On a Class of Conservation Laws in Linearized and Finite Elastostatics

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1. Introduction

Several years ago ESHELBY [1] (1956), in a paper devoted to the continuum theory of lattice defects, deduced a surface-integral representation for the "force on an elastic singularity or inhomogeneity", which – in the absence of such defects – gives rise to a conservation law for regular elastostatic fields appropriate to homogeneous but not necessarily isotropic solids in the presence of infinitesimal deformations. Morevoer, ESHELBY noted that his result, when suitably interpreted, remains strictly valid for finite deformations of elastic solids.

The two-dimensional analogue of the conservation law alluded to above, asserting the path-independence of a certain *line*-integral associated with *plane* elastostatic fields, was independently discovered by RICE [2] (1968), who adhered to infinitesimal deformations but admitted the possibility of nonlinear stress-strain relations. The physical interpretation of this line integral advanced by RICE is based on the energetics of quasi-static crack extension. The work contained in [2] is intimately related to earlier investigations by SANDERS [3] and CHEREPANOV [4].

Apart from its inherent theoretical interest, the conservation law made explicit in [2] is of practical importance in connection with the direct asymptotic analysis of geometrically induced singular stress concentrations, such as those occasioned by cracks and notches. Applications of this kind, most of which pertain to inelastic behavior, may be found in [2], [4], as well as in papers by HUTCHINSON [5], [6]. The two-dimensional conservation law given in [2] may roughly be stated as follows. Let (x_1, x_2) be rectangular cartesian coordinates and let Π be a domain of the (x_1, x_2)-plane. Suppose u_α, $\gamma_{\alpha\beta}$, $\sigma_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are suitably smooth functions defined on Π that represent, respectively, the cartesian components of a displacement field, its associated infinitesimal strains, and the components of an equilibrium field of stress corresponding to vanishing body forces. Thus assume

$$\gamma_{\alpha\beta} = u_{(\alpha,\beta)}, \quad \sigma_{\alpha\beta,\beta} = 0, \quad \sigma_{\beta\alpha} = \sigma_{\alpha\beta} \quad \text{on } \Pi. \qquad (1.1)^1$$

Further, let the stresses be derivable from an elastic potential in accordance with the constitutive law

$$\sigma_{\alpha\beta} = \partial\Gamma(\gamma)/\partial\gamma_{\alpha\beta} \quad \text{on } \Pi, \qquad (1.2)$$

[1] Here and throughout this paper summation over repeated subscripts is implied; subscripts preceded by a comma indicate partial differentiation with respect to the cartesian coordinates.

in which γ stands for the matrix $[\gamma_{\alpha\beta}]$, whereas

$$\Gamma(\gamma(\boldsymbol{x})) = W(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Pi \tag{1.3}$$

is the strain-energy density at the point with the position vector $\boldsymbol{x}$. Then, for every curve C that is the boundary of a finite regular closed subregion[1] of Π,

$$\int_C (W n_\alpha - s_\beta u_{\beta,\alpha})\, d\ell = 0, \tag{1.4}$$

provided $\boldsymbol{n}$ is the unit outward normal vector of C and $\boldsymbol{s}$ the traction vector, that is,

$$s_\beta = \sigma_{\beta\alpha} n_\alpha \quad \text{on } C. \tag{1.5}$$

It should be emphasized that although (1.2) include as special cases the constitutive laws underlying the conventional linearized theories of plane strain and generalized plane stress, the validity of (1.4) is not contingent upon the linearity of the stress-strain relations (1.2).

The conservation law (1.4) may be confirmed at once by means of (1.1), (1.2), (1.3), (1.5) together with the divergence theorem, and (1.4) is established in this manner in [2][2]. Such an *ad hoc* verification of (1.4), however, supplies no clue as to its analytical roots within the theory under contemplation and at the same time suggests the question whether there exist other such laws.

In this paper we show that the conservation law (1.4), as well as its three-dimensional analogue, may be generated systematically with the aid of a theorem due to NOETHER [7] on invariant variational principles in conjunction with the principle of stationary potential energy. This procedure, moreover, yields two additional conservation laws. The two-dimensional version of the first of these asserts that if (1.2), in particular, is form-invariant under a rotation of the coordinate frame, so that the material is *isotropic*, then also

$$\int_C \varepsilon_{\alpha\beta}(W x_\beta n_\alpha + s_\alpha u_\beta - s_\rho u_{\rho,\alpha} x_\beta)\, d\ell = 0. \tag{1.6}$$ [3]

Next, if the stress-strain law (1.2) is *linear* (though not necessarily isotropic), in which instance

$$W = \tfrac{1}{2} \sigma_{\alpha\beta} \gamma_{\alpha\beta}, \tag{1.7}$$

one has in addition

$$\int_C (W n_\alpha x_\alpha - s_\beta u_{\beta,\alpha} x_\alpha)\, d\ell = 0. \tag{1.8}$$

We show further that, in a sense made precise, (1.4), (1.6), (1.8) together with their spatial counterparts are the *only* nontrivial conservation laws deducible from Noether's theorem in *linear* elastostatics. Finally, the application of Noether's theorem to *finite elastostatics* confirms that (1.4), (1.6), as well as their three-dimensional analogues, continue to hold for the nonlinear theory provided W and $\boldsymbol{s}$ are suitably redefined in this context.

In Section 2, after some required preliminaries, we state and sketch a proof of a restricted version of Noether's theorem on invariant variational principles that

[1] Thus C may be the union of several disjoint piecewise smooth closed simple curves.

[2] Actually, RICE [2] deals with the special case of (1.4) corresponding to $\alpha = 1$.

[3] Here $\varepsilon_{\alpha\beta}$ stands for the components of the two-dimensional alternator.

is sufficiently broad to cover our needs. We then specialize the theorem with a view toward its subsequent application to elastostatics. The expository part of Section 2 has been included primarily in order to render the present paper sensibly self-contained.

In Section 3 we employ the results of Section 2 to deduce conservation laws in the three-dimensional equilibrium theory of infinitesimally deformed elastic solids. The three laws thus emerging, which reduce to (1.4), (1.6), (1.8) in two dimensions, are then proved independently by recourse to the divergence theorem. Here we also establish the completeness of the foregoing three laws for the fully linearized theory. Section 4 is devoted to conservation laws in nonlinear elastostatics.

We conclude these introductory remarks by noting that NOETHER's [7] scheme has previously been applied in other areas of mathematical physics[1]. In contrast, its implications for the theory of elasticity appear to have gone unexplored.

2. Preliminaries. A Special Case of Noether's Theorem and Some of its Implications

Throughout this paper the symbols $\mathscr{E}$ and $\mathscr{V}$ denote, respectively, a three-dimensional euclidean point space and its associated vector space. We call R a *regular region* if R is a bounded closed region in $\mathscr{E}$ and the divergence theorem holds for all suitably smooth vector fields defined on R. Accordingly, the regularity of a bounded closed region R is assured if its boundary ∂R is the union of a finite number of disjoint "closed regular surfaces", the latter term being used in the sense of KELLOGG [11].

Letters in boldface are reserved for tensors of positive order and for matrices. In the case of tensors of the first or second order such letters denote both the tensor and its matrix of scalar components in a fixed rectangular cartesian coordinate frame, unless they appear as arguments of functions; in the latter instance boldface letters are used exclusively to designate the appropriate component matrix. Specifically, if the symbol $\boldsymbol{v}$ stands for a vector, it also refers to the *column* matrix[2] $[v_i]$ of the components of $\boldsymbol{v}$ in the underlying coordinate frame. Similarly, if $\boldsymbol{t}$ represents a second-order tensor, it also denotes the 3×3 matrix $[t_{ij}]$ of the components of $\boldsymbol{t}$ in the frame under consideration. On the other hand, $\boldsymbol{v}=[v_i]$ and $\boldsymbol{t}=[t_{ij}]$ in the event that $\boldsymbol{v}$ or $\boldsymbol{t}$ are values of arguments of functions.

In what follows we have occasion to deal with functions that depend on the cartesian coordinates, the components of a vector, the components of a second-order tensor, and on a scalar parameter—or on a subset of the preceding array of arguments. We now adopt a uniform notational scheme for the partial differentiation of such functions. To this end, let $\mathscr{T}$ be the space of all second-order tensors, let L be an open linear interval, and suppose F is a real number-valued function defined on the product-space $\mathscr{E}\times\mathscr{V}\times\mathscr{T}\times L$, with the values

$$F(\boldsymbol{x},\boldsymbol{v},\boldsymbol{t};\eta) \text{ for all } (\boldsymbol{x},\boldsymbol{v},\boldsymbol{t};\eta)\in\mathscr{E}\times\mathscr{V}\times\mathscr{T}\times L. \tag{2.1}$$

[1] See, for example, BESSEL-HAGEN [8], KRUSKAL & ZABUSKY [9]. Reference may also be made to CASTEN [10].

[2] Latin subscripts are understood to range over the integers (1, 2, 3).

We then write

$$
\begin{aligned}
F_{,i}(x, v, t; \eta) &= \frac{\partial}{\partial x_i} F(x, v, t; \eta), \\
F_{,v_i}(x, v, t; \eta) &= \frac{\partial}{\partial v_i} F(x, v, t; \eta), \\
F_{,t_{ij}}(x, v, t; \eta) &= \frac{\partial}{\partial t_{ij}} F(x, v, t; \eta), \\
F'(x, v, t; \eta) &= \frac{\partial}{\partial \eta} F(x, v, t; \eta),
\end{aligned}
\tag{2.2}
$$

provided the foregoing differentiations are meaningful. Strictly analogous notation will be employed in connection with vector or tensor-valued functions and for higher-order derivatives.

