Universal Deformations in Isotropic Incompressible Hyperelastic Materials when the Deformation Tensor has Equal Proper Values

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1. Introduction

In 1954 (1954 [1])*, ERICKSEN gave the solution to a substantial part of the following fundamental problem: to determine the deformations that can be produced in *every* isotropic, incompressible, hyperelastic body, by the application of surface tractions when body forces are absent.

Suppose that the deformation from the undeformed state is determined by the (sufficiently smooth) mapping

$$x^a = x^a(X^A), \quad a = 1, 2, 3, \quad A = 1, 2, 3,^{**} \tag{1.1}$$

where x^a and X^A are the co-ordinates of the points of the body in its deformed and undeformed states, respectively. Suppose further that the metrics of the co-ordinates x^a and X^A are g_{ab} and G_{AB}.

In the absence of inertia and body forces the stress tensor must satisfy the Euler-Cauchy condition

$$t^{ab}_{,b} = 0. \tag{1.2}$$

The constitutive equation for an incompressible hyperelastic material is

$$t^a_b = -p\,\delta^a_b + 2\frac{\partial \Sigma}{\partial I}(c^{-1})^a_b - 2\frac{\partial \Sigma}{\partial II} c^a_b, \tag{1.3}$$

* See also (1965 [1], p. 337).

** We employ Latin indices to denote curvilinear co-ordinates and Greek indices to denote Cartesian co-ordinates.

where p is an arbitrary hydrostatic pressure, and

$$c_{ab}=G_{AB}\frac{\partial X^A}{\partial x^a}\frac{\partial X^B}{\partial x^b}, \tag{1.4}$$

$$(c^{-1})^{ab}=G^{AB}\frac{\partial x^a}{\partial X^A}\frac{\partial x^b}{\partial X^B}, \tag{1.5}$$

and

$$\det c^a_b=1. \tag{1.6}$$

The condition (1.6) ensures that the deformation tensor $\boldsymbol{c}$ determines deformations possible to an incompressible material. In (1.3), the scalars I, II, and III$(=1)$ are the elementary invariants of $\boldsymbol{c}^{-1}$, and $\Sigma=\Sigma(\mathrm{I},\mathrm{II})$ is the strain energy function. The matrix $\{c_{ab}\}$ must be positive definite.

The tensor $\boldsymbol{c}$ must be a metric tensor in Euclidean space. Thus the Riemann curvature based on $\boldsymbol{c}$ must vanish. When $\boldsymbol{c}$ is referred to Cartesian co-ordinates x^α, $\alpha=1,2,3$, thus

$$\boldsymbol{c}=c_{\alpha\beta}\,\boldsymbol{i}_\alpha\boldsymbol{i}_\beta, \tag{1.7}$$

the curvature tensor is given by

$$\begin{aligned}4R_{\alpha\beta\gamma\delta}=2&\left(\frac{\partial^2 c_{\alpha\delta}}{\partial x^\beta\partial x^\gamma}+\frac{\partial^2 c_{\beta\gamma}}{\partial x^\alpha\partial x^\delta}-\frac{\partial^2 c_{\alpha\gamma}}{\partial x^\beta\partial x^\delta}-\frac{\partial^2 c_{\beta\delta}}{\partial x^\alpha\partial x^\gamma}\right)\\ &+(c^{-1})^{\lambda\mu}\left[\left(\frac{\partial c_{\beta\lambda}}{\partial x^\gamma}+\frac{\partial c_{\gamma\lambda}}{\partial x^\beta}-\frac{\partial c_{\beta\gamma}}{\partial x^\lambda}\right)\left(\frac{\partial c_{\alpha\mu}}{\partial x^\delta}+\frac{\partial c_{\delta\mu}}{\partial x^\alpha}-\frac{\partial c_{\alpha\delta}}{\partial x^\mu}\right)\right.\\ &\left.-\left(\frac{\partial c_{\beta\lambda}}{\partial x^\delta}+\frac{\partial c_{\delta\lambda}}{\partial x^\beta}-\frac{\partial c_{\beta\delta}}{\partial x^\lambda}\right)\left(\frac{\partial c_{\alpha\mu}}{\partial x^\gamma}+\frac{\partial c_{\gamma\mu}}{\partial x^\alpha}-\frac{\partial c_{\alpha\gamma}}{\partial x^\mu}\right)\right].\end{aligned} \tag{1.8}$$

A necessary and sufficient condition for the vanishing of $R_{\alpha\beta\gamma\delta}$ is the vanishing of the six components R_{1212}, R_{2323}, R_{3131}, R_{1223}, R_{2331}, R_{3112}.

ERICKSEN determined necessary and sufficient conditions for the existence of a function p satisfying (1.2) and (1.3).

Suppose $\boldsymbol{c}$ to be written in terms of its proper values in the form

$$\boldsymbol{c}=c_1\,\boldsymbol{n}\boldsymbol{n}+c_2\,\boldsymbol{s}\boldsymbol{s}+c_3\,\boldsymbol{b}\boldsymbol{b}. \tag{1.9}$$

The invariants of $\boldsymbol{c}^{-1}$ are given by

$$\mathrm{I}=\frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_3},\quad \mathrm{II}=\frac{1}{c_2c_3}+\frac{1}{c_3c_1}+\frac{1}{c_1c_2},\quad \mathrm{III}=\frac{1}{c_1c_2c_3}=1. \tag{1.10}$$

ERICKSEN'S analysis disclosed the following two classes.

Class A. The Invariants I and II Are Both Constant

By (1.10), the constancy of the invariants I and II is equivalent to the constancy of the proper values c_1, c_2, c_3.

In this case the tensors $c^b_{a,be}$ and $(c^{-1})^b_{a,be}$ must be symmetric in the indices a and e.

While this category certainly includes homogeneous deformations, examples have shown that it includes other deformations as well*. Although the principal stretches are constant the local rotation need not be constant.

Class B. The Invariants I and II Are Not Both Constant

ERICKSEN proved that necessary and sufficient conditions for the existence of the function p satisfying (1.2) and (1.3) are the following.

1) The proper values of $\boldsymbol{c}$ are each functions of a non-constant scalar U:

$$c_1=c_1(U), \quad c_2=c_2(U), \quad c_3=c_3(U); \tag{1.11}$$

2)
$$c_1 c_2 c_3=1; \tag{1.12}$$

3)
$$\boldsymbol{c}=c_1\, \boldsymbol{n}\boldsymbol{n}+c_2\, \boldsymbol{s}\boldsymbol{s}+c_3\, \boldsymbol{b}\boldsymbol{b}; \tag{1.9}$$

4) the unit proper vectors $\boldsymbol{n}$, $\boldsymbol{s}$, and $\boldsymbol{b}$ are such that $\boldsymbol{n}$ may be taken to be normal to the surface $U(x^\alpha)=$constant, so that

$$\boldsymbol{n}=\psi \operatorname{grad} U, \quad \psi=\frac{1}{|\operatorname{grad} U|}, \tag{1.13}$$

and $\boldsymbol{s}$ and $\boldsymbol{b}$ span the tangent plane to the surface $U=$constant:

$$\boldsymbol{s}\cdot \operatorname{grad} U=0, \tag{1.14}$$

$$\boldsymbol{b}\cdot \operatorname{grad} U=0; \tag{1.15}$$

5)
$$(\operatorname{div}\boldsymbol{n})\boldsymbol{n}+\boldsymbol{n}\cdot \operatorname{grad}\boldsymbol{n}=F(U)\,|\operatorname{grad} U|\,\boldsymbol{n}, \quad \text{if } c_1 \neq c_3; \tag{1.16}$$

6)
$$(\operatorname{div}\boldsymbol{s})\boldsymbol{s}+\boldsymbol{s}\cdot \operatorname{grad}\boldsymbol{s}=H(U) \operatorname{grad} U, \quad \text{if } c_2 \neq c_3. \tag{1.17}$$

ERICKSEN proved that, provided (1.16) holds, that is to say, provided $c_1 \neq c_3$, the only universal deformations possible within Class B are the following four families of solutions:

1) Bending, extension and shear of a block.

2) Straightening, extension and shear of a sector of a cylindrical tube.

3) Inflation, bending, torsion, extension and shear of a sector of a cylindrical tube.

4) Inflation or eversion of a sphere.

These universal solutions are discussd in detail in (1965 [1], Section 57, p. 186).

ERICKSEN also considered the case $c_1=c_3$ under the special condition that the vector $\boldsymbol{s}$ is complex-lamellar:

$$\Omega_s=\boldsymbol{s}\cdot \operatorname{curl}\boldsymbol{s}=0. \tag{1.18}$$

* SINGH & PIPKIN (1965 [2]) give the universal solution

$$r=aR, \quad \theta=b\log R+c\Theta, \quad z=dZ, \quad a^2cd=1.$$

SINGH & PIPKIN note that a special case of this deformation: $a^2c=1$, $b^2+c^2=1$, was found by KLINGBEIL & SHIELD. A second special case due to FOSDICK,

$$r=aR, \quad \theta=b\Theta, \quad z=cZ, \quad a^2bc=1, \quad b\neq 1,$$

is noted by TRUESDELL & NOLL (1965 [1], p. 338). This is an example of a deformation in which the invariants of $\boldsymbol{c}$ are constant. However, this particular deformation is a special case of category 3 of the four families of deformations possible under Class B.

He proved that, in this case, the material must either be unstrained or else the deformation is of Class A.

The general case of equal proper values, $c_1 = c_3$, in which case the condition (1.18) does not hold, remains to be considered. This paper is devoted to an analysis of this case.

We shall establish

Theorem 1.1. *The only universal deformations possible in the case*

$$c_1 = c_3 = \eta \tag{1.19}$$

correspond either to

1) deformations in which η *is constant,*

or

2) deformations in which the vector-lines of s *are circular helices mounted on concentric circular cylinders, the cases where the helices reduce to concentric circles or straight lines, being excluded.*

Deformations for which η is constant belong to Class A. Deformations corresponding to the vector-line geometries encompassed by 2) were considered by ERICKSEN for the case of distinct proper values. The present case thus becomes a special case of that already considered. Theorem 1.1 thus tells us that the case

$$c_1 = c_3 = \eta \tag{1.19}$$

introduces no new families of deformations, provided η is not constant.

The present analysis, taken in conjunction with ERICKSEN'S work, will thus establish the following result:

Theorem 1.2. *The only deformations that can be produced in* all *homogeneous, isotropic, incompressible, hyperelastic bodies by application of suitable surface tractions alone, are homogeneous deformations, the four families of deformations cited above, and non-homogeneous deformations in which the principal stretches are constant and both* $c^b_{a,be}$ *and* $(c^{-1})^b_{a,be}$ *are symmetric in the indices a and e.*

Preliminary Discussion

A peculiar complication arises in the case when the proper values c_1 and c_3 are equal. We lose the condition (1.16). ERICKSEN established the capital result that (1.16) implied that the surfaces $U(x^\alpha)$ = constant must be concentric spheres, parallel planes or concentric circular cylinders*. In the absence of (1.16), we

* In particular, since the surfaces $U(x^\alpha)$ = constant are parallel, the vector-lines of n are rectilinear, and

$$\operatorname{curl} n = 0. \tag{a}$$

Thus by (1.22), and anticipating equation (2.6) of Chapter 2,

$$\operatorname{div} s = \theta_{ns} + \theta_{bs} = 0. \tag{b}$$

Anticipating equation (2.11) of Chapter 2, from (a) we have

$$\operatorname{curl} n = -\operatorname{div} b\, s + \theta_{ns} b = 0. \tag{c}$$

Thus, the parameters θ_{ns} and θ_{bs} each vanish. The parameter θ_{bs} is the geodesic curvature of the b-lines, its vanishing ensures that the surfaces U = constant are developables. They cannot be concentric spheres. ERICKSEN proves this result by a different method.

cannot in general discover the nature of the surfaces $U(x^\alpha)$ = constant, independently of the equations resulting from the vanishing of the curvature tensor, but must glean this information from these equations themselves.*

By (1.13) the unit vector $\boldsymbol{n}$ is complex-lamellar, and

$$\Omega_n = \boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n} = 0. \tag{1.20}$$

Taking the dot product of (1.17) with $\boldsymbol{s}$ and noting that $\boldsymbol{s}$ is a unit vector, so that

$$\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s} \cdot \boldsymbol{s} = 0, \tag{1.21}$$

we obtain

$$\operatorname{div} \boldsymbol{s} = H(U)\, \boldsymbol{s} \cdot \operatorname{grad} U = 0 \quad \text{by (1.14)}. \tag{1.22}$$

By (1.17) and (1.22)

$$\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s} = H(U) \operatorname{grad} U. \tag{1.23}$$

When $\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s}$ is not zero, so that the vector-lines of $\boldsymbol{s}$ are not rectilinear, from (1.13) and (1.23) we obtain

$$\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s} = H(U)\, |\operatorname{grad} U|\, \boldsymbol{n}. \tag{1.24}$$

At the same time

$$\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s} = \kappa_s \boldsymbol{n}, \tag{1.25}$$

where κ_s is the curvature of the vector-line of $\boldsymbol{s}$ and $\boldsymbol{n}$ is the principal normal. The unit vector $\boldsymbol{n}$ of (1.9) thus points along the principal normal to the vector-line of $\boldsymbol{s}$. The unit vector $\boldsymbol{b}$ points along the binormal.

