On a Theory of Phase Transitions with Interfacial Energy

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1. Introduction

Consider a fluid which has free energy[1] $\psi(\varrho)$ a prescribed function of density ϱ, and which occupies a fixed container Ω, with Ω a bounded region in $\mathbb{R}^3$. If we neglect all other contributions to the free energy, then the total energy $E_0(\varrho)$ corresponding to a density distribution $\varrho(x)$, $x \in \Omega$, is

$$E_0(\varrho) = \int_\Omega \psi(\varrho(x))\, dx. \tag{1.1}$$

If the fluid in Ω has mass m, and fluid is neither added to—nor removed from—Ω, then the allowable density distributions must be consistent with the constraint

$$\int_\Omega \varrho(x)\, dx = m. \tag{1.2}$$

Following GIBBS, we postulate that the stable configurations of the fluid are those which minimize (1.1) subject to (1.2). Thus we are led to the problem:

$\mathscr{P}_0$ { *minimize the energy* $E_0(\varrho)$ *over all sufficiently regular fields* ϱ *that satisfy the constraint* (1.2).

Here we shall be concerned with situations in which $\psi(\varrho)$ is *nonconvex*, of a form capable of supporting *two phases* (Figure 1). In this instance the solution $\mathscr{P}_0$ is most easily described in terms of the Maxwell-parameters α_0, β_0, and μ_0 defined by the conditions

$$\begin{aligned} \psi(\beta_0) - \psi(\alpha_0) &= \mu_0(\beta_0 - \alpha_0), \\ \mu_0 &= \psi'(\alpha_0) = \psi'(\beta_0). \end{aligned} \tag{1.3}$$

The line through $(\alpha_0, \psi(\alpha_0))$ and $(\beta_0, \psi(\beta_0))$ forms the convex envelope of the free energy between α_0 and β_0 (Figure 1).

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[1] We assume isothermal conditions. Here $\psi(\varrho)$ is the free energy per unit volume; $\psi(\varrho)/\varrho$ is the free energy per unit mass. It is important to note that $\psi(0^+) < \infty$.

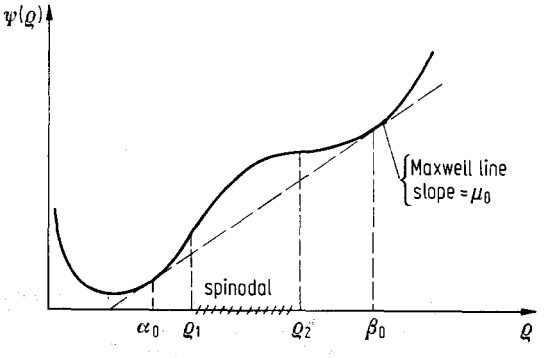

Fig. 1. Free energy $\psi(\varrho)$

It is not difficult to show that if we restrict our attention to continuous ϱ, then $\mathscr{P}_0$ has *no* solution for $\alpha_0 < m < \beta_0$. If we take piecewise continuous ϱ, or more generally ϱ, $\psi(\varrho) \in L^1(\Omega)$, and assume that the length scale is chosen with

$$\text{volume } (\Omega) = 1,$$

then the solution of $\mathscr{P}_0$ is as follows:

(i) for $m \leqq \alpha_0$ or $m \geqq \beta_0$ there is exactly one solution, the constant field

$$\varrho(x) \equiv m;$$

(ii) for $m \in (\alpha_0, \beta_0)$ *any* field ϱ of the form

$$\varrho(x) = \begin{cases} \alpha_0, & x \in \Omega_1 \\ \beta_0, & x \in \Omega_2, \end{cases}$$

$$\Omega = \Omega_1 \cup \Omega_2,$$

$$\text{volume } (\Omega_1) = \frac{\beta_0 - m}{\beta_0 - \alpha_0},$$

$$\text{volume } (\Omega_2) = \frac{m - \alpha_0}{\beta_0 - \alpha_0}$$

is a solution, and all solutions have this form.

Thus for each $m \in (\alpha_0, \beta_0)$ there are infinitely many solutions, some corresponding to sets Ω_i which are quite wild. This drastic loss of uniqueness occurs because interfaces—jumps in density—are allowed to form without a corresponding increase in energy.

A theory which attempts to overcome this difficulty was developed independently by VAN DER WAALS [1893] and CAHN & HILLIARD [1958] and is based on an energy of the form

$$\int_\Omega [\psi(\varrho(x)) + \varepsilon |\nabla \varrho(x)|^2] \, dx.$$

Thus jumps in density are not allowed, but rapid changes are, and such changes are penalized in energy by the presence of the term $|\nabla\varrho|^2$.

One problem with this theory is its difficulty, especially in space-dimensions larger than one. A second problem is that, because solutions are smooth, it is not a simple matter to locate—or even define—an interfacial zone between phases.

In this paper I present an alternative theory for problems of this type. This theory differs from its precedessors in two respects:

(1) Jumps in density are allowed, but are accompanied by an interfacial energy.

(2) The density distribution is not allowed to lie in the spinodal region $[\varrho_1, \varrho_2]$. In fact, all we need assume is that $\psi(\varrho)$ is defined and locally convex on the intervals $(0, \varrho_1)$ and (ϱ_2, ∞); nothing need be said about the behavior of ψ on $[\varrho_1, \varrho_2]$.

For fields ϱ consistent with (2) we define complementary subsets

$$\begin{aligned} \Omega_1(\varrho) &= \{x \in \Omega : \varrho(x) \leqq \varrho_1\}, \\ \Omega_2(\varrho) &= \{x \in \Omega : \varrho(x) \geqq \varrho_2\}, \end{aligned} \tag{1.4}$$

$\Omega_i(\varrho)$ being the region in which the fluid is in phase i. The surface

$$\mathscr{S}(\varrho) = \partial\Omega_1(\varrho) \cap \partial\Omega_2(\varrho)$$

then represents the *interface* between phases, and we endow $\mathscr{S}(\varrho)$ with *interfacial energy*

$$\sigma I(\varrho),$$

where

(i) $I(\varrho)$ is the area of $\mathscr{S}(\varrho)$,

(ii) σ is the interfacial energy per unit area.

We are therefore led to an energy of the form

$$E(\varrho) = \int_\Omega \psi(\varrho(x))\, dx + \sigma I(\varrho),$$

and to the following problem:

$\mathscr{P}$ *minimize the energy* $E(\varrho)$ *over all sufficiently regular fields* ϱ *which have range outside* $[\varrho_1, \varrho_2]$ *and which satisfy the constraint* (1.2).

In Section 2 we give a precise statement of problem $\mathscr{P}$ and list our assumptions underlying its solution. The main assumption we make is that σ be small, an assumption that should be in accord with the underlying physics. We make this assumption for convenience, as the solution for large σ is complicated and leads to instances of nonexistence.

We state our main results in Section 2, and prove them in Section 3. In particular, we establish an existence theorem for problem $\mathscr{P}$ and show that solutions $\varrho(x)$ are piecewise constant with constant chemical potential[1] $\mu = \psi'(\varrho(x))$, thereby reducing $\mathscr{P}$ to finding the global minimum of the energy $\mathscr{E}(\mu)$ expressed as a function of μ. We show further that, when the solution is considered a function of the parameter m, with μ_m the corresponding chemical potential and E_m

[1] This is the Euler-Lagrange equation for $\mathscr{P}$.

the associated minimal energy,

$$\mu_m = \frac{dE_m}{dm}.$$

We also prove that solutions ϱ of $\mathscr{P}$ have *minimal interface*; more precisely, we show that $\mathscr{S}(\varrho)$ has minimal area when compared to all other interfaces

$$\tilde{\mathscr{S}} = \partial\tilde{\Omega}_1 \cap \partial\tilde{\Omega}_2$$

with

$$\text{volume}\,(\tilde{\Omega}_i) = \text{volume}\,(\Omega_i(\varrho)).$$

In Section 4 we give the general solution of $\mathscr{P}$ in $\mathbb{R}^1$; in Section 5 we solve $\mathscr{P}$ completely for Ω the unit square in $\mathbb{R}^2$ and $\psi(\varrho)$ piecewise quadratic.

Our theory is also applicable to binary mixtures provided we identify ϱ with the concentration of one of the components, and restrict ϱ to the interval $0 < \varrho < 1$. To allow for this and other applications we take the domain of $\psi(\varrho)$ to be a set of the form $(\varkappa_1, \varrho_1) \cup (\varrho_2, \varkappa_2)$, with $-\infty \leqq \varkappa_1$, $\varkappa_2 \leqq +\infty$, and $[\varrho_1, \varrho_2]$ identified with the spinodal; and we do not assume that $\psi(\varrho) \to \infty$ as $\varrho \to \varkappa_i$, but rather allow these limits to be finite or infinite.

Because of the constraint (1.2), problem $\mathscr{P}$ is invariant when

$$\psi(\varrho) \text{ is replaced by } \psi(\varrho) - \psi(\alpha_0) - \mu_0(\varrho - \alpha_0).$$

We therefore assume, without loss in generality, that

$$\mu_0 = 0, \quad \psi(\alpha_0) = \psi(\beta_0) = 0,$$

$$\psi(\varrho) > 0, \quad \varrho \neq \alpha_0, \beta_0,$$

so that $\psi(\varrho)$ has the form shown in Figure 2.

