ such that

$$\frac{u f(u)}{F(u)} < c \qquad \text{for all } u \geqq u_0 . \tag{6.7}$$

It follows from (6.7) that for all $u \geqq u_0$

$$F(u) \leqq F(u_0) \left( \frac{u}{u_0} \right)^c .$$

Let us show that the boundary-initial value problem (0.1)–(0.3) has a global weak solution whenever the convex positive function $F$ satisfies the $\Delta_2$ condition (provided, of course, that the energy of the initial data is finite).

Let $G$ be the $N$-function complementary to $F$. The functions $F$ and $G$ are connected by the relation

$$|u| f(|u|) = F(u) + G(f(|u|)) . \tag{6.8}$$

The class of real-valued measurable functions on $\Omega_T$ ($T>0$) for which

$$\rho(u;F)=\int_0^T\int_\Omega F(u)\,dx\,dt$$

is finite is denoted by $L_F(\Omega_T)$ and is called an *Orlicz class.* We denote by $L_F^*(\Omega_T)$ the class of all measurable functions $u$ for which

$$\int_0^T\int_\Omega u\,v\,dx\,dt<+\infty$$

for all $v\in L_G(\Omega_T)$. The class $L_F^*(\Omega_T)$ contains the class $L_F(\Omega_T)$, and $L_F^*(\Omega_T)$ is a Banach space relative to the norm

$$\|u\|_F=\sup_{\rho(v;G)\leq 1}\left|\int_0^T\int_\Omega u\,v\,dx\,dt\right|.$$

The Banach space $L_F^*(\Omega_T)$ is called an Orlicz space.

Now let $\mathscr{F}^1(\Omega_T)$ denote the completion of $\hat{\mathscr{B}}^{1,2}(\Omega_T)$ (see section 1) under the norm

$$\|u\|_F^1=\left[\int_0^T\int_\Omega\sum_{i=0}^n\left(\frac{\partial u}{\partial x_i}\right)^2dx\,dx_0\right]^{\frac{1}{2}}+\|u\|_F.$$

The Banach space $\mathscr{F}^1(\Omega)$ consists of functions $u$ which possess weak derivaitves, which vanish on $\partial\Omega$, and which belong to the Orlicz space $L_F^*(\Omega_T)$.

A sequence of approximate solutions is now constructed as in section 5. The initial data is subject to the condition of finite energy $E(U,V)<+\infty$. The potential well in this case is infinitely deep. The approximate solutions $\{u_m(x,t)\}$ satisfy the energy equality

$$\int_\Omega\left\{\sum_{i=0}^n\left|\left(\frac{\partial u_m}{\partial x_i}\right)^2+F(u_m)\right\}dx=E(U_m,V_m).\tag{6.9}$$

An discussed in sections 4 and 5 a subsequence $\{u_{m'}\}$ is selected for which the derivatives converge weakly and which converges in $L_2(\Omega_T)$ for all $T>0$ to a limit function $u(x,t)$. It is assumed that $\lim u_{m'}(x,t)=u(x,t)$ a.e. in $\Omega_\infty$. In order to prove the existence theorem we must establish the convergence theorem

$$\lim_{m'\to\infty}\int_0^T\int_\Omega f(u_{m'})\,\eta\,dx\,dt=\int_0^T\int_\Omega f(u)\,\eta\,dx\,dt\tag{6.10}$$

for all bounded measurable $\eta$ and all $T>0$.

Now by (6.8) and (6.9) we see that the integrals

$$\int_0^T\int_\Omega G(|f(u_{m'})|)\,dx\,dt\tag{6.11}$$

are bounded as $m'$ tends to infinity provided that the integrals

$$\int_0^T\int_\Omega|f(u_{m'})|\cdot|u_{m'}|\,dx\,dt$$

are uniformly bounded. The boundedness of these integrals, however, follows from (6.7) and the energy equality (6.9).

The $N$-function $G$ satisfies the conditions of Lemma 1.3. Therefore from the boundedness of the integrals (6.11) it follows that the functions $\{f(u_{m'})\}$ have uniformly absolutely continuous integrals. The convergence theorem (6.13) is now proved in the same way that Lemma 1.4 was proved.

Thus (0.1)–(0.3) always has a global solution (for finite initial energy) whenever $F(u)$ is positive, convex, and satisfies the $\Delta_2$ condition. Note that the above analysis applies to arbitrary dimensions and that the growth condition (vii) has been considerably weakened.

The method indicated in this paper should be regarded as a general approach to existence theorems for non-linear equations of evolution. For example, the same method should work equally well for parabolic equations or systems. It is also possible that the existence of solutions of non-linear elliptic boundary value problems of the form

$$\nabla^2 u = f(u) \qquad u=0 \quad \text{on} \quad \partial\Omega$$

might also be proved by the method of approximation by systems with a finite number of degrees of freedom.

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