Since the principal normal to the vector-lines of $\boldsymbol{s}$ coincides with the normal to the surface U = constant, the vector-lines of $\boldsymbol{s}$ must be geodesics on the surface U = constant. The vector-lines of $\boldsymbol{b}$ are parallel curves on the surface.

Furthermore, from (1.24) and (1.25),

$$\kappa_s = H(U)\, |\operatorname{grad} U|. \tag{1.26}$$

When $\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s}$ vanishes, (1.14) and (1.17) give the two conditions

$$\operatorname{div} \boldsymbol{s} = 0, \tag{1.27}$$

and

$$H(U) = 0. \tag{1.28}$$

In this case, by (1.3), $\boldsymbol{n}$ is the normal to the surface $U(x^\alpha)$ = constant. The vector-lines of $\boldsymbol{s}$ are straight lines on the surface $U(x^\alpha)$ = constant, they are simultaneously geodesics and rectilinear asymptotic lines on the surface. The vector-lines of $\boldsymbol{b} = \boldsymbol{s} \times \boldsymbol{n}$ are again parallel curves on the surface.

The analysis we shall give will be valid even though the vector-lines of $\boldsymbol{s}$ are rectilinear. Since $\boldsymbol{s}$ is of unit magnitude,

$$\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s} = \boldsymbol{\omega} \times \boldsymbol{s}, \tag{1.29}$$

* The case of $\boldsymbol{s}$ complex-lamellar, $\Omega_s = 0$, is an exception. In this case, the surfaces $U(x^\alpha)$ = constant are concentric cylinders or parallel planes. This result was proved by PRIM (1952 [1]). ERICKSEN analysed the case $c_1 = c_3$ under the condition $\Omega_s = 0$.

where

$$\omega = \operatorname{curl} s. \tag{1.30}$$

From (1.23) and (1.28), we see that

$$\operatorname{curl}(\omega \times s) = 0. \tag{1.31}$$

The vector field of s is a steady, isochoric, circulation-preserving field of constant magnitude.

Collecting these results pertaining to the case $c_1 = c_3$, we have, from (1.9), $(1.10)_3$ and (1.19),

$$c = \frac{1}{\eta^2} s s + \eta\, n n + \eta\, b b, \tag{1.32}$$

where

$$\eta = \eta(U), \tag{1.33}$$

and where

1) n points along the normal to the surface $U(x^\alpha) = \text{constant}$,

2) $$\Omega_n = n \cdot \operatorname{curl} n = 0, \tag{1.20}$$

3) the vector-lines of s and b are respectively geodesics and parallel curves on the representative surface $U(x^\alpha) = \text{constant}$,

4) $$\operatorname{div} s = 0, \tag{1.22}$$

5) $$\operatorname{curl}(\omega \times s) = 0, \quad \text{where } \omega = \operatorname{curl} s. \qquad (1.31), (1.30)$$

We require that the Riemann curvature tensor based on c should vanish.

2. Background Material on Vector Fields

Before proceeding with the analysis, it is necessary to invoke some basic formulae appropriate to the vector field s. These results have been derived in earlier works (1969 [1], 1969 [2]). We employ the notation $\frac{\delta F}{\delta s}, \frac{\delta F}{\delta n}, \frac{\delta F}{\delta b}$ to denote the components of grad F in the preferred directions s, n and b.

We write

$$\Omega_s = s \cdot \operatorname{curl} s, \quad \Omega_n = n \cdot \operatorname{curl} n, \quad \Omega_b = b \cdot \operatorname{curl} b, \tag{2.1}$$

and our formulae pertain to the case

$$\Omega_n = n \cdot \operatorname{curl} n = 0, \tag{2.2}$$

or, equivalently,

$$n = \psi \operatorname{grad} U. \tag{2.3}$$

For the isochoric vector field s under consideration we also have the condition

$$\operatorname{div} s = \theta_{ns} + \theta_{bs} = 0, \tag{2.4}$$

where

$$\theta_{ns} = n \cdot \operatorname{grad} s \cdot n, \quad \theta_{bs} = b \cdot \operatorname{grad} s \cdot b. \tag{2.5}$$

The gradients of $\boldsymbol{s}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ are represented by*

$$\begin{aligned}
\operatorname{grad} \boldsymbol{s} = \quad & \boldsymbol{s}\boldsymbol{n}\kappa_s \\
& \boldsymbol{n}\boldsymbol{n}\theta_{ns} \quad + \boldsymbol{n}\boldsymbol{b}(\Omega_s - \tau_s) \\
& -\boldsymbol{b}\boldsymbol{n}\tau_s \quad + \boldsymbol{b}\boldsymbol{b}\theta_{bs},
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\operatorname{grad} \boldsymbol{n} = & -\boldsymbol{s}\boldsymbol{s}\kappa_s \quad + \boldsymbol{s}\boldsymbol{b}\tau_s \\
& -\boldsymbol{n}\boldsymbol{s}\theta_{ns} \quad - \boldsymbol{n}\boldsymbol{b}\operatorname{div}\boldsymbol{b} \\
& \boldsymbol{b}\boldsymbol{s}\tau_s \quad + \boldsymbol{b}\boldsymbol{b}(\kappa_s + \operatorname{div}\boldsymbol{n}),
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\operatorname{grad} \boldsymbol{b} = \quad & -\boldsymbol{s}\boldsymbol{n}\tau_s \\
& -\boldsymbol{n}\boldsymbol{s}(\Omega_s - \tau_s) + \boldsymbol{n}\boldsymbol{n}\operatorname{div}\boldsymbol{b} \\
& -\boldsymbol{b}\boldsymbol{s}\theta_{bs} \quad - \boldsymbol{b}\boldsymbol{n}(\kappa_s + \operatorname{div}\boldsymbol{n}).
\end{aligned} \tag{2.8}$$

In these formulae κ_s and τ_s are the curvature and the torsion of the s-lines. τ_s is related to Ω_s and Ω_b through

$$2\tau_s = \Omega_s - \Omega_b. \tag{2.9}$$

From (2.6), (2.7), (2.8), we obtain

$$\operatorname{curl} \boldsymbol{s} = \Omega_s \boldsymbol{s} + \kappa_s \boldsymbol{b}, \tag{2.10}$$

$$\operatorname{curl} \boldsymbol{n} = -\operatorname{div} \boldsymbol{b}\, \boldsymbol{s} + \theta_{ns} \boldsymbol{b}, \tag{2.11}$$

$$\operatorname{curl} \boldsymbol{b} = (\kappa_s + \operatorname{div} \boldsymbol{n})\boldsymbol{s} - \theta_{bs} \boldsymbol{n} + \Omega_b \boldsymbol{b}. \tag{2.12}$$

Since the divergence of each of these expressions must vanish, we obtain

$$\frac{\delta \Omega_s}{\delta s} + \frac{\delta \kappa_s}{\delta b} + \Omega_s(\theta_{ns} + \theta_{bs}) + \kappa_s \operatorname{div} \boldsymbol{b} = 0, \tag{2.13}$$

$$-\frac{\delta}{\delta s} \operatorname{div} \boldsymbol{b} + \frac{\delta}{\delta b} \theta_{ns} - \theta_{bs} \operatorname{div} \boldsymbol{b} = 0, \tag{2.14}$$

$$\frac{\delta}{\delta s}(\kappa_s + \operatorname{div} \boldsymbol{n}) - \frac{\delta}{\delta n} \theta_{bs} + \frac{\delta}{\delta b} \Omega_b + (\kappa_s + \operatorname{div} \boldsymbol{n}) \operatorname{div} \boldsymbol{s} - \theta_{bs} \operatorname{div} \boldsymbol{n} + \Omega_b \operatorname{div} \boldsymbol{b} = 0. \tag{2.15}$$

The vector-lines of $\boldsymbol{s}$ and $\boldsymbol{b}$ are, respectively, geodesics and geodesic parallels on the representative surface $U(x^\alpha) = \text{constant}$.

The parameter θ_{bs} is identified as the geodesic curvature of the $\boldsymbol{b}$-lines; thus

$$\theta_{bs} = \kappa_{bG}. \tag{2.16}$$

* In the ensuing analysis we shall assume that (2.2) holds. However, in the interest of giving results of general reference value we shall not eliminate θ_{bs} in favor of θ_{ns} by (2.4), unless we specifically state that we have done so. Thus, θ_{bs} is eliminated in the proof of Theorem 2.1 at the end of this section and in the analysis towards the final proof in Sections 4 and 5.

θ_{bs} is related to the Gaussian curvature of the surface $U(x^\alpha)=$constant through the Gauss equation

$$\frac{\delta\theta_{bs}}{\delta s}+\theta_{bs}^2=-G. \tag{2.17}$$

One also has

$$\kappa_s+\operatorname{div} n=-\kappa_{b_N}, \tag{2.18}$$

where κ_{b_N} is the normal curvature of the b-lines. κ_{b_N} is related to the Gaussian curvature of the surface through the equation

$$\kappa_s(\kappa_s+\operatorname{div}\boldsymbol{n})+\tau_s^2=-\kappa_s\kappa_{b_N}+\tau_s^2=-G. \tag{2.19}$$

The Mainardi-Codazzi equations are

$$\frac{\delta\tau_s}{\delta s}+\frac{\delta\kappa_s}{\delta b}+2\theta_{bs}\tau_s=0, \tag{2.20}$$

$$\frac{\delta\kappa_{b_N}}{\delta s}+\frac{\delta\tau_s}{\delta b}+\theta_{bs}(\kappa_{b_N}-\kappa_s)=0. \tag{2.21}$$

From (2.13) and (2.20) we obtain the relation

$$\frac{\delta}{\delta s}(\tau_s-\Omega_s)=\kappa_s\operatorname{div}\boldsymbol{b}-2\theta_{bs}\tau_s. \tag{2.22}$$

For a point function F, the condition

$$\operatorname{curl}\operatorname{grad}F=\operatorname{curl}\left(\boldsymbol{s}\frac{\delta F}{\delta s}+\boldsymbol{n}\frac{\delta F}{\delta n}+\boldsymbol{b}\frac{\delta F}{\delta b}\right)=0 \tag{2.23}$$

leads to the commutation formulae

$$\frac{\delta^2 F}{\delta b\,\delta n}-\frac{\delta^2 F}{\delta n\,\delta b}=\Omega_s\frac{\delta F}{\delta s}-\operatorname{div}\boldsymbol{b}\frac{\delta F}{\delta n}-\kappa_{b_N}\frac{\delta F}{\delta b}, \tag{2.24}$$

$$\frac{\delta^2 F}{\delta s\,\delta b}-\frac{\delta^2 F}{\delta b\,\delta s}=-\theta_{bs}\frac{\delta F}{\delta b}, \tag{2.25}$$

$$\frac{\delta^2 F}{\delta n\,\delta s}-\frac{\delta^2 F}{\delta s\,\delta n}=\kappa_s\frac{\delta F}{\delta s}+\theta_{ns}\frac{\delta F}{\delta n}+\Omega_b\frac{\delta F}{\delta b}. \tag{2.26}$$