2. Assumptions. Main results

Let Ω be a bounded, open region in $\mathbb{R}^n$ with $\partial\Omega$ Lipschitz-continuous, and assume that the length scale is chosen with

$$\mathscr{H}_n(\Omega) = 1. \tag{2.1}$$

Here and in what follows, $\mathscr{H}_k$ denotes k-dimensional Hausdorff measure.[1]

We assume that the domain of ψ has the form

$$\begin{gathered} \mathscr{D} = (\varkappa_1, \varrho_1) \cup (\varrho_2, \varkappa_2), \\ -\infty \leqq \varkappa_1 < \varrho_1 < \varrho_2 < \varkappa_2 \leqq \infty. \end{gathered} \tag{2.2}$$

[1] Roughly speaking, $\mathscr{H}_3$ = volume, $\mathscr{H}_2$ = area, $\mathscr{H}_1$ = length, $\mathscr{H}_0$ = number of points.

We assume[1] further that (Figure 2):

(A_1) $\psi \in C^2(\mathscr{D})$ with $\psi'' > 0$; $\psi(\alpha_0) = \psi(\beta_0) = 0$, $\psi(\varrho) > 0$ otherwise.

It then follows that

$$\alpha_0 \in (\varkappa_1, \varrho_1), \quad \beta_0 \in (\varrho_2, \varkappa_2).$$

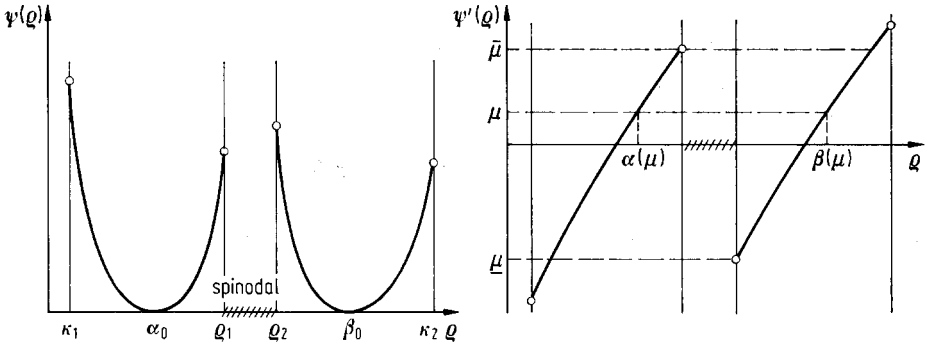

Fig. 2. Free energy $\psi(\varrho)$ and $\psi'(\varrho)$ as functions of density

By (A_1), ψ' *restricted to* $(\varkappa_1, \varrho_1)$ is invertible, as is ψ' *restricted to* $(\varrho_2, \varkappa_2)$; let α and β, respectively, denote the corresponding inverse functions (Figure 2), so that

$$\alpha(\psi'(\varrho)) = \varrho \text{ for } \varkappa_1 < \varrho < \varrho_2,$$

$$\beta(\psi'(\varrho)) = \varrho \text{ for } \varrho_2 < \varrho < \varkappa_2,$$

$$\alpha(0) = \alpha_0, \quad \beta(0) = \beta_0.$$

Further, for a nonempty interval $(\underline{\mu}, \bar{\mu})$ of values of μ the equation $\psi'(\varrho) = \mu$ has two solutions, namely $\varrho = \alpha(\mu)$ and $\varrho = \beta(\mu)$:

$$\psi'(\alpha(\mu)) = \psi'(\beta(\mu)) = \mu. \tag{2.3}$$

By an ***admissible field*** we mean a function[2] $\varrho \in BV(\Omega)$ with

$$\varrho(x) \in \mathscr{D} \text{ for all } x \in \Omega. \tag{2.4}$$

For ϱ admissible we write

$$\Omega_1(\varrho) := \{x \in \Omega : \varkappa_1 < \varrho(x) < \varrho_1\},$$

$$\Omega_2(\varrho) := \{x \in \Omega : \varrho_2 < \varrho(x) < \varkappa_2\},$$

$$v_i(\varrho) := \mathscr{H}_n(\Omega_i(\varrho)),$$

[1] Note that we make no assumptions concerning the behavior of ψ in the spinodal region $[\varrho_1, \varrho_2]$.

[2] $BV(\Omega)$ is the space of functions on Ω of bounded variation. The elements of $BV(\Omega)$ are actually equivalence classes of functions that coincide almost everywhere. Since we shall not bother with this distinction, some of our statements are modulo a rearrangement on a set of $\mathscr{H}_n$-measure zero.

and call ϱ ***single-phase*** if either $v_1(\varrho) = 0$ or $v_2(\varrho) = 0$, ***two-phase*** if $v_1(\varrho) > 0$, $v_2(\varrho) > 0$. Since $\varrho \in BV(\Omega)$, the sets $\Omega_i(\varrho)$ have finite perimeter. We write[1]

$$\mathscr{S}(\varrho) := \partial\Omega_1(\varrho) \cap \partial\Omega_2(\varrho) \tag{2.5}$$

for the ***interface*** between phases and

$$I(\varrho) := \mathscr{H}_{n-1}(\mathscr{S}(\varrho)) \tag{2.6}$$

for the corresponding ***interfacial measure***. Finally, we call the field $\psi'(\varrho(x))$ the ***chemical potential*** for ϱ.

As before, the ***energy*** $E(\varrho)$ has the form

$$E(\varrho) = \int_{\Omega} \psi(\varrho(x))\, dx + \sigma I(\varrho) \tag{2.7}$$

with σ the interfacial energy per unit $\mathscr{H}_{n-1}$-measure, and we consider fields ϱ consistent with the ***constraint***

$$\int_{\Omega} \varrho(x)\, dx = m. \tag{2.8}$$

Here σ and m are prescribed with

$$\sigma > 0, \quad m \in (\varkappa_1, \varkappa_2).$$

Within this framework, problem $\mathscr{P}$ has the following form:

$\mathscr{P}$ {*minimize the energy* $E(\varrho)$ *over all admissible fields* ϱ *that satisfy the constraint* (2.8).

In discussing this problem we shall reserve the term ***solution*** for *global minimizer*.

Note that for $m \leqq \alpha_0$ or $m \geqq \beta_0$ the constant $\varrho(x) \equiv m$ is the unique solution of problem $\mathscr{P}_0$ (*cf.* the discussion in Section 1) and, since this field satisfies $I(\varrho) = 0$, it must also be the unique solution of $\mathscr{P}$. We therefore have the following result, which allows us to concentrate our efforts on $\alpha_0 < m < \beta_0$.

Theorem 2.1. *For* $m \leqq \alpha_0$ *or* $m \geqq \beta_0$, *problem* $\mathscr{P}$ *has exactly one solution, the constant* $\varrho(x) \equiv m$.

We henceforth restrict our attention to the range

$$\alpha_0 < m < \beta_0. \tag{2.9}$$

As we shall see, problem $\mathscr{P}$ leads, in a natural manner, to the following ***minimal-surface problem***:

$\mathscr{O}_v$ {*minimize* $\mathscr{H}_{n-1}(\partial\Gamma \cap \mathring{\Omega})$ *over all sets* $\Gamma \subset \Omega$ *which have finite perimeter and which satisfy the constraint* $\mathscr{H}_n(\Gamma) = v$,

[1] Here $\partial\Omega_i(\varrho)$ indicates the measure-theoretic boundary of $\Omega_i(\varrho)$.

with $v \in [0, 1]$ prescribed. When $\mathcal{O}_v$ has a solution, we write $a(v)$ for the common value of $\mathcal{H}_{n-1}(\partial\Gamma \cap \mathring{\Omega})$ on solutions Γ. We call $a(v)$ the ***minimal-interface measure***. Clearly,

$$\begin{aligned} a(0) = a(1) = 0, \quad a(v) > 0 \text{ on } (0, 1), \\ a(v) = a(1 - v), \quad 0 \leqq v \leqq 1. \end{aligned} \tag{2.10}$$

We say that an admissible field ϱ has ***minimal interface*** if $\Omega_1(\varrho)$ is a solution of $\mathcal{O}_v$ with $v = v_1(\varrho)$. Since

$$\mathcal{S}(\varrho) = \partial\Omega_1(\varrho) \cap \mathring{\Omega}$$

(*cf.* (2.4)), this definition makes sense: admissible fields ϱ with minimal interface are exactly those fields with

$$I(\varrho) \leqq I(\gamma) \tag{2.11}$$

for all admissible γ with $v_i(\gamma) = v_i(\varrho)$.

Remark. Problem $\mathcal{O}_v$ can have many solutions. For example, let $\Omega \subset \mathbb{R}^2$ be the unit square. Then (Figure 3):

(i) for $0 < v < 1/\pi$ there are four solutions, each of which is a quarter-disc consisting of the intersection of Ω with a disc centered at a corner of Ω;

(ii) for $1/\pi < v \leqq \frac{1}{2}$ there are four solutions, each of which is the rectangular region lying between a side of Ω and the line a distance v from the side;

(iii) for $v = 1/\pi$ there are eight solutions, four of type (i) and four of type (ii).

