On the Solutions of a Class of Equations Occurring in Continuum Mechanics, with Application to the Stokes Paradox

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1. Introduction

In this paper we apply methods developed by FINN in [1] to study asymptotic properties of solutions of equations

$$(1.1\,a)\qquad \Delta\vec{w} - \vec{b}\cdot\nabla\vec{w} - \nabla p = 0, \qquad \vec{b} = \text{const.},$$

$$(1.1\,b)\qquad \nabla\cdot\vec{w} = -c\,p, \qquad c = \text{const.},$$

for a vector field $\vec{w}(x) = \{w_1, \ldots, w_n\}$ and scalar field $p(x)$ which are defined in a neighborhood $\mathscr{E}$ of infinity in n-dimensional Euclidian space. The notation in (1.1) is the usual one of vector analysis. By choosing particular values of the constant c and the constant vector $\vec{b}$, we obtain from (1.1) the following equations of fluid and solid mechanics (see, *e.g.*, [2, 3]):

i) $c = 0$, $\vec{b} = 0$ Stokes equations of hydrodynamics

ii) $c = 0$ Oseen equations of hydrodynamics

iii) $\vec{b} = 0$ Equations of linear elasticity.

We assume in most of this paper, as is the case in the problems of mechanics from which the equations are derived, that $c \geqq 0$. However, in some contexts, notably the basic Theorems 1 and 4, it is sufficient to assume $c > -1$.

The main result of this study consists in showing that under a surprisingly weak growth condition at infinity, the solution admits a representation in terms of a single surface integral over an inner bounding surface Σ. From this representation we conclude that the solution tends to a limit $(\vec{w}_0, p_0)$ at infinity, and the rate of decay of $(\vec{w}, p)$ to this limit is controlled by the asymptotic properties of the fundamental solution tensor associated with (1.1). We determine this tensor explicitly in §3. In §4 we apply our result to a discussion of solutions for which $\vec{w}(x)$ vanishes on Σ. These solutions are of importance physically as steady fluid flows or as deformations exterior to a rigid body. We obtain in particular a new clarification of the Stokes Paradox of Hydrodynamics, which states essentially that in two dimensions a solution of the (time-independent) Stokes equations in $\mathscr{E}$ which vanishes on Σ is necessarily unbounded. Our result is sharp in all dimensions and shows that the singular behavior of the solution is intimately connected with the forces exerted on Σ in the motion. We obtain as special cases all earlier results on this and on related problems which are known to us [*cf.* *3*, p. 361, *4*, *5*, *6*, *7*]; the demonstrations are in our opinion simpler and more natural than those previously employed.

The methods of this paper are not restricted to equations of the form (1.1), but are applicable to any elliptic system for which the fundamental solution tensor exhibits the qualitative properties needed in our demonstrations[1]. Equations with variable coefficients can also be treated, and it seems likely that a general description of asymptotic behavior can be constructed along lines suggested by N. G. MEYERS [*8*]. The particular choice of the equations (1.1) has been based on the consideration that they are general enough to indicate the scope of the methods, but not so general as to cause the ideas to be obscured by technical detail.

We remark that in the range of c considered the system (1.1) is elliptic $\left(\text{the characteristic determinant is } (1+c)\sum_{i=1}^{n}\xi_i^2>0 \text{ for } c>-1,\ \sum\xi_i^2>0\right)$ but not strongly elliptic, hence not of a type for which boundary value problems have been studied in recent literature. The properly set boundary value problem requires prescribing $\vec{w}(\boldsymbol{x})$ on the boundary, but not the pressure $p(\boldsymbol{x})$, this function being determined by the solution[2]. Correspondingly, in the material

[1] It would suffice, for example, for the fundamental solution tensor to have the property that for any prescribed positive integer k, all derivatives of sufficiently high order (depending on k) of each component of this tensor behave at infinity as $O(r^{-k})$. Professor L. HÖRMANDER has communicated to us a proof of the following equivalent condition, valid for any system with constant coefficients: Let $P(D)$, where $D=(-i\,D_1,\ldots,-i\,D_n)$, $i=$ imaginary unit, denote the determinant of the differential operators occurring in the system, and let, respectively, $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)$, $\boldsymbol{\zeta}=(\zeta_1,\ldots,\zeta_n)$ be vectors in real or complex Euclidean n-space. Then the fundamental solution tensor has the above stated property if and only if both the conditions

a) $P(\boldsymbol{\xi})\neq 0$ for $\boldsymbol{\xi}\neq 0$,

and

b) $\operatorname{Im}\boldsymbol{\zeta}\to\infty$ if $\boldsymbol{\zeta}\to\infty$ on the surface $P(\boldsymbol{\zeta})=0$,

are satisfied by the polynomial P. The representation theorems of this paper, and the corresponding asymptotic estimates of the solutions, can be extended without essential change to all systems which satisfy the conditions of HÖRMANDER.

Such systems include, for example, the linearized magnetohydrodynamic equations

$$\Delta\vec{w}-M^2\left[\vec{w}-\frac{(\vec{w}\cdot\vec{w}_0)}{|\vec{w}_0|}\,\frac{\vec{w}_0}{|\vec{w}_0|}\right]=0,$$
$$\nabla\cdot\vec{w}=0$$

for which the essential features of Theorem 1 are easily seen to apply. In this case, there is no Stokes Paradox, since the fundamental solution tensor vanishes at infinity for all $n\geqq 2$.

[2] The existence of a solution corresponding to such boundary data has to our knowledge been proved in the literature only in the special case $c=\vec{b}=0$ (Stokes equations). It seems likely that, starting with the solution of ODQVIST [*9*] for this case, the existence in the general case can be demonstrated by the continuity method, using estimates of the type studied in [*10*] and in [*11*]. A direct demonstration by the method of surface potentials in the case $c=0$, $\vec{b}$ arbitrary, has been given by J. SANDERS in a work to appear shortly.

of this paper no assumption is made on the behavior of the pressure, and the regularity of the pressure at infinity is proved as a consequence of growth assumptions on $\vec{w}(x)$ alone.