We write

$$\boldsymbol{\omega}=\operatorname{curl}\boldsymbol{s}. \tag{2.27}$$

From the general representation for $\operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{v})$ given in (1969 [1]), we have, for any vector field $\boldsymbol{s}$ of unit magnitude for which Ω_n is zero,

$$\operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{s})=\operatorname{div}(\Omega_s\boldsymbol{s})\boldsymbol{s}+[\operatorname{div}(\kappa_s\boldsymbol{s})-\theta_{bs}\kappa_s]\boldsymbol{b}=0. \tag{2.28}$$

Invoking the condition

$$\operatorname{div}\boldsymbol{s}=\theta_{ns}+\theta_{bs}=0, \tag{2.4}$$

we obtain

$$\frac{\delta \Omega_s}{\delta s} = 0 \tag{2.29}$$

and

$$\frac{\delta \kappa_s}{\delta s} = -\theta_{ns} \kappa_s . \tag{2.30}$$

Also, by (2.13)

$$\frac{\delta \kappa_s}{\delta b} = -\kappa_s \operatorname{div} \boldsymbol{b} , \tag{2.31}$$

and by (2.4), (2.22) and (2.29)

$$\frac{\delta \tau_s}{\delta s} = 2\theta_{ns} \tau_s + \kappa_s \operatorname{div} \boldsymbol{b} . \tag{2.32}$$

When κ_s and τ_s are non-vanishing, we see from (2.30) and (2.32) that the vanishing of θ_{ns} and div $\boldsymbol{b}$ is equivalent to the vanishing $\delta \kappa_s / \delta s$ and $\delta \tau_s / \delta s$. The latter condition is necessary and sufficient for an individual s-line to be a circular-helix. Again, by (2.31) the vanishing of div $\boldsymbol{b}$ is equivalent to the vanishing of $\delta \kappa_s / \delta b$. By (2.4) the vanishing of θ_{ns} is equivalent to the vanishing of θ_{bs}; thus by (2.17) the Gaussian curvature G of the surface $U(x^\alpha) = \text{constant}$ is zero. By (2.19) $\delta \kappa_{b_N} / \delta s$ is zero, and by (2.21) $\delta \tau_s / \delta b$ vanishes. The vanishing of θ_{ns} and div $\boldsymbol{b}$ is thus a sufficient condition to ensure that κ_s and τ_s are each constant over the surface $U(x^\alpha)$; that is, to ensure that these surfaces are coaxial circular cylinders. The vanishing of θ_{ns} and div $\boldsymbol{b}$ is also a necessary condition, for if these parameters did not vanish, an s-line would not be a circular helix, nor indeed, by (2.11), could the surfaces $U(x^\alpha) = \text{constant}$ be parallel.

We collect these results as follows:

Theorem 2.1. *Let s be a steady, isochoric, circulation-preserving vector field of constant unit magnitude; then the vector-lines of s are geodesics on the one parameter family of surfaces $U(x^\alpha) = constant$, and*

$$\frac{\delta \Omega_s}{\delta s} = 0 , \tag{2.29}$$

$$\frac{\delta \kappa_s}{\delta s} = -\theta_{ns} \kappa_s , \tag{2.30}$$

$$\frac{\delta \kappa_s}{\delta b} = -\kappa_s \operatorname{div} \boldsymbol{b} , \tag{2.31}$$

$$\frac{\delta \tau_s}{\delta s} = 2\theta_{ns} \tau_s + \kappa_s \operatorname{div} \boldsymbol{b} . \tag{2.32}$$

Provided κ_s and τ_s do not vanish, the vector-lines of s will be circular helices and the surfaces $U = constant$ will be coaxial circular cylinders, if and only if θ_{ns} and div $\boldsymbol{b}$ *each vanish.*

3. Further Results for a Vector Field s for which $\Omega_n = 0$

The vector field defined by the title of this section is characterized by the condition that its vector-lines are geodesics on the surfaces associated with a normal congruence of n-lines. The binormals are tangent to parallel curves on these surfaces. These conditions are of local validity only.

Let $\boldsymbol{i}_\alpha$, $\alpha = 1, 2, 3$ be Cartesian unit vectors. Let $\boldsymbol{s}$, $\boldsymbol{n}$, and $\boldsymbol{b}$, denoting the tangent to the vector-line, the principal normal, and binormal, be represented alternatively by s_β, $\beta = 1, 2, 3$, with the additional convention that the gradient components $\delta/\delta s$, $\delta/\delta n$, $\delta/\delta b$ are represented alternatively by $\delta/\delta s^\beta$, $\beta = 1, 2, 3$, respectively. We may write

$$\boldsymbol{i}_\alpha = a_{\alpha\beta}\, s_\beta, \tag{3.1}$$

where

$$a_{\alpha\beta} = a_{\alpha\beta}(x^\gamma), \tag{3.2}$$

where the matrix $\{a_{\alpha\beta}\}$ is orthogonal. We may assert that $\{a_{\alpha\beta}\}$ is the unit matrix at the origin, but not in a neighbourhood. Thus we may take $a_{\alpha\beta} = \delta_{\alpha\beta}$ *after* taking gradients. From (3.1)

$$\begin{aligned} 0 &= \operatorname{grad} \boldsymbol{i}_\alpha = \operatorname{grad}(a_{\alpha\beta}\, s_\beta) \\ &= (\operatorname{grad} a_{\alpha\beta})\, s_\beta + a_{\alpha\beta} \operatorname{grad} s_\beta . \end{aligned} \tag{3.3}$$

From (3.3)

$$\begin{aligned} s_\lambda \cdot (\operatorname{grad} a_{\alpha\beta})\, s_\beta &= -a_{\alpha\beta}\, s_\lambda \cdot \operatorname{grad} s_\beta \\ &= -a_{\alpha\mu} \frac{\delta s_\mu}{\delta s^\lambda} . \end{aligned} \tag{3.4}$$

From (3.4)

$$\frac{\delta}{\delta s^\lambda} a_{\alpha\beta} = -a_{\alpha\mu} \frac{\delta s_\mu}{\delta s^\lambda} \cdot s_\beta . \tag{3.5}$$

Setting $a_{\alpha\mu} = \delta_{\alpha\mu}$, from (3.5) we obtain

$$\frac{\delta}{\delta s^\lambda} a_{\alpha\beta} = -\frac{\delta s_\alpha}{\delta s^\lambda} \cdot s_\beta . \tag{3.6}$$

From (2.6), (2.7), (2.8), (2.18) and (3.6), we obtain the following relations:

$$\left\{\frac{\delta a_{\alpha\beta}}{\delta s}\right\} = \begin{bmatrix} 0 & -\kappa_s & 0 \\ \kappa_s & 0 & -\tau_s \\ 0 & \tau_s & 0 \end{bmatrix}, \tag{3.7}$$

$$\left\{\frac{\delta a_{\alpha\beta}}{\delta n}\right\} = \begin{bmatrix} 0 & -\theta_{ns} & -(\Omega_s - \tau_s) \\ \theta_{ns} & 0 & \operatorname{div} \boldsymbol{b} \\ \Omega_s - \tau_s & -\operatorname{div} \boldsymbol{b} & 0 \end{bmatrix}, \tag{3.8}$$

$$\left\{\frac{\delta a_{\alpha\beta}}{\delta b}\right\} = \begin{bmatrix} 0 & \tau_s & -\theta_{bs} \\ -\tau_s & 0 & \kappa_{b_N} \\ \theta_{bs} & -\kappa_{b_N} & 0 \end{bmatrix}. \tag{3.9}$$

From (3.4)

$$\left(\frac{\delta}{\delta s^\mu} a_{\alpha\beta}\right) s_\beta = -a_{\alpha\beta}\, s_\mu \cdot \operatorname{grad} s_\beta ,$$

so that

$$\frac{\delta^2 a_{\alpha\beta}}{\delta s^\lambda \delta s^\mu} s_\beta = -\left[\frac{\delta a_{\alpha\theta}}{\delta s^\mu}\frac{\delta s_\theta}{\delta s^\lambda} + \frac{\delta a_{\alpha\theta}}{\delta s^\lambda} s_\mu \cdot \operatorname{grad} s_\theta + \frac{\delta s_\mu}{\delta s^\lambda} \cdot \operatorname{grad} s_\alpha + s_\mu \cdot \frac{\delta s_\varphi}{\delta s^\lambda}\frac{\delta s_\alpha}{\delta s^\varphi} + \frac{\delta^2 s_\alpha}{\delta s^\lambda \delta s^\mu}\right],$$

or

$$\frac{\delta^2 a_{\alpha\beta}}{\delta s^\lambda \delta s^\mu} = -\left[\frac{\delta a_{\alpha\theta}}{\delta s^\mu}\frac{\delta s_\theta}{\delta s^\lambda} + \frac{\delta a_{\alpha\theta}}{\delta s^\lambda}\frac{\delta s_\theta}{\delta s^\mu} + \frac{\delta s_\mu}{\delta s^\lambda} \cdot \operatorname{grad} s_\alpha + s_\mu \cdot \frac{\delta s_\varphi}{\delta s^\lambda}\frac{\delta s_\alpha}{\delta s^\varphi} + \frac{\delta^2 s_\alpha}{\delta s^\lambda \delta s^\mu}\right] \cdot s_\beta. \tag{3.10}$$

Since

$$(\operatorname{grad} s_\alpha) \cdot s_\alpha = 0, \tag{3.11}$$

we note that the third and fourth terms on the right hand side of (3.10) vanish in the case $\alpha = \beta$.

From (2.6), (2.7), (2.8), (2.18) and (3.10) we obtain the following relations:

$$\left\{\frac{\delta^2 a_{\alpha\beta}}{\delta s^2}\right\} = \begin{bmatrix} -\kappa_s^2 & -\frac{\delta \kappa_s}{\delta s} & \tau_s \kappa_s \\ \frac{\delta \kappa_s}{\delta s} & -(\kappa_s^2 + \tau_s^2) & -\frac{\delta \tau_s}{\delta s} \\ \tau_s \kappa_s & \frac{\delta \tau_s}{\delta s} & -\tau_s^2 \end{bmatrix}, \tag{3.12}$$

$$\left\{\frac{\delta^2}{\delta n \delta s} a_{\alpha\beta}\right\} = \begin{bmatrix} -\theta_{ns}\kappa_s & -\left(\frac{\delta \kappa_s}{\delta n} + (\Omega_s - \tau_s)\tau_s\right) & \theta_{ns}\tau_s \\ \frac{\delta \kappa_s}{\delta n} & -\theta_{ns}\kappa_s & -\frac{\delta \tau_s}{\delta n} \\ -\operatorname{div} \boldsymbol{b}\, \kappa_s & \frac{\delta \tau_s}{\delta n} - \kappa_s(\Omega_s - \tau_s) & \operatorname{div} \boldsymbol{b}\, \tau_s \end{bmatrix}, \tag{3.13}$$

$$\left\{\frac{\delta^2}{\delta b \delta s} a_{\alpha\beta}\right\} = \begin{bmatrix} \tau_s \kappa_s & -\left(\frac{\delta \kappa_s}{\delta b} + \theta_{bs}\tau_s\right) & -\tau_s^2 \\ \frac{\delta \kappa_s}{\delta b} & (\kappa_s + \kappa_{b_N})\tau_s & -\frac{\delta \tau_s}{\delta b} \\ -\kappa_s \kappa_{b_N} & \frac{\delta \tau_s}{\delta b} - \theta_{bs}\kappa_s & \kappa_{b_N}\tau_s \end{bmatrix}, \tag{3.14}$$