Further, $a(v)$ will generally not be C^1. Indeed, in the above example

$$a(v) = \begin{cases} (\pi v)^{\frac{1}{2}}, & 0 \leqq v < 1/\pi \\ 1, & 1/\pi \leqq v \leqq \frac{1}{2}. \end{cases} \tag{2.12}$$

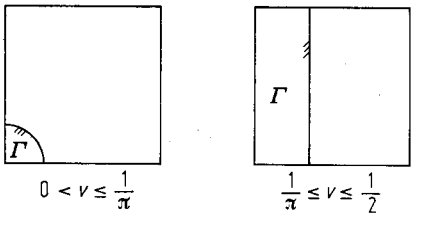

Fig. 3. Solutions of $\mathcal{O}_v$ for Ω the unit square in $\mathbb{R}^2$

The next theorem shows that problem $\mathcal{O}_v$ is well-posed. The first sentence is due to MASSARI & PEPE [1974], GIUSTI [1981], and GONZALEZ, MASSARI, & TAMANINI [1983]. The remaining assertions—as well as all other theorems stated in this section—will be proved in Section 3.

Theorem 2.2. (Properties of the minimal interface). *Problem $\mathcal{O}_v$ has a solution for each $v \in [0, 1]$, and each solution Γ has $\partial\Gamma \cap \mathring{\Omega}$ a C^∞ surface. Moreover:*

(i) *for $n = 1$, $a(v) = 1$ on $(0, 1)$;*

(ii) *for $n \geqq 2$, $a(v)$ is Hölder continuous on $[0, 1]$ with exponent $(n-1)/n$;*

(iii) *there is a constant $C = C(\Omega, n) > 0$ such that*

$$a(v) \geqq Cv^{\frac{n-1}{n}}$$

for all sufficiently small $v \geqq 0$.

Note that, by (i) and (ii),

$$a(v) \text{ is lower semi-continuous on } [0, 1] \tag{2.13}$$

and that

$$a_M := \sup_{0 \leq v \leq 1} a(v) < \infty. \tag{2.14}$$

To avoid tedious considerations of special cases, and to insure that problem $\mathcal{P}$ have a solution, we will henceforth asusme that σ *is small.* More precisely, we suppose that

$$(A_2) \quad \sigma a_M < \psi(\alpha(\mu)), \psi(\beta(\mu)) \quad \text{for } \mu = \underline{\mu}^+, \bar{\mu}^-,$$

an assumption which holds automatically when $\psi(\varrho)$ is infinite at the end points $\varkappa_1$, ϱ_1, ϱ_2, $\varkappa_2$. We are now in a position to state our main results.

Theorem 2.3. *Problem $\mathcal{P}$ has a solution.*

More important, we will show that each solution ϱ of $\mathcal{P}$ has constant chemical potential $\mu = \mu(\varrho)$; that the energy $E(\varrho)$ can be expressed as a function $\mathcal{E}(\mu)$ with

$$\begin{aligned} \mathcal{E}(\mu) &:= v_1(\mu)\, \psi(\alpha(\mu)) + v_2(\mu)\, \psi(\beta(\mu)) + \sigma a(v_1(\mu)), \\ v_1(\mu) &:= \frac{\beta(\mu) - m}{\beta(\mu) - \alpha(\mu)}, \qquad v_2(\mu) := \frac{m - \alpha(\mu)}{\beta(\mu) - \alpha(\mu)}; \end{aligned} \tag{2.15}$$

and that problem $\mathcal{P}$ reduces to minimizing $\mathcal{E}$ over the set

$$U = \{\mu \in (\underline{\mu}, \bar{\mu}) : \alpha(\mu) \leqq m \leqq \beta(\mu)\}. \tag{2.16}$$

These facts are expressed precisely as

Theorem 2.4. (Properties of solutions). *Let ϱ be a solution of $\mathcal{P}$. Then:*

(i) *ϱ has minimal interface;*

(ii) *ϱ has constant chemical potential μ, with μ a global minimizer of $\mathcal{E}$ over U;*

(iii) *$E(\varrho) = \mathcal{E}(\mu)$;*

(iv) ϱ *is piecewise constant and of the form*

$$\varrho(x) = \begin{cases} \alpha(\mu), & x \in \Omega_1(\varrho) \\ \beta(\mu), & x \in \Omega_2(\varrho), \end{cases}$$
$$v_i(\varrho) = v_i(\mu). \tag{2.17}$$

Conversely, any global minimizer of $\mathscr{E}$ *over* U *is the chemical potential of a solution of* $\mathscr{P}$.

As a consequence of this theorem:

$$\text{all single-phase solutions are of the form } \varrho(x) \equiv m. \tag{2.18}$$

Thus for $m \in [\varrho_1, \varrho_2]$ all solutions of $\mathscr{P}$ are two-phase, as $m \notin \mathscr{D}$ (*cf.* (2.4)). Less trivial consequences are the next two corollaries. There, for convenience, we call a solution ϱ regular if $a(v)$ is differentiable at $v = v_1(\varrho)$.

Corollary 1. *If* μ *is the chemical potential of a regular two-phase solution of* $\mathscr{P}$, *then*

$$\psi(\beta(\mu)) - \psi(\alpha(\mu)) - \mu[\beta(\mu) - \alpha(\mu)] = \sigma a'(v_1(\mu)). \tag{2.19}$$

Remark. Equation (2.19) is the analog of the Weierstrass-Erdmann "corner-condition" in the classical calculus of variations.

Remark. Since $\psi(\varrho)$ is the free energy per unit volume,

$$p := -\psi(\varrho) + \varrho\psi'(\varrho)$$

is the *pressure.* For the problem without interfacial energy the Maxwell condition (1.3) yields the continuity of the pressure in Ω. In the present theory the pressure is constant in each phase, but need not be continuous across the interface. Indeed, if we define p_i to be the pressure in $\Omega_i(\varrho)$, then (2.19) yields

$$p_1 - p_2 = \sigma a'(v), \qquad v = v_1(\varrho).$$

Note that for the example shown in Figure 3,

$$a'(v) = \frac{1}{r} \qquad \left(0 < v < \frac{1}{\pi}\right),$$

with r the radius of the disc, and

$$p_1 - p_2 = \frac{\sigma}{r},$$

which is the *classical relation between surface tension and pressure.* For the specific problem considered in Section 5 there is a range of values of v for which the minimal interface is *flat.* For this range it turns out that $a'(v) = 0$, which yields $p_1 = p_2$,

a result to be expected from equilibrium of forces. I believe that this result is generic. More precisely, I offer the following

Conjecture. *Suppose that* $a(v)$ *is differentiable at* $v = v_0 \in (0, 1)$. *Then* $a'(v_0) = 0$ *if and only if every solution of the minimal-interface problem* $\mathcal{O}_{v_0}$ *is flat.*

To facilitate the statement of our next result, we make explicit the dependence of $\mathcal{P}$ on m by writing $\mathcal{P} = \mathcal{P}(m)$, and, for each fixed m, we let

$$E_m = \text{the } \textbf{\textit{minimal energy}},$$

that is, the common value of $E(\varrho)$ on solutions ϱ of $\mathcal{P}(m)$.

Suppose that $\mathcal{P}(m)$ has a single-phase solution—with chemical potential μ_m—for all m in some open set. Then, by (2.18), $E_m = \psi(m)$, so that, trivially

$$\mu_m = \frac{dE_m}{dm}.$$

The next result shows that this relation holds also for two-phase solutions.

Corollary 2. *Suppose that for all* m *in some open interval* J, μ_m *is the chemical potential of a regular two-phase solution of* $\mathcal{P}(m)$, *with* μ_m *a differentiable function of* m. *Then*

$$\mu_m = \frac{dE_m}{dm} \tag{2.20}$$

for all $m \in J$.

In the theory without interfacial energy the solution is single-phase for $m \leqq \alpha_0$ and $m \geqq \beta_0$, two-phase otherwise. The next result shows that one effect of interfacial energy is to increase this single-phase range.[1]

Theorem 2.5. *There exists a* $\delta > 0$ *such that two-phase solutions of* $\mathcal{P}$ *are possible only in the interval*

$$\alpha_0 + \delta < m < \beta_0 - \delta. \tag{2.21}$$

3. Analysis

Proof of Theorem 2.2. As noted previously, the chief assertion of the theorem (the first sentence) has already been established. Moreover, (i) is obvious. Thus we need only prove (ii).

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[1] *Cf.* CARR, GURTIN, & SLEMROD [1984], Theorem 7.1, for a similar result within the VAN DER WAALS-CAHN-HILLIARD theory.

Assume $n \geqq 2$. Trivially, there is a constant $C = C_n > 0$ such that, for any ball $B \subset \mathbb{R}^n$,

$$\mathscr{H}_{n-1}(\partial B) = C\mathscr{H}_n(B)^q, \quad q = \frac{n-1}{n}.$$

Choose $v \in [0, 1)$ and let Γ be a solution of $\mathscr{O}_v$. Then for all sufficiently small $\varepsilon > 0$, $\Omega \setminus \Gamma$ contains in its interior a ball B_ε of $\mathscr{H}_n$-measure ε, and $Q_\varepsilon = \Gamma \cup B_\varepsilon$ satisfies

$$\mathscr{H}_n(Q_\varepsilon) = v + \varepsilon, \quad \mathscr{H}_{n-1}(\partial Q_\varepsilon \cap \mathring{\Omega}) = a(v) + C\varepsilon^q.$$

Thus

$$a(v + \varepsilon) \leqq a(v) + C\varepsilon^q, \tag{3.1}$$

and, since $a(0) = 0$, we have Hölder continuity with exponent q at $v = 0$.