2. Notation and Definitions; the Integral Representation

The following notation will be used. We choose the origin O of our coordinate system to be a point interior to Σ. The set of all points exterior to Σ is denoted by $\mathscr{E}$. Let Σ_R denote a sphere of radius R centered at O and enclosing Σ. We denote by $\mathscr{I}_R$ the annular region bounded between Σ and Σ_R. The region exterior to Σ_R is denoted by $\mathscr{E}_R$.

Let $\vec{w}(x)=[w_i(x)]$ and $T(x)=[T_{ij}(x)]$ be, respectively, vector and tensor fields defined in $\mathscr{E}$. We define

$$|\vec{w}(x)| \equiv \left[\sum_{i=1}^{n} w_i^2(x)\right]^{\frac{1}{2}},$$

$$|T(x)| \equiv \left[\sum_{i,j=1}^{n} T_{ij}^2(x)\right]^{\frac{1}{2}}.$$

We write $T_{ij}=O(R^{\alpha})$ if there exists a constant A so that $|T_{ij}(x)|\leqq A|x|^{\alpha}$ as $x\to\infty$; we write $T_{ij}=o(R^{\alpha})$ if $|x|^{-\alpha}|T_{ij}(x)|\to 0$ as $x\to\infty$. Similar definitions apply to vector and higher-order tensor fields.

Corresponding to a given vector field $\vec{u}(x)$ and scalar $p(x)$, we define the second order tensor field

$$T\vec{u}=\{T\vec{u}\}_{ij}=-(1-c)\,p\,\delta_{ij}+\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{2.1}$$

where

$$\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

We also define the linear differential operators $L\vec{u}$ and $M\vec{v}$ by the expressions

$$L\vec{u}=\Delta\vec{u}-\nabla p-\vec{b}\cdot\nabla\vec{u},$$

$$M\vec{v}=\Delta\vec{v}-\nabla q+\vec{b}\cdot\nabla\vec{v}.$$

For any vector fields $\vec{u}$ and $\vec{v}$ and corresponding scalars p, q satisfying the conditions

$$\nabla\cdot\vec{u}=-c\,p,\qquad \nabla\cdot\vec{v}=-c\,q \tag{2.2}$$

in a region V with sufficiently smooth boundary S, we have always the identity

$$\int_V(\vec{v}\cdot L\vec{u}-\vec{u}\cdot M\vec{v})\,dV=\int_S[\vec{v}\cdot T\vec{u}-\vec{u}\cdot T\vec{v}-(\vec{u}\cdot\vec{v})\,(\vec{b}\cdot\vec{n})]\,dS \tag{2.3}$$

where by $\vec{v}\cdot T\vec{u}$ is understood the expression

$$\vec{v}\cdot T\vec{u}=v_i\{T\vec{u}\}_{ij}\,n_j,$$

$\vec{n}=\{n_j\}$ is the unit exterior directed normal on S, and summation is extended over repeated indices.

Let $\boldsymbol{\chi}(\boldsymbol{x}-\boldsymbol{y})=\{\chi_{ij}\}$, $\boldsymbol{\psi}(\boldsymbol{x}-\boldsymbol{y})=\{\psi_i\}$ denote an $n\times n$ matrix and an n-vector, respectively, which become singular at $\boldsymbol{x}=\boldsymbol{y}$ in such a way that

$$\delta_{ik}=\lim_{r\to 0}\int_{\Sigma_r}[T\boldsymbol{\chi}+\boldsymbol{\chi}(\vec{b}\cdot\vec{n})]\,dS$$

where Σ_r denotes the surface of a sphere of radius r about $\boldsymbol{x}$ as center and $T\boldsymbol{\chi}$ is formed by interpreting the components of $\boldsymbol{\psi}$ as pressures. We obtain from (2.3) the integral representation

$$\vec{u}(\boldsymbol{x})=\int_V(\boldsymbol{\chi}\cdot L\vec{u}-\vec{u}\cdot M\boldsymbol{\chi})\,dV-\int_S[\boldsymbol{\chi}\cdot T\vec{u}-\vec{u}\cdot T\boldsymbol{\chi}-(\vec{u}\cdot\boldsymbol{\chi})\,(\vec{b}\cdot\vec{n})]\,dS. \tag{2.4}$$

In (2.4), $\boldsymbol{x}$ appears as a parameter, the differentiation and integration being performed with respect to $\boldsymbol{y}$.

Let now $\vec{w}(\boldsymbol{x})$ and $\boldsymbol{\chi}(\boldsymbol{x}-\boldsymbol{y})$ be vector and tensor fields which satisfy (2.2) and also the equations

$$L(\vec{w})=0, \qquad M(\boldsymbol{\chi})=0$$

for each column vector in the matrix (χ_{ij}), with $\boldsymbol{y}$ as independent variable. We have then the integral representation for the solutions of (1.1),

$$\vec{w}(\boldsymbol{x})=-\int_S[\boldsymbol{\chi}\cdot T\vec{w}-\vec{w}\cdot T\boldsymbol{\chi}-(\vec{w}\cdot\boldsymbol{\chi})\,(\vec{b}\cdot\vec{n})]\,dS. \tag{2.5}$$

A similar reasoning yields the representation for the scalar $p(\boldsymbol{x})$,

$$p(\boldsymbol{x})=-\int_S[\boldsymbol{\psi}\cdot T\vec{w}-\vec{w}\cdot T\boldsymbol{\psi}-(\vec{w}\cdot\boldsymbol{\psi})\,(\vec{b}\cdot\vec{n})]\,dS. \tag{2.6}$$

In forming $T\boldsymbol{\psi}$ we have introduced a "pressure", $\psi^*=\vec{b}\cdot\nabla\left(\frac{1}{r}\right)$.

