A Universal Connexion for Waves in Anisotropic Media

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1. Introduction

This note is concerned with the propagation of waves in a variety of media – perfectly elastic, elastic, thermo-elastic and visco-elastic. For each of these media it is shown that there is a very simple yet quite general connexion existing between the speeds of the waves which may propagate. This connexion is also universal, in the sense that it is independent of the symmetry of the material, so that the same connexion holds for all materials of the given class.

Of course, such connexions are of value to the experimentalist. For, he may use them as a check on his measured data. Or, he may test them in a material without knowing the values of the parameters which describe that material. If the experimentalist finds for a particular material that this universal connexion is not satisfied then he may conclude that the constitutive equation used as a basis in the derivation of the universal connexion does not adequately describe the deformations of the particular material.

Previously [1, 2] universal connexions have been obtained for waves in finitely deformed isotropic elastic and viscoelastic solids. These connexions have related to principal waves. There is no such limitation on the connexion derived here nor is it restricted to isotropic materials.

It is of interest to note that in general the universal connexion of the type derived here is not valid for incompressible materials, and hence *a fortiori* is not valid, in general, for internally constrained media.

2. Classical Linear Elasticity

First, homogeneous anisotropic elastic bodies are considered and the universal connexion derived in some detail. Then the bases for the result in other media are briefly sketched.

In the classical linear theory of homogeneous elastic bodies the stress-deformation relation takes the form

$$\sigma_{ij} = c_{ijkl}\,\partial u_k/\partial X_l, \qquad (1)$$

all quantities being referred to a fixed rectangular Cartesian coordinate system x. In this system the components of the (symmetric) Cauchy stress tensor are denoted by σ_{ij} and the components of the constant elasticity tensor by c_{ijkl}. These constants satisfy

$$c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}. \tag{2}$$

The displacement $\boldsymbol{u}$ has components u_i given by

$$\boldsymbol{u}=\boldsymbol{x}-\boldsymbol{X}, \quad u_i(\boldsymbol{X},t)=x_i-X_i, \tag{3}$$

where $\boldsymbol{x}$ is the position vector at time t of the particle initially at $\boldsymbol{X}$. The equations of motion are

$$\partial\sigma_{ij}/\partial X_j=\rho\,\partial^2 u_i/\partial t^2, \tag{4}$$

or, using (1) give

$$c_{ijkl}\,\partial^2 u_k/\partial X_l\,\partial X_j=\rho\,\partial^2 u_i/\partial t^2, \tag{5}$$

where ρ is the density, and body forces are neglected.

The displacement

$$u_i=A_i\exp\imath(N_jX_j-ct), \quad N_jN_j=1, \tag{6}$$

represents a plane wave polarised along $\boldsymbol{A}$, propagating in the direction $\boldsymbol{N}$ with speed c. Inserting (6) into (5) gives the propagation condition:

$$(c_{ijkl}N_jN_l-\rho c^2\delta_{ik})A_k=0. \tag{7}$$

Hence in order that $\boldsymbol{A}$ is not zero trivially, the secular equation

$$\det(c_{ijkl}N_jN_l-\rho c^2\delta_{ik})=0 \tag{8}$$

must be satisfied. This cubic in c^2 has three real roots [3] in view of the symmetry

$$c_{ijkl}N_jN_l=c_{kjil}N_jN_l, \tag{9}$$

which follows from (2). Moreover, the corresponding amplitudes $\boldsymbol{A}$ are orthogonal [3]. Thus, in general, three plane waves, polarised in three mutually perpendicular directions, may propagate in every direction in the material.

Denote the squared speeds of propagation of the three waves which can propagate in the direction $\boldsymbol{N}$ by $c_\alpha^2(\boldsymbol{N})$, $\alpha=1,2,3$. For any given $\boldsymbol{N}$, the sum $\sum_{\alpha=1}^{3}c_\alpha^2(\boldsymbol{N})$ may be read off from (8). In fact,

$$\rho\sum_{\alpha=1}^{3}c_\alpha^2(\boldsymbol{N})=\operatorname{trace}(c_{ijkl}N_jN_l)=c_{ijil}N_jN_l. \tag{10}$$