Next, for $v \in (0, 1]$, (3.1) applied at $1 - v$ in conjunction with the last of (2.10) yield

$$a(v - \varepsilon) \leqq a(v) + C\varepsilon^q.$$

Hence we have Hölder continuity with exponent q also at $v = 1$, and, given any $v \in (0, 1)$, (3.1) must hold for all sufficiently small ε, positive or negative.

Next, each solution Γ of $\mathscr{O}_v$ has $\mathscr{H}_n(\Gamma) = v$ and $\partial\Gamma \cap \mathring{\Omega}$ a C^∞ surface. Thus given any $v_0 \in (0, 1]$ there is a $\delta > 0$ such that, for $v_0 \leqq v \leqq 1$, each solution Γ of $\mathscr{O}_v$ contains in its interior a ball $B = B(\Gamma)$ with $\mathscr{H}_n(B) = \delta$. Thus, given $v \in (0, 1)$, for all sufficiently small $\varepsilon > 0$ some solution $\Gamma_{v+\varepsilon}$ of $\mathscr{O}_{v+\varepsilon}$ contains in its interior a ball B_ε with $\mathscr{H}_n(B_\varepsilon) = \varepsilon$. Taking $Q_\varepsilon = \Gamma_{v+\varepsilon} \setminus B_\varepsilon$ yields

$$\mathscr{H}_n(Q_\varepsilon) = v, \quad \mathscr{H}_{n-1}(\partial Q_\varepsilon \cap \mathring{\Omega}) = a(v + \varepsilon) + C\varepsilon^q.$$

Therefore

$$a(v) \leqq a(v + \varepsilon) + C\varepsilon^q$$

and, by replacing v by $1 - v$, we see that this inequality holds for all small ε, positive or negative. This fact with (3.1) yields the desired degree of Holder continuity on $(0, 1)$.

To prove[1] (iii) note first that the Sobolev-Gagliardo-Nirenberg inequality[2] implies the existence of a constant $C_0 = C_0(\Omega, n) > 0$ such that

$$\left(\int_\Omega |f(x) - \bar{f}|^{\frac{1}{q}}\, dx\right)^q \leqq C_0 \int_\Omega |\nabla f(x)|\, dx$$

for any $f \in BV(\Omega)$, where

$$q = \frac{n-1}{n}, \quad \bar{f} = \int_\Omega f(x)\, dx.$$

[1] This proof is due to R. KOHN (private communication).

[2] *Cf.*, *e.g.*, MEYERS & ZIEMER [1977].

(Recall that $\mathscr{H}_n(\Omega) = 1$.) Further, if we choose f to be the characteristic function of a solution Γ of the minimal-surface problem $\mathscr{O}_v$, then

$$\int_\Omega f(x)\,dx = \mathscr{H}_n(\Gamma) = v,$$

$$\int_\Omega |\nabla f(x)|\,dx = a(v),$$

$$\left(\int_\Omega |f(x) - \bar{f}|^{\frac{1}{q}}\,dx\right)^q = \left[v(1-v)^{\frac{1}{q}} + v^{\frac{1}{q}}(1-v)\right]^q \leqq C_0\,a(v),$$

which yields the desired inequality in (iii). □

Our proof of the remaining results will proceed as a series of lemmas.

Let us agree to use the terms: ***m-admissible field*** for an admissible field that satisfies the constraint (2.8); ***phasewise-constant field*** for an admissible field ϱ that is constant on $\Omega_1(\varrho)$ and on $\Omega_2(\varrho)$. Consider the problem:

$\mathscr{P}^*$ *minimize $E(\varrho)$ over all m-admissible phasewise-constant fields τ which have minimal interface.*

Lemma 3.1. *ϱ solves $\mathscr{P}$ $\Leftrightarrow$ ϱ solves $\mathscr{P}^*$.*

Proof. Our first step will be to show that:

> given any m-admissible field γ which is not phasewise constant there is an m-admissible, phasewise-constant field ϱ with $E(\varrho) < E(\gamma)$. (3.2)

Thus let γ be as in (3.2), write $\Omega_i = \Omega_i(\gamma)$, let

$$m_i = v_i(\gamma)^{-1} \int_{\Omega_i} \gamma(x)\,dx, \tag{3.3}$$

and take

$$\varrho(x) = \begin{cases} m_1, & x \in \Omega_1 \\ m_2, & x \in \Omega_2. \end{cases}$$

Then ϱ is m-admissible, since γ is m-admissible, and

$$I(\varrho) = I(\gamma).$$

Further, by (3.3) and the convexity of ψ on $(\varkappa_1, \varrho_1)$,

$$\int_{\Omega_1} [\psi(\gamma) - \psi(\varrho)] = \int_{\Omega_1} [\psi(\gamma) - \psi(m_1) - \psi'(m_1)(\gamma - m_1)] \geqq 0.$$

Similarly,

$$\int_{\Omega_2} [\psi(\gamma) - \psi(\varrho)] \geqq 0.$$

Moreover, since γ is not phasewise constant, one of the above inequalities is strict, and $E(\varrho) < E(\gamma)$. Thus (3.2) is valid.

Let ϱ be m-admissible and phasewise constant. If, in addition, ϱ is two-phase, then

$$\varrho(x) = \begin{cases} \tau_1, & x \in \Omega_1(\varrho) \\ \tau_2, & x \in \Omega_2(\varrho) \end{cases} \tag{3.4a}$$

with

$$\varkappa_1 < \tau_1 < \varrho_1, \quad \varrho_2 < \tau_2 < \varkappa_2.$$

Further, by (2.1), (2.6), and (2.8),

$$v_1(\varrho)\,\tau_1 + v_2(\varrho)\,\tau_2 = m,$$

$$v_1(\varrho) + v_2(\varrho) = 1;$$

hence

$$\begin{aligned} v_1(\varrho) &= \frac{\tau_2 - m}{\tau_2 - \tau_1} =: v_1(\tau_1, \tau_2), \\ v_2(\varrho) &= \frac{m - \tau_1}{\tau_2 - \tau_1} =: v_2(\tau_1, \tau_2), \end{aligned} \tag{3.4b}$$

and, since $v_i(\varrho) > 0$,

$$\tau_1 < m < \tau_2. \tag{3.5}$$

On the other hand, if ϱ is single-phase, then, necessarily, $\varrho(x) \equiv m$ with $m \in (\alpha_0, \varrho_1)$ or $m \in (\varrho_2, \beta_0)$. In the former case $v_1(\varrho) = 1$, $v_2(\varrho) = 0$, and ϱ trivially has the form (3.4) with $\tau_1 = m$, τ_2 arbitrary. Similarly, for $m \in (\varrho_2, \beta_0)$ we take τ_1 abritrary, $\tau_2 = m$. Thus, letting

$$T = \{(\tau_1, \tau_2) \colon \varkappa_1 < \tau_1 < \varrho_1, \varrho_2 < \tau_2 < \varkappa_2, \tau_1 \leqq m \leqq \tau_2\},$$

we have the following result:

$$\left.\begin{array}{l} \text{every } m\text{-admissible, phasewise-constant field } \varrho \text{ admits} \\ \text{the representation (3.4) with } (\tau_1, \tau_2) \in T. \end{array}\right\} \tag{3.6}$$

Note further that, for such ϱ, the energy (2.7) may be written as

$$E(\varrho) = \psi(\tau_1, \tau_2) + \sigma I(\varrho) \tag{3.7}$$

with

$$\psi(\tau_1, \tau_2) := v_1(\tau_1, \tau_2)\,\psi(\tau_1) + v_2(\tau_1, \tau_2)\,\psi(\tau_2). \tag{3.8}$$

Next, for each $(\tau_1, \tau_2) \in T$, let $\mathscr{A}(\tau_1, \tau_2)$ designate the class of all m-admissible fields of the form (3.4a), so that (3.4b) and (3.7) hold. Clearly, $\mathscr{A}(\tau_1, \tau_2)$ contains at least one ϱ with minimal interface. Further, since $v_1(\varrho)$ does not vary over $\mathscr{A}(\tau_1, \tau_2)$, (3.7) and the remark containing (2.11) imply that:

$$\left.\begin{array}{l} \text{the global minimum of } E \text{ over } \mathscr{A}(\tau_1, \tau_2) \text{ is assumed at, and} \\ \text{only at, those } \varrho \text{ with minimal interface.} \end{array}\right\} \tag{3.9}$$

The results (3.2) and (3.9) together yield the desired conclusion. □

Lemma 3.1 allows us to confine our attention to minimal interface fields ϱ of the form (3.4). On such ϱ

$$I(\varrho) = a(v_1(\varrho)) = a(v_2(\varrho)), \tag{3.10}$$

with a the minimal-interface measure, and the energy (3.7) reduces to a function $e(\tau_1, \tau_2)$ of (τ_1, τ_2):

$$E(\varrho) = \psi(\tau_1, \tau_2) + \sigma a(v_1(\tau_1, \tau_2)) =: e(\tau_1, \tau_2). \tag{3.11}$$

Thus we are led to problem

$$(\mathscr{P}_e) \quad \textit{minimize } e \textit{ over } T.$$

In fact, we have

Lemma 3.2. *Problem $\mathscr{P}$ reduces to problem $\mathscr{P}_e$:*

(i) *If ϱ solves $\mathscr{P}$, then ϱ has the form* (3.4) *with (τ_1, τ_2) a solution of $\mathscr{P}_e$.*

(ii) *If (τ_1, τ_2) solves $\mathscr{P}_e$, then any minimal-interface field ϱ of the form* (3.4) *solves $\mathscr{P}$.*

Remark. The proofs of Theorems 2.3 and 2.4 are quite simple for the special case in which $\psi(\varrho)$ is infinite at the end points $\varkappa_1, \varrho_1, \varrho_2, \varkappa_2$. In this instance $e(\tau_1, \tau_2)$ is lower semi-continuous, so that $\mathscr{P}_e$ has a solution. Further, for any two-phase solution (τ_1, τ_2), we write $c_i = v_i(\tau_1, \tau_2)^{-1}$, note that $v_i(\tau_1 + c_1\varepsilon, \tau_2 - c_2\varepsilon)$ is independent of ε, and equate to zero the derivative, at $\varepsilon = 0$, of $e(\tau_1 + c_1\varepsilon, \tau_2 - c_2\varepsilon)$. This leads to $\psi'(\tau_1) = \psi'(\tau_2)$ and, subsequently, to the results in Theorem 2.4.