3. The Fundamental Solution

The fundamental solution $(\boldsymbol{\chi},\boldsymbol{\psi})$ is determined from the equations

$$\frac{\partial\chi_{ki}}{\partial y_i}=-c\,\psi_k, \tag{3.1a}$$

$$\Delta\chi_{ki}-\frac{\partial\psi_k}{\partial y_i}+\vec{b}\cdot\nabla\chi_{ki}=\delta_{ki}\,\delta(\boldsymbol{y}-\boldsymbol{x}). \tag{3.1b}$$

One puts (*cf.* [*12*, p. 31])

$$\psi_k=-\frac{\partial}{\partial y_k}(\Delta+\vec{b}\cdot\nabla)\,\Phi, \tag{3.2a}$$

$$\chi_{ki}=\delta_{ki}[(1+c)\,\Delta+c\,\vec{b}\cdot\nabla]\,\Phi-\frac{\partial^2}{\partial y_i\,\partial y_k}\Phi. \tag{3.2b}$$

Equation (3.1a) is then identically satisfied, while equation (3.1b) reduces to

$$(\Delta+\vec{b}\cdot\nabla)\,[(1+c)\,\Delta+c\,\vec{b}\cdot\nabla]\,\Phi=\delta(\boldsymbol{y}-\boldsymbol{x}). \tag{3.3}$$

If $\vec{b}\neq 0$, let $b=|\vec{b}|$, $\tau=\frac{\vec{b}\cdot(\boldsymbol{y}-\boldsymbol{x})}{b}$. A solution of (3.3) can be determined in the form

$$\Phi(\boldsymbol{y}-\boldsymbol{x})=\frac{1}{b}\int^{\tau}[(1+c)\,\Phi_2-\Phi_1]\,d\tau \tag{3.4}$$

where Φ_1 and Φ_2 are solutions of the equations

$$(3.5)\qquad (\Delta + \vec{b}\cdot\nabla)\,\Phi_1 = \delta(y-x); \qquad [(1+c)\,\Delta + c\,\vec{b}\cdot\nabla]\,\Phi_2 = \delta(y-x).$$

If $c \neq 0$,

$$(3.6a)\qquad \Phi_1 = -\frac{1}{2\pi}\left(\frac{b}{4\pi r}\right)^{\frac{n-2}{2}} K_{\frac{n-2}{2}}\left(\frac{b\,r}{2}\right) e^{-\frac{1}{2}\vec{b}\cdot(y-x)},$$

$$(3.6b)\qquad \Phi_2 = -\frac{1}{2\pi(1+c)}\left(\frac{b|c|}{4\pi|1+c|\,r}\right)^{\frac{n-2}{2}} K_{\frac{n-2}{2}}\left(\frac{b|c|}{2|1+c|}\,r\right) e^{-\frac{c}{2(1+c)}\vec{b}\cdot(y-x)},$$

while if $c=0$, the expression for Φ_2 is replaced by

$$(3.6c)\qquad \Phi_2 = \frac{1}{2\pi}\log r, \qquad n=2,$$

$$(3.6d)\qquad \Phi_2 = -\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}(n-2)\,r^{n-2}}, \qquad n \geqq 3,$$

where $r=|y-x|$ and $K_{\frac{n-2}{2}}$ is a modified Bessel function of the second kind (*cf.* [*12*, p. 36]); if $\vec{b}=0$, equation (3.3) reduces to

$$(3.7a)\qquad (1+c)\,\Delta\Delta\Phi = \delta(y-x),$$

and we may take

$$(3.7b)\qquad \Phi = \frac{(-1)^{\frac{n-2}{2}}\, r^{4-n}\log r}{8\pi^{\frac{n}{2}}\left(2-\frac{n}{2}\right)!\,(1+c)}, \qquad n=2 \text{ and } 4$$

and

$$(3.7c)\qquad \Phi = \frac{\Gamma\left(\frac{n}{2}-2\right) r^{4-n}}{16\pi^{\frac{n}{2}}(1+c)}, \qquad n = \text{odd and } n = \text{even} > 4.$$

When these expressions are substituted into (3.4) and (3.2), the fundamental solution tensor (χ, Ψ) is obtained explicitly. From (3.6) and (3.7) we may compute formally all asymptotic properties of this tensor. However, for the principal purposes of this paper we need only the following crude estimates as $r\to\infty$, easily obtained from the form of the expression[3]:

a) $\vec{b}=0$:

$$(3.8a)\qquad |\chi| = O(\log r), \quad |\Psi| = O(r^{-1}), \qquad n=2,$$

$$(3.8b)\qquad |\chi| = O(r^{2-n}), \quad |\Psi| = O(r^{1-n}), \qquad n>2,$$

$$(3.8c)\qquad |\nabla^{(k)}\chi| = O(r^{2-n-k}), \quad |\nabla^{(k)}\Psi| = O(r^{1-n-k}), \qquad k\geqq 1, \quad n\geqq 2$$

where $\nabla^{(k)}$ denotes a k^{th} iterate of the gradient operator.