Now choose $\boldsymbol{M}$ and $\boldsymbol{P}$ to be two unit vectors such that $\boldsymbol{N}$, $\boldsymbol{M}$, and $\boldsymbol{P}$ form an orthogonal triad. Then

$$\rho\sum_{\alpha=1}^{3}\{c_\alpha^2(\boldsymbol{N})+c_\alpha^2(\boldsymbol{M})+c_\alpha^2(\boldsymbol{P})\}=c_{ijil}(N_jN_l+M_jM_l+P_jP_l)=c_{ijij}, \tag{11}$$

since

$$\delta_{jl}=N_jN_l+M_jM_l+P_jP_l. \tag{12}$$

The right hand side of (11) is independent of $\boldsymbol{N}$, $\boldsymbol{M}$ and $\boldsymbol{P}$. Hence if $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$ is another orthogonal triad of unit vectors, then

$$\sum_{\alpha=1}^{3}\{c_\alpha^2(\boldsymbol{N})+c_\alpha^2(\boldsymbol{M})+c_\alpha^2(\boldsymbol{P})\}=\sum_{\alpha=1}^{3}\{c_\alpha^2(\boldsymbol{R})+c_\alpha^2(\boldsymbol{S})+c_\alpha^2(\boldsymbol{T})\}. \tag{13}$$

Equation (13) gives a relation between the wave speeds corresponding to two pairs of triples of orthogonal directions. It is a universal connexion since it is not dependent upon the c_{ijkl} which determine the properties of the material. It holds for all homogeneous elastic materials, irrespective of their symmetry, whose deformations satisfy the constitutive equation (1).

Clearly, to check equation (13) the experimentalist does not need to determine the elastic constants of the material. If equation (13) is found to be invalid for a particular material it may be concluded that the deformations of the material in question do not satisfy the constitutive equation (1). On the other hand, it should be noted that if equation (13) is found to be valid experimentally for a given material it may not be concluded that the deformations of that material satisfy the constitutive equation (1). For, it has not been shown that the materials whose constitutive equation is (1) are the only ones for which the relation (13) is satisfied. In fact it will be seen in the following sections that (13) holds for a wider class of materials.

Clearly, from (13) if $(\boldsymbol{N}, \boldsymbol{M})$ is *any* pair of orthogonal directions and $(\boldsymbol{R}, \boldsymbol{S})$ is *any* other pair of orthognal directions coplanar with $(\boldsymbol{N}, \boldsymbol{M})$, there follows the universal connexion

$$\sum_{\alpha=1}^{3} \{c_\alpha^2(\boldsymbol{N})+c_\alpha^2(\boldsymbol{M})\} = \sum_{\alpha=1}^{3} \{c_\alpha^2(\boldsymbol{R})+c_\alpha^2(\boldsymbol{S})\}. \tag{14}$$

3. Small Deformations Superimposed on Large

Turning now to the theory of small deformations superimposed on large, suppose that a perfectly-elastic material of arbitrary elastic symmetry is in static equilibrium in a state of arbitrarily large static deformation. If the body is now subjected to an infinitesimal time-dependent deformation $\varepsilon \boldsymbol{u}$, where ε is a constant so small that its squares and higher powers may be neglected in comparison with first degree terms, the equations governing the superposed deformation are of the form [4, 5]

$$\frac{\partial}{\partial X_j}\left(d_{ijkl}\frac{\partial u_k}{\partial X_l}\right)=\rho\frac{\partial^2 u_i}{\partial t^2}, \tag{15}$$

in the absence of body forces. Here d_{ijkl} are functions of the large static deformation and of the strain-energy function of the material and have the symmetry [5]

$$d_{ijkl}=d_{klij}, \tag{16}$$

and ρ is the material density in the state of large deformation.

Consider first the case when the body is initially homogeneous and the large deformation is homogeneous. Then d_{ijkl} and ρ are constants and the equations (15) reduce to

$$d_{ijkl}\partial^2 u_k/\partial X_l \partial X_j=\rho\partial^2 u_i/\partial t^2, \tag{17}$$

which are of the same form as (5). If it is now assumed that u_i is given by (6), then from (16)

$$d_{ijkl}N_j N_l=d_{kjil}N_j N_l, \tag{18}$$

and the roots c^2 of the secular equation are real. The universal connexion (13) and its corollary (14) follow as in § 2. The connexion is now universal in the sense that it is valid for *any* strain-energy function and for *any* basic static large homogeneous deformation.