We now divide the set $(\varkappa_1, \varrho_1)\times(\varrho_2, \varkappa_2)$ into three sets:

$$\begin{aligned} \mathscr{D}_{12} &= \{(\tau_1, \tau_2)\colon \psi'(\tau_1) > \psi'(\tau_2)\}, \\ \mathscr{D}_{21} &= \{(\tau_1, \tau_2)\colon \psi'(\tau_2) > \psi'(\tau_1)\}, \\ \Sigma &= \{(\tau_1, \tau_2)\colon \psi'(\tau_1) = \psi'(\tau_2)\}. \end{aligned} \tag{3.12}$$

Note that, by the discussion leading to (2.3),

$$\Sigma = \{(\alpha(\mu), \beta(\mu))\colon \underline{\mu} < \mu < \bar{\mu}\}. \tag{3.13}$$

Further, since $\psi'' > 0$, if we differentiate (2.3) we find that

$$\dot{\alpha} > 0, \quad \dot{\beta} > 0, \tag{3.14}$$

where we have used the notation

$$\dot{f} = \frac{df}{d\mu}. \tag{3.15}$$

Thus, on Σ, τ_2 increases with increasing τ_1.

Our next step will be to analyze $\psi(\tau_1, \tau_2)$ on certain line segments in T on which $v_i(\tau_1, \tau_2)$ have constant values. This will facilitate our study of $e(\tau_1, \tau_2)$, since the interfacial energy $\sigma a(v_1(\tau_1, \tau_2))$ is invariant on such segments.

Choose $(\lambda_1, \lambda_2) \in \mathring{T}$ and consider the line segment l consisting of all $(\tau_1, \tau_2) \in T$ of the form

$$\tau_2 - \lambda_2 = -k(\tau_1 - \lambda_1) \tag{3.16}$$

with

$$k = \frac{v_1(\lambda_1, \lambda_2)}{v_2(\lambda_1, \lambda_2)}.$$

We will refer to l as the *v-**constant line*** through (λ_1, λ_2).

Let

$$\mathscr{M} = \{(\tau_1, \tau_2) \in T: \tau_1 = m \text{ or } \tau_2 = m\},$$

so that $\mathscr{M}$ is vertical for $m \in (\alpha_0, \varrho_1)$, horizontal for $m \in (\varrho_1, \beta_0)$, and empty for $m \in [\varrho_1, \varrho_2]$. Further

$$\mathscr{M} \cap \Sigma \neq \emptyset \quad \text{for } \mathscr{M} \neq \emptyset. \tag{3.17}$$

Lemma 3.3.

(i) *There is exactly one v-constant line through any given point of* $\mathring{T}$.

(ii) *The v-constant lines do not intersect* $\mathscr{M}$.

(iii) *Let l be a v-constant line. Then* $e(\tau_1, \tau_2)$*—restricted to l—strictly decreases with* τ_1 *on* $l \cap \mathscr{D}_{21}$, *strictly increases with* τ_1 *on* $l \cap \mathscr{D}_{12}$, *and, if* $l \cap \Sigma \neq \emptyset$, *has a global minimum at* $l \cap \Sigma$.

(iv) $e(\tau_1, \tau_2) \equiv \psi(m)$ *on* $\mathscr{M}$.

Proof. Assertion (i) is trivial, while (ii) follows from the fact that one of $v_i(\tau_1, \tau_2)$ vanishes on $\mathscr{M}$. To verify (iii), let l be the v-constant line (3.16) and write $\mathring{v}_i = v_i(\lambda_1, \lambda_2)$. Then

$$\tau_1 \mathring{v}_1 + \tau_2 \mathring{v}_2 = \lambda_1 \mathring{v}_1 + \lambda_2 \mathring{v}_2 = m;$$

hence

$$v_i(\tau_1, \tau_2) = \mathring{v}_i \tag{3.18}$$

for all $(\tau_1, \tau_2) \in l$.

Next, $\psi(\tau_1, \tau_2)$—restricted to l and considered as a function of τ_1—has the form

$$\psi_l(\tau_1) = \psi(\tau_1, f(\tau_1)), \quad f(\tau_1) = \lambda_2 - \varkappa(\tau_1 - \lambda_1).$$

Therefore, using (3.8) and (3.18),

$$\frac{d}{d\tau_1} \psi_l(\tau_1) = \mathring{v}_1[\psi'(\tau_1) - \psi'(\tau_2)], \quad \tau_2 = f(\tau_1),$$

and, by (3.12), (iii) holds with e replaced by ψ. In view of (3.11) and (3.18), this, in turn, implies (iii) for e itself.

Finally, (2.10), (3.4b), and (3.11) yield (iv). □

The proof of the next lemma will make use of the inequality

$$e(\alpha_0, \beta_0) \leqq \sigma a_M, \tag{3.19}$$

which follows from (2.10), (3.4b), and (3.11).

Lemma 3.4. *$\mathscr{P}_e$ has a solution, and all solutions lie on $(\Sigma \cap T) \cap \mathscr{M}$.*

Proof. By (3.14) and the properties of ψ,

$$f(\mu) := \min \{\psi(\alpha(\mu)), \psi(\beta(\mu))\}$$

is strictly decreasing on $(\underline{\mu}, 0)$, strictly increasing on $(0, \bar{\mu})$, and zero at $\mu = 0$. We may therefore use (A_2) to establish the existence of numbers $\underline{u} \in (\underline{\mu}, 0)$, $\bar{u} \in (0, \bar{\mu})$ such that

$$f(\underline{u}) = f(\bar{u}) = \sigma a_M.$$

Let

$$\underline{\alpha} = \alpha(\underline{u}), \quad \underline{\beta} = \beta(\underline{u}), \quad \bar{\alpha} = \alpha(\bar{u}), \quad \bar{\beta} = \beta(\bar{u})$$

and write

$$\Sigma^* = \{(\tau_1, \tau_2) \in \Sigma \cap T : \underline{\alpha} \leqq \tau_1 \leqq \bar{\alpha}\}.$$

We will show that: given any $(\tau_1, \tau_2) \in T \setminus (\Sigma^* \cup \mathscr{M})$, there is a point $(\mathring{\tau}_1, \mathring{\tau}_2) \in \Sigma^*$ such that

$$e(\mathring{\tau}_1, \mathring{\tau}_2) < e(\tau_1, \tau_2). \tag{3.20}$$

Since e is constant on $\mathscr{M}$ (*cf.* (iv) of Lemma 3.3), Σ^* is compact, and e is lower semi-continuous (*cf.* (2.13)), this would imply that e have a global minimizer, and that all such global minimizers lie on $\Sigma^* \cup \mathscr{M}$; and this, in turn, would yield the validity of Lemma 3.4.

Assume that

$$m \in (\alpha_0, \bar{\alpha}]. \tag{3.21}$$

We then have the situation shown in Figure 4, where T is the region $(\varkappa_1, m] \times (\varrho_2, \varkappa_2)$.

Let

$$\mathscr{P} = (\varkappa_1, \underline{\alpha}] \times (\varrho_2, \underline{\beta}].$$

Our first step will be to show that

$$e(\lambda_1, \lambda_2) > e(\alpha_0, \beta_0) \text{ for } (\lambda_1, \lambda_2) \in \mathscr{P}. \tag{3.22}$$

Choose $(\lambda_1, \lambda_2) \in \mathscr{P}$. Since ψ decreases on $(\varkappa_1, \alpha_0)$ and on (ϱ_2, β_0),

$$\psi(\lambda_1) \geqq \psi(\underline{\alpha}), \quad \psi(\lambda_2) \geqq \psi(\underline{\beta}),$$

and we may use (3.8) and the identity $v_1 + v_2 = 1$ to conclude that

$$\psi(\lambda_1, \lambda_2) \geqq f(\underline{u}) = \sigma a_M.$$

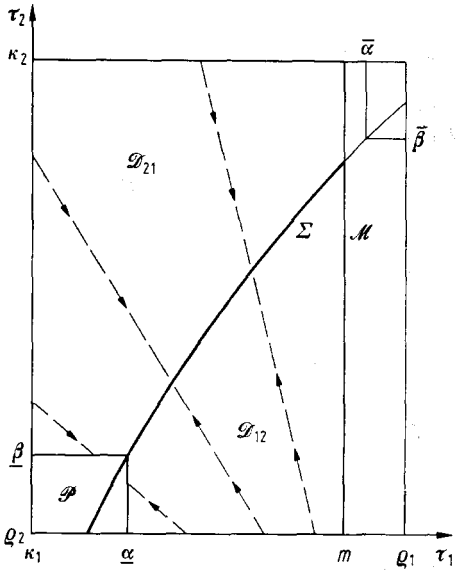

Fig. 4. The set T for the case $\alpha_0 < m < \underline{\alpha}$. The v-constant lines have arrows indicating the direction of decreasing $\psi(\tau_1, \tau_2)$

Further, since $\lambda_1, \lambda_2 \neq m$, $v_1(\lambda_1, \lambda_2) \neq 0, 1$ and $a(v_1(\lambda_1, \lambda_2)) > 0$; hence, using (3.19),

$$e(\lambda_1, \lambda_2) > \psi(\lambda_1, \lambda_2) \geqq e(\alpha_0, \beta_0),$$

which is (3.22).