$$\left\{\frac{\delta^2}{\delta s \delta n} a_{\alpha\beta}\right\} = \begin{bmatrix} -\theta_{ns}\kappa_s & -\frac{\delta \theta_{ns}}{\delta s} & -\left(\frac{\delta}{\delta s}(\Omega_s - \tau_s) + \kappa_s \operatorname{div} \boldsymbol{b}\right) \\ \frac{\delta \theta_{ns}}{\delta s} - (\Omega_s - \tau_s)\tau_s & \tau_s \operatorname{div} \boldsymbol{b} - \theta_{ns}\kappa_s & \frac{\delta}{\delta s}\operatorname{div} \boldsymbol{b} - (\Omega_s - \tau_s)\kappa_s \\ \frac{\delta}{\delta s}(\Omega_s - \tau_s) + \theta_{ns}\tau_s & -\frac{\delta}{\delta s}\operatorname{div} \boldsymbol{b} & \operatorname{div} \boldsymbol{b}\, \tau_s \end{bmatrix}, \tag{3.15}$$

$$\left\{\frac{\delta^2}{\delta n^2}a_{\alpha\beta}\right\}=$$

$$\begin{bmatrix} -(\theta_{ns}^2+(\Omega_s-\tau_s)^2) & -\left(\frac{\delta\theta_{ns}}{\delta s}-(\Omega_s-\tau_s)\operatorname{div}\boldsymbol{b}\right) & -\left(\frac{\delta}{\delta n}(\Omega_s-\tau_s)+\theta_{ns}\operatorname{div}\boldsymbol{b}\right) \\ \frac{\delta\theta_{ns}}{\delta n}+(\Omega_s-\tau_s)\operatorname{div}\boldsymbol{b} & -((\operatorname{div}\boldsymbol{b})^2+\theta_{ns}^2) & \frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b}-\theta_{ns}(\Omega_s-\tau_s) \\ \frac{\delta}{\delta n}(\Omega_s-\tau_s)-\theta_{ns}\operatorname{div}\boldsymbol{b} & -\left(\frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b}+\theta_{ns}(\Omega_s-\tau_s)\right) & -((\Omega_s-\tau_s)^2+(\operatorname{div}\boldsymbol{b})^2) \end{bmatrix}, \quad (3.16)$$

$$\left\{\frac{\delta^2}{\delta b\,\delta n}a_{\alpha\beta}\right\}=$$

$$\begin{bmatrix} \theta_{ns}\tau_s-\theta_{bs}(\Omega_s-\tau_s) & -\left(\frac{\delta\theta_{ns}}{\delta b}-\theta_{bs}\operatorname{div}\boldsymbol{b}\right) & -\left(\frac{\delta}{\delta b}(\Omega_s-\tau_s)-\tau_s\operatorname{div}\boldsymbol{b}\right) \\ \frac{\delta\theta_{ns}}{\delta b}+\kappa_{b_N}(\Omega_s-\tau_s) & \theta_{ns}\tau_s-\kappa_{b_N}\operatorname{div}\boldsymbol{b} & \frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}+\tau_s(\Omega_s-\tau_s) \\ \frac{\delta}{\delta b}(\Omega_s-\tau_s)-\theta_{ns}\kappa_{b_N} & -\left(\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}+\theta_{ns}\theta_{bs}\right) & -(\theta_{bs}(\Omega_s-\tau_s)+\kappa_{b_N}\operatorname{div}\boldsymbol{b}) \end{bmatrix}, \quad (3.17)$$

$$\left\{\frac{\delta^2}{\delta s\,\delta b}a_{\alpha\beta}\right\}=\begin{bmatrix} \tau_s\kappa_s & \frac{\delta\tau_s}{\delta s} & -\left(\frac{\delta\theta_{bs}}{\delta s}+\kappa_s\kappa_{b_N}\right) \\ -\left(\frac{\delta\tau_s}{\delta s}+\theta_{bs}\tau_s\right) & (\kappa_s+\kappa_{b_N})\tau_s & \frac{\delta}{\delta s}\kappa_{b_N}-\theta_{bs}\kappa_s \\ \frac{\delta\theta_{bs}}{\delta s}-\tau_s^2 & -\frac{\delta\kappa_{b_N}}{\delta s} & \tau_s\kappa_{b_N} \end{bmatrix}, \quad (3.18)$$

$$\left\{\frac{\delta^2}{\delta n\,\delta b}a_{\alpha\beta}\right\}=$$

$$\begin{bmatrix} \theta_{ns}\tau_s-\theta_{bs}(\Omega_s-\tau_s) & \frac{\delta\tau_s}{\delta n}+(\Omega_s-\tau_s)\kappa_{b_N} & -\left(\frac{\delta\theta_{bs}}{\delta n}+\theta_{ns}\kappa_{b_N}\right) \\ -\left(\frac{\delta\tau_s}{\delta n}-\theta_{bs}\operatorname{div}\boldsymbol{b}\right) & \theta_{ns}\tau_s-\kappa_{b_N}\operatorname{div}\boldsymbol{b} & \frac{\delta}{\delta n}\kappa_{b_N}-\theta_{ns}\theta_{bs} \\ \frac{\delta}{\delta n}\theta_{bs}+\tau_s\operatorname{div}\boldsymbol{b} & -\left(\frac{\delta}{\delta n}\kappa_{b_N}-(\Omega_s-\tau_s)\tau_s\right) & -(\theta_{bs}(\Omega_s-\tau_s)+\kappa_{b_N}\operatorname{div}\boldsymbol{b}) \end{bmatrix}, \quad (3.19)$$

$$\left\{\frac{\delta^2}{\delta b^2}a_{\alpha\beta}\right\}=\begin{bmatrix} -(\theta_{bs}^2+\tau_s^2) & \frac{\delta\tau_s}{\delta b}+\theta_{bs}\kappa_{b_N} & -\left(\frac{\delta\theta_{bs}}{\delta b}-\tau_s\kappa_{b_N}\right) \\ -\left(\frac{\delta\tau_s}{\delta b}-\theta_{bs}\kappa_{b_N}\right) & -(\tau_s^2+\kappa_{b_N}^2) & \frac{\delta\kappa_{b_N}}{\delta b}+\theta_{bs}\tau_s \\ \frac{\delta\theta_{bs}}{\delta b}+\kappa_{b_N}\tau_s & -\left(\frac{\delta}{\delta b}\kappa_{b_N}-\theta_{bs}\tau_s\right) & -(\theta_{bs}^2+\kappa_{b_N}^2) \end{bmatrix}. \quad (3.20)$$

From (2.24)

$$\left\{\left(\frac{\delta^2}{\delta b\,\delta n}-\frac{\delta^2}{\delta n\,\delta b}\right)a_{\alpha\beta}\right\}=\Omega_s\left\{\frac{\delta a_{\alpha\beta}}{\delta s}\right\}-\operatorname{div}\boldsymbol{b}\left\{\frac{\delta a_{\alpha\beta}}{\delta n}\right\}-\kappa_{b_N}\left\{\frac{\delta a_{\alpha\beta}}{\delta b}\right\}. \quad (3.21)$$

From (3.7), (3.8), (3.9), (3.17) and (3.19), we find that (3.21) is equivalent to the three conditions

$$\frac{\delta \theta_{ns}}{\delta b}+\frac{\delta \tau_s}{\delta n}+(\Omega_s-2\tau_s)\kappa_{b_N}-\Omega_s\kappa_s+(\theta_{ns}-\theta_{bs})\operatorname{div}\boldsymbol{b}=0, \tag{3.22}$$

$$\frac{\delta}{\delta b}(\Omega_s-\tau_s)-\frac{\delta}{\delta n}\theta_{bs}-(\theta_{ns}-\theta_{bs})\kappa_{b_N}+(\Omega_s-2\tau_s)\operatorname{div}\boldsymbol{b}=0, \tag{3.23}$$

$$\frac{\delta}{\delta n}\kappa_{b_N}-\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}-\theta_{ns}\theta_{bs}-(2\Omega_s-\tau_s)\tau_s-(\operatorname{div}\boldsymbol{b})^2-\kappa_{b_N}^2=0. \tag{3.24}$$

Equation (3.23) is not a new condition but may be derived from (2.4), (2.9), (2.15), (2.18) and (2.21). Thus by (2.4), (2.9), (2.15), (2.18)

$$-\frac{\delta}{\delta s}\kappa_{b_N}-\frac{\delta}{\delta n}\theta_{bs}+\frac{\delta}{\delta b}(\Omega_s-2\tau_s)-\kappa_{b_N}(\theta_{ns}+\theta_{bs})$$
$$+\theta_{bs}(\kappa_s+\kappa_{b_N})+(\Omega_s-2\tau_s)\operatorname{div}\boldsymbol{b}=0,$$

and (3.23) follows when $\delta\kappa_{b_N}/\delta s$ is eliminated in favour of $\delta\tau_s/\delta b$ through the Mainardi-Codazzi equation (2.21).

From (2.25)

$$\left\{\left(\frac{\delta^2}{\delta s\,\delta b}-\frac{\delta^2}{\delta b\,\delta s}\right)a_{\alpha\beta}\right\}=-\theta_{bs}\left\{\frac{\delta a_{\alpha\beta}}{\delta b}\right\}. \tag{3.25}$$

From (3.9), (3.14), and (3.18) we find that (3.25) is equivalent to the three conditions

$$\frac{\delta \tau_s}{\delta s}+\frac{\delta \kappa_s}{\delta b}+2\theta_{bs}\tau_s=0, \tag{3.26}$$

$$\frac{\delta \theta_{bs}}{\delta s}+\kappa_s\kappa_{b_N}-\tau_s^2+\theta_{bs}^2=0, \tag{3.27}$$

$$\frac{\delta \kappa_{b_N}}{\delta s}+\frac{\delta \tau_s}{\delta b}+\theta_{bs}(\kappa_{b_N}-\kappa_s)=0. \tag{3.28}$$

Equation (3.26) and (3.28) are identified with the Mainardi-Codazzi relations (2.20) and (2.21), while by (2.18) and (2.19) equation (3.27) is the Gauss equation (2.17). While (3.25) offers a proof of the Gauss and Mainardi-Codazzi equations, it gives no new information.

From (2.9), (2.26)

$$\left\{\left(\frac{\delta^2}{\delta n\,\delta s}-\frac{\delta^2}{\delta s\,\delta n}\right)a_{\alpha\beta}\right\}=\kappa_s\left\{\frac{\delta a_{\alpha\beta}}{\delta s}\right\}+\theta_{ns}\left\{\frac{\delta}{\delta n}a_{\alpha\beta}\right\}+(\Omega_s-2\tau_s)\left\{\frac{\delta}{\delta b}a_{\alpha\beta}\right\}. \tag{3.29}$$

From (3.7), (3.8), (3.9), (3.13), and (3.15) we find that (3.29) is equivalent to the three conditions

$$\frac{\delta \kappa_s}{\delta n}-\frac{\delta \theta_{ns}}{\delta s}+(2\Omega_s-3\tau_s)\tau_s-\kappa_s^2-\theta_{ns}^2=0, \tag{3.30}$$

$$\frac{\delta}{\delta s}(\Omega_s-\tau_s)+\theta_{ns}\Omega_s+\theta_{bs}(\Omega_s-2\tau_s)+\kappa_s \operatorname{div} \boldsymbol{b}=0, \tag{3.31}$$

$$\frac{\delta \tau_s}{\delta n}+\frac{\delta}{\delta s}\operatorname{div} \boldsymbol{b}-\kappa_s\Omega_s+\theta_{ns}\operatorname{div} \boldsymbol{b}+\kappa_{b_N}(\Omega_s-2\tau_s)=0. \tag{3.32}$$

Equation (3.31) is obtainable by eliminating $\delta\kappa_s/\delta b$ between (2.13) and (2.20). Also, equation (3.32) is obtained from (3.22) by eliminating $\delta\tau_{ns}/\delta b$ in favour of $\delta/\delta s \operatorname{div} \boldsymbol{b}$ by means of (2.14).

