Uniqueness of the Ground State Solution for $\Delta u - u + u^3 = 0$ and a Variational Characterization of Other Solutions

CHARLES V. COFFMAN

Communicated by J. SERRIN

1. Introduction

It is known [17] that the equation

$$\Delta u - u + u^3 = 0 \tag{1.1}$$

in R^3 has a positive radially symmetric solution $u = \varphi_1 \in C^2 \cap L^4$. (All function spaces considered here consist of real valued functions on R^3; except that we omit the argument R^3, our notation for these spaces is standard and follows that of [11].) In this note we show that φ_1 is unique, that is, there is precisely one positive radially symmetric solution of (1.1) which belongs to $C^2 \cap L^4$. (Here radial symmetry is to be understood only as radial symmetry with respect to the origin in R^3.) Moreover we show that for $u \in H^1$, $u \neq 0$

$$J(\varphi_1) < J(u) \tag{1.2}$$

unless

$$u(x) = \lambda \varphi_1(x + x_0) \tag{1.3}$$

for some non-zero real λ and $x_0 \in R^3$. Here J is the Rayleigh quotient associated with (1.1),

$$J(u) = \left(\int (|\operatorname{grad} u|^2 + u^2)\, dx\right)^2 / \int u^4\, dx \tag{1.4}$$

(when not explicitly denoted, the range of integration is always understood to be R^3). The expression on the right in (1.4) is meaningful for $u \in H^1$, $u \neq 0$; such functions will henceforth be referred to as *admissible* functions.

The equation (1.1) is considered in [10], where it is asserted that there exist functions $v_n(r) \in C^2[0, \infty)$, $n = 1, 2, \ldots$, such that for each n, v_n has exactly $n-1$ isolated zeros in $[0, \infty)$, decays exponentially as $r \to \infty$, and $\varphi_n(x) = v_n(|x|)$ is a solution of (1.1). A rigorous proof of the existence of v_1 was given by NEHARI [17]; the proof of the existence of v_n, $n \geqq 1$, was given by RYDER [21]. BERGER [1] has proved the same results using the Lyusternik-Schnirelman theory. We have proved the main result of this note in order to answer some questions which are raised in [20] but are not satisfactorily answered there. We also demonstrate the equivalence of several variational characterizations of the non-positive radially symmetric solutions of (1.1).

2. Preliminaries

We seek solutions of (1.1) subject to the "boundary condition at infinity"

$$u \in L^4. \tag{2.1}$$

The problem (1.1), (2.1) is equivalent to the integral equation

$$u(x) = \int g(x-t)\, u^3(t)\, dt, \tag{2.2}$$

in L^4, where

$$g(x) = (4\pi)^{-1} |x|^{-1} e^{-|x|}. \tag{2.3}$$

We list below a number of facts, most of which are standard or are obtained routinely by standard methods. The details, as well as a more complete bibliography concerning equation (1.1), will be found in [3].

First, concerning the space H^1, we have the following results:

a) C_0^∞ *is dense in* H^1.

b) If $u \in H^1$ *then* $v = |u| \in H^1$ *and*

$$|u|_{1,2} = |v|_{1,2}.$$

c) If $u \in H^1$ *then* $u \in L^4$ *and*

$$|u|_{0,4} \leqq 2^{-\frac{1}{2}} |u|_{1,2}. \tag{2.4}$$

d) Let V *denote the subspace of* H^1 *consisting of radially symmetric functions. The embedding* $V \to L^4$ *is compact.*

Except for the constant, the inequality (2.4) follows from [14, Lemma 2, p. 11], or from a more general inequality of NIRENBERG [18], which is quoted as [11, Theorem 9.3, p. 24]. One can obtain the constant in (2.4) by using the representation $u = g * w$ where $w = -\Delta u + u$; by *a)* it suffices to prove (2.4) for $u \in C_0^\infty$. The assertion *d)* follows in a straightforward way from the Sobolev imbedding theorem and the inequality

$$4\pi \int_{|x| \geqq \rho} |v(x)|^4 dx \leqq 2\rho^{-1} |v|_{1,2}^4$$

for $v \in V$, $\rho > 0$.

Concerning the convolution operator $\tau: u \to g * u$, where

$$(g * u)(x) = \int g(x-t)\, u(t)\, dt$$

and g is given by (2.3), we have the following results:

e) If $u \in L^{4/3}$ *then* $v = g * u \in H^1 \subseteq L^4$, $\int u\, v\, dx > 0$ *unless* $u = 0$, *and* v *is a weak solution of*

$$-\Delta v + v = u. \tag{2.5}$$

f) If $u \in L^1 \cap L^\infty$ then $v = g * u$ has bounded continuous first derivatives and

$$\lim_{|x| \to \infty} v(x) = 0.$$

g) If $u \in L^1 \cap L^\infty \cap C^1$ then $v = g * u \in C^2$ and v satisfies (2.5).

h) Let X and Y denote the subspaces of $L^{4/3}$ and L^4, respectively, consisting of radially symmetric functions. Then $Y = X^*$ and $\tau: X \to Y$ is compact.