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[3] Somewhat more detailed estimates in the case $c=0$ are given in [*11*] and [*13*].

b) $\vec{b} \neq 0$:

(3.9a) $$|\chi| = O(r^{-\frac{1}{2}}), \quad |\Psi| = O(r^{-1}), \quad n = 2,$$

(3.9b) $$|\nabla^{(k)}\chi| = O\left(r^{-\frac{1+k}{2}}\right), \quad |\nabla^{(k)}\Psi| = O\left(r^{-\frac{2+k}{2}}\right), \quad n = 2,$$

(3.9c) $$|\chi| = O\left(r^{-\frac{n-1}{2}}\right), \quad |\Psi| = O\left(r^{-\frac{n}{2}}\right), \quad n \geqq 3,$$

(3.9d) $$|\nabla^{(k)}\chi| = O\left(r^{-\frac{n-1+k}{2}}\right), \quad |\nabla^{(k)}\Psi| = O\left(r^{-\frac{n+k}{2}}\right), \quad k \geqq 1, \quad n \geqq 3.$$

4. Statement of Results

In this section we state the main theorems of our paper. We prove these theorems in §6. Because of the qualitative difference in asymptotic behavior which occurs, we discuss separately the case $\vec{b} = 0$. It is the singular behavior of the solutions in this case which gives rise to the Stokes Paradox. The case $\vec{b} \neq 0$, and some theorems valid for every choice of $\vec{b}$, are introduced later.

Case 1. $\vec{b} = 0$.

Theorem 1 (Representation theorem). *Let $\vec{w}(x)$ be a vector field defined in an n-dimensional neighborhood $\mathscr{E}$ of infinity and which satisfies* (1.1) *for some scalar field $p(x)$. Suppose that as $r \to \infty$, either*

i) $\vec{w}(x) = o(r)$

or

ii) $\int_{\mathscr{S}_R} |\vec{w}|^2 r^{-(n+2)} dV = o(\log R)$

or

iii) $\int_{\mathscr{S}_R} |\nabla \vec{w}|^2 r^{-n} dV = o(\log R)$.

Then there exist vector and scalar constants $\vec{w}_0$ and p_0 such that throughout $\mathscr{E}$

(4.1a) $$\vec{w}(x) = \vec{w}_0 - \int_{\Sigma} [\chi \cdot T\vec{w} - \vec{w} \cdot T\chi]\, dS,$$

(4.1b) $$p(x) = p_0 - \int_{\Sigma} [\psi \cdot T\vec{w} - \vec{w} \cdot T\psi]\, dS.$$

The fields $\vec{w}(x)$ and $p(x)$ admit the asymptotic expressions

(4.2a) $$\vec{w}(x) = \vec{w}_0 + \chi(x) \cdot \vec{\mathfrak{T}} + O(r^{1-n}),$$

(4.2b) $$p(x) = p_0 + \psi(x) \cdot \vec{\mathfrak{T}} + \ldots {}^{-n})$$

where

$$\vec{\mathfrak{T}} = -\int_{\Sigma} T\vec{w}\, dS.$$

Theorem 1 is best possible, in the sense that "o" cannot be replaced by "O" in the hypotheses. We remark that the vector $\vec{\mathfrak{T}}$ can be interpreted physically as the force exerted on the contour Σ in the motion or displacement.

A particular consequence of Theorem 1 is that

$$|\vec{w}(x) - \vec{w}_0| = \begin{cases} O(\log r), & n=2, \\ O(r^{2-n}), & n>2 \end{cases}$$

with an improved behavior at infinity in case $\vec{\mathfrak{T}}=0$. This observation bears on our next theorem, in which we introduce a boundary condition on Σ.

Theorem 2. *Under the hypotheses of Theorem* 1, *if* $\vec{w}(x)=0$ *on* Σ, *then* $\vec{\mathfrak{T}} \neq 0$ *unless* $\vec{w}(x) \equiv 0$ *in* $\mathscr{E}$. *If* $n \geqq 3$, *then*

$$\vec{\mathfrak{T}} \cdot \vec{w}_0 = 2\int_{\mathscr{E}} (\operatorname{def} \vec{w})^2 \, dV + c(1-c) \int_{\mathscr{E}} p^2 \, dV \tag{4.3}$$

where $\operatorname{def} \vec{w}$ *denotes the matrix,* $\frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$.

The physical content of (4.3) is that in any steady fluid motion past Σ with adherence on the boundary, the work done at the boundary per unit time equals the rate at which energy is converted into heat in the motion. Thus, in three or more dimensions no energy is introduced into the flow at infinity. This is not the case in two dimensions when $\vec{b}=0$. In the interpretation of $\vec{w}(x)$ as an elastic deformation, (4.3) asserts that the work done at the boundary due to a constant deformation $-\vec{w}_0$ on Σ equals the potential energy of deformation in the medium. A particular consequence of (4.3) is that in any fluid flow past a rigid body Σ, or in any elastic deformation of a medium which surrounds Σ and is stationary at infinity, the body necessarily experiences a "drag" force in the direction of $\vec{w}_0$.

Theorem 3. *Let* $\vec{w}(x)$ *be a solution of* (1.1) *in* $\mathscr{E}$. *If for some* $\vec{w}_0$,

$$\vec{w}(x) - \vec{w}_0 = \begin{cases} o(\log r), & n=2, \\ o(r^{2-n}), & n>2, \end{cases}$$

then $\vec{\mathfrak{T}}=0$.

From Theorems 2 and 3 we obtain in particular the Stokes Paradox of Hydrodynamics. For if $\vec{w}(x)$ represents a two-dimensional fluid motion exterior to Σ, and if $\vec{w}(x)$ is bounded, then necessarily $\vec{\mathfrak{T}}=0$. But if the fluid adheres to Σ, then $\vec{w}(x)=0$ on Σ, hence $\vec{\mathfrak{T}} \neq 0$, a contradiction. If $c=0$ our result is a strengthened form of the results in [*3, 4, 5, 6*]; if $c>0$, we obtain a result which includes the extension to this case by DUFFIN & NOLL [*7*].

Case 2. $\vec{b} \neq 0$.