We are at present in a position to assemble the chief ingredients of the theorem that constitutes our immediate objective. Thus, suppose H is a real number-valued function defined on $\mathscr{E} \times \mathscr{V} \times \mathscr{T}$ that has the values

$$
H(x, v, t) \text{ for all } (x, v, t) \in \mathscr{E} \times \mathscr{V} \times \mathscr{T}. \tag{2.3}
$$

Assume further $H \in \mathscr{C}^\infty(\mathscr{E} \times \mathscr{V} \times \mathscr{T})$, so that H possesses continuous partial derivatives of all orders with respect to the elements of its matrix arguments on its domain of definition. Next, let D henceforth be a fixed open region in $\mathscr{E}$ and for every regular subregion Ω of D and all vector fields $w \in \mathscr{C}^2(D)$, let Φ be the functional defined by

$$
\Phi\{w\} = \int_\Omega H\big(x, w(x), \nabla w(x)\big)\, dv, \tag{2.4}
$$

in which ∇w denotes the component matrix $[w_{i,j}]$. We shall refer to Φ so defined as an *admissible functional for D generated by H.*

We now proceed to embed Φ in a one-parameter family of functionals. For this purpose we first subject x and v in (2.3) each to a one-parameter family of transformations. We call f a *regular family of coordinate mappings on D* if f is a vector-valued function defined on $D \times L$ such that

$$
\xi = f(x; \eta) \text{ for all } (x, \eta) \in D \times L, \qquad L = (-\eta_0, \eta_0), \tag{2.5}
$$

and f has the properties:

(i) $f \in \mathscr{C}^2(D \times L), \quad f(x; 0) = x$ for each $x \in D$; (2.6)

(ii) $f(\cdot\,; \eta)$ is one-to-one on D for each $\eta \in L$ with

$$
\varphi(\xi; \eta) = x \qquad \text{for all } (\xi, \eta) \in \Lambda, \varphi \in \mathscr{C}^2(\Lambda), \tag{2.7}
$$

where

$$
\Lambda = \{(\xi, \eta) \mid \xi \in D_\eta, \eta \in L\}, \tag{2.8}
$$

while D_η is the image of D under the mapping $f(\cdot, \eta)$. It follows from (i), (ii) that the Jacobian determinant of f satisfies

$$
\begin{aligned}
\Delta(x; \eta) \equiv \det[f_{i,j}(x; \eta)] &> 0 \qquad \text{for all } (x, \eta) \in D \times L, \\
\Delta(x; 0) &= 1 \qquad \text{for all } x \in D.
\end{aligned}
\tag{2.9}
$$

Next, we call $\boldsymbol{h}$ a *regular family of vector transformations*, provided $\boldsymbol{h}$ is a vector-valued function defined on $\mathscr{V} \times L$ with

$$\boldsymbol{h} \in \mathscr{C}^2(\mathscr{V} \times L), \quad L = (-\eta_0, \eta_0), \quad \boldsymbol{h}(\boldsymbol{v}; 0) = \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in \mathscr{V}. \tag{2.10}$$

Finally, let $\boldsymbol{\psi}$ be the vector-valued function defined by means of the composite mapping

$$\boldsymbol{\psi}(\boldsymbol{\xi}; \eta) = \boldsymbol{h}(\boldsymbol{w}(\boldsymbol{\varphi}(\boldsymbol{\xi}; \eta)); \eta) \quad \text{for all } (\boldsymbol{\xi}, \eta) \in \Lambda \tag{2.11}$$

and, corresponding to each regular subregion Ω of D, introduce a one-parameter family of functionals Φ_η by setting

$$\Phi_\eta\{\boldsymbol{w}\} = \int_{\Omega_\eta} H(\boldsymbol{\xi}, \boldsymbol{\psi}(\boldsymbol{\xi}; \eta), \underset{\boldsymbol{\xi}}{\nabla} \boldsymbol{\psi}(\boldsymbol{\xi}; \eta))\, dv_{\boldsymbol{\xi}} \quad (-\eta_0 < \eta < \eta_0) \tag{2.12}$$

for every vector field $\boldsymbol{w} \in \mathscr{C}^2(D)$. Here

$$\underset{\boldsymbol{\xi}}{\nabla} \boldsymbol{\psi}(\boldsymbol{\xi}; \eta) = [\psi_{i, \xi_j}(\boldsymbol{\xi}; \eta)], \tag{2.13}$$

whereas Ω_η is the image of Ω under the mapping $\boldsymbol{f}(\cdot\,; \eta)$. From (2.5), (2.6), (2.7), (2.10), (2.11), in conjunction with (2.4), (2.12), one infers easily that

$$\underset{\boldsymbol{\xi}}{\nabla} \boldsymbol{\psi}(\boldsymbol{\xi}; \eta)|_{\eta=0} = \nabla \boldsymbol{w}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in D \tag{2.14}$$

and that

$$\Phi_\eta\{\boldsymbol{w}\}|_{\eta=0} = \Phi\{\boldsymbol{w}\} \quad \text{for every } \boldsymbol{w} \in \mathscr{C}^2(D). \tag{2.15}$$

Hence Φ is the member of the family Φ_η that corresponds to $\eta = 0$ in (2.12). We shall address Φ_η as *the family of functionals for D associated with the functional Φ and induced by the families of mappings $\boldsymbol{f}$ and $\boldsymbol{h}$*. Moreover, we shall say that Φ is *invariant at $\boldsymbol{w}$ with respect to $\boldsymbol{f}$ and $\boldsymbol{h}$* whenever

$$\Phi_\eta\{\boldsymbol{w}\} = \Phi\{\boldsymbol{w}\} \quad (-\eta_0 < \eta < \eta_0) \quad \text{for every regular } \Omega \subset D \tag{2.16}$$

and that Φ is *infinitesimally invariant* at $\boldsymbol{w}$ with respect to the given pair of mapping families, provided

$$\Phi_0'\{\boldsymbol{w}\} \equiv \frac{\partial}{\partial \eta} \Phi_\eta\{\boldsymbol{w}\}|_{\eta=0} = 0 \quad \text{for every regular } \Omega \subset D. \tag{2.17}$$

Evidently, Φ is necessarily *infinitesimally* invariant if it satisfies the stronger invariance requirement (2.16).

The preceding auxiliary notions enable us to state concisely the subsequent restricted version of Noether's theorem [1].

Theorem 2.1. *Let D be a domain in $\mathscr{E}$ and let Φ be an admissible functional for D generated by H. Let $\boldsymbol{f}$ be a regular family of coordinate mappings on D and $\boldsymbol{h}$ a regular family of vector transformations. Suppose $\boldsymbol{w} \in \mathscr{C}^2(D)$ is a vector field satisfying the Euler equations*

$$H_{,v_k}(X) - \frac{\partial}{\partial x_j} H_{,t_{kj}}(X) = 0 \quad \text{for all } \boldsymbol{x} \in D, \tag{2.18}$$

[1] See Noether [7] and Gelfand-Fomin [12] (p. 176) for a more comprehensive theorem of the same type, which rests on a broader class of admissible functionals and involves more general mappings.

where

$$X=(x, w(x), \nabla w(x)). \tag{2.19}$$

Then Φ is infinitesimally invariant at w with respect to f and h if and only if w satisfies

$$\frac{\partial}{\partial x_j}[H(X)\,a_j(x)+m_k(x)\,H_{,t_{kj}}(X)]=0 \quad \text{for all } x\in D, \tag{2.20}$$

where

$$\begin{aligned} a_j(x)&=f_j'(x;0) \quad \text{for all } x\in D,\\ b_j(v)&=h_j'(v;0) \quad \text{for all } v\in\mathscr{V},\\ m_k(x)&=b_k(w(x))-a_j(x)\,w_{k,j}(x) \quad \text{for all } x\in D. \end{aligned} \tag{2.21}$$

Further, (2.20) *is equivalent to the conservation law in integral form asserting that*

$$\int_S [H(X)\,a_j(x)+m_k(x)\,H_{,t_{kj}}(X)]\,n_j(x)\,da=0 \tag{2.22}$$

for every surface S that is the boundary of a regular subregion of D, provided n is the outward unit normal vector of S.

The hypothesis that w is a solution of the Euler equations (2.18) is, in turn, equivalent to the requirement that the functional $\Phi\{\cdot\}$ introduced in (2.4) be stationary at w with respect to a suitable class of variations of w. Indeed, as is readily confirmed, $w\in\mathscr{C}^2(D)$ satisfies (2.18) if and only if, for every fixed regular region $\Omega\subset D$,

$$\delta\Phi\{w\}=0, \tag{2.23}$$

the first variation of Φ being taken over the class of all vector fields twice continuously differentiable on D that coincide with w on the boundary $\partial\Omega$. The usefulness of Theorem 5.1 as a device for generating conservation laws in any particular branch of mathematical physics thus depends on the availability of an appropriate variational principle as well as on the existence of regular mapping families f and h with respect to which the stationary functional is infinitesimally invariant. We now sketch a

Proof of Theorem 2.1. From (2.12), (2.11), (2.7), and (2.9) follows, for every $\eta\in L$,

$$\Phi_\eta\{w\}=\int_\Omega H(\xi, h(w(x);\eta), \nabla_\xi\psi(\xi;\eta))\,\Delta(x;\eta)\,dv_x, \tag{2.24}$$

where ξ is related to x through (2.5). Differentiating (2.24) with respect to η and then setting $\eta=0$, we infer, on making use of the abridged notation introduced in (2.19) and in view of (2.5), (2.10), (2.14), that

$$\begin{aligned} \Phi_0'\{w\}=\int_\Omega [&H(X)\,\Delta'(x;0)+H_{,j}(X)f_j'(x;0)\\ &+H_{,v_j}(X)\,h_j'(w(x);0)+H_{,t_{ij}}(X)\,\psi_{i,\xi_j}'(x;0)]\,dv. \end{aligned} \tag{2.25}$$

On the other hand, (2.5) to (2.9) together with (2.10), (2.11), after elementary computations, furnish for all $x\in D$,

$$\begin{aligned} \Delta'(x;0)&=f_{j,j}'(x;0),\\ \psi_{i,\xi_j}'(x;0)&=h_{i,v_k}'(w(x);0)\,w_{k,j}(x)-w_{i,k}(x)\,f_{k,j}'(x;0). \end{aligned} \tag{2.26}$$

Substituting from (2.26) into (2.25) and noting that the resulting integrand is continuous on D, one finds that $\Phi_0'\{w\}$ vanishes for *every* regular $\Omega\subset D$ if and only if for each $x\in D$,

$$\begin{aligned} &H(X)f_{j,j}'(x;0)+H_{,j}(X)f_j'(x;0)+H_{,v_j}(X)\,h_j'(w(x);0)\\ &\quad+H_{,t_{ij}}(X)\left[h_{i,v_k}'(w(x);0)\,w_{k,j}(x)-w_{i,k}(x)f_{k,j}'(x;0)\right]=0. \end{aligned} \tag{2.27}$$

Next, because of (2.19) one has the identity

$$\begin{aligned} &\frac{\partial}{\partial x_j}\left[H(X)f_j'(x;0)\right]\\ &\quad=H(X)f_{j,j}'(x;0)+f_j'(x;0)\left[H_{,j}(X)+H_{,v_k}(X)\,w_{k,j}(x)+H_{,t_{ki}}(X)\,w_{k,ij}(x)\right]. \end{aligned} \tag{2.28}$$

With the aid of (2.28) and the auxiliary notation appearing in (2.21) one may write (2.27) in the form

$$\begin{aligned} &\frac{\partial}{\partial x_j}\left[H(X)\,a_j(x)+m_k(x)\,H_{,t_{kj}}(X)\right]\\ &\quad+m_k(x)\left[H_{,v_k}(X)-\frac{\partial}{\partial x_j}H_{,t_{kj}}(X)\right]=0\quad\text{for all } x\in D. \end{aligned} \tag{2.29}$$

Evidently, if w is a solution of the Euler equations (2.18), equation (2.29) yields (2.20) as a necessary and sufficient condition that Φ be infinitesimally invariant at w with respect to the mapping families f and h. Finally, the equivalence of the conservation laws (2.20) and (2.22) is immediate from the divergence theorem. This completes the proof.

Our next task is to show that if H in the preceding theorem is subjected to certain additional restrictions, there do in fact exist mapping families f and h with respect to which Φ is infinitesimally invariant at w. Further, we shall deduce the explicit form of the conservation laws emerging from these particular mappings.