Choose

$$(\tau_1, \tau_2) \in T \setminus (\Sigma^* \cup \mathscr{M}),$$

and let l designate the v-constant line through (τ_1, τ_2). If $(\tau_1, \tau_2) \in \mathscr{P}$, then (3.22) trivially implies (3.20). If $(\tau_1, \tau_2) \notin \mathscr{P}$, then one of the following is true:

(a) $l \cap \Sigma^* \neq \emptyset$;

(b) $l \cap \mathscr{P} \neq \emptyset$.

In case (a), (iii) of Lemma 3.3 implies (3.20) with $(\mathring{\tau}_1, \mathring{\tau}_2)$ the point at which l intersects Σ^*. In case (b), (3.22) and (iii) of Lemma 3.3 imply (3.20) with $(\mathring{\tau}_1, \mathring{\tau}_2) = (\alpha_0, \beta_0)$. This completes the proof for $m \in (\alpha_0, \bar{\alpha}]$. The proof for m in the intervals $(\bar{\alpha}, \varrho_1)$, $[\varrho_1, \varrho_2]$, $(\varrho_2, \underline{\beta})$, and $[\underline{\beta}, \beta_0)$ is completely analogues and can safely be omitted. □

Proof of Theorem 2.3. This follows directly from Theorem 2.2 and Lemmas 3.2, 3.4. □

Proof of Theorem 2.4. Let ϱ be a solution of $\mathscr{P}$. Then (i) is a consequence of Lemma 3.2. Further, by Lemmas 3.2 and 3.3, ϱ has the form (3.4) with $(\tau_1, \tau_2) \in \mathscr{M}$ or $(\tau_1, \tau_2) \in \Sigma \cap T$. By (3.17), and since all points on $\mathscr{M}$ correspond to the same field (the constant with value m), we may assume that $(\tau_1, \tau_2) \in \Sigma \cap T$. Therefore,

by $(3.12)_3$, ϱ has constant chemical potential, and, by (2.3) and (3.13),

$$\tau_1 = \alpha(\mu), \quad \tau_2 = \beta(\mu), \quad \mu \in (\underline{\mu}, \bar{\mu}),$$

with μ the chemical potential. Thus ϱ has the form (2.17), and $\mu \in U$. Further, for fields of the form (2.17) the energy $E(\varrho)$ reduces to

$$\mathscr{E}(\mu) := e(\alpha(\mu), \beta(\mu)),$$

which is (2.15). Comparing $(3.12)_3$ and (3.13) we see that μ must be a global minimizer of $\mathscr{E}$ over U, since (τ_1, τ_2) minimizes e over Σ. Thus (ii)–(iv) are valid, and tracing the above steps backward, we also have the converse assertion in Theorem 2.4. □

The next lemma will be useful in proving the remaining results of Section 2. There (*cf.* (2.15))

$$\begin{aligned} \Psi(\mu) &:= \psi(\alpha(\mu), \beta(\mu)) = v_1(\mu)\, \psi(\alpha(\mu)) + v_2(\mu)\, \psi(\beta(\mu)), \\ G(\mu) &:= \psi(\alpha(\mu)) - \mu\alpha(\mu), \\ g(\mu) &:= \psi(\beta(\mu)) - \mu\beta(\mu), \end{aligned} \tag{3.23}$$

and we use the notation (3.15).

Lemma 3.5. *On* $\overset{\circ}{U}$:

(i) $\dot{v}_1 = -\dot{v}_2 > 0$;

(ii) $\dot{\Psi} = (G - g)\, \dot{v}_1$;

(iii) $\Psi(\mu)$ *has a global minimum at* $\mu = 0$ *and is strictly decreasing for* $\mu < 0$, *strictly increasing for* $\mu > 0$.

Proof. By (2.15), (3.14), and the identity $v_1 + v_2 = 1$,

$$\dot{v}_1 = -\dot{v}_2 = (\beta - \alpha)^{-1} (v_1\dot{\alpha} + v_2\dot{\beta}) > 0,$$

which is (i). Next, by (2.3) and $(3.23)_{2,3}$,

$$\dot{G} = -\alpha, \quad \dot{g} = -\beta.$$

Also, since $v_1\alpha + v_2\beta = m$,

$$\Psi(\mu) = v_1(\mu)\, G(\mu) + v_2(\mu)\, g(\mu) + \mu m,$$

which leads to (ii). Finally $G(0) = g(0)$, as $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$, while

$$(G - g)^{\cdot} = \beta - \alpha > 0.$$

Thus

$$G(\mu) - g(\mu) \begin{cases} < 0 & \text{for } \mu < 0 \\ > 0 & \text{for } \mu > 0, \end{cases}$$

and, since $\dot{v}_1 > 0$, (iii) follows from (ii). □

Note that, by (2.15) and $(3.23)_1$,

$$\mathscr{E}(\mu) = \Psi(\mu) + \sigma a(v_1(\mu)). \tag{3.24}$$

Proof of Corollary 1. Since ϱ is two-phase, $\alpha \in \overset{\circ}{U}$ and

$$\dot{\mathscr{E}}(\mu) = 0. \tag{3.25}$$

In view of (3.24) and (i), (ii) of Lemma 3.5, this implies (2.19). □

In proving the remaining results of Section 2 it is convenient to make explicit the dependence of $\mathscr{P}$, $\mathscr{E}$, Ψ, U, and v_i on m by writing

$$\mathscr{P} = \mathscr{P}(m), \quad \mathscr{E}(\mu) = \mathscr{E}(\mu, m), \quad \Psi(\mu) = \Psi(\mu, m), \quad U = U(m), \quad v_i(\mu) = v_i(\mu, m).$$

By (2.16) the functions of (μ, m) have domain

$$\{(\mu, m): \underline{\mu} < \mu < \bar{\mu}, \quad \alpha(\mu) \leqq m \leqq \beta(\mu)\}. \tag{3.26}$$

Note that, by (ii) of Lemma 3.5 and the fact that $\Psi(\mu, m) = \psi(m)$ for $m = \alpha(\mu)$ or $m = \beta(\mu)$,

$$\left.\begin{array}{l}\Psi(\mu, m) \text{ is bounded away from zero on any subset of (3.26)} \\ \text{with } \mu \text{ bounded away from zero.}\end{array}\right\} \tag{3.27}$$

Proof of Corollary 2. By (3.25) and $(2.15)_{2,3}$,

$$\frac{\partial \mathscr{E}(\mu, m)}{\partial \mu} = 0 \quad \text{at } \mu = \mu_m,$$

$$\frac{\partial v_1(\mu, m)}{\partial m} = -\frac{\partial v_2(\mu, m)}{\partial m} = -[\beta(\mu) - \alpha(\mu)]^{-1}.$$

Thus, since

$$E_m = \mathscr{E}(\mu_m, m),$$

if we write $\mu_m = \alpha(\mu_m)$, $\beta_m = \beta(\mu_m)$ and use $(2.15)_1$ and (2.19), we find that

$$\begin{aligned}\frac{dE_m}{dm} &= \left.\frac{\partial \mathscr{E}(\mu, m)}{\partial m}\right|_{\mu=\mu_m}, \\ &= -(\beta_m - \alpha_m)^{-1}\,[\psi(\alpha_m) - \psi(\beta_m) + \sigma a'(v_1(\mu_m, m))], \\ &= \mu_m. \quad \square\end{aligned}$$

Proof of Theorem 2.5. By $(3.23)_1$ and the identity $v_1 = 1 - v_2$,

$$\Psi(\mu, m) \geqq [1 - v_2(\mu, m)]\,\psi(\alpha(\mu)).$$

Thus and by $(2.15)_3$, (3.24), we are led to the inequality

$$\mathscr{E}(\mu, m) - \psi(m) \geqq v_2 \left[-(\beta - \alpha)\frac{\psi(m) - \psi(\alpha)}{m - \alpha} - \psi(\alpha) + \sigma \frac{a(v_2)}{v_2}\right] \tag{3.28}$$

for $v_2 > 0$, where we have written $\alpha = \alpha(\mu)$, $\beta = \beta(\mu)$, $v_2 = v_2(\mu, m)$).