Theorem 4. *Under the hypotheses of Theorem* 1, *there exist vector and scalar constants* $\vec{w}_0$ *and* p_0 *such that throughout* $\mathscr{E}$,

$$\vec{w}(x) = \vec{w}_0 - \int_{\Sigma} [\chi \cdot T\vec{w} - \vec{w} \cdot T\chi - (\vec{w} \cdot \chi)(\vec{b} \cdot \vec{n})\, dS, \tag{4.4a}$$

$$p(x) = p_0 - \int_{\Sigma} [\psi \cdot T\vec{w} - \vec{w} \cdot T\psi - (\vec{w} \cdot \psi)(\vec{b} \cdot \vec{n})]\, dS. \tag{4.4b}$$

The fields $\vec{w}(x)$ and $p(x)$ admit the asymptotic expressions

$$\vec{w}(x) = \vec{w}_0 + \chi \cdot \vec{\mathfrak{M}} + O(r^{-1}), \qquad n = 2, \tag{4.5a}$$

$$\vec{w}(x) = \vec{w}_0 + \chi \cdot \vec{\mathfrak{M}} + O\left(r^{-\frac{n}{2}}\right), \qquad n > 2, \tag{4.5b}$$

$$p(x) = p_0 + \psi \cdot \vec{\mathfrak{M}}^* + O\left(r^{-\frac{n+1}{2}}\right), \tag{4.5c}$$

where

$$\mathfrak{M} = -\int_{\Sigma} [T\vec{w} - \vec{w}(\vec{b} \cdot \vec{n})]\, dS$$

and

$$\mathfrak{M}^* = -\int_{\Sigma} [T\vec{w} - \vec{b}(\vec{w} \cdot \vec{n}) - \vec{w}(\vec{b} \cdot \vec{n})]\, dS.$$

Theorem 5. *Under the hypotheses of Theorem 1, if $\vec{w}(x) = 0$ on Σ, then $\vec{\mathfrak{T}} \neq 0$ unless $\vec{w}(x) \equiv 0$ in $\mathscr{E}$. Also, for all $n \geqq 2$,*

$$\vec{\mathfrak{T}} \cdot \vec{w}_0 = 2 \int_{\mathscr{E}} (\operatorname{def} \vec{w})^2\, dV + c(1-c) \int_{\mathscr{E}} p^2\, dV. \tag{4.6}$$

The following theorems are valid for every choice of $\vec{b}$. The first of them includes the work of Finn & Noll [6] and of Duffin & Noll [7] on the same question.

Theorem 6 (Uniqueness theorem). *Let $\vec{w}(x)$ be a solution of (1.1) in $\mathscr{E}$ for some scalar field $p(x)$. Let $\vec{w}(x) = 0$ on Σ and suppose that as $x \to \infty$ either*

i) $\vec{w}(x) \to 0$

or

ii) $\int_{\mathscr{S}_R} |\vec{w}|^2 r^{-n}\, dV = o(\log R)$.

Then $\vec{w}(x) \equiv 0$ in $\mathscr{E}$.

With any fluid motion for which $\vec{w}(x) \to \vec{w}_0$ at infinity there is associated the kinetic energy of disturbance from the uniform flow, $E = \int_{\mathscr{E}} |\vec{w} - \vec{w}_0|^2 dV$. In any potential fluid motion, this quantity is always finite and can be used as the basis for a variational approach to the study of such motions. The situation is, however, quite different in the case of solutions of (1.1), as can be seen from the following theorem[4]:

Theorem 7. *Let $n = 2$ or 3, and let $\vec{w}(x)$ be a solution of (1.1) which satisfies the hypotheses of Theorem 1 and for which $\vec{w}(x) = 0$ on Σ, $\vec{w}(x) \not\equiv 0$ in $\mathscr{E}$. Then $\int_{\mathscr{E}} |\vec{w}(x) - \vec{w}_0|^2 dV = \infty$ for every choice of $\vec{w}_0$.*

5. Preliminary Lemmas

Lemma 1. *Let $u(x)$ be any Cartesian component of a vector field $\vec{w}(x)$ which satisfies (1.1) in $\mathscr{E}$ for some scalar field $p(x)$. Suppose that for all sufficiently large R,*

$$\int_{\mathscr{S}_{R+1}} u^2 r^{-s}\, dV < K(R) \tag{5.1}$$

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[4] A theorem of this type has been proved also for the (non-linear) Navier-Stokes hydrodynamical equations, in two dimensions by W. Wolibner [14], and for the three-dimensional case by Finn [15]. Important extensions of Wolibner's results to time-dependent motions of compressible fluids in two and in three dimensions have been given by A. Krzywicki [16].

for some real s. Then for any non-negative integer k,

$$|\nabla^{(k)} u(\boldsymbol{x})| = O[r^{s/2} K^{\frac{1}{2}}(r)]$$

where $\nabla^{(k)}$ denotes the k^{th} iterative of the gradient operator.

Proof (*cf.* [17]). One finds from (1.1) the following equation governing $u(\boldsymbol{x})$:

$$L_1 u = \{(1+c)\Delta - c\vec{\boldsymbol{b}}\cdot\nabla\}\{\Delta - \vec{\boldsymbol{b}}\cdot\nabla\} u = 0.$$

If one defines the adjoint operator L_2 by

$$L_2 = \{(1+c)\Delta + c\vec{\boldsymbol{b}}\cdot\nabla\}\{\Delta + \vec{\boldsymbol{b}}\cdot\nabla\},$$

one obtains the following identity for any scalar functions φ and ψ in a region $\mathscr{I}$:

$$\varphi L_1 \psi - \psi L_2 \varphi = \operatorname{div} \vec{\boldsymbol{W}} \tag{5.2}$$

where

$$\begin{aligned}\vec{\boldsymbol{W}} = {} & (1+c)\,[\varphi\nabla(\Delta\psi) - \Delta\psi\,\nabla\varphi + \Delta\varphi\,\nabla\psi - \psi\nabla(\Delta\varphi)] - \\ & - (1+2c)\,[\varphi\nabla(\vec{\boldsymbol{b}}\cdot\nabla\psi) + \psi\nabla(\vec{\boldsymbol{b}}\cdot\nabla\varphi) - \vec{\boldsymbol{b}}(\nabla\varphi\cdot\nabla\psi)] + \\ & + c\,\vec{\boldsymbol{b}}[\varphi\vec{\boldsymbol{b}}\cdot\nabla\psi - \psi\vec{\boldsymbol{b}}\cdot\nabla\varphi].\end{aligned}$$