Theorem 2.2. *Let D, Φ, H and w satisfy the same hypotheses as in Theorem 2.1. In addition, let*

$$H(x,v,t)=M(t)\quad \textit{for all } (x,v,t)\in\mathscr{E}\times\mathscr{V}\times\mathscr{T}. \tag{2.30}$$

Then the Euler equations satisfied by w reduce to

$$\frac{\partial}{\partial x_j}M_{,t_{kj}}(\nabla w(x))=0\quad \textit{for all } x\in D, \tag{2.31}$$

Φ is invariant at w with respect to the pair of mapping families

$$\begin{aligned} f(x;\eta)&=x+\eta\alpha\quad \textit{for all } x\in D\ (\alpha=\textit{constant}),\\ h(v;\eta)&=v\quad \textit{for all } v\in\mathscr{V}\ (-\eta_0<\eta<\eta_0), \end{aligned} \tag{2.32}$$

and the corresponding conservation law becomes

$$\int_S\left[M(\nabla w(x))\,n_i(x)-w_{k,i}(x)\,M_{,t_{kj}}(\nabla w(x))\,n_j(x)\right]da=0. \tag{2.33}$$

Moreover, if M for every orthogonal matrix $\boldsymbol{q}$ obeys

$$M(\boldsymbol{q}\boldsymbol{t}\boldsymbol{q}^T)=M(\boldsymbol{t}) \quad \text{for all } \boldsymbol{t}\in\mathscr{T}, \tag{2.34}$$ [1]

then Φ is invariant at $\boldsymbol{w}$ also with respect to

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x};\eta)&=\boldsymbol{q}(\eta)\boldsymbol{x} \quad \text{for all } \boldsymbol{x}\in D,\\ \boldsymbol{h}(\boldsymbol{v};\eta)&=\boldsymbol{q}(\eta)\boldsymbol{v} \quad \text{for all } \boldsymbol{v}\in\mathscr{V}\ (-\eta_0<\eta<\eta_0), \end{aligned} \tag{2.35}$$

where

$$\boldsymbol{q}\in\mathscr{C}^2(L), \quad \boldsymbol{q}\boldsymbol{q}^T=\mathbf{1} \ \text{ on } L, \quad \boldsymbol{q}(0)=\mathbf{1}, \quad L=(-\eta_0,\eta_0), \tag{2.36}$$ [2]

and this pair of mapping families gives rise to the conservation law

$$\begin{aligned} \int_S \varepsilon_{ijk}\big[M(\nabla\boldsymbol{w}(\boldsymbol{x}))\,x_k n_j(\boldsymbol{x})+w_k(\boldsymbol{x})\,M_{,t_{jl}}(\nabla\boldsymbol{w}(\boldsymbol{x}))\,n_l(\boldsymbol{x})\\ -w_{p,j}(\boldsymbol{x})\,x_k M_{,t_{pl}}(\nabla\boldsymbol{w}(\boldsymbol{x}))\,n_l(\boldsymbol{x})\big]\,da=0. \end{aligned} \tag{2.37}$$ [3]

Finally, if (2.30) *holds and M for every real constant κ obeys*

$$M(\kappa\boldsymbol{t})=\mathscr{K}^2 M(\boldsymbol{t}) \quad \text{for all } \boldsymbol{t}\in\mathscr{T}, \tag{2.38}$$

so that M is a homogeneous function of the second degree, then Φ is infinitesimally *invariant with respect to*

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x};\eta)&=(1+\eta)\boldsymbol{x} \quad \text{for all } \boldsymbol{x}\in D,\\ \boldsymbol{h}(\boldsymbol{v};\eta)&=(1-\tfrac{1}{2}\eta)\boldsymbol{v} \quad \text{for all } \boldsymbol{v}\in\mathscr{V}\ (-\eta_0<\eta<\eta_0), \end{aligned} \tag{2.39}$$

and the corresponding conservation law takes the form

$$\int_S \{M(\nabla\boldsymbol{w}(\boldsymbol{x}))\,x_i n_i(\boldsymbol{x})-[x_i w_{k,i}(\boldsymbol{x})+\tfrac{1}{2}w_k(\boldsymbol{x})]\,M_{,t_{kj}}(\nabla\boldsymbol{w}(\boldsymbol{x}))\,n_j(\boldsymbol{x})\}\,da=0. \tag{2.40}$$

The three pairs of mappings introduced in the foregoing theorem trivially involve *regular* families of coordinate mappings on D and of accompanying vector transformations. Evidently, $\boldsymbol{f}$ in (2.31) represents a family of *translations*, $\boldsymbol{f}$ in (2.34) defines a family of *rotations* of the coordinate system, while $\boldsymbol{f}$ in (2.38) corresponds to a family of *scale changes*. We now turn to a

Proof of Theorem 2.2. The restriction on H imposed by (2.30) clearly reduces (2.18) to (2.31). Also, for the particular mappings (2.32) one draws from (2.9), (2.7), (2.11) that

$$\begin{aligned} \Delta(\boldsymbol{x};\eta)&=1 && \text{for all } (\boldsymbol{x},\eta)\in D\times(-\eta_0,\eta_0),\\ \boldsymbol{\varphi}(\boldsymbol{\xi};\eta)&=\boldsymbol{\xi}-\eta\boldsymbol{\alpha} && \text{for all } (\boldsymbol{\xi},\eta)\in\Lambda,\\ \boldsymbol{\psi}(\boldsymbol{\xi};\eta)&=\boldsymbol{w}(\boldsymbol{\varphi}(\boldsymbol{\xi};\eta)) && \text{for all } (\boldsymbol{\xi},\eta)\in\Lambda. \end{aligned} \tag{2.41}$$

Now (2.30), (2.41), together with (2.4), (2.12) yield, for every regular $\Omega\subset D$ and each $\eta\in(-\eta_0,\eta_0)$,

$$\Phi_\eta\{\boldsymbol{w}\}=\int_{\Omega_\eta} M(\nabla_{\boldsymbol{\xi}}\boldsymbol{\psi}(\boldsymbol{\xi};\eta))\,dv_{\xi}=\int_{\Omega} M(\nabla\boldsymbol{w}(\boldsymbol{x}))\,dv=\Phi\{\boldsymbol{w}\}, \tag{2.42}$$

[1] If $\boldsymbol{p}$ is a second-order tensor or a 3×3 matrix, we write $\boldsymbol{p}^T$ for the transpose of $\boldsymbol{p}$.

[2] The symbol $\mathbf{1}$ stands for the second-order unit tensor and for the 3×3 idem matrix $[\delta_{ij}]$.

[3] ε_{ijk} stands for the usual three-dimensional alternator.

so that Φ is invariant at every $w \in \mathscr{C}^2(D)$ with respect to the particular mappings (2.32). Further, from (2.32), (2.21) follows

$$a_j = \delta_{ij}, \quad m_k = -\delta_{ij} w_{k,j} \quad \text{on } D \quad (i \text{ fixed}), \tag{2.43}$$

if $\boldsymbol{\alpha}$ is chosen to be the unit base vector in the x_i-direction, and substitution from (2.43), (2.30) into (2.20) furnishes (2.33).

Next, suppose (2.30), (2.34), (2.35), and (2.36) hold. Proceeding as above one arrives in the present instance at

$$\begin{aligned} &\Delta(\boldsymbol{x};\eta) = 1 && \text{for all } (\boldsymbol{x},\eta) \in D \times (-\eta_0, \eta_0), \\ &\boldsymbol{\varphi}(\boldsymbol{\xi};\eta) = \boldsymbol{q}^T(\eta)\,\boldsymbol{\xi} && \text{for all } (\boldsymbol{\xi},\eta) \in \Lambda, \\ &\boldsymbol{\psi}(\boldsymbol{\xi};\eta) = \boldsymbol{q}(\eta)\,\boldsymbol{w}(\boldsymbol{\varphi}(\boldsymbol{\xi};\eta)) && \text{for all } (\boldsymbol{\xi},\eta) \in \Lambda. \end{aligned} \tag{2.44}$$

On the other hand, here

$$\Phi_\eta\{\boldsymbol{w}\} = \int_\Omega M(\boldsymbol{q}(\eta)\,\nabla \boldsymbol{w}(\boldsymbol{x})\,\boldsymbol{q}^T(\eta))\,dv = \int_\Omega M(\nabla \boldsymbol{w}(\boldsymbol{x}))\,dv = \Phi\{\boldsymbol{w}\} \tag{2.45}$$

for all regular $\Omega \subset D$ and each $\eta \in (-\eta_0, \eta_0)$. Hence Φ is invariant at every $w \in \mathscr{C}^2(D)$ with respect to the pair of mapping families (2.35). By virtue of (2.36),

$$q'_{jk}(0) = -q'_{kj}(0), \tag{2.46}$$

so that $q'_{jk}(0)$ admits the representation

$$q'_{jk}(0) = \varepsilon_{pjk}\, e_p, \tag{2.47}$$

where $\boldsymbol{e}$ is a constant vector. Upon choosing $\boldsymbol{e}$ coincident with the unit base vector in the x_i-direction, one finds from (2.35), (2.21) that

$$\begin{aligned} &a_j(\boldsymbol{x}) = \varepsilon_{ijk}\, x_k, \\ &m_k(\boldsymbol{x}) = \varepsilon_{ijp}\, x_j\, w_{k,p}(\boldsymbol{x}) - \varepsilon_{ijk}\, w_j(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in D. \end{aligned} \tag{2.48}$$

Substitution from (2.48) into (2.22), after elementary manipulations, leads to (2.37).

Lastly, assume (2.38) holds in addition to (2.30) and let $\boldsymbol{f}$, $\boldsymbol{h}$ be given by (2.39). Then (2.9), (2.7), (2.11) imply

$$\begin{aligned} &\Delta(\boldsymbol{x};\eta) = (1+\eta)^3 && \text{for all } (\boldsymbol{x},\eta) \in D \times (-\eta_0, \eta_0), \\ &\boldsymbol{\varphi}(\boldsymbol{\xi};\eta) = \frac{\boldsymbol{\xi}}{1+\eta} && \text{for all } (\boldsymbol{\xi},\eta) \in \Lambda, \\ &\boldsymbol{\psi}(\boldsymbol{\xi};\eta) = (1-\tfrac{1}{2}\eta)\,\boldsymbol{w}(\boldsymbol{\xi}/(1+\eta)) && \text{for all } (\boldsymbol{\xi},\eta) \in \Lambda. \end{aligned} \tag{2.49}$$ [1]

In these circumstances (2.12) gives

$$\begin{aligned} \Phi_\eta\{\boldsymbol{w}\} &= \int_{\Omega_\eta} M(\nabla_{\boldsymbol{\xi}} \boldsymbol{\psi}(\boldsymbol{\xi};\eta))\,dv_{\xi} \\ &= \int_\Omega M\left(\frac{1-\eta/2}{1+\eta}\,\nabla \boldsymbol{w}(\boldsymbol{x})\right)(1+\eta)^3\,dv \quad (-\eta_0 < \eta < \eta_0), \end{aligned} \tag{2.50}$$

[1] Here we suppose, say, $\eta_0 < 1$.

and thus

$$\Phi_0'\{w\}=\tfrac{3}{2}\int_{\Omega}[2M(\nabla w(x))-M_{,t_{ij}}(\nabla w(x))\,w_{i,j}(x)]\,dv. \tag{2.51}$$

But (2.38) assures that

$$2M(t)-M_{,t_{ij}}(t)\,t_{ij}=0 \quad \text{for all } t\in\mathcal{T}. \tag{2.52}$$

Consequently, Φ is *infinitesimally* invariant at every $w\in\mathscr{C}^2(D)$ with respect to the two mapping families (2.39). Incidentally, it is evident from (2.51), (2.52), and Euler's theorem on homogeneous functions that the hypothesis (2.38) is also a *necessary* condition for the infinitesimal invariance of Φ with respect to the mappings f and h under consideration.