Our analysis has yielded three equations, which we believe to be new, expressing the normal gradients $\delta\kappa_s/\delta n$, $\delta\tau_s/\delta n$, $\delta\kappa_{b_N}/\delta n$ in terms of the vector field parameters.* We have

Theorem 3.1. *Given a vector field* **s** *such that the principal normal* **n** *to the vector-lines is such that*

$$\Omega_n=\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}=0,$$

with the consequence that the s-lines are geodesics and the b-lines are parallel curves on the surfaces associated with the normal congruence of n-lines. For this vector field, the normal gradients of the principal curvature κ_s *of the s-lines, the torsion* τ_s *of the s-lines, and the normal curvature* κ_{b_N} *of the b-lines, are given respectively by*

$$\frac{\delta\kappa_s}{\delta n}=\frac{\delta\theta_{ns}}{\delta s}-(2\Omega_s-3\tau_s)\tau_s+\kappa_s^2+\theta_{ns}^2, \tag{3.30}$$

$$\frac{\delta\tau_s}{\delta n}=-\frac{\delta}{\delta s}\operatorname{div}\boldsymbol{b}+\kappa_s\Omega_s-\theta_{ns}\operatorname{div}\boldsymbol{b}-(\Omega_s-2\tau_s)\kappa_{b_N}, \tag{3.32}$$

$$\frac{\delta\kappa_{b_N}}{\delta n}=\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}+\theta_{ns}\theta_{bs}+(2\Omega_s-\tau_s)\tau_s+(\operatorname{div}\boldsymbol{b})^2+\kappa_{b_N}^2. \tag{3.24}$$

4. The Curvature Tensor

Consider now the deformation tensor

$$\boldsymbol{c}=\frac{1}{\eta^2}\boldsymbol{s}\boldsymbol{s}+\eta\,\boldsymbol{n}\boldsymbol{n}+\eta\,\boldsymbol{b}\boldsymbol{b}, \tag{1.32}$$

where

$$\eta=\eta(U). \tag{1.33}$$

The Cartesian components of $\boldsymbol{c}$ are given by

$$c_{\alpha\beta}=\boldsymbol{i}_\alpha\cdot\boldsymbol{c}\cdot\boldsymbol{i}_\beta, \tag{4.1}$$

* The relations (3.30) and (3.24) may also be derived by setting $\boldsymbol{v}=\boldsymbol{s}$ and $\boldsymbol{v}=\boldsymbol{b}$ in the vector identity

$$\operatorname{div}(\boldsymbol{v}\cdot\operatorname{grad}\boldsymbol{v}-(\operatorname{div}\boldsymbol{v})\boldsymbol{v})=-2\mathrm{II}_{\boldsymbol{d}}-\tfrac{1}{2}\operatorname{curl}\boldsymbol{v}\cdot\operatorname{curl}\boldsymbol{v}, \tag{a}$$

where $\mathrm{II}_{\boldsymbol{d}}$ is the second principal invariant of

$$\boldsymbol{d}=\tfrac{1}{2}(\operatorname{grad}\boldsymbol{v}+(\operatorname{grad}\boldsymbol{v})^T). \tag{b}$$

where $\boldsymbol{i}_\alpha$ are given by (3.1). Then

$$c_{\alpha\beta}=a_{\alpha 1}\frac{1}{\eta^2}a_{\beta 1}+a_{\alpha 2}\eta a_{\beta 2}+a_{\alpha 3}\eta a_{\beta 3}. \tag{4.2}$$

Our aim in this section is to derive, in intrinsic form, the six equations necessary and sufficient for the vanishing of $R_{\alpha\beta\gamma\delta}$, as given by (1.8), when $c_{\alpha\beta}$ is given by (4.2).

We require that the vector field of s satisfy the following specific conditions:

$$\operatorname{div} s=\theta_{ns}+\theta_{bs}=0, \quad \theta_{ns}=-\theta_{bs}, \tag{2.4}$$

$$\Omega_n=0, \tag{2.2}$$

$$\frac{\delta\Omega_s}{\delta s}=0, \tag{2.29}$$

$$\frac{\delta\kappa_s}{\delta s}=-\theta_{ns}\kappa_s, \tag{2.30}$$

$$\frac{\delta\kappa_s}{\delta b}=-\kappa_s \operatorname{div}\boldsymbol{b}, \tag{2.31}$$

$$\frac{\delta\tau_s}{\delta s}=2\theta_{ns}\tau_s+\kappa_s \operatorname{div}\boldsymbol{b}, \tag{2.32}$$

$$\frac{\delta\eta}{\delta s}=0, \quad \frac{\delta\eta}{\delta b}=0. \tag{4.3}$$

We now introduce the notation

$$c^*_{11}=\frac{1}{\eta^2}, \quad c^*_{22}=\eta, \quad c^*_{33}=\eta, \tag{4.4}$$

and we represent the gradients $\delta/\delta s$, $\delta/\delta n$, $\delta/\delta b$ alternatively by $\delta/\delta s^\alpha$, $\alpha=1, 2, 3$. We write (4.2) in the form*

$$c_{\alpha\beta}=a_{\alpha 1}c^*_{11}a_{\beta 1}+a_{\alpha 2}c^*_{22}a_{\beta 2}+a_{\alpha 3}c^*_{33}a_{\beta 3}=a_{\alpha\mu}c^*_{\underline{\mu}\underline{\mu}}a_{\beta\mu}. \tag{4.5}$$

From (3.1) and (4.5)

$$\begin{aligned}\frac{\partial}{\partial x^\gamma}c_{\alpha\beta}&=\boldsymbol{i}_\gamma\cdot\operatorname{grad}c_{\alpha\beta}=a_{\gamma\lambda}\frac{\delta}{\delta s^\lambda}c_{\alpha\beta}=\frac{\delta}{\delta s^\gamma}c_{\alpha\beta}\\&=\frac{\delta a_{\alpha\mu}}{\delta s^\lambda}c^*_{\underline{\mu}\underline{\mu}}a_{\beta\mu}+a_{\alpha\mu}\frac{\delta}{\delta s^\lambda}c^*_{\underline{\mu}\underline{\mu}}a_{\beta\mu}+a_{\alpha\mu}c^*_{\underline{\mu}\underline{\mu}}\frac{\delta}{\delta s^\lambda}a_{\beta\mu}.\end{aligned} \tag{4.6}$$

* The underscore in (4.5) means that no further summation is implied. When a Greek subscript without underscore occurs repeated in a term, it is understood to represent a summation over the range 1, 2, 3 as in (4.5).

In (4.14) a term like $\frac{\delta}{\delta s^\gamma}a_{\varepsilon\lambda}\left(\frac{\delta a_{\alpha\beta}}{\delta s^\lambda}\right)c^*_{\underline{\beta}\underline{\beta}}$ is summed on λ, but not on β. Again, in (4.14) a term like $\left(\frac{\delta^2}{\delta s^\gamma\delta s^\varepsilon}a_{\alpha\beta}\right)c^*_{\underline{\beta}\underline{\beta}}$ is a single term, and in (4.15) a term like $\left(\frac{\delta^2}{\delta s^\gamma\delta s^\varepsilon}a_{\underline{\alpha}\underline{\alpha}}\right)c^*_{\underline{\alpha}\underline{\alpha}}$ is a single term.

Since we may take $a_{\alpha\beta}=\delta_{\alpha\beta}$ at the reference point, from (4.6) we obtain

1) $\alpha=\beta$.

$$\begin{aligned}\frac{\partial}{\partial x^\gamma} c_{\underline{\alpha\alpha}} &= 2c^*_{\underline{\alpha\alpha}} \frac{\delta a_{\underline{\alpha\alpha}}}{\delta s^\lambda} + \frac{\delta c^*_{\underline{\alpha\alpha}}}{\delta s^\lambda}, \\ &= \frac{\delta c^*_{\underline{\alpha\alpha}}}{\delta s^\lambda}, \quad \text{by (3.15), (3.16), (3.17);}\end{aligned} \tag{4.7}$$

2) $\alpha \neq \beta$.

$$\frac{\partial}{\partial x^\gamma} c_{\alpha\beta} = \frac{\delta a_{\alpha\beta}}{\delta s^\lambda} c^*_{\underline{\beta\beta}} + \frac{\delta a_{\beta\alpha}}{\delta s^\lambda} c^*_{\underline{\alpha\alpha}}. \tag{4.8}$$

From (4.3) and (4.4)

$$\frac{\delta}{\delta s} c^*_{\underline{\beta\beta}} = 0, \quad \frac{\delta}{\delta b} c^*_{\underline{\beta\beta}} = 0, \tag{4.9}$$

$$\frac{\delta}{\delta n} c^*_{11} = -\frac{2}{\eta^3}\frac{\delta\eta}{\delta n}, \quad \frac{\delta}{\delta n} c^*_{22} = \frac{\delta}{\delta n} c^*_{33} = \frac{\delta\eta}{\delta n}. \tag{4.10}$$

From (3.15), (3.16), (3.17), (4.7), (4.8), (4.9) and (4.10) we obtain

$$\left\{\frac{\partial}{\partial x^1} c_{\alpha\beta}\right\} = \begin{bmatrix} 0 & -\left(\eta-\frac{1}{\eta^2}\right)\kappa_s & 0 \\ -\left(\eta-\frac{1}{\eta^2}\right)\kappa_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.11}$$

$$\left\{\frac{\partial}{\partial x^2} c_{\alpha\beta}\right\} = \begin{bmatrix} -\frac{2}{\eta^3}\frac{\delta\eta}{\delta n} & -\theta_{ns}\left(\eta-\frac{1}{\eta^2}\right) & -(\Omega_s-\tau_s)\left(\eta-\frac{1}{\eta^2}\right) \\ -\theta_{ns}\left(\eta-\frac{1}{\eta^2}\right) & \frac{\delta\eta}{\delta n} & 0 \\ -(\Omega_s-\tau_s)\left(\eta-\frac{1}{\eta^2}\right) & 0 & \frac{\delta\eta}{\delta n} \end{bmatrix}, \tag{4.12}$$

$$\left\{\frac{\partial}{\partial x^3} c_{\alpha\beta}\right\} = \begin{bmatrix} 0 & \left(\eta-\frac{1}{\eta^2}\right)\tau_s & \left(\eta-\frac{1}{\eta^2}\right)\theta_{ns} \\ \left(\eta-\frac{1}{\eta^2}\right)\tau_s & 0 & 0 \\ \left(\eta-\frac{1}{\eta^2}\right)\theta_{ns} & 0 & 0 \end{bmatrix}, \tag{4.13}$$

where we have eliminated θ_{bs} in favour of θ_{ns} by (2.4). From (3.1), and (4.5),

$$\begin{aligned}\frac{\partial^2}{\partial x^\gamma \partial x^\varepsilon} c_{\alpha\beta} &= \boldsymbol{i}_\gamma \cdot \operatorname{grad}(\boldsymbol{i}_\varepsilon \cdot \operatorname{grad} c_{\alpha\beta}), \\ &= \frac{\delta}{\delta s^\gamma}\left[a_{\varepsilon\lambda}\frac{\delta}{\delta s^\lambda}(a_{\alpha\mu} c^*_{\underline{\mu\mu}} a_{\beta\mu})\right], \\ &= \left(\frac{\delta}{\delta s^\gamma} a_{\varepsilon\lambda}\right)\frac{\delta}{\delta s^\lambda}(a_{\alpha\mu} c^*_{\underline{\mu\mu}} a_{\beta\mu}) + \frac{\delta^2}{\delta s^\gamma \delta s^\varepsilon}(a_{\alpha\mu} c^*_{\underline{\mu\mu}} a_{\beta\mu}),\end{aligned}$$