Let P be an arbitrary point of large magnitude in $\mathscr{E}$. Construct a unit sphere S with P as center and let Q be an arbitrary point of S. Let $\zeta(Q)$ be an infinitely differentiable function defined in S such that

$$\zeta(Q) = \begin{cases} 1 & \text{for} \quad |Q-P| \leqq \frac{1}{4}, \\ 0 & \text{for} \quad |Q-P| \geqq \frac{3}{4}. \end{cases}$$

Let us now put $\psi = u$ and $\varphi = \zeta\Phi$ in (5.2); u and Φ being the solutions of

$$L_1 u = 0, \qquad L_2 \Phi = \delta(P-Q).$$

Integrating (5.2) over the unit sphere S, we find

$$u(P) = -\int_{\mathscr{A}} u(Q)\, L_2[\zeta(Q)\,\Phi(P-Q)]\, dV_Q \tag{5.3}$$

where $\mathscr{A}$ is the annular region $\frac{1}{4} \leqq |P-Q| \leqq \frac{3}{4}$. We can differentiate the right-hand side of (5.3) with respect to P, thereby obtaining a representation for the derivatives of $u(P)$:

$$\nabla^{(k)} u(P) = -\int_{\mathscr{A}} u(Q)\, L_2[\zeta(Q)\,\nabla_P^{(k)}\Phi(P-Q)]\, dV_Q.$$

Thus, letting r denote distance to the origin of a reference system,

$$|\nabla^{(k)} u(P)|^2 \leqq \int_{\mathscr{A}} u^2 r^{-s}\, dV \int_{\mathscr{A}} r^s |L_2[\zeta(Q)\,\nabla_P^{(k)}\Phi(P-Q)|^2\, dV_Q$$

by Schwarz' Inequality. Thus for a suitable constant C,

$$\begin{aligned}|\nabla^{(k)} u(P)|^2 &\leqq C(r_P+1)^s \int_{\mathscr{I}_{R+1}} u^2 r^{-s}\, dV \\ &= O[r^s K(r)], \qquad \text{q.e.d.}\end{aligned}$$

Lemma 2. *Let $P(x)$ be a polynomial of degree m in $\mathscr{E}$, and let s be a real number such that*

$$\int_{\mathscr{S}_R} P^2(x)\, r^{-s}\, dV = o(\log R).$$

Then $P(x) \equiv 0$ if $s \leqq n$, and $m < \frac{1}{2}(s-n)$ if $s > n$.

Proof. We have

$$P(x) = a_m(\Omega)\, r^m + \cdots + a_0,$$

where the coefficients depend only on the position Ω on a unit sphere. We obtain

$$\int_{\mathscr{S}_R} P^2(x)\, r^{-s}\, dV = \int_{\mathscr{S}_R} a_m^2\, r^{2m-s}\, r^{n-1}\, dr\, d\Omega + O\Big[\int_{\mathscr{S}_R} r^{2m-s-1}\, r^{n-1}\, dr\, d\Omega\Big]$$
$$= k^2 \int^r r^{2m-s+n-1}\, dr + O(1) + o\left[r^{2m-s+n}\right]$$

for some positive constant k^2.

Suppose $s \leqq n$. If then $m \geqq 0$, we shall have $\int_{\mathscr{S}_R} P^2(x)\, r^{-s} dV > k^2 \log R + O(1)$, a contradiction. Thus in this case $P(x) \equiv 0$. If on the other hand $s > n$, then in order that $\int^R r^{2m-s+n-1} dr = o(\log R)$ we must have $2m < s - n$, q.e.d.

Lemma 3. *Let $\vec{w}(x)$ be a continuously differentiable vector-valued function on and exterior to a unit n-sphere Σ, and let $\vec{w}(x) = 0$ on Σ. Then*

$$\int_{\mathscr{S}_R} |\vec{w}|^2\, r^{-(n+2)}\, dV \leqq \int_{\mathscr{S}_R} |\nabla \vec{w}|^2\, r^{-n}\, dV.$$

Proof. On an arbitrary ray through the center of Σ, we have

$$\int_1^R |\vec{w}|^2\, r^{-(n+2)}\, r^{n-1}\, dr = -\frac{|\vec{w}|^2}{2r^2}\bigg|_1^R + \int_1^R (\vec{w} \cdot \vec{w}_s)\, r^{-2}\, dr$$
$$\leqq \sqrt{\int_1^R |\vec{w}|^2\, r^{-3}\, dr}\ \sqrt{\int_1^R |\vec{w}_s|^2\, r^{-1}\, dr}$$

so that

$$\int_1^R |\vec{w}|^2\, r^{-(n+2)}\, r^{n-1}\, dr \leqq \int_1^R |\nabla \vec{w}|^2\, r^{-n}\, r^{n-1}\, dr,$$

and the result then follows by an integration over concentric spheres.