Noting that now

$$a_j(x)=x_j, \quad m_k(x)=-\tfrac{1}{2}w_k(x)-x_j w_{k,j}(x) \quad \text{for all } x\in D, \tag{2.53}$$

one easily confirms the conservation law (2.40) on the basis of (2.22), (2.30), and this completes the proof.

The preceding proof of (2.40) suggests that one may generate a more comprehensive conservation law by replacing (2.38) with the broader assumption that M is a homogeneous function of *arbitrary* non-zero degree and by modifying the mappings (2.39) accordingly. While this is indeed true, such a generalization of (2.40) is of no particular interest for our purposes.

3. Conservation Laws for Infinitesimal Deformations of Elastic Solids

With a view toward applying the results established in Section 2 to the equilibrium theory of elastic bodies in the presence of infinitesimal deformations, we summarize here the pertinent field equations and constitutive relations governing the theory to be considered. In this connection we shall confine our attention to mechanically homogeneous solids and shall assume that the body forces vanish identically.

Thus let $D\subset\mathscr{E}$ at present be the open region occupied by the interior of the body in its undeformed configuration and call $[u, \gamma, \sigma]$, in this order, the displacement vector-field, the infinitesimal strain tensor-field, and the stress tensor-field, all of which are taken to be defined on D. The strain-displacement relations and stress equations of equilibrium then become

$$\gamma_{ij}=u_{(i,j)}, \quad \sigma_{ij,j}=0, \quad \sigma_{ij}=\sigma_{ji} \quad \text{on } D. \qquad (3.1)^1$$

To these field equations we adjoin a constitutive law by postulating the existence of a scalar-valued elastic potential Γ, defined on the space $\mathscr{S}$ of all *symmetric* second-order tensors, such that $\Gamma\in\mathscr{C}^\infty(\mathscr{S})$ and

$$\sigma_{ij}=\Gamma_{,\gamma_{ij}}(\gamma) \quad \text{on } D, \quad \Gamma_{,\gamma_{ij}}(0)=0, \qquad (3.2)^2$$

with the understanding that the elements γ_{ij} and γ_{ji} of the symmetric matrix argument $\gamma=[\gamma_{ij}]$ are to be treated as mutually independent as far as (3.2) is con-

[1] If t is a second-order tensor, we write $t_{(ij)}$ and $t_{[ij]}$ for the components of the symmetric and of the skew-symmetric part of t, respectively.

[2] The second of (3.2) serves to insure that the stresses vanish in the undeformed state.

cerned. Further, we adopt the notation

$$\Gamma(\gamma(x)) = W(x) \quad \text{for all } x \in D, \tag{3.3}$$

so that W represents the *strain-energy density* (as a function of position) *associated with the elastic potential* Γ. It should be emphasized that while, in accordance with our initial agreement in Section 2, σ_{ij} and γ_{ij} in (3.2) refer to the components of stress and infinitesimal strain in the fixed underlying cartesian coordinate frame, the function Γ is frame-dependent unless the mechanical response of the material is isotropic. In the latter instance Γ is form-invariant under orthogonal transformations of the cartesian coordinates and this invariance is equivalent to the requirement that

$$\Gamma(q \gamma q^T) = \Gamma(\gamma) \quad \text{for every orthogonal } q \text{ and all } \gamma \in \mathscr{S}. \tag{3.4}$$

We say that $[u, \gamma, \sigma]$ is *an elastic state with infinitesimal deformations on* D, *corresponding to the elastic potential* Γ, if $\Gamma \in \mathscr{C}^\infty(\mathscr{S})$, $u \in \mathscr{C}^2(D)$ and (3.1), (3.2) hold[1]; we call such a state *isotropic* if, in addition, (3.4) is satisfied.

For future reference we cite next the restrictions upon the structure of the stress-strain law (3.2) arising from the isotropy requirement (3.4). When (3.4) is imposed, $\Gamma(\gamma)$ can involve the strain tensor only via its principal scalar invariants, so that in this case

$$\Gamma(\gamma) = \Psi(I_1(\gamma), I_2(\gamma), I_3(\gamma)) \quad \text{for all } \gamma \in \mathscr{S}, \tag{3.5}$$

where

$$\begin{gathered} I_1(\gamma) = \gamma_{kk}, \quad I_2(\gamma) = \tfrac{1}{2}(\gamma_{jj}\gamma_{kk} - \gamma_{jk}\gamma_{jk}), \\ I_3(\gamma) = \det \gamma = \tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{prs}\gamma_{ip}\gamma_{jr}\gamma_{ks}. \end{gathered} \tag{3.6}$$

On the other hand, from (3.5), (3.6), and (3.2) one finds readily that

$$\sigma_{ij} = \Lambda_1(I(\gamma))\delta_{ij} + \Lambda_2(I(\gamma))(\delta_{ij}\gamma_{kk} - \gamma_{ij}) + \tfrac{1}{2}\Lambda_3(I(\gamma))\varepsilon_{ikl}\varepsilon_{jpr}\gamma_{kp}\gamma_{lr} \quad \text{on } D, \tag{3.7}$$

with

$$\Lambda_k = \partial\Psi/\partial I_k, \quad I = (I_1, I_2, I_3), \quad \Lambda_1(0) = 0, \tag{3.8}$$

the last of (3.8) being a consequence of the second of (3.2).

Although the constitutive law (3.2) is not necessarily linear, it includes as a special case the stress-strain relations of the classical linear theory of elasticity, where

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} \quad \text{on } D, \qquad c_{ijkl} = c_{jikl} = c_{klij} \tag{3.9}$$

and $\Gamma(\gamma)$ is the quadratic form

$$\Gamma(\gamma) = \tfrac{1}{2} c_{ijkl}\gamma_{ij}\gamma_{kl} \quad \text{for all } \gamma \in \mathscr{S}. \tag{3.10}$$

Here c is the elasticity tensor, which is constant by virtue of our assumption of homogeneity and is required to be invertible[2]. If, further, the response is isotropic, c is an isotropic fourth-order tensor and admits the representation

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{3.11}$$

[1] The state so defined might more appropriately be referred to as *hyperelastic* (in the terminology of TRUESDELL & NOLL [13]) since we presuppose the existence of an elastic potential.

[2] Thus, the determinant of the 6×6 coefficient matrix appropriate to the system of linear algebraic equations appearing in (3.9) is assumed to be non-zero.

in which λ and μ denote Lamé's modulus and the shear modulus, respectively. In this particular instance (3.9) reduces to

$$\sigma_{ij}=\lambda\delta_{ij}\gamma_{kk}+2\mu\gamma_{ij} \quad \text{on } D, \tag{3.12}$$

whereas (3.10) becomes

$$\Gamma(\gamma)=\tfrac{1}{2}\lambda\gamma_{jj}\gamma_{kk}+\mu\gamma_{ij}\gamma_{ij} \qquad \text{for all } \gamma\in\mathscr{S}. \tag{3.13}$$

Observe on the basis of (3.6) that (3.13) may be written as

$$\Gamma(\gamma)=\tfrac{1}{2}(\lambda+2\mu)I_1^2(\gamma)-2\mu I_2(\gamma) \qquad \text{for all } \gamma\in\mathscr{S}, \tag{3.14}$$

which is the form assumed by (3.5) in the present circumstances. Consequently (3.8) here yields

$$\Lambda_1(I(\gamma))=(\lambda+2\mu)\gamma_{kk}, \qquad \Lambda_2(I(\gamma))=-2\mu, \qquad \Lambda_3(I(\gamma))=0, \tag{3.15}$$

and hence (3.12) is consistent with (3.7).

We agree to call $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ a *linear elastic state on D corresponding to the elasticity tensor* $\boldsymbol{c}$, provided $\boldsymbol{u}\in\mathscr{C}^2(D)$ and (3.1), (3.9) hold; we shall say that such a state is *isotropic* if, in addition, (3.11) is in force.

At this stage we turn to the following theorem on conservation laws pertaining to elastostatic fields with infinitesimal deformations.

Theorem 3.1. *Let D be a domain in $\mathscr{E}$ and let $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ be an elastic state with infinitesimal deformations on D, corresponding to the elastic potential Γ. Let W be the strain-energy density associated with Γ. Then, for every surface S, with the outward unit normal vector $\boldsymbol{n}$, that is the boundary of a regular subregion of D,*

$$\int_S (W n_i - s_j u_{j,i})\, da = 0, \tag{3.16}$$

where $\boldsymbol{s}$ is the traction vector on S, i.e.,

$$s_i=\sigma_{ij}n_j \quad \text{on } S. \tag{3.17}$$

If, moreover, $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is isotropic, *then also*

$$\int_S \varepsilon_{ijk}(W x_k n_j + s_j u_k - s_p u_{p,j} x_k)\, da = 0. \tag{3.18}$$

Finally, if $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is a linear *(not necessarily isotropic) elastic state on D, one has*

$$\int_S (W x_i n_i - s_j u_{j,i} x_i - \tfrac{1}{2} s_i u_i)\, da = 0. \tag{3.19}$$

Proof of Theorem 3.1. The foregoing theorem is an almost immediate consequence of Theorem 2.2. To confirm this claim, choose M and $\boldsymbol{w}$ in Theorem 2.2 in accordance with

$$M(\boldsymbol{t})=\Gamma(\operatorname{sym}\boldsymbol{t}) \quad \text{for all } \boldsymbol{t}\in\mathscr{T}, \qquad \boldsymbol{w}=\boldsymbol{u} \quad \text{on } D, \tag{3.20}$$ [1]

as is evidently legitimate. Then, from (3.2) together with the first and the last of (3.1) follows

$$M_{,t_{kj}}(\nabla\boldsymbol{w})=\Gamma_{,\gamma_{kj}}(\gamma)=\sigma_{kj} \quad \text{on } D. \tag{3.21}$$

[1] Throughout the remainder of this paper sym $\boldsymbol{t}$ and skw $\boldsymbol{t}$ are used to denote the symmetric and the skew-symmetric parts of a second-order tensor $\boldsymbol{t}$.

Therefore the Euler equations (2.31) now reduce to the stress equations of equilibrium

$$\sigma_{kj,j}=0 \quad \text{on } D, \tag{3.22}$$

which hold true by virtue of the second of (3.1) since $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is an elastic state on D. Next, note on the basis of (3.4) that M defined in (3.20) satisfies condition (2.34) if $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is *isotropic*. Also, M meets the requirement (2.38) whenever $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is a *linear* elastic state on D, as is clear from the first of (3.20) and (3.10).

Finally, observe that (3.2), (3.3), (3.16), (3.20), (3.21) imply

$$M(\nabla w)=W \quad \text{on } D, \qquad M_{,t_{kj}}(\nabla w)\,n_j=s_k \quad \text{on } S. \tag{3.23}$$

These equations, in turn, in conjunction with the second of (3.20), reduce the conservation laws (2.33), (2.37), and (2.40) to (3.16), (3.18), and (3.19), respectively. The proof is now complete. Note that *all* three laws are applicable to isotropic, linear elastic states.