$$
\begin{aligned}
&=\left(\frac{\delta}{\delta s^{\gamma}}a_{\varepsilon\lambda}\right)\left[\left(\frac{\delta a_{\alpha\mu}}{\delta s^{\gamma}}\right)c^{*}_{\underline{\mu\mu}}a_{\beta\mu}+a_{\alpha\mu}\left(\frac{\delta}{\delta s^{\lambda}}c^{*}_{\underline{\mu\mu}}\right)a_{\beta\mu}+a_{\alpha\mu}c^{*}_{\underline{\mu\mu}}\frac{\delta}{\delta s^{\lambda}}a_{\beta\mu}\right]\\
&\quad+\frac{\delta}{\delta s^{\gamma}}\left[\left(\frac{\delta a_{\alpha\mu}}{\delta s^{\varepsilon}}\right)c^{*}_{\underline{\mu\mu}}a_{\beta\mu}+a_{\alpha\mu}\left(\frac{\delta}{\delta s^{\varepsilon}}c^{*}_{\underline{\mu\mu}}\right)a_{\beta\mu}+a_{\alpha\mu}c^{*}_{\underline{\mu\mu}}\frac{\delta}{\delta s^{\varepsilon}}a_{\beta\mu}\right],\\
&=\left(\frac{\delta}{\delta s^{\gamma}}a_{\varepsilon\lambda}\right)\left[\left(\frac{\delta a_{\alpha\beta}}{\delta s^{\lambda}}\right)c^{*}_{\underline{\beta\beta}}\right.\\
&\quad\left.+a_{\alpha\mu}\left(\frac{\delta}{\delta s^{\lambda}}c^{*}_{\mu\mu}\right)a_{\beta\mu}+c^{*}_{\underline{\alpha\alpha}}\frac{\delta a_{\beta\alpha}}{\delta s^{\lambda}}\right]+\left(\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}a_{\alpha\beta}\right)c^{*}_{\underline{\beta\beta}}\\
&\quad+\frac{\delta a_{\alpha\beta}}{\delta s^{\varepsilon}}\frac{\delta}{\delta s^{\gamma}}c^{*}_{\underline{\beta\beta}}+\left(\frac{\delta a_{\alpha\mu}}{\delta s^{\varepsilon}}\right)c^{*}_{\underline{\mu\mu}}\frac{\delta a_{\beta\mu}}{\delta s^{\lambda}}\\
&\quad+\frac{\delta a_{\alpha\beta}}{\delta s^{\gamma}}\frac{\delta}{\delta s^{\varepsilon}}c^{*}_{\underline{\beta\beta}}+a_{\alpha\mu}\left(\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}c^{*}_{\underline{\mu\mu}}\right)a_{\beta\mu}+\frac{\delta}{\delta s^{\varepsilon}}c^{*}_{\underline{\alpha\alpha}}\frac{\delta a_{\beta\alpha}}{\delta s^{\gamma}}\\
&\quad+\left(\frac{\delta a_{\alpha\mu}}{\delta s^{\gamma}}\right)c^{*}_{\underline{\mu\mu}}\frac{\delta a_{\beta\mu}}{\delta s^{\varepsilon}}+\frac{\delta}{\delta s^{\gamma}}c^{*}_{\underline{\alpha\alpha}}\frac{\delta a_{\beta\alpha}}{\delta s^{\varepsilon}}+c^{*}_{\underline{\alpha\alpha}}\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}a_{\beta\alpha}.
\end{aligned}
\tag{4.14}
$$

Once again there are two cases.

1) $\alpha=\beta$. By (3.15), (3.16), (3.17), (4.9), and (4.14),

$$
\begin{aligned}
\frac{\partial^{2}c_{\underline{\alpha\alpha}}}{\partial x^{\gamma}\partial x^{\varepsilon}}&=\frac{\delta a_{\varepsilon 2}}{\delta s^{\gamma}}\frac{\delta c^{*}_{\underline{\alpha\alpha}}}{\delta n}+\left(\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}a_{\underline{\alpha\alpha}}\right)c^{*}_{\underline{\alpha\alpha}}+\left(\frac{\delta}{\delta s^{\varepsilon}}a_{\underline{\alpha}\mu}\right)c^{*}_{\underline{\mu\mu}}\frac{\delta}{\delta s^{\gamma}}a_{\underline{\alpha}\mu}\\
&\quad+\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}c^{*}_{\underline{\alpha\alpha}}+\left(\frac{\delta}{\delta s^{\gamma}}a_{\underline{\alpha}\mu}\right)c^{*}_{\underline{\mu\mu}}\frac{\delta}{\delta s^{\varepsilon}}a_{\underline{\alpha}\mu}+c^{*}_{\underline{\alpha\alpha}}\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}a_{\underline{\alpha\alpha}}\\
&=\frac{\delta a_{\varepsilon 2}}{\delta s^{\gamma}}\frac{\delta c^{*}_{\underline{\alpha\alpha}}}{\delta n}+\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}c^{*}_{\underline{\alpha\alpha}}\\
&\quad+2\left(c^{*}_{\underline{\alpha\alpha}}\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}a_{\underline{\alpha\alpha}}+c^{*}_{\underline{\mu\mu}}\frac{\delta a_{\underline{\alpha}\mu}}{\delta s^{\gamma}}\frac{\delta a_{\underline{\alpha}\mu}}{\delta s^{\varepsilon}}\right).
\end{aligned}
\tag{4.15}
$$

2) $\alpha\neq\beta$.

$$
\begin{aligned}
\frac{\partial^{2}c_{\alpha\beta}}{\partial x^{\gamma}\partial x^{\varepsilon}}&=\left(\frac{\delta}{\delta s^{\gamma}}a_{\varepsilon\lambda}\right)\left(\frac{\delta a_{\alpha\beta}}{\delta s^{\lambda}}c^{*}_{\underline{\beta\beta}}+\frac{\delta a_{\beta\alpha}}{\delta s^{\lambda}}c^{*}_{\underline{\alpha\alpha}}\right)\\
&\quad+\left(\frac{\delta^{2}}{\delta s^{\gamma}\delta s^{\varepsilon}}a_{\alpha\beta}\right)c^{*}_{\underline{\beta\beta}}+\left(\frac{\delta^{2}a_{\beta\alpha}}{\delta s^{\gamma}\delta s^{\varepsilon}}\right)c^{*}_{\underline{\alpha\alpha}}\\
&\quad+\frac{\delta a_{\alpha\beta}}{\delta s^{\gamma}}\frac{\delta}{\delta s^{\varepsilon}}c^{*}_{\underline{\beta\beta}}+\frac{\delta a_{\alpha\beta}}{\delta s^{\varepsilon}}\frac{\delta}{\delta s^{\gamma}}c^{*}_{\underline{\beta\beta}}+\frac{\delta a_{\beta\alpha}}{\delta s^{\gamma}}\frac{\delta}{\delta s^{\varepsilon}}c^{*}_{\underline{\alpha\alpha}}\\
&\quad+\frac{\delta a_{\beta\alpha}}{\delta s^{\varepsilon}}\frac{\delta}{\delta s^{\gamma}}c^{*}_{\underline{\alpha\alpha}}+c^{*}_{\underline{\mu\mu}}\left(\frac{\delta a_{\alpha\mu}}{\delta s^{\gamma}}\frac{\delta a_{\beta\mu}}{\delta s^{\varepsilon}}+\frac{\delta a_{\alpha\mu}}{\delta s^{\varepsilon}}\frac{\delta a_{\beta\mu}}{\delta s^{\gamma}}\right).
\end{aligned}
\tag{4.16}
$$

The relations (3.7), (3.8), (3.9), (3.12) to (3.20), may now be substituted in (4.15) and (4.16), to yield the expressions for $\partial^{2}c_{\alpha\beta}/\partial x^{\gamma}\partial x^{\delta}$, required by the formula (1.8) for the curvature tensor.

We tabulate the expressions

$$\left\{\frac{\partial^2}{\partial x^{1^2}} c_{\alpha\beta}\right\} = \begin{bmatrix} -2\kappa_s^2\left(\frac{1}{\eta^2}-\eta\right)-\kappa_s\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}\right) & -2\kappa_s\theta_{ns}\left(\frac{1}{\eta^2}-\eta\right) & -\kappa_s(\Omega_s-2\tau_s)\left(\frac{1}{\eta^2}-\eta\right) \\ - & 2\kappa_s^2\left(\frac{1}{\eta^2}-\eta\right)-\kappa_s\frac{\delta\eta}{\delta n} & 0 \\ - & - & -\kappa_s\frac{\delta\eta}{\delta n} \end{bmatrix}, \tag{4.17}$$

$$\left\{\frac{\partial^2 c_{\alpha\beta}}{\partial x^{2^2}}\right\} = \begin{bmatrix} -2[\theta_{ns}^2+(\Omega_s-\tau_s)^2]\left(\frac{1}{\eta^2}-\eta\right) + \frac{\delta^2}{\delta n^2}\left(\frac{1}{\eta^2}\right) & \left[\frac{\delta\theta_{ns}}{\delta n}+\theta_{ns}\kappa_s+(\Omega_s-2\tau_s)\operatorname{div}\boldsymbol{b}\right]\cdot\left(\frac{1}{\eta^2}-\eta\right)+2\theta_{ns}\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}-\eta\right) & \left[\frac{\delta}{\delta n}(\Omega_s-\tau_s)-2\theta_{ns}\operatorname{div}\boldsymbol{b}\right]\left(\frac{1}{\eta^2}-\eta\right)+2(\Omega_s-\tau_s)\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}-\eta\right) \\ - & 2\theta_{ns}^2\left(\frac{1}{\eta^2}-\eta\right)+\frac{\delta^2\eta}{\delta n^2} & 2\theta_{ns}(\Omega_s-\tau_s)\left(\frac{1}{\eta^2}-\eta\right) \\ - & - & 2(\Omega_s-\tau_s)^2\left(\frac{1}{\eta^2}-\eta\right)+\frac{\delta^2\eta}{\delta n^2} \end{bmatrix}, \tag{4.18}$$

$$\left\{\frac{\partial^2 c_{\alpha\beta}}{\partial x^{3^2}}\right\} = \begin{bmatrix} -2(\tau_s^2+\theta_{ns}^2)\left(\frac{1}{\eta^2}-\eta\right) - \kappa_{b_N}\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}\right) & -\left[\frac{\delta\tau_s}{\delta b}+\theta_{ns}(\kappa_s+2\kappa_{b_N})\right]\left(\frac{1}{\eta^2}-\eta\right) & -\left[\frac{\delta\theta_{ns}}{\delta b}+\kappa_{b_N}(\Omega_s-2\tau_s)\right]\left(\frac{1}{\eta^2}-\eta\right) \\ - & 2\tau_s^2\left(\frac{1}{\eta^2}-\eta\right)-\kappa_{b_N}\frac{\delta\eta}{\delta n} & 2\tau_s\theta_{ns}\left(\frac{1}{\eta^2}-\eta\right) \\ - & - & 2\theta_{ns}^2\left(\frac{1}{\eta^2}-\eta\right)-\kappa_{b_N}\frac{\delta\eta}{\delta n} \end{bmatrix}, \tag{4.19}$$

$$\left\{\frac{\partial^2 c_{\alpha\beta}}{\partial x^1 \partial x^2}\right\}=\begin{bmatrix} -2\theta_{ns}\kappa_s\left(\frac{1}{\eta^2}-\eta\right)-\theta_{ns}\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}\right) & \left[\kappa_s^2+\tau_s^2+\frac{\delta\theta_{ns}}{\delta s}-(\Omega_s-\tau_s)\right] \cdot\left(\frac{1}{\eta^2}-\eta\right)+\kappa_s\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}-\eta\right) & \left[2\tau_s\theta_{ns}+\frac{\delta}{\delta s}(\Omega_s-\tau_s)\right]\left(\frac{1}{\eta^2}-\eta\right) \\ - & 2\theta_{ns}\kappa_s\left(\frac{1}{\eta^2}-\eta\right)-\theta_{ns}\frac{\delta\eta}{\delta n} & \kappa_s(\Omega_s-\tau_s)\left(\frac{1}{\eta^2}-\eta\right) \\ - & - & -\theta_{ns}\frac{\delta\eta}{\delta n} \end{bmatrix}, \quad (4.20)$$