6. Proofs of Theorems

Proof of Theorem 1. By (2.5) $\vec{w}(x)$ has the representation $\vec{w}(x) = \vec{w}_1(x) + \vec{w}_2(x)$ where

$$\vec{w}_1(x) = -\int_\Sigma [\chi \cdot T\vec{w} - \vec{w} \cdot T\chi]\, dS, \tag{6.1}$$

$$\vec{w}_2(x) = -\int_{\Sigma_R} [\chi \cdot T\vec{w} - \vec{w} \cdot T\chi]\, dS. \tag{6.2}$$

We show first that $\vec{w}_1(x) = \chi \cdot \vec{\mathfrak{T}} + O(r^{1-n})$. We have, in fact,

$$\vec{w}_1(x) = -\chi(x) \cdot \int_\Sigma T\vec{w}(y)\, dS_y - \int_\Sigma \{[\chi(x-y) - \chi(x)]\, T\vec{w}(y) - \vec{w}(y)\, T\chi(x-y)\}\, dS_y$$
$$= \chi(x) \cdot \vec{\mathfrak{T}} + O(r^{1-n})$$

since by the estimates (3.8c)

$$|T\chi| = O(r^{1-n}) \quad \text{for} \quad \boldsymbol{y} \text{ on } \Sigma,$$

and

$$|\chi(\boldsymbol{x}-\boldsymbol{y}) - \chi(\boldsymbol{x})| = |\boldsymbol{y}\cdot\nabla\chi(\boldsymbol{\xi})|$$

where

$$\boldsymbol{\xi} = \boldsymbol{x} - \alpha\boldsymbol{y}, \qquad 0 \leqq \alpha \leqq 1,$$

so that

$$|\chi(\boldsymbol{x}-\boldsymbol{y}) - \chi(\boldsymbol{x})| = O(r^{1-n})$$

again by the estimates (3.8c).

To estimate $\vec{\boldsymbol{w}}_2(\boldsymbol{x})$, we observe first that by Lemmas 1 and 3,

$$|\nabla^{(k)}\vec{\boldsymbol{w}}| = O(r^{s/2})$$

under any of the hypotheses of Theorem 1 for any $s > n+2$. (In applying Lemma 3, we may clearly assume that $\vec{\boldsymbol{w}}(\boldsymbol{x})$ vanishes in a suitable sphere.) From the equation (1.1) we then find $|\nabla p| = |\Delta\vec{\boldsymbol{w}}| = O(r^{s/2})$ for any such s, so that $|p(\boldsymbol{x})| = O(r^{(s+2)/2})$. Now

$$\vec{\boldsymbol{w}}_2(\boldsymbol{x}) = -\int_{\Sigma_R} [\chi\cdot T\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}\cdot T\chi]\, dS \tag{6.3}$$

for any sufficiently large R. Since the left-hand side in (6.3) is independent of R, so is the right-hand side. Similarly, we find

$$\nabla^{(k)}\vec{\boldsymbol{w}}_2(\boldsymbol{x}) = -\int_{\Sigma_R} [\nabla^{(k)}\chi\cdot T\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}\cdot T\nabla^{(k)}\chi]\, dS$$

for any integer k, the integral being independent of R. But by the above estimates and the estimates (3.8c), we see that for any $k > \frac{s+4}{2}$, the integral over Σ_R tends to zero as $R\to\infty$. It follows that for any such k, $\nabla^{(k)}\vec{\boldsymbol{w}}_2(\boldsymbol{x}) \equiv 0$, from which we find that $\vec{\boldsymbol{w}}_2(\boldsymbol{x})$ is a polynomial of degree at most $k+1$, $\vec{\boldsymbol{w}}_2(\boldsymbol{x}) = P_m(\boldsymbol{x})$, $m \leqq k+1$. Thus, $\vec{\boldsymbol{w}}(\boldsymbol{x}) \equiv \vec{\boldsymbol{w}}_1(\boldsymbol{x}) + \vec{\boldsymbol{w}}_2(\boldsymbol{x}) = P_m(\boldsymbol{x}) + o(r)$ by the estimates (3.8a, b). If $\vec{\boldsymbol{w}}(\boldsymbol{x}) = o(r)$, we conclude immediately $P_m(\boldsymbol{x}) \equiv \text{const.} = \vec{\boldsymbol{w}}_0$, from which the stated representations of $\vec{\boldsymbol{w}}(\boldsymbol{x})$ follow. If

$$\int_{\mathcal{S}_R} |\vec{\boldsymbol{w}}|^2 r^{-(n+2)}\, dV = o(\log R),$$

we observe (since $|P_m(\boldsymbol{x})|^2 > c r^2$ in a semi-infinite cone with vertex at the origin) that necessarily

$$\int_{\mathcal{S}_R} |P_m(\boldsymbol{x})|^2 r^{-(n+2)}\, dV = o(\log R).$$

By Lemma 2, we have $m < \frac{n+2-n}{2} = 1$, i.e. $P_m(\boldsymbol{x}) \equiv \text{const.} = \vec{\boldsymbol{w}}_0$. Finally, if $\int_{\mathcal{S}_R} |\nabla\vec{\boldsymbol{w}}|^2 r^{-n}\, dV = o(\log R)$, the result follows from the above discussion by use of Lemma 3. The stated representations for $p(\boldsymbol{x})$ are obtained similarly.

Proof of Theorem 2. Let us suppose $\vec{\mathfrak{T}} = 0$. Then by Theorem 1, $\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 + O(r^{1-n})$, $p(\boldsymbol{x}) = p_0 + O(r^{-n})$, and (by the same proof) $\nabla\vec{\boldsymbol{w}}(\boldsymbol{x}) = O(r^{-n})$, as $r\to\infty$. Formal integration by parts, applied to the solution $\vec{\boldsymbol{w}}(\boldsymbol{x}) - \vec{\boldsymbol{w}}_0$, establishes the relation

$$\vec{\mathfrak{T}}\cdot\vec{w}_0 = 0 = 2\int_{\mathcal{I}_R}(\operatorname{def}\vec{w})^2 dV + c(1-c)\int_{\mathcal{I}_R} p^2 dV + \oint_{\Sigma_R}(\vec{w}-\vec{w}_0)\cdot T\vec{w}\,dS.$$

Because of the above estimates, the outer surface integral vanishes in the limit, and we find

$$2\int_{\mathcal{E}}(\operatorname{def}\vec{w})^2 dV + c(1-c)\int_{\mathcal{E}} p^2 dV = 0. \tag{6.4}$$