It is apparent from (2.4), (2.30), (3.1), (3.3) and the first of (3.20) that the admissible functional for D generated by the particular choice of H underlying Theorem 3.1 is given by

$$\Phi\{w\}=\Phi\{u\}=\int_{\Omega}\Gamma(\gamma(x))\,dv=\int_{\Omega}W(x)\,dv \tag{3.24}$$

for every regular region $\Omega\subset D$. Consequently, $\Phi\{u\}$ is the total strain energy stored in Ω as a result of the deformations associated with the elastic state at hand. The variational principle (2.23) in the present instance is accordingly a restricted version of the principle of stationary potential energy.

For the purpose of arriving at a convenient coordinate-free form of the three conservation laws asserted in Theorem 3.1 we introduce the rotation vector field belonging to $\boldsymbol{u}$ by setting

$$\omega=\tfrac{1}{2}\,\mathrm{curl}\,\boldsymbol{u} \quad \text{on } D \tag{3.25}$$

and note the identity

$$s_j u_{j,i}=\gamma_{ij}s_j+\varepsilon_{ijk}s_j\omega_k \quad \text{on } S, \tag{3.26}$$

which follows easily from (3.25) and the first of (3.1). In view of (3.26), the vectorial conservation laws (3.16), (3.18) may now be written as

$$\int_S (W\boldsymbol{n}-\gamma\boldsymbol{s}-\boldsymbol{s}\wedge\omega)\,da=0, \tag{3.27}$$ [1]

$$\int_S [(W\boldsymbol{n}-\gamma\boldsymbol{s}-\boldsymbol{s}\wedge\omega)\wedge\boldsymbol{x}+\boldsymbol{s}\wedge\boldsymbol{u}]\,da=0, \tag{3.28}$$

while the scalar law (3.19) is equivalent to

$$\int_S [(W\boldsymbol{n}-\gamma\boldsymbol{s}-\boldsymbol{s}\wedge\omega)\cdot\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{s}\cdot\boldsymbol{u}]\,da=0. \tag{3.29}$$

Clearly, (3.16), (3.27) are the three-dimensional counterpart of the two-dimensional conservation law (1.4); on the other hand, (3.18), (3.28) and (3.19), (3.29) are found to be reducible to (1.6) and (1.8), respectively, in the special case of plane strain.

[1] If α and β are vectors, $\alpha\wedge\beta$ and $\alpha\cdot\beta$ stand for their vector product and scalar product, respectively; if α is a second-order tensor and β a vector, $\alpha\beta$ denotes the vector with the components $\alpha_{ij}\beta_j$.

Although, as we have seen, the three conservation laws under consideration may be generated systematically with the aid of Theorem 2.1, it would appear useful to include here also a confirmation of these laws that is entirely independent of Noether's theorem on invariant variational principles. This leads us to the subsequent.

Direct Proof of Theorem 3.1. Let R denote the regular region in D of which S is the boundary and let Q_i first stand for the surface integral appearing in the left-hand member of (3.16). Then, from (3.17) and the divergence theorem,

$$Q_i = \int_R \{W_{,i} - [\sigma_{jk} u_{j,i}]_{,k}\} \, dv. \tag{3.30}$$

But, by hypothesis and (3.1), (3.2), (3.3),

$$\begin{aligned} W_{,i} - [\sigma_{jk} u_{j,i}]_{,k} &= \sigma_{jk} \gamma_{jk,i} - \sigma_{jk} u_{j,ik} = \sigma_{jk} [\gamma_{jk} - u_{j,k}]_{,i} \\ &= \sigma_{jk} [\gamma_{jk} - u_{(j,k)}]_{,i} = 0 \quad \text{on } D, \end{aligned} \tag{3.31}$$

whence $Q_i = 0$ and thus (3.16) holds[1].

Next assume $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is isotropic and let Q_i now designate the left-hand member in (3.18). Consequently, (3.17) and the divergence theorem furnish

$$Q_i = \int_R \varepsilon_{ijk} \{[W x_k]_{,j} + [\sigma_{jm} u_k - \sigma_{pm} u_{p,j} x_k]_{,m}\} \, dv. \tag{3.32}$$

Bearing in mind (3.1), (3.2), (3.3), as well as the skew-symmetry of the alternator, one finds that (3.32) reduces to

$$Q_i = \int_R \varepsilon_{ijk} (\sigma_{mj} u_{k,m} - \sigma_{mk} u_{m,j}) \, dv. \tag{3.33}$$

Since $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is at present isotropic, (3.7) must hold. Therefore, writing Λ_k in place of $\Lambda_k(I(\gamma))$, one has

$$\begin{aligned} \sigma_{mj} u_{k,m} - \sigma_{mk} u_{m,j} &= \Lambda_2 (\gamma_{mk} u_{m,j} - \gamma_{mj} u_{k,m}) \\ &\quad + \tfrac{1}{2} \Lambda_3 \varepsilon_{mpq} \gamma_{pr} \gamma_{qs} (\varepsilon_{jrs} u_{k,m} - \varepsilon_{krs} u_{m,j}) \quad \text{on } D. \end{aligned} \tag{3.34}$$

Further, using the first of (3.1) and the skew-symmetry of ε_{ijk} one concludes after elementary computations that

$$\begin{aligned} \varepsilon_{ijk} (\gamma_{mk} u_{m,j} - \gamma_{mj} u_{k,m}) &= 2 \varepsilon_{ijk} \gamma_{mk} \gamma_{mj} = 0, \\ \varepsilon_{ijk} \varepsilon_{mpq} \gamma_{pr} \gamma_{qs} (\varepsilon_{jrs} u_{k,m} - \varepsilon_{krs} u_{m,j}) &= 4 \varepsilon_{mpq} \gamma_{mr} \gamma_{pr} \gamma_{qs} = 0 \quad \text{on } D. \end{aligned} \tag{3.35}$$

Equations (3.33), (3.34), (3.35), in turn, yield $Q_i = 0$ and this confirms (3.18).

Finally, suppose the elastic state $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is linear and let Q stand for the left member of (3.19). Then, in view of (3.17) and the divergence theorem,

$$Q = \int_R \{[W x_i]_{,i} - [\sigma_{jk} u_{j,i} x_i + \tfrac{1}{2} \sigma_{ik} u_i]_{,k}\} \, dv. \tag{3.36}$$

Under the present hypothesis,

$$W = \tfrac{1}{2} \sigma_{jk} \gamma_{jk} \quad \text{on } D \tag{3.37}$$

[1] This is in essence the same argument as that employed by RICE [2] to verify (1.4).

because of (3.3), (3.9), (3.10). Moreover, (3.37) and (3.1) enable one to replace (3.36) with

$$Q=\tfrac{1}{2}\int_R (\sigma_{jk,i}\gamma_{jk}-\sigma_{jk}\gamma_{jk,i})\,x_i\,dv. \tag{3.38}$$

But the integrand in (3.38) vanishes identically on D as a consequence of the linear stress-strain law (3.9). Hence $Q=0$ and (3.19) is valid. This completes the independent direct proof of Theorem 3.1.

It is clear from the *first* proof of Theorem 3.1 that the three conservation laws for an elastic state $[\boldsymbol{u}, \gamma, \sigma]$ with infinitesimal deformations on D, asserted in Theorem 3.1, may be generated systematically with the aid of Noether's theorem (Theorem 2.1) on invariant variational principles. This method of deducing (3.16), (3.18), (3.19) depends on choosing the functional Φ in Theorem 2.1 as the total strain energy stored in any regular subregion of D and upon the fact that Φ is infinitesimally invariant at $\boldsymbol{u}$ with respect to the three special pairs of mapping families introduced in Theorem 2.2 under the assumptions concerning $[\boldsymbol{u}, \gamma, \sigma]$ made in Theorem 3.1. The question then arises whether there exist mappings other than those listed in Theorem 2.2 with respect to which Φ is infinitesimally invariant at $\boldsymbol{u}$. If so, there would exist additional conservation laws[1], beyond those claimed in Theorem 3.1. The completeness theorem to which we proceed presently supplies a partial answer to the question just raised: it says, roughly speaking, that for *linear* elastic states the answer is negative but for a trivial exception.

Theorem 3.2. *Let $\boldsymbol{f}$ be a regular family of coordinate mappings on $\mathscr{E}$ and let $\boldsymbol{h}$ be a regular family of vector transformations. Let D be a domain in $\mathscr{E}$ and assume $\boldsymbol{c}$ is an invertible fourth-order tensor satisfying the symmetry relations*

$$c_{ijkl}=c_{jikl}=c_{klij}. \tag{3.39}$$

Suppose H is the function defined by

$$H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{t})=\Gamma(\operatorname{sym}\boldsymbol{t}) \qquad \text{for all } (\boldsymbol{x},\boldsymbol{v},\boldsymbol{t})\in\mathscr{E}\times\mathscr{V}\times\mathscr{T}, \tag{3.40}$$

where

$$\Gamma(\operatorname{sym}\boldsymbol{t})=\tfrac{1}{2}c_{ijkl}t_{ij}t_{kl} \qquad \text{for all } \boldsymbol{t}\in\mathscr{T}, \tag{3.41}$$

and let Φ be the admissible functional for D generated by H. Suppose, further, Φ is infinitesimally invariant at $\boldsymbol{u}$ with respect to $\boldsymbol{f}$ and $\boldsymbol{h}$ for every linear *elastic state on D corresponding to the elasticity tensor $\boldsymbol{c}$, and that such is the case for* every *$D\subset\mathscr{E}$ and for* every *isotropic invertible $\boldsymbol{c}$ consistent with* (3.39).

Then $\boldsymbol{f}$ and $\boldsymbol{h}$ must obey

$$\begin{aligned} \boldsymbol{f}'(\boldsymbol{x};0)&\equiv\boldsymbol{a}(\boldsymbol{x})=\boldsymbol{\alpha}+\boldsymbol{\phi}\boldsymbol{x}+\kappa\boldsymbol{x} && \text{for all } \boldsymbol{x}\in\mathscr{E},\\ \boldsymbol{h}'(\boldsymbol{v};0)&\equiv\boldsymbol{b}(\boldsymbol{v})=\boldsymbol{\beta}+\boldsymbol{\phi}\boldsymbol{v}-\frac{\kappa}{2}\boldsymbol{v} && \text{for all } \boldsymbol{v}\in\mathscr{V}, \end{aligned} \tag{3.42}$$

where κ is a scalar constant, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectorial constants, while $\boldsymbol{\phi}$ is a skew-symmetric second-order tensor. Moreover, $\boldsymbol{\phi}=\boldsymbol{0}$ if $\boldsymbol{c}$ is not necessarily isotropic.

[1] We leave aside the broader question as to the existence of conservation laws that follow from generalizations of Theorem 2.1 and of such laws that are not encompassed by NOETHER's [7] original theorem.