$$\left\{\frac{\partial^2 c_{\alpha\beta}}{\partial x^2 \partial x^3}\right\}=\begin{bmatrix} 2\theta_{ns}\Omega_s\left(\frac{1}{\eta^2}-\eta\right) -\operatorname{div}\boldsymbol{b}\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}\right) & \left[(\Omega_s-\tau_s)\kappa_s-2\theta_{ns}\operatorname{div}\boldsymbol{b}-\frac{\delta\tau_s}{\delta n}\right]\cdot\left(\frac{1}{\eta^2}-\eta\right)-\tau_s\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}-\eta\right) & \left[-\operatorname{div}\boldsymbol{b}(\Omega_s-2\tau_s)-\frac{\delta\theta_{ns}}{\delta n}\right]\cdot\left(\frac{1}{\eta^2}-\eta\right)-\theta_{ns}\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}-\eta\right) \\ - & -2\tau_s\theta_{ns}\left(\frac{1}{\eta^2}-\eta\right)-\operatorname{div}\boldsymbol{b}\frac{\delta\eta}{\delta n} & -[\theta_{ns}^2+\tau_s(\Omega_s-\tau_s)]\left(\frac{1}{\eta^2}-\eta\right) \\ - & - & -2\theta_{ns}(\Omega_s-\tau_s)\left(\frac{1}{\eta^2}-\eta\right)-\operatorname{div}\boldsymbol{b}\frac{\delta\eta}{\delta n} \end{bmatrix}, \quad (4.21)$$

$$\left\{\frac{\partial^2 c_{\alpha\beta}}{\partial x^3 \partial x^1}\right\}=\begin{bmatrix} 2\tau_s\kappa_s\left(\frac{1}{\eta^2}-\eta\right)+\tau_s\frac{\delta}{\delta n}\left(\frac{1}{\eta^2}\right) & \frac{\delta\kappa_s}{\delta b}\left(\frac{1}{\eta^2}-\eta\right) & [\tau_s(\Omega_s-\tau_s)-\theta_{ns}^2-\kappa_s\kappa_{b_N}]\left(\frac{1}{\eta^2}-\eta\right) \\ - & -2\tau_s\kappa_s\left(\frac{1}{\eta^2}-\eta\right)+\tau_s\frac{\delta\eta}{\delta n} & -\kappa_s\theta_{ns}\left(\frac{1}{\eta^2}-\eta\right) \\ - & - & \tau_s\frac{\delta\eta}{\delta n} \end{bmatrix}. \quad (4.22)$$

From the relations (4.11), (4.12), and (4.13), and (4.17) to (4.22), we now obtain, in intrinsic form, the six equations necessary and sufficient for the vanishing of the curvature tensor given by (1.8):

$R_{1212}=0$

$$\left(\frac{1}{\eta^3}+3\right)\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2+4\left[\frac{\delta\theta_{ns}}{\delta s}+\theta_{ns}^2-3\tau_s(\Omega_s-\tau_s)\right]\left(\frac{1}{\eta^2}-\eta\right) -\frac{10\kappa_s}{\eta^3}\frac{\delta\eta}{\delta n}-\frac{10}{\eta^4}\left(\frac{\delta\eta}{\delta n}\right)^2+\frac{4}{\eta^3}\frac{\delta^2\eta}{\delta n^2}=0, \tag{4.23}$$

$R_{2323}=0$

$$-(3+\eta^3)\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2+4\left(\tau_s(\Omega_s-\tau_s)-\theta_{ns}^2\right)\eta^3\left(\frac{1}{\eta^2}-\eta\right)+2\kappa_{b_N}\frac{\delta\eta}{\delta n} +\frac{2}{\eta}\left(\frac{\delta\eta}{\delta n}\right)^2-2\frac{\delta^2\eta}{\delta n^2}=0, \tag{4.24}$$

$R_{3131}=0$

$$\left(\frac{1}{\eta^3}-1\right)\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2+4(\tau_s\Omega_s-\kappa_s\kappa_{b_N})\left(\frac{1}{\eta^2}-\eta\right)+\frac{2}{\eta^3}(\kappa_s-2\kappa_{b_N})\frac{\delta\eta}{\delta n} +\frac{2}{\eta^3}\left(\frac{\delta\eta}{\delta n}\right)^2=0, \tag{4.25}$$

$R_{1223}=0$

$$\left(\frac{\delta\Omega_s}{\delta n}-2\kappa_s\Omega_s\right)\left(\frac{1}{\eta^2}-\eta\right)-\frac{4\Omega_s}{\eta^3}\frac{\delta\eta}{\delta n}-2(\Omega_s-2\tau_s)\frac{\delta\eta}{\delta n}=0, \tag{4.26}$$

$R_{2331}=0$

$$\left[\frac{\delta\tau_s}{\delta b}-\frac{\delta\theta_{ns}}{\delta n}+2\kappa_{b_N}\theta_{ns}-\operatorname{div}\boldsymbol{b}(\Omega_s-2\tau_s)\right]\left(\frac{1}{\eta^2}-\eta\right)+4\theta_{ns}\frac{\delta\eta}{\delta n}=0,$$

or, equivalently, by (2.4), (2.9), (2.15), (2.18), and (2.21),

$$\frac{\delta\Omega_s}{\delta b}\left(\frac{1}{\eta^2}-\eta\right)+4\theta_{ns}\frac{\delta\eta}{\delta n}=0, \tag{4.27}$$

$R_{3112}=0$

$$\left[2\theta_{ns}(\Omega_s-\tau_s)+\frac{\delta\tau_s}{\delta s}-\frac{\delta\kappa_s}{\delta b}\right]\left(\frac{1}{\eta^2}-\eta\right)+\frac{2\operatorname{div}\boldsymbol{b}}{\eta^3}\frac{\delta\eta}{\delta n}=0,$$

or, equivalently, by (2.4) and (2.20),

$$\left[\frac{\delta\tau_s}{\delta s}+\theta_{ns}(\Omega_s-2\tau_s)\right]\left(\frac{1}{\eta^2}-\eta\right)+\frac{\operatorname{div}\boldsymbol{b}}{\eta^3}\frac{\delta\eta}{\delta n}=0. \tag{4.28}$$

5. Proof of Theorem 1.1

Since the conditions (4.23) to (4.28) must hold in a neighbourhood, from (4.27) we have

$$\frac{\delta^2\Omega_s}{\partial s\,\delta b}\left(\frac{1}{\eta^2}-\eta\right)+4\frac{\delta\theta_{ns}}{\delta s}\frac{\delta\eta}{\delta n}+4\theta_{ns}\frac{\delta^2\eta}{\delta s\,\delta n}=0, \tag{5.1}$$

since

$$\frac{\delta \eta}{\delta s}=0, \tag{4.3$_1$}$$

Again, since

$$\frac{\delta \Omega_s}{\delta s}=0, \tag{2.29}$$

$$\frac{\delta \eta}{\delta b}=0, \tag{4.3$_2$}$$

we obtain, from (2.4), (2.25), (2.26), and (5.1),

$$\theta_{ns}\frac{\delta \Omega_s}{\delta b}\left(\frac{1}{\eta^2}-\eta\right)+4\frac{\delta \theta_{ns}}{\delta s}\frac{\delta \eta}{\delta n}-4\theta_{ns}^2\frac{\delta \eta}{\delta n}=0, \tag{5.2}$$

From (4.27) and (5.2),

$$\left(\frac{\delta \theta_{ns}}{\delta s}-2\theta_{ns}^2\right)\frac{\delta \eta}{\delta n}=0. \tag{5.3}$$

Thus, either

$$\frac{\delta \eta}{\delta n}=0, \tag{5.4}$$

in which case, by $(4.3)_1$ and $(4.3)_2$, η is constant, or else

$$\frac{\delta \theta_{ns}}{\delta s}=2\theta_{ns}^2. \tag{5.5}$$

We assume that η is not constant. By (2.4), (2.17) and (5.5) the Gaussian curvature of the surfaces $U(x^\alpha)=\text{constant}$ must be given by

$$G=\theta_{ns}^2, \tag{5.6}$$

so that by (2.19),

$$\kappa_s \kappa_{b_N}=\tau_s^2+\theta_{ns}^2. \tag{5.7}$$

From (4.23) and (5.5),

$$\begin{aligned}\left(\frac{1}{\eta^3}+3\right)\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2+12(\theta_{ns}^2+\tau_s^2-\Omega_s\tau_s)\left(\frac{1}{\eta^2}-\eta\right)\\ -\frac{10\kappa_s}{\eta^3}\frac{\delta \eta}{\delta n}-\frac{10}{\eta^4}\left(\frac{\delta \eta}{\delta n}\right)^2+\frac{4}{\eta^3}\frac{\delta^2 \eta}{\delta n^2}=0.\end{aligned} \tag{5.8}$$

By (4.24),

$$\begin{aligned}\left(\frac{6}{\eta^3}+2\right)\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2+8(\theta_{ns}^2+\tau_s^2-\Omega_s\tau_s)\left(\frac{1}{\eta^2}-\eta\right)\\ -\frac{4\kappa_{b_N}}{\eta^3}\frac{\delta \eta}{\delta n}-\frac{4}{\eta^4}\left(\frac{\delta \eta}{\delta n}\right)^2+\frac{4}{\eta^3}\frac{\delta^2 \eta}{\delta n^2}=0.\end{aligned} \tag{5.9}$$

Eliminating $\delta^2\eta/\delta n^2$ between (5.8) and (5.9), we obtain

$$\begin{aligned}\left(1-\frac{5}{\eta^3}\right)\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2+4(\theta_{ns}^2+\tau_s^2-\Omega_s\tau_s)\left(\frac{1}{\eta^2}-\eta\right)\\ -\frac{10\kappa_s}{\eta^3}\frac{\delta \eta}{\delta n}+\frac{4\kappa_{b_N}}{\eta^3}\frac{\delta \eta}{\delta n}-\frac{6}{\eta^4}\left(\frac{\delta \eta}{\delta n}\right)^2=0.\end{aligned} \tag{5.10}$$

From (4.25) and (5.7),

$$-\left(1-\frac{1}{\eta^3}\right)\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2-4(\theta_{ns}^2+\tau_s^2-\Omega_s\tau_s)\left(\frac{1}{\eta^2}-\eta\right)$$
$$+\frac{2\kappa_s}{\eta^3}\frac{\delta\eta}{\delta n}-\frac{4\kappa_{b_N}}{\eta^3}\frac{\delta\eta}{\delta n}+\frac{2}{\eta^4}\left(\frac{\delta\eta}{\delta n}\right)^2=0. \tag{5.11}$$

From (5.10) and (5.11),

$$\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2+2\kappa_s\frac{\delta\eta}{\delta n}+\frac{1}{\eta}\left(\frac{\delta\eta}{\delta n}\right)^2=0. \tag{5.12}$$

Taking the directional derivative of (5.12) with respect to s and invoking (2.29) and $(4.3)_1$, we obtain

$$\frac{\delta\kappa_s}{\delta s}\frac{\delta\eta}{\delta n}+\kappa_s\frac{\delta^2\eta}{\delta s\,\delta n}+\frac{1}{\eta}\frac{\delta\eta}{\delta n}\frac{\delta^2\eta}{\delta s\,\delta n}=0. \tag{5.13}$$

whence, by (2.26), (2.30), (4.3), and (5.13),

$$\theta_{ns}\left(2\kappa_s+\frac{1}{\eta}\frac{\delta\eta}{\delta n}\right)\frac{\delta\eta}{\delta n}=0. \tag{5.14}$$

Since η is assumed to be not constant, (5.14) requires that either

$$\theta_{ns}=0 \tag{5.15}$$

or

$$\left(2\kappa_s+\frac{1}{\eta}\frac{\delta\eta}{\delta n}\right)=0, \tag{5.16}$$

or both these conditions hold.