If $c<3$, we verify easily, using (6.4), that $\operatorname{def}\vec{w}\equiv 0$ in $\mathcal{E}$. Hence the motion is a pure rotation. But $\vec{w}=0$ on Σ, hence $\vec{w}(\boldsymbol{x})\equiv 0$ in $\mathcal{E}$. This result, however, is true for every non-negative c. To see this, we observe that the divergence theorem and the above asymptotic properties of $\boldsymbol{w}(\boldsymbol{x})$ imply (summation on repeated indices)

$$\oint_{\Sigma}\left(\frac{\partial w_j}{\partial x_j}n_i - \frac{\partial w_j}{\partial x_i}n_j\right)dS = 0,$$

$$\oint_{\Sigma}(w_i - w_{0i})\left(\frac{\partial w_j}{\partial x_j}n_i - \frac{\partial w_j}{\partial x_i}n_j\right)dS = \int_{\mathcal{E}}\left(\frac{\partial w_i}{\partial x_i}\frac{\partial w_j}{\partial x_j} - \frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i}\right)dV = 0$$

since $w_i=0$ on Σ. Adding this relation to (6.4), we find, using (1.1b),

$$\int_{\mathcal{E}}(\nabla\vec{w})^2 dV + c\int_{\mathcal{E}} p^2 dV = 0,$$

and we conclude again $\vec{w}\equiv 0$ in $\mathcal{E}$.

If $n\geqq 3$, the relation (4.3) can be obtained by the method used to prove (6.4), the outer surface integral vanishing in the limit because of the asymptotic properties of the solution. For $n=2$ the volume integrals which occur are necessarily infinite when $\vec{\mathfrak{T}}\neq 0$, as can be shown by using Theorem 1 and the explicit knowledge of $\chi(\boldsymbol{x},\boldsymbol{y})$.

Proof of Theorem 3. By Theorem 1, it is sufficient to observe that if $\vec{\mathfrak{T}}\neq 0$, then

$$\chi\cdot\vec{\mathfrak{T}} = \begin{cases} O(\log r), & n=2,\\ O(r^{2-n}), & n>2.\end{cases}$$

This property is easily checked, using the explicit knowledge of $\chi(\boldsymbol{x},\boldsymbol{y})$.

Proof of Theorems 4 and 5. These theorems are proved by the methods used for Theorems 1 and 2, the differing estimates arising because of the different properties of the fundamental solution tensor in the case $\vec{b}\neq 0$. We omit details.

Proof of Theorem 6. As in the proof of Theorem 2, we use the identity

$$\int_{\mathcal{I}_R}(\nabla\vec{w})^2 dV + c\int_{\mathcal{I}_R} p^2 dV = \oint_{\Sigma_R}\left[\vec{w}\cdot\frac{\partial\vec{w}}{\partial n} - p(\vec{w}\cdot\vec{n})\right]dS.$$

According to our assumptions, the estimates of Theorem 1 or of Theorem 4 must hold with $\vec{w}_0=0$. It follows that the outer surface integral vanishes in the limit. Hence $\nabla\vec{w}\equiv 0$ in $\mathcal{E}$, from which $\vec{w}\equiv\text{const}$. But $\vec{w}=0$ on Σ, therefore $\vec{w}(\boldsymbol{x})\equiv 0$ in $\mathcal{E}$, *q.e.d.*

Proof of Theorem 7. We first observe that by Theorem 1 or Theorem 4 ($\vec{w}=0$ on Σ_n)

$$\vec{w}(x) = \vec{w}_0 + \chi(x) \cdot \vec{\mathfrak{T}} + \vec{w}_1(x)$$

where $\vec{w}_1(x)$ has finite square integral over $\mathscr{E}$ except in the case $n=2$, $\vec{b}=0$. (The proof of this fact requires estimates on $\chi(x, y)$ which are slightly more delicate than those indicated in §3, but is not difficult.) Putting the exceptional case aside for the moment, we see that $\int_{\mathscr{E}} |\vec{w}-\vec{w}_0|^2\, dV$ is finite or infinite according as $\int_{\mathscr{E}} |\chi \cdot \vec{\mathfrak{T}}|^2\, dV$ is finite or infinite. Denoting the components of $\vec{\mathfrak{T}}$ by $\{f_i\}$, the integrand becomes a quadratic form, $|\chi \cdot \vec{\mathfrak{T}}|^2 = \chi_{ij}\chi_{ik} f_j f_k$. We adopt spherical coordinates and observe that because of the symmetry properties of χ_{ij}, the integral over a sphere centered at the origin of any term for which $j \neq k$ must vanish. Thus, we need only consider an integral of a sum of squares. Each of these terms has, however, an infinite volume integral unless $\vec{\mathfrak{T}}=0$, as is easily verified from the explicit expression for χ. If $\vec{\mathfrak{T}}=0$ on the other hand, we conclude from Theorem 2 or Theorem 5 that $\vec{w}(x) \equiv 0$ in $\mathscr{E}$.

It remains to discuss the exceptional case $n=2$, $\vec{b}=0$. In this case we find that if $\vec{\mathfrak{T}} \neq 0$, the ratio of the integrals, $\int_{\mathscr{S}_R} |\chi \cdot \vec{\mathfrak{T}}|^2 dV$ and $\int_{\mathscr{S}_R} |\vec{w}_1|^2 dV$, tends to infinity with R. Thus we conclude again that $\int_{\mathscr{E}} |\vec{w}|^2 dV = \infty$ unless $\vec{\mathfrak{T}}=0$, but again $\vec{\mathfrak{T}}=0$ implies by Theorem 2 that $\vec{w}(x) \equiv 0$ in $\mathscr{E}$, *q.e.d.*

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