For homogeneously deformed homogeneous isotropic compressible elastic materials ERICKSEN* has shown that the squared speeds of propagation of the three principal longitudinal waves are

$$\rho^{-1}\,\partial t_\alpha/\partial \log \lambda_\alpha, \qquad \alpha=1,2,3, \text{ no sum}, \tag{19}$$

whilst the squared speeds of propagation of the six principal transverse waves are

$$\lambda_\alpha^2(t_\alpha-t_\beta)/\rho(\lambda_\alpha^2-\lambda_\beta^2), \qquad \alpha,\beta=1,2,3, \text{ no sum}. \tag{20}$$

Here λ_α are the principal extension ratios of the basic homogeneous deformation and t_α are the principal stresses necessary to maintain the body in the state of homogeneous deformation. It follows from (13), (19) and (20) that if $(\boldsymbol{N},\boldsymbol{M},\boldsymbol{P})$ is *any* triad of mutually orthogonal unit vectors, then

$$\rho\sum_{\alpha=1}^{3}\{c_\alpha^2(\boldsymbol{N})+c_\alpha^2(\boldsymbol{M})+c_\alpha^2(\boldsymbol{P})\}=\sum_{\alpha=1}^{3}\partial t_\alpha/\partial\log\lambda_\alpha+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}\lambda_\alpha^2(t_\alpha-t_\beta)/(\lambda_\alpha^2-\lambda_\beta^2). \tag{21}$$

It may be noted that for Cauchy-elastic bodies the symmetry relation (16) is not valid in general [6]. Hence (18) no longer holds and consequently there is no assurance that these plane waves can propagate in every direction [6]. The universal connexions (13) and (14) still hold, but clearly only make sense for those pairs and triples of directions along which three waves may propagate.

4. Acceleration Waves

Now consider the case when the d_{ijkl} are not constants but functions of position as for example when the body is inhomogeneous or when the large static deformation is inhomogeneous. Of course, in general, sinusoidal plane waves cannot propagate in the material [6], but acceleration waves may do so.

Let $\Phi(X_A,t)=0$ be the wave-front across which $\partial^2 u_i/\partial t^2$ has a discontinuity as the particle X_A is traversed, whilst u_i, $\partial u_i/\partial t$, $\partial u_i/\partial X_j$ are continuous. Then from (15), in order that an acceleration wave propagate in the material, the secular equation

$$\det\{d_{ijkl}(X)\,(\partial\Phi/\partial X_j)\,(\partial\Phi/\partial X_l)-\rho(\partial\Phi/\partial t)^2\,\delta_{ik}\}=0 \tag{22}$$

must be satisfied [7]. Note that the local speed V of the acceleration wave, *i.e.* its speed relative to the material, and the unit normal N_i to the wave-front are given by [8]

$$\begin{aligned} V&=-(\partial\Phi/\partial t)\,\{\partial\Phi/\partial X_r\,\partial\Phi/\partial X_r\}^{-\frac{1}{2}},\\ N_i&=\partial\Phi/\partial X_i\{\partial\Phi/\partial X_r\,\partial\Phi/\partial X_r\}^{-\frac{1}{2}}. \end{aligned} \tag{23}$$

Hence equation (22) may be written

$$\det(d_{ijkl}N_jN_l-\rho V^2\,\delta_{ik})=0, \tag{24}$$

* See reference [1].

which is similar to (8). The argument goes through as in § 2, and the universal connexions (13) and (14) are also valid at each particle of the material for acceleration waves.

5. Linear Viscoelasticity

Next consider the propagation of a damped linearly polarised sinusoidal wave in a linear viscoelastic solid. Let the stress σ_{ij} in the material at time t be given by

$$\sigma_{ij} = g_{ijkl} e_{kl} + \int_{-\infty}^{t} f_{ijkl}(t-\tau)\, e_{kl}(\tau)\, d\tau, \tag{25}$$

where $e_{kl}(\tau)$ is the infinitesimal strain at time τ, defined in terms of the displacement components $u_i(\tau)$ through

$$2e_{kl}(\tau) = \partial u_k(\tau)/\partial X_l + \partial u_l(\tau)/\partial X_k. \tag{26}$$

e_{kl} denotes $e_{kl}(t)$, g_{ijkl} are material constants and $f_{ijkl}(t-\tau)$ are material functions. It can be shown [9] that the damped sinusoidal wave

$$u_i(\tau) = U_i \exp \imath\omega(S_j X_j - \tau), \tag{27}$$

where $\boldsymbol{S}$ is the slowness and $\boldsymbol{U}$ is a constant, will propagate in the material provided

$$(h_{ijkl} S_j S_l - \rho\,\delta_{ik})\, U_k = 0, \tag{28}$$

where

$$h_{ijkl} = g_{ijkl} + \int_{-\infty}^{t} f_{ijkl}(t-\tau) \exp \imath\omega(t-\tau)\, d\tau. \tag{29}$$