The first assertion of *h)* is obvious, and the second follows immediately from *d)* and *e)*.

Remark. For consideration of the equation

$$\Delta u - u + |u|^{p-2} u = 0,$$

one replaces L^4 by L^p and $L^{4/3}$ by L^q where $p^{-1} + q^{-1} = 1$. If $2 < p < 6$, then *c), d), e),* and *h)* remain valid in this more general case (except for a change of the constant in (2.4)); *e)* and *h)* of course fail for $p = 2$.

3. Minimization of J

For $u \in L^4$, $u \neq 0$, we define $\sigma(u)$ by

$$(3.1) \qquad (\sigma(u))(x) = c \int g(x-t)\, u^3(t)\, dt$$

where $c > 0$ is chosen so that $v = \sigma(u)$ satisfies

$$(3.2) \qquad \int v^4\, dx = 1.$$

This is possible since $u \in L^4$ implies $u^3 \in L^{4/3}$; thus by *e)*, $g * (u^3) \in L^4$ and is non-zero. It is clear that up to positive factors the fixed points of σ are precisely the non-trivial solutions of (2.2). From *e)* above it follows that σ actually maps L^4 into H^1, and thus by *c)*, σ can also be regarded as an operator in $H^1 \backslash \{0\}$. In particular it follows that an L^4 solution of (2.2) must belong to H^1.

Lemma 3.1. *Let u be an admissible function with*

$$(3.3) \qquad \int u^4\, dx = 1.$$

Then $\sigma(u)$ is admissible and

$$(3.4) \qquad J(\sigma(u)) \leqq J(u)$$

with equality only if $\sigma(u) = u$. Moreover $\sigma(u) \in L^\infty$, and $v = \sigma^2(u)$ has bounded continuous derivatives and satisfies

$$(3.5) \qquad \lim_{|x| \to \infty} v(x) = 0;$$

finally $\sigma^3(u) \in C^2$.

Proof. The admissibility of $\sigma(u)$ follows from *e)*. By *e)*, (3.1) and (3.3), $w = \sigma(u)$ satisfies

$$c = c \int u^4\, dx = \int (\operatorname{grad} w \cdot \operatorname{grad} u + w u)\, dx \leqq |u|_{1,2}\, |w|_{1,2}.$$

By *e)*, (3.2) and (3.3)

$$\int (|\operatorname{grad} w|^2 + w^2)\, dx = c \int w u^3\, dx \leqq c |w|_{0,4} |u|_{0,4}^3 = c. \tag{3.6}$$

Combining the above inequalities we get

$$|w|_{1,2} \leqq |u|_{1,2},$$

which, in view of (3.1) and (3.3), implies (3.4). From the way in which the Schwarz inequality was used, it follows that equality can hold only if w and u are proportional. By (3.2), (3.3) and (3.6) the constant of proportionality in this case must be 1. To obtain the boundedness of $\sigma(u)$, one applies the Schwarz inequality to (3.1) and then uses the inequality [14, p. 12]

$$\int u^6\, dx \leqq 48 \left(\int |\operatorname{grad} u|^2\, dx\right)^3;$$

see also [11, Theorem 9.3, p. 24]. The remaining assertions follow immediately from *f)* and *g)*.

Lemma 3.1 has the following corollaries.

Lemma 3.2. *If $v \in L^4$ is a solution of* (2.2) *then $v \in C^2$, v has bounded first derivatives, and v satisfies* (3.5).

Lemma 3.3. *If u is any (radially symmetric) admissible function, then there is a (radially symmetric) admissible function $v \in C^2$ which is positive, has bounded first derivatives and satisfies* (3.5) *and*

$$J(v) \leqq J(u). \tag{3.7}$$

Moreover, unless u itself has the same properties and is a solution of (2.2) *(to within a positive factor), then v can be chosen so that inequality* (3.7) *is strict.*

Proof of Lemma 3.3. Except for positivity and the assertion about radial symmetry, the result follows immediately from Lemma 3.1. It suffices to prove the positivity assertion for continuous u; we replace such a u by $|u|$, whence by *b)*

$$J(|u|) = J(u).$$

By continuity u and $|u|$ must vanish at some point of R^3 unless u is already of one sign. If the latter is not the case, then since g is positive, $\sigma(|u|)$ will be positive; we cannot then have $\sigma(|u|) = |u|$, and therefore $J(\sigma(|u|)) < J(|u|) = J(u)$. The assertion concerning radial symmetry follows from the observation that σ preserves radial symmetry.

Theorem 3.1. *Let*

$$\lambda_1 = \inf\{J(u) \colon u \text{ admissible}\}.$$

There exists a $\varphi_1 \in V$ with

$$J(\varphi_1) = \lambda_1.$$

For $u \in H^1$, $J(u) > \lambda_1$ unless u is of the form (1.3).

Proof. That J attains an infimum in the class of radially symmetric admissible functions was shown by NEHARI [17]. This also follows from the assertion *d)* of Section 2.