Suppose that θ_{ns} does not vanish; then since η is not constant, so that $\delta\eta/\delta n$ is not zero, by (5.16)

$$2\kappa_s\frac{\delta\eta}{\delta n}+\frac{1}{\eta}\left(\frac{\delta\eta}{\delta n}\right)^2=0, \tag{5.17}$$

and by (5.12) and (5.17),

$$\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2=0$$

so that either η is unity or Ω_s vanishes. Equation (4.27) then shows that

$$4\theta_{ns}\frac{\delta\eta}{\delta n}=0, \tag{5.18}$$

so that if θ_{ns} does not vanish η must be constant.

We conclude that, for equations (4.23) to (4.28) to be satisfied in a neighbourhood, for non-vanishing θ_{ns}, η must be constant. The only possible solution for non-constant η must be associated with a vector field for which θ_{ns} vanishes.

When θ_{ns} vanishes, (2.30) requires that

$$\frac{\delta\kappa_s}{\delta s}=0. \tag{5.19}$$

Also, by (2.32)

$$\frac{\delta\tau_s}{\delta s}=\kappa_s\operatorname{div}\boldsymbol{b}. \tag{5.20}$$

By (4.28) and (5.20)

$$\operatorname{div} \boldsymbol{b}\left[\frac{\delta \eta}{\delta n}+\kappa_s \eta^3\left(\frac{1}{\eta^2}-\eta\right)\right]=0, \tag{5.21}$$

so that either

$$\operatorname{div} \boldsymbol{b}=0 \tag{5.22}$$

or

$$\frac{\delta \eta}{\delta n}+\kappa_s \eta^3\left(\frac{1}{\eta^2}-\eta\right)=0 \tag{5.23}$$

or both these conditions hold.

When div $\boldsymbol{b}$ vanishes, (5.20) shows that

$$\frac{\delta \tau_s}{\delta s}=0, \tag{5.24}$$

so that the vector-lines of $\boldsymbol{s}$ are circular helices and the surfaces $U(x^\alpha)=$constant are concentric circular cylinders.

When (5.23) holds, (5.12) gives

$$\left(\frac{1}{\eta^2}-\eta\right)\Omega_s^2-2\kappa_s^2 \eta^3\left(\frac{1}{\eta^2}-\eta\right)+\kappa_s^2 \eta^5\left(\frac{1}{\eta^2}-\eta\right)^2=0,$$

or, if η is not constant,

$$\Omega_s^2=\kappa_s^2 \eta^3(1+\eta^3). \tag{5.25}$$

Again by (4.27), since θ_{ns} is zero and η is not constant,

$$\frac{\delta \Omega_s}{\delta b}=0. \tag{5.26}$$

Taking the gradient of (5.25) with respect to $\boldsymbol{b}$, and using $(4.3)_2$ and (5.26), we obtain

$$\eta^3 \kappa_s \frac{\delta \kappa_s}{\delta b}(1+\eta^3)=0,$$

so that by (2.31)

$$\eta^3 \kappa_s^2 \operatorname{div} \boldsymbol{b}(1+\eta^3)=0. \tag{5.27}$$

From (5.27) we see that if η is not constant, or zero, either κ_s or div $\boldsymbol{b}$ must vanish. When κ_s vanishes, the vector-lines of $\boldsymbol{s}$ are straight lines. According to the convention discussed in the Introduction our basic formalism holds for $\kappa_s=0$; however this case requires special consideration. When κ_s does not vanish, the vanishing of div $\boldsymbol{b}$ and θ_{ns} implies once again that the vector-lines of $\boldsymbol{s}$ must be circular helices and the surfaces $U(x^\alpha)=$constant must be circular cylinders.

So far, we have shown that either η must be constant, or the vector-lines of $\boldsymbol{s}$ must be straight lines or circular helices mounted on concentric circular cylinders. The latter case includes the case $\tau_s=0$, $\kappa_s \neq 0$, in which the helices become concentric circles perpendicular to the generators. We may easily show that η has to be constant in this case.

Our argument above (which is not predicated on τ_s not vanishing) requires that div $\boldsymbol{b}$ and θ_{ns} each vanish. From (5.7), since κ_s is not zero,

$$\kappa_{b_N}=0. \tag{5.28}$$

Then by (3.32)

$$\Omega_s = 0. \tag{5.29}$$

ERICKSEN analysed this case. We may confirm his result. Thus by (5.10), for θ_{ns}, τ_s, Ω_s and κ_{b_N} each zero,

$$5\kappa_s \frac{\delta\eta}{\delta n} + \frac{3}{\eta}\left(\frac{\delta\eta}{\delta n}\right)^2 = 0, \tag{5.30}$$

while, from (5.12)

$$2\kappa_s \frac{\delta\eta}{\delta n} + \frac{1}{\eta}\left(\frac{\delta\eta}{\delta n}\right)^2 = 0. \tag{5.31}$$

It follows from (5.30) and (5.31) that $\delta\eta/\delta n$ must vanish.

It remains to consider the case in which κ_s vanishes.

If η is not constant, equation (5.7) requires

$$\tau_s^2 + \theta_{ns}^2 = 0, \tag{5.32}$$

so that θ_{ns} and τ_s must each be zero. By (4.28),

$$\frac{\operatorname{div} \boldsymbol{b}}{\eta^3} \frac{\delta\eta}{\delta n} = 0, \tag{5.33}$$

so that if η is not constant, div $\boldsymbol{b}$ must vanish. By (3.32) when θ_{ns}, τ_s, κ_s and div $\boldsymbol{b}$ vanish,

$$\kappa_{b_N} \Omega_s = 0, \tag{5.34}$$

so that either Ω_s or κ_{b_N} vanish, or both. When Ω_s and κ_s are zero, equation (5.12) shows that $\delta\eta/\delta n$ must be zero, so that η is constant. When κ_{b_N}, κ_s, θ_{ns}, and τ_s vanish, equations (5.10) and (5.11) reduce, respectively, to

$$\left(1 - \frac{5}{\eta^3}\right)\left(\frac{1}{\eta^2} - \eta\right)\Omega_s^2 - \frac{6}{\eta^4}\left(\frac{\delta\eta}{\delta n}\right)^2 = 0, \tag{5.35}$$

and

$$-\left(1 - \frac{1}{\eta^3}\right)\left(\frac{1}{\eta^2} - \eta\right)\Omega_s^2 + \frac{2}{\eta^4}\left(\frac{\delta\eta}{\delta n}\right)^2 = 0. \tag{5.36}$$

By (5.35) and (5.36),

$$\left(1 + \frac{1}{\eta^3}\right)\left(\frac{\delta\eta}{\delta n}\right)^2 = 0, \tag{5.37}$$

so that $\delta\eta/\delta n$ must be zero. Thus η is constant.

We have thus established

Theorem 1.1. *The only universal deformations of a homogeneous isotropic incompressible hyperelastic material due to surface tractions only, which correspond to the case*

$$c_1 = c_3 = \eta, \tag{1.19}$$

are

1) deformations in which η is constant,

2) deformations in which the vector-lines of s are circular helices mounted on concentric circular cylinders, the cases where the helices reduce to circles or straight lines being excluded.

Theorem 1.1 shows that the only possible vector-line geometry corresponding to non-constant η is the circular helical configuration which was analysed by ERICKSEN for the case when the proper values are distinct (1954 [1], p. 480). The case represented by (1.19) is thus a special case of that already considered.

We obtain

Theorem 1.2. *The only deformations that can be produced in* all *homogeneous, isotropic, incompressible, hyperelastic bodies by application of suitable surface tractions alone, are homogeneous deformations, the four families of deformations cited in the Introduction, and non-homogeneous deformations in which the principal stretches are constant and both* $c^b_{a,be}$ *and* $(c^{-1})^b_{a,be}$ *are symmetric in the indices a and e.*

Professor ERICKSEN has pointed out to us that our analysis may be applied to one particular category of deformations of the class defined by Theorem 1.2.

We require that

$$\operatorname{div} \boldsymbol{c} = \operatorname{grad} \chi . \tag{5.38}$$

In the case

$$c_1 = c_3 = \eta , \tag{1.19}$$

we have, by (1.32),

$$\boldsymbol{c} = \left(\frac{1}{\eta^2} - \eta\right) \boldsymbol{s}\boldsymbol{s} + \eta \boldsymbol{I} , \tag{5.39}$$

where $\boldsymbol{I}$ is the unit dyadic, and η is constant.

By (5.38) and (5.39)

$$\operatorname{div} \boldsymbol{c} = \left(\frac{1}{\eta^2} - \eta\right) [\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s} + \boldsymbol{s} \operatorname{div} \boldsymbol{s}] = \operatorname{grad} \chi . \tag{5.40}$$

In this case we do not know that the vector-lines of $\boldsymbol{s}$ lie on the surfaces $\chi(x^\alpha) =$ constant. However, in the special case when they do so,

$$\boldsymbol{s} \cdot \operatorname{grad} \chi = 0; \tag{5.41}$$

then, since η is constant, (5.40) yields the conditions

$$\operatorname{div} \boldsymbol{s} = \theta_{ns} + \theta_{bs} = 0 , \tag{2.4}$$

$$\operatorname{curl}(\boldsymbol{\omega} \times \boldsymbol{s}) = 0 , \quad \boldsymbol{\omega} = \operatorname{curl} \boldsymbol{s} . \qquad (1.31), (1.30)$$

Also

$$\boldsymbol{s} \cdot \operatorname{grad} \boldsymbol{s} = \kappa_s \boldsymbol{n} = \left(\frac{1}{\eta^2} - \eta\right)^{-1} \operatorname{grad} \chi , \quad \text{for } \eta \neq 1 . \tag{5.42}$$

When $\eta = 1$, the material is unstrained. From (5.42), we see that $\boldsymbol{n}$ is parallel to grad χ and that the vector-lines of $\boldsymbol{b} = \boldsymbol{s} \times \boldsymbol{n}$ lie on the surfaces $\chi(x^\alpha) =$ constant. Our formalism holds for this particular case.

From (2.4), (2.17), and (2.19)

$$\frac{\delta \theta_{ns}}{\delta s} = \theta_{ns}^2 + \kappa_s \kappa_{b_N} - \tau_s^2 . \tag{5.43}$$

For $\delta\eta/\delta n=0$, and $\eta\neq 1$, (4.23), (4.24), (4.25), and (5.43), give

$$\left(\frac{1}{\eta^3}+3\right)\Omega_s^2+4[2\theta_{ns}^2+\kappa_s\kappa_{b_N}+\tau_s(2\tau_s-3\Omega_s)]=0, \tag{5.44}$$

$$\left(\frac{3}{\eta^3}+1\right)\Omega_s^2+4[\theta_{ns}^2+\tau_s^2-\Omega_s\tau_s]=0, \tag{5.45}$$

$$\left(\frac{1}{\eta^3}-1\right)\Omega_s^2+4[\Omega_s\tau_s-\kappa_s\kappa_{b_N}]=0, \tag{5.46}$$

Equations (5.44), (5.45), and (5.46), imply

$$\Omega_s=\theta_{ns}=\tau_s=0 \tag{5.47}$$

and

$$\kappa_s\kappa_{b_N}=0. \tag{5.48}$$

The three remaining curvature tensor equations, (4.26), (4.27) and (4.28), are satisfied by (5.47) for $\delta\eta/\delta n=0$.

By (5.48) either κ_s or κ_{b_N} or both these curvatures vanish. When κ_s vanishes, by (2.6) and (5.47), grad $s=0$, so that the s-lines are parallel straight lines. If κ_s does not vanish, by (2.30) and (5.47)

$$\frac{\delta\kappa_s}{\delta s}=0, \quad \tau_s=0,$$

so that the s-lines are concentric circles.

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