Equation (28) leads to the secular equation

$$\det(h_{ijkl} S_j S_l - \rho\,\delta_{ik}) = 0. \tag{30}$$

If attention is restricted to the case when planes of constant phase are also planes of constant amplitude, then $\boldsymbol{S}$ is of the form

$$S_i = S N_i = (1/c)\, N_i, \tag{31}$$

and equation (30) becomes

$$\det(h_{ijkl} N_j N_l - \rho c^2 \delta_{ik}) = 0. \tag{32}$$

The universal connexions (13) and (14) are obtained again. Of course these universal connexions are meaningful only for those pairs and triples of directions along each of which three waves may propagate.

Similar results hold for sinusoidal small amplitude waves propagating in a finitely deformed homogeneous isotropic viscoelastic solid. The results may be read off from Equation (31) of the paper by Hayes & Rivlin [2].

6. Linear Thermoelasticity

In the classical linear theory of thermoelasticity the Helmholtz free energy A measured per unit mass may be written [10]

$$2A = a\theta^2/T_0 + 2\theta\, b_{ij}\, \partial u_i/\partial X_j + m_{ijkl} (\partial u_i/\partial X_j)(\partial u_k/\partial X_l), \tag{33}$$

where a, $b_{ij}=b_{ji}$ are constants, and m_{ijkl} are constants with the symmetries (2). $T_0+\theta$ is the temperature of the body, initially at uniform temperature T_0. The heat conduction equation and the stress deformation relation are given by

$$\begin{aligned} q_i &= -a_{ij}\partial\theta/\partial X_j, \\ \sigma_{ij} &= b_{ij}\theta + m_{ijkl}\partial u_k/\partial X_l, \end{aligned} \tag{34}$$

where $\boldsymbol{q}$ is the heat flux vector and $a_{ij}=a_{ji}$ are constants. In the absence of heat sources and body forces the equation of motion and the energy equation are [10]

$$\begin{aligned} a\,\partial\theta/\partial t + T_0\, b_{ij}\partial^2 u_i/\partial X_j\,\partial t &= a_{ij}\partial^2\theta/\partial X_i\,\partial X_j, \\ b_{ij}\partial\theta/\partial X_j + m_{ijkl}\partial^2 u_k/\partial X_l\,\partial X_j &= \rho\,\partial^2 u_i/\partial t^2. \end{aligned} \tag{35}$$

Now taking

$$\begin{aligned} u_i &= A_i \exp\imath\,\omega(SN_j X_j - t), \\ \theta &= \Theta\exp\imath\,\omega(SN_j X_j - t) \end{aligned} \tag{36}$$

in (35), the propagation conditions

$$\begin{aligned} & T_0\,S\,\omega\, b_{ij}A_i N_j + (\omega\, a_{ij} N_i N_j S^2 - \imath\, a)\,\Theta = 0, \\ & (S^2\, m_{ijkl} N_j N_l - \rho\,\delta_{ik})\,A_k - (\imath S\, b_{ik} N_j/\omega)\,\Theta = 0 \end{aligned} \tag{37}$$

are obtained. The secular equation is

$$\det \Psi_{\Gamma\Phi} = 0, \qquad \Gamma, \Phi = 1, 2, 3, 4, \tag{38}$$

where

$$\begin{aligned} \Psi_{ik} &= m_{ijkl} N_j N_l - \rho\, c^2\,\delta_{ik}, \qquad i, j, k, l = 1, 2, 3; \\ \Psi_{i4} &= \Psi_{4i} = -\imath\, b_{ij} N_j\, c/\omega, \qquad i, j = 1, 2, 3, \\ \Psi_{44} &= -\imath(a_{ij} N_i N_j\,\omega - a\,\imath\, c^2)/\omega^2\, T_0, \end{aligned} \tag{39}$$

and I have written $c=1/S$.