If $\boldsymbol{\beta}=\mathbf{0}$, $\boldsymbol{\phi}=\mathbf{0}$, $\kappa=0$, the functions $\boldsymbol{a}$ and $\boldsymbol{b}$ appearing in (3.42) reduce to the "infinitesimal ingredients" $\boldsymbol{f}'(\cdot\,;0)$ and $\boldsymbol{h}'(\cdot\,;0)$ of the two mappings $\boldsymbol{f}$ and $\boldsymbol{h}$ defined in (2.32), which lead to the conservation law[1] (3.16). When $\boldsymbol{\alpha}=\boldsymbol{\beta}=\mathbf{0}$ and $\kappa=0$, one recovers from (3.42), on setting $\boldsymbol{\phi}=\boldsymbol{q}'(0)$, the infinitesimal ingredients of the mapping pair (2.35) that gives rise to the law (3.18). Further, for $\boldsymbol{\alpha}=\boldsymbol{\beta}=\mathbf{0}$, $\boldsymbol{\phi}=\mathbf{0}$, and $\kappa=1$, (3.42) yields the infinitesimal ingredients of the two mapping families (2.39) by means of which (3.19) is generated. This leaves only $\boldsymbol{\alpha}=\mathbf{0}$, $\boldsymbol{\phi}=\mathbf{0}$, $\kappa=0$ in (3.43) to be accounted for. Choosing $\boldsymbol{\beta}$ coincident with the unit base vector in the x_i-direction, and bearing in mind (3.2), (3.17), (3.40), (3.41), one finds that (2.22) at present furnish

$$\int_S \boldsymbol{s}\, da=\mathbf{0}. \tag{3.43}$$

Since the vanishing of the resultant force of the tractions acting on S is immediate from the hypothesis that $[\boldsymbol{u},\boldsymbol{\gamma},\boldsymbol{\sigma}]$ is an elastic state on D and hence conforms to the stress equations of equilibrium, the conservation law (3.43) is both trivial and redundant[2].

It is clear from the preceding observations that Theorem 3.2 indeed assures the essential completeness of the three pairs of mapping families introduced in Theorem 2.2 within the context of *linear* elastostatics. We turn now to a

Proof of Theorem 3.2. Take H in Theorem 2.1 to be the function characterized by (3.40), (3.41), (3.39) and take $\boldsymbol{w}=\boldsymbol{u}$ on D. Then, for every $D\subset\mathscr{E}$ and every linear elastic state $[\boldsymbol{u},\boldsymbol{\gamma},\boldsymbol{\sigma}]$ on D corresponding to the elasticity tensor $\boldsymbol{c}$, as well as for every invertible $\boldsymbol{c}$ consistent with (3.39), the vector field $\boldsymbol{u}$ satisfies the Euler equations (2.18) on D because of (2.19), (3.1), (3.2), and (3.10). Further, since, by hypothesis, the admissible functional for D generated by H is infinitesimally invariant at $\boldsymbol{u}$ with respect to $\boldsymbol{f}$ and $\boldsymbol{h}$, (2.20) must hold, provided $\boldsymbol{w}$ in (2.21) is replaced by $\boldsymbol{u}$.

Now choose, in particular, $D=\mathscr{E}$ and take $[\boldsymbol{u},\boldsymbol{\gamma},\boldsymbol{\sigma}]$ to be the linear elastic state on $\mathscr{E}$ corresponding to $\boldsymbol{c}$ given by

$$u_i(\boldsymbol{x})=\rho_{ik}x_k,\quad \gamma_{ij}(\boldsymbol{x})=\rho_{(ij)},\quad \sigma_{ij}(\boldsymbol{x})=c_{ijkl}\rho_{(kl)}\quad \text{for all } \boldsymbol{x}\in\mathscr{E}, \tag{3.44}$$ [3]

where $\boldsymbol{\rho}$ is an *arbitrary* constant 3×3 matrix, so that $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}$ are a homogeneous field of infinitesimal strain and stress, respectively. In this instance (2.20) are readily found to assume the form

$$c_{ijkl}\rho_{(kl)}\left[\tfrac{1}{2}\rho_{(ij)}a_{r,r}(\boldsymbol{x})+b_{i,v_r}(\boldsymbol{\rho x})\rho_{rj}-a_{r,j}(\boldsymbol{x})\rho_{ir}\right]=0. \tag{3.45}$$

Next, suppose (3.45) holds for every invertible *isotropic* $\boldsymbol{c}$ with the symmetries (3.39). Then, in particular, (3.45) must hold if $\boldsymbol{c}$ is given by (3.11) with $\lambda=0$ and

[1] Note from (2.21) that only the infinitesimal ingredients of $\boldsymbol{f}$ and $\boldsymbol{h}$ enter (2.22).

[2] In order to arrive at the equally redundant conclusion that

$$\int_S \boldsymbol{x}\wedge\boldsymbol{s}\, da=\mathbf{0}$$

on the basis of NOETHER's scheme, it is necessary to generalize Theorem 2.1 slightly by permitting $\boldsymbol{h}$ to be position-dependent.

[3] Throughout the remainder of this proof the position vector $\boldsymbol{x}$ is understood to range over the entire space $\mathscr{E}$.

$\mu>0$. This special choice of c, in turn, reduces (3.45) to

$$\rho_{(ij)}[\rho_{(ij)}a_{r,r}(x)-2\rho_{ir}a_{r,j}(x)+2\rho_{rj}b_{i,v_r}(\rho x)]=0. \tag{3.46}$$

Setting

$$\rho_{ij}=\varepsilon\delta_{ij} \quad (\varepsilon>0) \tag{3.47}$$

in (3.46) and subsequently dividing by ε^2, one arrives at

$$a_{r,r}(x)+2b_{r,v_r}(\varepsilon x)=0. \tag{3.48}$$

Also, in view of the continuous differentiability[1] of b on $\mathscr{E}$,

$$\lim_{\varepsilon\to 0} b_{p,v_q}(\varepsilon x)=b_{p,v_q}(\mathbf{0})\equiv \mathring{b}_{p,v_q}, \tag{3.49}$$

the second of which merely defines convenient auxiliary notation. Hence passing to the limit as $\varepsilon\to 0$ in (3.48) and alternatively taking $\varepsilon=1$, one is led to the relation

$$a_{r,r}(x)=-2b_{r,v_r}(x)=-2\mathring{b}_{r,v_r}, \tag{3.50}$$

between the traces of the matrices $[a_{i,j}]$ and $[b_{i,v_j}]$, both traces being now known to be constant on $\mathscr{E}$.

Equation (3.46) is an identity in ρ and may accordingly be differentiated with respect to ϱ_{pq}. Carrying out this differentiation and thereafter once again adopting the special choice (3.47) of ρ, one obtains upon division by ε,

$$2\delta_{pq}a_{r,r}-a_{p,q}(x)-3a_{q,p}(x)+b_{p,v_q}(\varepsilon x)+3b_{q,v_q}(\varepsilon x)=0. \tag{3.51}$$

From (3.51), (3.49), in turn, follows

$$2\delta_{pq}a_{r,r}-a_{p,q}(x)-3a_{q,p}(x)+\mathring{b}_{p,v_q}+3\mathring{b}_{q,v_p}=0, \tag{3.52}$$

whence

$$a_{[p,q]}(x)=\mathring{b}_{[p,v_q]}. \tag{3.53}$$

Combining (3.52) with (3.53) and (3.50), one draws

$$a_{p,q}(x)=\mathring{b}_{p,v_q}-\delta_{pq}\mathring{b}_{r,v_r}, \tag{3.54}$$

which implies the constancy of $[a_{i,j}]$. Further, (3.51) with $\varepsilon=1$, in conjunction with (3.54), furnishes

$$b_{p,v_q}(x)+3b_{q,v_p}(x)=\mathring{b}_{p,v_q}+3\mathring{b}_{q,v_p} \tag{3.55}$$

and thus

$$b_{p,v_q}(x)=\mathring{b}_{p,v_q}. \tag{3.56}$$

Consequently $[b_{p,v_q}]$ is also constant.

At this stage we substitute from (3.54) into (3.46) and, after trivial computations, are led to

$$(\rho_{ij}\rho_{rj}-\rho_{ji}\rho_{jr})b_{i,v_r}=0 \quad \text{for all constant } \rho. \tag{3.57}$$

[1] Recall from (2.21) that $b(x)=h'(x;0)$ and bear in mind the smoothness requirement imposed in Section 2 on "regular families of vector transformations".

Differentiation of this identity with respect to ρ_{pq} gives the matrix equation

$$\boldsymbol{d}\boldsymbol{\rho}-\boldsymbol{\rho}\boldsymbol{d}=\boldsymbol{0} \quad \text{for all } \boldsymbol{\rho}, \qquad \boldsymbol{\rho}=[\rho_{ij}], \qquad \boldsymbol{d}=[b_{(i,v_j)}]. \tag{3.58}$$

Since $\boldsymbol{d}$ is symmetric, there exists an *orthogonal* matrix $\boldsymbol{q}$ such that $\boldsymbol{q}\boldsymbol{d}\boldsymbol{q}^T$ is diagonal. Choosing $\boldsymbol{\rho}=\boldsymbol{q}^T$, we gather from (3.58) that

$$\boldsymbol{d}\boldsymbol{q}^T=\boldsymbol{q}^T\boldsymbol{d}, \qquad \boldsymbol{q}\boldsymbol{d}\boldsymbol{q}^T=\boldsymbol{d}, \tag{3.59}$$

whence $\boldsymbol{d}$ is diagonal. Thus $[b_{(i,v_j)}]$ is a diagonal matrix. This conclusion, together with (3.58), implies that

$$b_{(i,v_i)}=b_{(j,v_j)} \quad \text{(no sums)}, \qquad b_{(i,v_j)}=0 \quad (i\neq j), \tag{3.60}$$

and (3.60), (3.54) furnish

$$a_{(i,i)}=a_{(j,j)} \quad \text{(no sums)}, \qquad a_{(i,j)}=0 \quad (i\neq j). \tag{3.61}$$

In view of (3.61), we may set

$$a_{(i,j)}=\kappa\delta_{ij}, \qquad a_{[i,j]}=\phi_{ij}, \qquad \phi_{ji}=-\phi_{ij}, \tag{3.62}$$

where κ is a scalar constant and $\boldsymbol{\phi}$ a constant skew-symmetric matrix. Moreover, combining (3.62) with (3.54), (3.56), one has

$$b_{(i,v_j)}=-\frac{\kappa}{2}\delta_{ij}, \qquad b_{[i,v_j]}=\phi_{ij}. \tag{3.63}$$

Therefore,

$$\begin{aligned} a_{i,j}(\boldsymbol{x})&=\kappa\delta_{ij}+\phi_{ij} && \text{for all } \boldsymbol{x}\in\mathscr{E}, \\ b_{i,v_j}(\boldsymbol{v})&=-\frac{\kappa}{2}\delta_{ij}+\phi_{ij} && \text{for all } \boldsymbol{v}\in\mathscr{V}. \end{aligned} \tag{3.64}$$

Finally, upon integrating (3.64), one arrives at

$$\begin{aligned} a_i(\boldsymbol{x})&=\alpha_i+\phi_{ij}x_j+\kappa x_i && \text{for all } \boldsymbol{x}\in\mathscr{E}, \\ b_i(\boldsymbol{v})&=\beta_i+\phi_{ij}v_j-\frac{\kappa}{2}v_i && \text{for all } \boldsymbol{v}\in\mathscr{V}, \end{aligned} \tag{3.65}$$

in which α_i and β_i are real constants. This confirms the conclusions (3.42).