Suppose that there exists a sequence $\{m_k\}$, $m_k \to \alpha_0$ as $k \to \infty$, such that $\mathscr{P}(m_k)$ has a two-phase solution ϱ_k for each k. Let μ_k be the chemical potential of ϱ_k. Then, since ϱ_k must have lower energy than the single-phase solution, $\varrho \equiv m_k$,

$$0 \leqq \Psi(\mu_k, m_k) \leqq \Psi(m_k).$$

But $\psi(m_k) \to 0$ as $k \to \infty$, since $m_k \to \alpha_0$; hence $\Psi(\mu_k, m_k) \to 0$, which with (3.27) lead to the conclusion that $\mu_k \to 0$. Thus $\alpha(\mu_k) \to \alpha_0$ and $(2.15)_3$ implies $v_2(\mu_k, m_k) \to 0$; hence (iii) of Theorem 2.2 yields

$$\frac{a(v_2(\mu_k, m_k))}{v_2(\mu_k, m_k))} \to \infty.$$

The other terms on (3.28)—evaluated at (μ_k, m_k)—have finite limits as $k \to \infty$. Thus

$$\psi(m_k) < \mathscr{E}(\mu_k, m_k)$$

for k sufficiently large, which contradicts the assumption that ϱ_k solve $\mathscr{P}(m_k)$. Thus there exists a $\delta_1 > 0$ with no two-phase solutions of $\mathscr{P}(m)$ for $m \in (\alpha_0, a_0 + \delta_1]$. A similar argument yields the nonexistence of two-phase solutions for $m \in [\beta_0 - \delta_2, \delta_0)$, for some $\delta_2 > 0$. □

4. Solution in $\mathbf{R}^1$

When $n = 1$ problem $\mathscr{P}$ is so simple we need not assume σ small. In view of Lemma 3.3 and the subsequent remark, $\mathscr{P}$ consists in minimizing $e(\tau_1, \tau_2)$ over T, where

$$e(\tau_1, \tau_2) = \begin{cases} \psi(\tau_1, \tau_2) + \sigma, & \tau_1, \tau_2 \neq m \\ \psi(m), & \tau_1 = m \text{ or } \tau_2 = m \end{cases}$$

(*cf.* $(2.10)_1$, (3.11), (i) of Theorem 2.2, and (iv) of Lemma 3.3). The minimum value of $\psi(\tau_1, \tau_2) + \sigma$ is σ at $(\tau_1, \tau_2) = (\alpha_0, \beta_0)$. Thus

(i) for $m \in (\alpha_0, \varrho_1) \cup (\varrho_2, \beta_0)$ the solutions are
  (a) the single-phase field $\varrho(x) \equiv m$ if $\psi(m) \leqq \sigma$,
  (b) the two-phase fields corresponding to (α_0, β_0) if $\psi(m) \geqq \sigma$;

(ii) for $m \in [\varrho_1, \varrho_2]$ the solutions are
  (a) the two-phase fields corresponding to (α_0, β_0).

Here for

$$\Omega = (0, 1)$$

the two-phase fields corresponding to (α_0, β_0) are

$$\varrho(x) = \begin{cases} \alpha_0, & 0 < x < s \\ \beta_0, & s < x < 1, \end{cases}$$

$$s = \frac{\beta_0 - m}{\beta_0 - \alpha_0},$$

and $\varrho(1 - x)$.

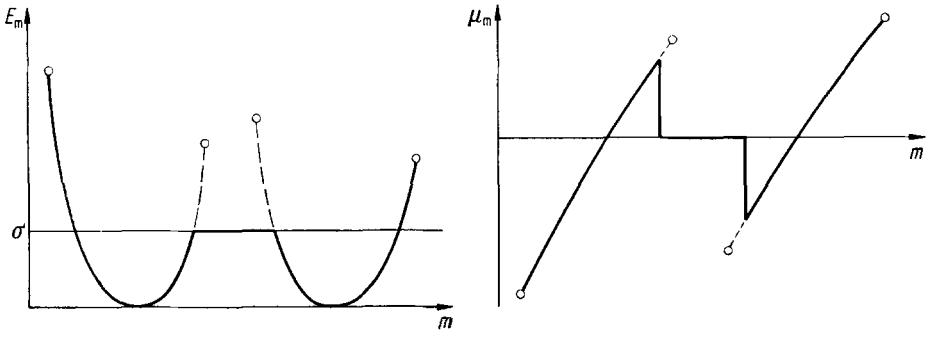

Fig. 5. The energy E_m and chemical potential μ_m (solid lines) of the global minimizer as functions of m

Figure 5 shows the energy E_m and chemical potential μ_m of the global minimizers as functions of m, for the case in which $\psi(\varkappa_i), \psi(\varrho_i) > \sigma$.

5. A simple example in $\mathbb{R}^2$

Let Ω be the unit square

$$\Omega = (0, 1)\times(0, 1),$$

so that the minimal-interface measure $a(v)$ is given by (2.12).

For the free energy we take the piecewise-quadratic function (Figure 6)

$$\psi(\varrho) = \begin{cases} \dfrac{\varrho^2}{2}, & -\frac{1}{4} < \varrho < \frac{1}{4} \\ \psi(\varrho - 1), & \frac{3}{4} < \varrho < \frac{5}{4}. \end{cases} \tag{5.1}$$

Thus

$$\varkappa_1 = -\tfrac{1}{4}, \quad \varrho_1 = \tfrac{1}{4}, \quad \varrho_2 = \tfrac{3}{4}, \quad \varkappa_2 = \tfrac{5}{4},$$

while[1]

$$\begin{aligned} \alpha(\mu) &= \mu, \quad \beta(\mu) = 1 + \mu, \\ v_1(\mu, m) &= 1 + \mu - m, \quad v_2(\mu, m) = m - \mu, \end{aligned} \tag{5.2}$$

so that

$$\Psi(\mu) = \frac{\mu^2}{2} \tag{5.3}$$

with domain

$$U(m) = (-\tfrac{1}{4}, \tfrac{1}{4}) \cap [m - 1, m]. \tag{5.4}$$

[1] Here, to make our discussion unambiguous, we shall use the notation (3.26) in which dependences on m are made explicit.

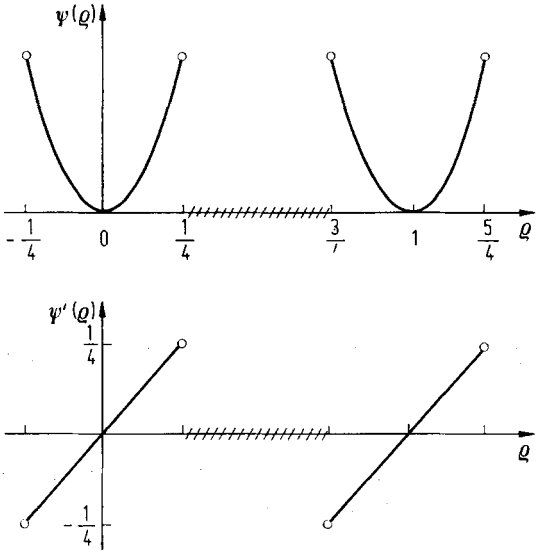

Fig. 6. Free energy $\psi(\varrho)$ and $\psi'(\varrho)$ for the example of Section 5

We shall assume that σ is consistent with (A_2), which in the present circumstances reduces to

$$\sigma < 1/32. \tag{5.5}$$

We shall restrict our attention to $m \in (0, 1) = (\alpha_0, \beta_0)$, since we know the form of the solution otherwise (Theorem 2.1). Further, the symmetry of ψ tells us that: if $\varrho(x)$ solves $\mathscr{P}(m)$, then $1 - \varrho(x)$ solves $\mathscr{P}(1 - m)$. Thus we may further restrict our attention to

$$0 < m \leqq \tfrac{1}{2}.$$

Next, to write an explitit relation for $a(v_2(\mu, m))$ we note that, by (2.10) and (2.12), $a(v)$ is C^1 on each of the intervals $(0, 1/\pi)$, $(1/\pi, 1 - 1/\pi)$, $(1 - 1/\pi, 1)$, but not differentiable across their mutual boundaries. Labeling these intervals by V_1, V_2, V_3 respectively, it is clear that $v_2(\mu, m) \in V_i$ if and only if $(\mu, m) \in A_i$, where (Figure 7)

$$\begin{aligned} A_1 &= \{(\mu, m) \in A : \mu \in [m - \pi^{-1}, m]\}, \\ A_2 &= \{(\mu, m) \in A : \mu \in [m + \pi^{-1} - 1, m - \pi^{-1}]\}, \\ A_3 &= \{(\mu, m) \in A : \mu \in [m - 1, m + \pi^{-1} - 1]\}, \\ A &= (-\tfrac{1}{4}, \tfrac{1}{4}) \times (0, \tfrac{1}{2})]. \end{aligned} \tag{5.6}$$

Thus $a(\mu, m) = a(v_2(\mu, m))$ is given by

$$a(\mu, m) = \begin{cases} [\pi(m - \mu)]^{\frac{1}{2}}, & (\mu, m) \in A_1 \\ 1, & (\mu, m) \in A_2 \\ [\pi(1 - m + \mu)]^{\frac{1}{2}}, & (\mu, m) \in A_3. \end{cases} \tag{5.7}$$