The secular equation (38) is a quartic in c^2 so that four waves may possibly propagate in any direction $\boldsymbol{N}$. Denote their squared speeds by $c_\Gamma^2(\boldsymbol{N})$, $\Gamma=1, 2, 3, 4$. From use of (39) it is easily seen that

$$\sum_{\Gamma=1}^{4} c_\Gamma^2(\boldsymbol{N}) = \Lambda_{jl} N_j N_l, \tag{40}$$

where

$$\rho\, a\, \Lambda_{jl} = a\, m_{ijil} + T_0\, b_{ij}\, b_{il} - \imath\,\rho\,\omega\, a_{jl} = \rho\, a\, \Lambda_{lj}. \tag{41}$$

By use of the same argument as before (§ 2) it follows that

$$\sum_{\Gamma=1}^{4} \{c_\Gamma^2(\boldsymbol{N}) + c_\Gamma^2(\boldsymbol{M}) + c_\Gamma^2(\boldsymbol{P})\} = \sum_{\Gamma=1}^{4} \{c_\Gamma^2(\boldsymbol{R}) + c_\Gamma^2(\boldsymbol{S}) + c_\Gamma^2(\boldsymbol{T})\}, \tag{42}$$

where $(\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{P})$ and $(\boldsymbol{R}, \boldsymbol{S}, \boldsymbol{T})$ are *any* two triads of mutually orthogonal unit vectors. Also

$$\sum_{\Gamma=1}^{4} \{c_\Gamma^2(\boldsymbol{N}) + c_\Gamma^2(\boldsymbol{M})\} = \sum_{\Gamma=1}^{4} \{c_\Gamma^2(\boldsymbol{R}) + c_\Gamma^2(\boldsymbol{S})\}, \tag{43}$$

where (N, M) is *any* pair of orthogonal unit vectors and (R, S) is *any* other pair of orthogonal unit vectors coplanar with the pair (N, M).

7. Internally Constrained Media

It is now shown that universal connexions similar to (13) and (14) are not valid in general for incompressible materials and hence *a fortiori* are not valid in general for materials subjected to internal constraints.

The constitutive equation for an incompressible homogeneous elastic body in classical linear elasticity theory is

$$\sigma_{ij} = -p\delta_{ij} + k_{ijkl}\,\partial u_k/\partial X_l, \tag{44}$$

where p is an arbitrary hydrostatic pressure to be determined from the equations of motion and boundary conditions, and k_{ijkl} are constants with the symmetries (2). Since the material is incompressible

$$\partial u_i/\partial X_i = 0. \tag{45}$$

Consider the plane wave

$$\begin{aligned} u_i &= A_i \exp\imath(N_j X_j - ct), \\ p &= P \exp\imath(N_j X_j - ct). \end{aligned} \tag{46}$$

The equations of motion (4) with (44) give

$$k_{ijkl} A_k N_l N_j + \imath P N_i = \rho c^2 A_i, \tag{47}$$

whilst by (45)

$$A_i N_i = 0. \tag{48}$$

From (47) and (48) the propagation condition is

$$(k_{ijkl} N_j N_l - k_{rjkl} N_i N_r N_j N_l - \rho c^2 \delta_{ik}) A_k = 0. \tag{49}$$

This leads to the secular equation

$$\det\{k_{rjkl} N_j N_l (\delta_{ir} - N_i N_r) - \rho c^2 \delta_{ik}\} = 0. \tag{50}$$

Of course

$$\det\{k_{rjkl} N_j N_l (\delta_{ir} - N_i N_r)\} = 0, \tag{51}$$

since

$$\det(\delta_{ir} - N_i N_r) = 0, \tag{52}$$

identically. Thus there are only two possible non-zero speeds for a given direction N. Their sum of squares is

$$\rho \sum_{\alpha=1}^{2} c_\alpha^2(N) = k_{rjrl} N_j N_l - k_{ijkl} N_i N_j N_k N_l. \tag{53}$$

Clearly this sum depends upon N in a quartic fashion. The previous argument is not valid here. A simple example will illustrate the point.