It remains to be shown merely that $\phi_{ij}=0$ if (3.45) must hold for every (invertible) *not necessarily isotropic* elasticity tensor $\boldsymbol{c}$. To this end suppose, in particular, $\boldsymbol{c}$ is the anisotropic fourth-order tensor with the components

$$c_{ijkl}=\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\varepsilon\delta_{im}\delta_{jm}\delta_{km}\delta_{lm} \quad \text{(no sum)}, \tag{3.66}$$

where $\mu>0$, $\varepsilon>0$. This special choice of $\boldsymbol{c}$ evidently conforms to the symmetry requirements (3.39); moreover, for sufficiently small values of ε, the tensor $\boldsymbol{c}$ so chosen is invertible[1]. Substitution from (3.64) and (3.66) into (3.45) reduces the latter equation to

$$\rho_{mm}\rho_{(mr)}\phi_{rm}=0 \text{ for } \quad \text{every constant } \boldsymbol{\rho} \text{ (no sum on } m). \tag{3.67}$$

[1] Observe that (3.66) with $\mu>0$ defines a perturbation of an isotropic elasticity tensor that is positive definite and hence invertible.

Now take $m=1$ and choose ρ in accordance with

$$[\rho_{ij}]=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{3.68}$$

From (3.67), (3.68) and the skew-symmetry of $\boldsymbol{\phi}$ follows $\phi_{12}=0$. One finds by strictly analogous means that ϕ_{23} and ϕ_{31} must also vanish. Hence $\boldsymbol{\phi}=\mathbf{0}$ in the present circumstances and this completes the proof.

The statement of Theorem 3.2 involves the implicit assumption that the two mapping families $\boldsymbol{f}$ and $\boldsymbol{h}$ are independent of the elastic constants. It is not difficult to show that the conclusions (3.42) appropriate to *isotropic* linear elastic states remain valid even if $\boldsymbol{f}$ and $\boldsymbol{h}$ are permitted to depend on the shear modulus μ and on Lamé's modulus λ, provided λ and μ satisfy certain inequalities that are automatically fulfilled if the associated elastic potential is positive definite.

4. Conservation Laws for Finite Deformations of Elastic Solids

In preparation for a derivation of conservation laws applicable to elastostatic fields involving *finite* deformations we cite here certain results from the nonlinearized equilibrium theory of homogeneous hyperelastic solids. As far as the requisite kinematics[1] is conerned we again denote by D the open region occupied by the interior of the body in its undeformed configuration and call $\boldsymbol{x}$ the position vector of a generic point in D. A deformation of the body is taken to be given by the transformation

$$\boldsymbol{y}=\hat{\boldsymbol{y}}(\boldsymbol{x})=\boldsymbol{x}+\boldsymbol{u}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x}\in D, \tag{4.1}$$

in which $\boldsymbol{u}$ is the displacement field and the mapping $\hat{\boldsymbol{y}}\in\mathscr{C}^2(D)$. Further, we assume that this mapping is one-to-one and that its inverse $\hat{\boldsymbol{x}}\in\mathscr{C}^2(\overset{*}{D})$, where $\overset{*}{D}$ is the image of D under $\hat{\boldsymbol{y}}$. Accordingly,

$$\boldsymbol{x}=\hat{\boldsymbol{x}}(\boldsymbol{y})=\boldsymbol{y}-\boldsymbol{u}(\hat{\boldsymbol{x}}(\boldsymbol{y})) \quad \text{for all } \boldsymbol{y}\in\overset{*}{D}, \tag{4.2}$$

with the understanding that $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{x}}$ are referred to a common rectangular cartesian coordinate frame. Let $\boldsymbol{F}$ and $\overset{*}{\boldsymbol{F}}$ designate the deformation-gradient tensor fields associated with $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{x}}$, respectively, so that

$$\boldsymbol{F}=\underset{y}{\nabla}\hat{\boldsymbol{y}} \quad \text{on } D, \qquad \overset{*}{\boldsymbol{F}}=\underset{y}{\nabla}\hat{\boldsymbol{x}} \quad \text{on } \overset{*}{D}. \tag{4.3}$$

Further, put

$$J=\det\boldsymbol{F} \quad \text{on } D, \qquad \overset{*}{J}=\det\overset{*}{\boldsymbol{F}} \quad \text{on } \overset{*}{D}, \tag{4.4}$$

whence J and $\overset{*}{J}$ stand for the Jacobian determinants of the mapping $\hat{\boldsymbol{y}}$ and of its inverse $\hat{\boldsymbol{x}}$. Then,

$$\boldsymbol{F}(\boldsymbol{x})\overset{*}{\boldsymbol{F}}(\hat{\boldsymbol{y}}(\boldsymbol{x}))=\mathbf{1}, \qquad J(\boldsymbol{x})=1/\overset{*}{J}(\hat{\boldsymbol{y}}(\boldsymbol{x}))>0 \quad \text{for all } \boldsymbol{x}\in D. \tag{4.5}$$

[1] See, for example, TRUESDELL & TOUPIN [14], Chapter B.

From (4.4), (4.5) follows

$$\overset{*}{F}_{ij}(\hat{y}(x))=\frac{1}{2J(x)}\varepsilon_{ipq}\varepsilon_{jrs}F_{rp}(x)F_{sq}(x) \quad \text{for all } x\in D, \tag{4.6}$$

which implies the familiar identity

$$\frac{\partial}{\partial x_j}[J(x)\overset{*}{F}_{jk}(\hat{y}(x))]=0 \quad \text{for all } x\in D. \tag{4.7}$$

Next, we define the symmetric deformation tensor fields $\boldsymbol{C}$ and $\boldsymbol{G}$ by setting

$$\begin{aligned} \boldsymbol{C}(x)&=\boldsymbol{F}^T(x)\boldsymbol{F}(x) && \text{for all } x\in D,\\ \boldsymbol{G}(y)&=\boldsymbol{F}(\hat{x}(y))\boldsymbol{F}^T(\hat{x}(y)) && \text{for all } y\in \overset{*}{D}, \end{aligned} \tag{4.8}$$ [1]

and introduce the Green-St. Venant strain-tensor field γ through

$$\gamma=\tfrac{1}{2}(\boldsymbol{C}-\boldsymbol{1}) \quad \text{on } D. \tag{4.9}$$ [2]

In view of (4.1), (4.3), (4.8), and (4.9), one has the usual *displacement-strain relations*

$$\gamma_{ij}=u_{(i,j)}+\tfrac{1}{2}u_{k,i}u_{k,j} \quad \text{on } D. \tag{4.10}$$

Next, we recall that the *stress equations of equilibrium*, which express the balance of linear and angular momentum, in the absence of body forces take the form

$$\tau_{ij,j}=0, \quad \tau_{ji}=\tau_{ij} \quad \text{on } \overset{*}{D}, \tag{4.11}$$

provided τ is the conventional Cauchy stress tensor, regarded as a function of position on $\overset{*}{D}$. The preceding field equations are accompanied by the *constitutive law* appropriate to a hyperelastic solid, which may be written as[3]

$$\tau_{ij}(y)=\frac{1}{J(\hat{x}(y))}\Gamma_{,\gamma_{pq}}(\gamma(\hat{x}(y)))F_{ip}(\hat{x}(y))F_{jq}(\hat{x}(y)) \quad \text{for all } y\in\overset{*}{D}. \tag{4.12}$$

Here Γ is the stored strain energy (elastic potential) per unit undeformed volume as a function of the strain tensor defined in (4.9). We assume $\Gamma\in\mathscr{C}^\infty(\mathscr{S})$, where $\mathscr{S}$, as before, stands for the space of all symmetric second-order tensors, and emphasize that *all* elements of the matrix $\gamma=[\gamma_{ij}]$ in (4.12) are to be treated as mutually independent in performing the required partial differentiations. Moreover, because of (4.12), (4.3), and the second of (4.11), one has the symmetry relations

$$\Gamma_{,\gamma_{pq}}(\gamma)=\Gamma_{,\gamma_{qp}}(\gamma) \quad \text{for all } \gamma\in\mathscr{S}. \tag{4.13}$$

In the special case of an isotropic medium Γ is form-invariant with respect to orthogonal transformations of its matrix argument, i.e.,

$$\Gamma(\boldsymbol{q}\gamma\boldsymbol{q}^T)=\Gamma(\gamma) \quad \text{for every orthogonal } \boldsymbol{q} \text{ and all } \gamma\in\mathscr{S}. \tag{4.14}$$

[1] Note that $\boldsymbol{C}$ here stands for Green's deformation tensor.

[2] Observe that γ was the infinitesimal strain field in Section 3. The fact that some of the symbols introduced there are now given a different meaning ought not to cause confusion since the current section is entirely independent of Section 3.

[3] See, for example, RIVLIN [15], p. 174.

In these circumstances (4.9) furnish

$$\Gamma(\gamma)=\Psi\big(I_1(C), I_2(C), I_3(C)\big) \quad \text{for all } \gamma\in\mathscr{S}, \qquad C=2\gamma+1, \tag{4.15}$$

where I_1, I_2, I_3 are the principal scalar invariants of a symmetric second-order tensor defined in (3.6). Also, (4.8), (4.15), (4.12) lead to the well-known representation of the stress-deformation law for isotropic hyperelastic solids[1]:

$$\tau_{ij}=\Sigma_1(I)\,G_{ij}+\Sigma_2(I)\,G_{ik}G_{jk}+\Sigma_3(I)\,\delta_{ij} \quad \text{on } \overset{*}{D}, \tag{4.16}$$

where

$$I\equiv\big(I_1(C), I_2(C), I_3(C)\big), \qquad C\equiv C\big(\hat{x}(y)\big) \quad \text{for all } y\in\overset{*}{D}, \tag{4.17}$$

while the scalar coefficients Σ_i are given by

$$\Sigma_1=\frac{2}{\sqrt{I_3}}\left[\frac{\partial\Psi}{\partial I_1}+I_1\frac{\partial\Psi}{\partial I_2}\right], \quad \Sigma_2=-\frac{2}{\sqrt{I_3}}\frac{\partial\Psi}{\partial I_2}, \quad \Sigma_3=2\sqrt{I_3}\frac{\partial\Psi}{\partial I_3}. \tag{4.18}$$

For our purposes it is essential to introduce, in addition to the Cauchy stress field τ on $\overset{*}{D}$, the Piola stress field[2] σ on D, which is related to τ by means of

$$\sigma_{ij}(x)=J(x)\overset{*}{F}_{jk}\big(\hat{y}(x)\big)\tau_{ik}\big(\hat{y}(x)\big) \quad \text{for all } x\in D. \tag{4.19}$$

On account of (4.19), (4.5), and (4.13), the constitutive law (4.12) implies

$$\sigma_{ij}=\Gamma_{,\gamma_{jk}}(\gamma)\,F_{ik} \quad \text{on } D. \tag{4.20}$$

From (4.19), (4.3), (4.5), and (4.7) one draws the known identity

$$\sigma_{ij,j}(x)=J(x)\,\tau_{ij,j}\big(\hat{y}(x)\big) \quad \text{for all } x\in D. \tag{4.21}$$

According to (4.21) and (4.5), the first of the equilibrium equations (4.11) gives

$$\sigma_{ij,j}=0 \quad \text{on } D. \tag{4.22}$$

In contrast to τ, however, the stress field σ is in general not symmetric.