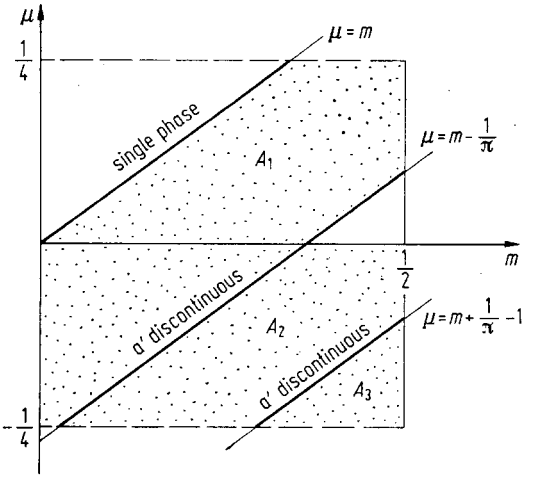

Fig. 7. The sets A_i

By Theorem 2.4, (for each fixed m) problem $\mathscr{P}(m)$ reduces to minimizing $\mathscr{E}(\mu, m)$ over $U(m)$, where, by (5.3),

$$\mathscr{E}(\mu, m) = \tfrac{1}{2}\mu^2 + \sigma a(\mu, m). \tag{5.8}$$

Our first step will be to show that

$$\text{for any } m \in (0, \tfrac{1}{2}], \text{ the global minimizer necessarily has } \mu \geqq 0. \tag{5.9}$$

To see this note that $a(\mu, m)$ is constant in A_2 and strictly increases with decreasing μ in A_1; thus, by (5.8), $\mathscr{E}(\mu, m)$, for m fixed, cannot have a minimum in A_1 or A_2. The only other possibility is a point (μ, m) in the interior of A_3, and this, in turn, would require that $\dot{\mathscr{E}}(\mu, m) := (\partial/\partial\mu)\,\mathscr{E}(\mu, m) = 0$ at that m. If, in (5.8), we use the expression for $a(\mu, m)$ appropriate to A_3, and set $\dot{\mathscr{E}}(\mu, m) = 0$, we find that

$$\mu = -\frac{\sigma\sqrt{\pi}}{2}(1 - m + \mu)^{-\frac{1}{2}}. \tag{5.10}$$

But in A_3,

$$-\tfrac{1}{2} + \pi^{-1} \geqq \mu, \quad 1 - m + \mu \geqq \tfrac{1}{4};$$

hence (5.5) and (5.10) yield the contradictionary inequalities

$$\tfrac{1}{2} - \pi^{-1} \leqq \sigma\sqrt{\pi} < \frac{\sqrt{\pi}}{32}.$$

Thus (5.9) is valid.

We may therefore restrict our search for global minimizers to those portions of A_1 and A_2 that lie in $\mu \geqq 0$. We begin by determining the points of A_1 at which $\dot{\mathscr{E}} = 0$. Within A_1,

$$\mathscr{E}(\mu, m) = \tfrac{1}{2}\mu^2 + \sigma[\pi(m - \mu)]^{\frac{1}{2}}, \tag{5.11}$$

and $\dot{\mathscr{E}}(\mu, m) = 0$ on the curve

$$\mathscr{C} := \left\{(\mu, m) \in A_1 : m = \mu + \frac{4\lambda^3}{\mu^2}\right\}, \tag{5.12}$$

where

$$\lambda := \left(\frac{\sqrt{\pi}\,\sigma}{4}\right)^{\frac{2}{3}}.$$

The curve $\mathscr{C}$ is asymptotic to $m = \mu$ and $m = 4\lambda^3/\mu^2$ and has a minimum value of m at

$$\mu = 2\lambda, \quad m = 3\lambda.$$

Further,

$$\dot{\mathscr{E}}(\mu, m) < 0 \quad \text{for } m < \mu + \frac{4\lambda^3}{\mu^2},$$

$$\dot{\mathscr{E}}(\mu, m) > 0 \quad \text{for } m > \mu + \frac{4\lambda^3}{\mu^2}.$$

Finally, on A_2,

$$\mathscr{E}(\mu, m) = \tfrac{1}{2}\mu^2 + \sigma$$

and $\dot{\mathscr{E}}(\mu, m) > 0$ for $\mu > 0$.

Thus we have the situation shown[1] in Figure 8, and it is clear that the only candidates for global minimizers are:

(i) the line $\mu = m$ (with $\mu < \frac{1}{4}$),
(ii) the portion of the curve $\mathscr{C}$ that lies below $\mu = 2\lambda$,
(iii) the portion of the m-axis between $m = 1/\pi$ and $m = \frac{1}{2}$.

Thus for each value of m we have only to see which of these curves has lowest energy.

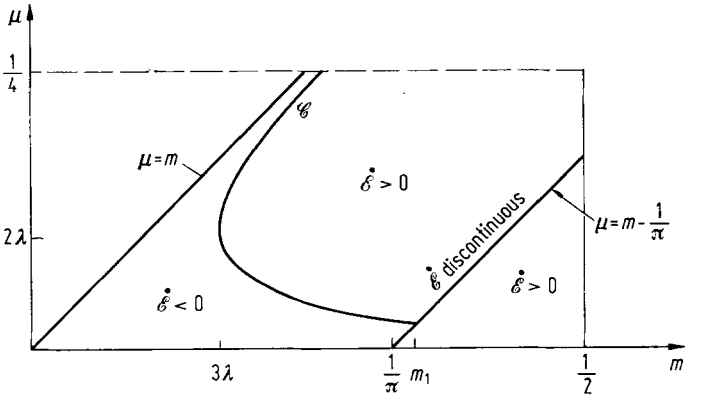

Fig. 8. The behavior of $\dot{\mathscr{E}}$

[1] Assumption (5.5) ensures that $3\lambda < \pi^{-1}$ and that $\mathscr{C}$ intersects the line $\mu = m - \pi^{-1}$.

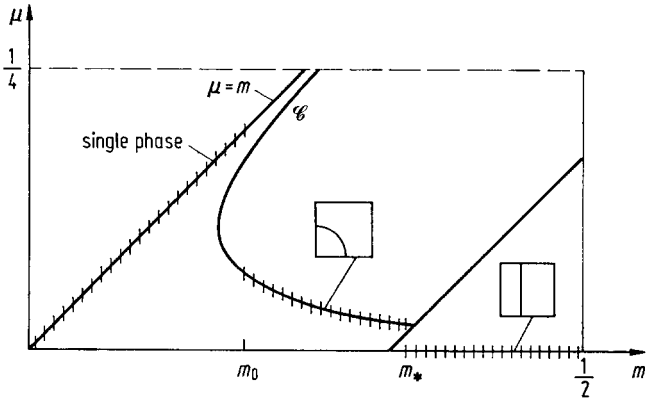

Fig. 9. The chemical potential which globally minimizes $\mathscr{P}$ consists of the cross-hatched curves

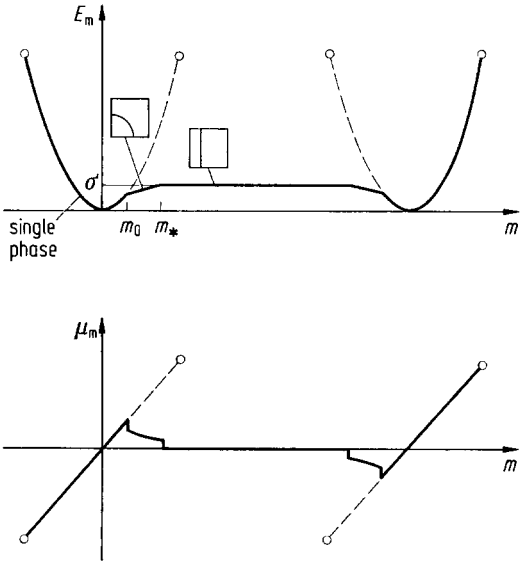

Fig. 10. The energy E_m and chemical potential μ_m (solid line) of the global minimizer as functions of m

Case 1: $m \in (0, 3\lambda]$. Here it is clear that the single-phase solution $\mu = m$ has least energy.

Case 2: $m \in (3\lambda, 1/\pi]$. For this range the candidates are the lower portion of the curve $\mathscr{C}$ and the line $\mu = m$. Comparing the energy on these curves, we find, after some algebra, that $\mu = m$ has least energy for $m \in (3\lambda, m_0)$, they have equal energy for $m = m_0$, and the lower portion of $\mathscr{C}$ has least energy for $m \in (m_0, 1/\pi)$, where

$$m_0 = 2^{\frac{1}{3}} 3\lambda.$$

Case 3: $m \in (1/\pi, m_1)$, where

$$m_1 = \pi^{-1} + 2\lambda\sqrt{\pi}$$

is the value of m at which $\mathscr{C}$ intersects the line $\mu = m - \pi^{-1}$ (Figure 8). Here the candidates are the lower part of $\mathscr{C}$ and the m-axis; one can show that $\mathscr{C}$ has lower energy for $m \in (1/\pi, m_*)$, the m-axis has lower energy for $m \in (m_*, m_1)$, and they have equal energy at $m = m_*$, where $m_* \in (1/\pi, m_1)$ is the solution of a complicated equation.

Case 4: $m \in [m_1, \frac{1}{2}]$. Here the m-axis obviously has least energy.

Thus we have found the global minimizer of $\mathscr{E}$ for each choice of the parameter m; the result is shown in Figure 9. The corresponding minimal energy is shown in Figure 10.

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