Consider the incompressible elastic material with constitutive equation

$$\begin{aligned}
\sigma_{11} &= -p + k_{11} u_{1,1} + k_{12} u_{2,2} + k_{13} u_{3,3}, \\
\sigma_{22} &= -p + k_{12} u_{1,1} + k_{11} u_{2,2} + k_{13} u_{3,3}, \\
\sigma_{33} &= -p + k_{13} u_{1,1} + k_{13} u_{2,2} + k_{33} u_{3,3}, \\
\sigma_{12} &= k_{66}(u_{1,2} + u_{2,1}), \\
\sigma_{13} &= k_{55}(u_{1,3} + u_{3,1}), \\
\sigma_{23} &= k_{55}(u_{2,3} + u_{3,2}),
\end{aligned} \tag{54}$$

where the k's are constants and $u_{i,j} \equiv \partial u_i / \partial X_j$. Choose two sets (N_1, N_2) and (N_3, N_4), of orthogonal coplanar unit vectors, where

$$N_1 = (1,0,0), \quad N_2 = (0,1,0), \quad \sqrt{2} N_3 = (1,-1,0), \quad \sqrt{2} N_4 = (1,1,0). \tag{55}$$

It is easily seen that two waves can propagate along each of these directions. The corresponding amplitudes A and speeds of propagation are given by

$$\begin{aligned}
N_1 &: A = (0,1,0),\ \rho c^2 = k_{66}; \quad A = (0,0,1),\ \rho c^2 = k_{55}; \\
N_2 &: A = (1,0,0),\ \rho c^2 = k_{66}; \quad A = (0,0,1),\ \rho c^2 = k_{55}; \\
N_3 &: \sqrt{2} A = (1, \ 1, 0),\ 2\rho c^2 = k_{11} - k_{12}; \quad A = (0,0,1),\ \rho c^2 = k_{55}; \\
N_4 &: \sqrt{2} A = (1, -1, 0),\ 2\rho c^2 = k_{11} - k_{12}; \quad A = (0,0,1),\ \rho c^2 = k_{55}.
\end{aligned} \tag{56}$$

Thus

$$\begin{aligned}
\sum_{\alpha=1}^{2} \{c_\alpha^2(N_1) + c_\alpha^2(N_2)\} &= 2k_{55} + 2k_{66}, \\
\sum_{\alpha=1}^{2} \{c_\alpha^2(N_3) + c_\alpha^2(N_4)\} &= k_{11} - k_{12} + 2k_{55},
\end{aligned} \tag{57}$$

and hence

$$\sum_{\alpha=1}^{2} \{c_\alpha^2(N_1) + c_\alpha^2(N_2)\} \neq \sum_{\alpha=1}^{2} \{c_\alpha^2(N_3) + c_\alpha^2(N_4)\}. \tag{58}$$

Thus universal connexions similar to (14) do not hold in general.

8. Concluding Remarks

The universal connexions (13) and (14) obtained here for mechanical waves may also hold good for other types of waves.

For example, in the case of light propagation in non-magnetic biaxial crystals [11] the secular equation for the phase velocity V is Fresnel's equation:

$$\frac{N_1^2}{V^2 - d_1^2} + \frac{N_2^2}{V^2 - d_2^2} + \frac{N_3^2}{V^2 - d_3^2} = 0. \tag{59}$$

Here N is the unit vector in the direction of propagation of the wave and d_i are constants. It is easy to obtain the universal connexion

$$\sum_{\alpha=1}^{2}\{V_\alpha^2(N)+V_\alpha^2(M)\}=\sum_{\alpha=1}^{2}\{V_\alpha^2(R)+V_\alpha^2(T)\}, \tag{60}$$

where (N, M) and (R, T) are two pairs of coplanar orthogonal unit vectors.

The situation for the ray velocity $\bar{V}$ is slightly different. It satisfies

$$\frac{N_1^2 a^2}{\bar{V}^2-a^2}+\frac{N_2^2 b^2}{\bar{V}^2-b^2}+\frac{N_3^2 c^2}{\bar{V}^2-c^2}=0, \tag{61}$$

where N is now the direction of the ray and a, b, c are constants. A relation of the form (60) is not valid in general. Rather, write $\bar{V}=1/\bar{S}$. Then

$$\sum_{\alpha=1}^{2}\{\bar{S}_\alpha^2(N)+\bar{S}_\alpha^2(M)\}=\sum_{\alpha=1}^{2}\{\bar{S}_\alpha^2(R)+\bar{S}_\alpha^2(T)\}. \tag{62}$$

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