For future convenience we now adopt the following definition. We say that $[u, \gamma, \sigma]$ is a *finite elastic state on D, corresponding to the elastic potential Γ*, if $\Gamma\in\mathscr{C}^\infty(\mathscr{S})$, $u\in\mathscr{C}^2(D)$ and (4.10), (4.13), (4.20), (4.22) hold with F given by (4.1), (4.3); we call such a state *isotropic* if, in addition, (4.14) is satisfied. Further, we refer to W as the *strain-energy density associated with the elastic potential Γ*, provided

$$W(x)=\Gamma\big(\gamma(x)\big) \quad \text{for all } x\in D. \tag{4.23}$$

The foregoing definition of a finite elastic state derives its motivation from the fact that the Cauchy stress field τ belonging to σ in the sense of (4.19) necessarily conforms to the equilibrium equations (4.11), as well as to the constitutive law (4.12). To see this, note from (4.19), (4.5) that

$$\tau_{ij}(y)=\overset{*}{J}(y)\,F_{jk}\big(\hat{x}(y)\big)\sigma_{ik}\big(\hat{x}(y)\big) \quad \text{for all } y\in\overset{*}{D}. \tag{4.24}$$

[1] See, for instance, Rivlin [15], p. 180.

[2] See Truesdell & Toupin [14], p. 553.

Substituting from (4.20) into (4.24) and thereafter invoking (4.13) and the last of (4.5), one recovers (4.12). The second of (4.11) is immediate from (4.12) and the postulated symmetry relations (4.13). Finally, the first of (4.11) is a consequence of (4.22) and of the identity (4.21).

Having disposed of these expository preliminaries, we may proceed to

Theorem 4.1. *Let D be a domain in $\mathscr{E}$ and let $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ be a finite elastic state on D, corresponding to the elastic potential Γ. Let W be the strain-energy density associated with Γ. Then, for every surface S, with the outward unit normal vector $\boldsymbol{n}$, that is the boundary of a regular subregion on D,*

$$\int_S (W n_i - s_j u_{j,i})\, da = 0, \tag{4.25}$$

where $\boldsymbol{s}$ is the Piola traction vector on S defined by

$$s_i = \sigma_{ij} n_j \quad \text{on } S. \tag{4.26}$$

If, moreover, $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ *is* isotropic, *then also*

$$\int_S \varepsilon_{ijk}(W x_k n_j + s_j u_k - s_p u_{p,j} x_k)\, da = 0. \tag{4.27}$$

Proof of Theorem 4.1. This theorem, which is evidently an analogue for *finite* deformations of Theorem 3.1, like the latter, is readily established through a suitable specialization of Theorem 2.2. Thus choose M and $\boldsymbol{w}$ in Theorem 2.2 by setting

$$M(\boldsymbol{t}) = \Gamma(\tfrac{1}{2}[\boldsymbol{t}^T \boldsymbol{t} - \boldsymbol{1}]) \quad \text{for all } \boldsymbol{t} \in \mathscr{T}, \qquad \boldsymbol{w} = \hat{\boldsymbol{y}} \quad \text{on } D, \tag{4.28}$$

where $\hat{\boldsymbol{y}}$ is the deformation belonging to the displacement field $\boldsymbol{u}$ in the sense of (4.1). Then, in view of (4.3), (4.8), (4.9), (4.13), and (4.20),

$$M_{,t_{kj}}(\nabla \boldsymbol{w}) = \Gamma_{,\gamma_{jm}}(\gamma)\, F_{km} = \sigma_{kj} \quad \text{on } D. \tag{4.29}$$

Since the Piola stress field $\boldsymbol{\sigma}$ satisfies the equilibrium equations (4.22), equations (4.29) guarantee that the Euler equations (2.31) hold true for the choice of M and $\boldsymbol{w}$ specified in (4.28). Further, by virtue of (4.14), (4.28), M conforms to (2.34) if $[\boldsymbol{u}, \gamma, \boldsymbol{\sigma}]$ is *isotropic*.

Next, from (4.28), (4.3), (4.8), (4.9), (4.23), and (4.26), (4.29) one infers that

$$M(\nabla \boldsymbol{w}) = W \quad \text{on } D, \qquad M_{,t_{kj}}(\nabla \boldsymbol{w})\, n_j = s_i \quad \text{on } S. \tag{4.30}$$

Now, substitution from (4.30) and the second of (4.28) into (2.33), because of (4.3), leads to the conservation law

$$\int_S (W n_i - F_{ji} s_j)\, da = 0. \tag{4.31}$$

Similarly, the law (2.37) yields for the isotropic solid

$$\int_S \varepsilon_{ijk}(W x_k n_j + s_j \hat{y}_k - F_{pj} s_p x_k)\, da = 0. \tag{4.32}$$

Finally, invoke (4.1), (4.3), as well as (4.22), (4.26), and the divergence theorem to verify

$$F_{ji} = \delta_{ji} + u_{j,i} \quad \text{on } D, \qquad \int_S s_i\, da = 0. \tag{4.33}$$

Equations (4.33) reduce the conservation law (4.31) to the desired form (4.25). At the same time, the law (4.32), which is valid only for the isotropic elastic solid, may be written in the form (4.27), because of (4.1) and (4.33). The proof is now complete.

We note that (4.25) and (4.27) are identical with (3.16) and (3.18), respectively, but for the change in the meaning of the symbols W and $\boldsymbol{s}$. Indeed, as has become apparent, the two conservation laws (3.16), (3.18) in Theorem 3.1, which pertain to elastic states with *infinitesimal* deformations, remain strictly applicable to *finite* deformations of hyperelastic bodies provided W is interpreted as the appropriate strain-energy density per unit undeformed volume, whereas $\boldsymbol{s}$ is taken to stand for the Piola traction vector. On the other hand, (3.19) in Theorem 3.1 has no immediate counterpart in finite elastostatics. The two vectorial conservation laws (4.25) and (4.27) clearly admit the coordinate-free representation:

$$\int_S [W\boldsymbol{n}-(\nabla \boldsymbol{u})^T \boldsymbol{s}]\, da=0, \tag{4.34}$$

$$\int_S \{[W\boldsymbol{n}-(\nabla \boldsymbol{u})^T \boldsymbol{s}]\wedge \boldsymbol{x}+\boldsymbol{s}\wedge \boldsymbol{u}\}\, da=\boldsymbol{0}. \tag{4.35}$$

It would seem useful to establish the foregoing results independently, without recourse to Noether's theorem. For this purpose we include the subsequent

Direct Proof of Theorem 4.1. Denote by R the regular subregion of D that has S as its boundary and let Q_i temporarily designate the left-hand member in (4.25). Then, from (4.26) and the divergence theorem,

$$Q_i=\int_R \{W_{,i}-[\sigma_{jk} u_{j,i}]_{,k}\}\, dv. \tag{4.36}$$

By virtue of (4.23), (4.20), (4.22), together with (4.1), (4.3) and (4.8), (4.9), one has the identity

$$\begin{aligned} W_{,i}-[\sigma_{jk} u_{j,i}]_{,k} &= \Gamma_{,\gamma_{jk}}(\gamma)\,\gamma_{jk,i}-\Gamma_{,\gamma_{km}}(\gamma)\, F_{jm} F_{ji,k} \\ &= \Gamma_{,\gamma_{jk}}(\gamma)\,[\gamma_{jk,i}-F_{mk} F_{mi,j}] \\ &= \tfrac{1}{2}\Gamma_{,\gamma_{jk}}(\gamma)\,[F_{mj} F_{mk,i}-F_{mk} F_{mj,i}] \quad \text{on } D. \end{aligned} \tag{4.37}$$

Since the function within brackets in the extreme right-hand member of (4.37) is skew-symmetric with respect to (j, k), while its coefficient – according to (4.13) – is symmetric in these two indices, one concludes that the integrand in (4.36) vanishes on D. Hence $Q_i=0$ and thus (4.25) follows.

Suppose next $[\boldsymbol{u}, \boldsymbol{\gamma}, \boldsymbol{\sigma}]$ is isotropic, so that (4.14), (4.16) hold, and let Q_i at present stand for the surface integral in (4.27). Appealing once again to the divergence theorem one draws, on account of (4.22) and the skew-symmetry of the alternator, that

$$Q_i=\int_R \varepsilon_{ijk}[x_k(W_{,j}-\sigma_{pm} u_{p,jm})+\sigma_{jm} u_{k,m}-\sigma_{pk} u_{p,j}]\, dv. \tag{4.38}$$

On the other hand, (4.23), (4.20), (4.10), together with (4.3), (4.1), justify the identity

$$\begin{aligned} W_{,j}-\sigma_{pm} u_{p,jm} &= \Gamma_{,\gamma_{pq}}(\gamma)\,[\gamma_{pq,j}-F_{rq} u_{r,jp}] \\ &= \tfrac{1}{2}\Gamma_{,\gamma_{pq}}(\gamma)\,[u_{p,qj}-u_{q,pj}+u_{k,p} u_{k,qj}-u_{k,q} u_{k,pj}]. \end{aligned} \tag{4.39}$$

The extreme right-hand member of (4.39) vanishes on D because of (4.13) and the skew-symmetry of the function inside the brackets. Consequently, (4.38) may be replaced by

$$Q_i = \int_R \varepsilon_{ijk}(\sigma_{jm} u_{k,m} - \sigma_{mk} u_{m,j})\, dv. \tag{4.40}$$

For convenience set

$$\mu_{jk} = \frac{1}{J}(\sigma_{jm} u_{k,m} - \sigma_{mk} u_{m,j}) \quad \text{on } D, \tag{4.41}$$

where J is the Jacobian determinant defined in (4.4). Then (4.19), (4.1), (4.3), and (4.5) furnish

$$\begin{aligned} \mu_{jk} &= \overset{*}{F}_{mp} \tau_{jp} u_{k,m} - \overset{*}{F}_{kp} \tau_{mp} u_{m,j} \\ &= F_{km} \overset{*}{F}_{mp} \tau_{jp} - F_{mj} \overset{*}{F}_{kp} \tau_{mp} = \tau_{jk} - F_{mj} \overset{*}{F}_{kp} \tau_{mp}. \end{aligned} \tag{4.42}$$

Further, (4.16), (4.8), (4.5) imply

$$F_{mj} \overset{*}{F}_{kp} \tau_{mp} = \Sigma_1 F_{mj} F_{mk} + \Sigma_2 F_{mj} F_{mr} F_{sr} F_{sk} + \Sigma_3 \delta_{jk}, \tag{4.43}$$

where $\Sigma_i \equiv \Sigma_i(I)$. Now, in view of (4.42), (4.43) and the symmetry of the stress field $\boldsymbol{\tau}$ asserted in (4.11), one has

$$\mu_{jk} = \mu_{kj} \quad \text{on } D, \tag{4.44}$$

whereas (4.40), (4.41), (4.44) give $Q_i = 0$. The confirms (4.27) and concludes the direct proof of the theorem under consideration.

The completeness issue associated with the two conservation laws supplied by Theorem 4.1 appears to be considerably more complicated than the analogous question in the linearized theory, which is answered by Theorem 3.2.

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