Suppose now that $u \in H^1$ but that no translate of u is essentially radially symmetric. For the purpose of showing that $J(u) > \lambda_1$, we can suppose by Lemma 3.3 that u is positive and of class C^2, and that $u(x) \to 0$ as $|x| \to \infty$. By the Schwarz symmetrization procedure (see [19] or [16, Section 8]) we can produce a radially symmetric admissible function v with $J(v) < J(u)$. This shows that the infimum of J over all admissible functions is the same as its infimum over the radially symmetric ones and therefore that this infimum is attained. The final assertion of the theorem follows from Theorem 4.1 below.

Remarks. 1. The problem of minimizing J is essentially that of finding the "best constant" for the imbedding $H^1 \to L^4$. Our application here of the Schwarz symmetrization is similar to that of MOSER in his paper [15] on imbedding of Sobolev spaces in Orlicz spaces and the determination of a best imbedding constant.

2. Let Ω be a proper subregion of R^3 which contains balls of arbitrarily large radius, that is, such that

$$\sup_{\Omega} \operatorname{dist}(x, \partial\Omega) = \infty.$$

For the purpose of finding non-trivial solutions of (1.1) which belong to $H_0^1(\Omega)$, the direct method must fail. This is perhaps a trivial observation; it is obvious that the infimum of J over non-zero functions in $H_0^1(\Omega)$ is λ_1, but, by the trivial extension of functions on Ω to all of R^3, $H_0^1(\Omega)$ can be regarded as a subspace of H^1. It is then an immediate corollary of Theorem 3.1 that J does not attain its infimum in $H_0^1(\Omega)$; of course this already follows merely from the fact that a necessary condition for J to attain its minimum in $H^1 \backslash \{0\}$ at u is that u satisfy (1.1).

3. For consideration of the more general equation

$$\Delta u - u + |u|^{p-2} u = 0$$

(see the remark at the end of the preceding section), we note that the principal difference is in the general case of Lemma 3.1. That is, when $u \in L^p$, $2 < p < 6$, and p is near 6, boundedness is not necessarily obtained with a single iteration of σ. However it is obtained after some finite number of iterations for any p on the indicated range (an upper bound for this number depends only on p).

4. A Uniqueness Theorem

It is easily seen that the C^2 radially symmetric solutions of (1.1) are of the form

$$u(x) = |x|^{-1} w(|x|), \qquad x \neq 0,$$

where w is of class C^2 on $[0, \infty)$, $w(0) = 0$, and w is a solution of

$$w'' - w + r^{-2} w^3 = 0, \qquad (' = d/dr). \tag{4.1}$$

To prove the uniqueness assertion concerning φ_1 it suffices therefore to prove that (4.1) has at most one positive solution satisfying

$$0 < \lim_{r \to 0} r^{-1} w(r) < \infty, \qquad \lim_{r \to \infty} w(r) = 0. \tag{4.2}$$

The technique by which this will be proved is similar to that used in [6] and [7] to prove similar uniqueness theorems; the general approach was originally suggested by a work of KOLODNER [13].

We shall consider the "initial value problem"

$$\lim_{r\to 0} r^{-1} w(r) = a \tag{4.3}$$

for (4.1). The problem (4.1), (4.3) manifests most of the features of a regular initial value problem. The basic facts concerning (4.1), (4.3) are summarized below; the proofs are more or less routine and will be omitted.

Lemma 4.1. *For each $a>0$ the equation* (4.1) *has a unique solution $w=w(r,a)$ which is of class C^2 on $(0,\infty)$ and satisfies* (4.3). *The partial derivatives $\partial w(r,a)/\partial a$ and $\partial w'(r,a)/\partial a$ exist for all positive r and a. $\partial w(r,a)/\partial a$ coincides on $(0,\infty)$ with the solution $\delta=\delta(r,a)$ of the regular initial value problem*

$$\delta'' - \delta + 3r^{-2} w^2 \delta = 0, \qquad \delta(0)=0, \qquad \delta'(0)=1, \tag{4.4}$$

with $w=w(r,a)$; $\partial w'(r,a)/\partial a = \delta'(r,a)$.

It is clear that a solution of (4.1) which satisfies (4.2) belongs to the one-parameter family $w=w(r,a)$, $a>0$; we therefore formulate our uniqueness result as follows.

Theorem 4.1. *There is at most one positive value of a for which*

$$w(r,a)>0, \qquad 0<r<\infty, \tag{4.5}$$

and

$$\lim_{r\to\infty} w(r,a) = 0. \tag{4.6}$$

Theorem 4.1 is implied by the following lemma.

Lemma 4.2. *i) If $a>0$ and $w(r,a)>0$ on $(0,z_1)$ with $w(z_1,a)=0$, then $\delta(z_1,a)<0$.*

ii) If $a>0$ and $w(r,a)$ satisfies (4.5) *and* (4.6), *then*

$$\lim_{r\to\infty} e^{-r} \delta(r,a) < 0. \tag{4.7}$$

Proof of Theorem 4.1. We assume Lemma 4.2. Let A denote the set of positive values of a for which $w(r,a)$ has at least one zero in $(0,\infty)$. If $a\in A$ and $z_1=z_1(a)$ is the least positive zero of $w(r,a)$, then, by Lemma 4.1 and the implicit function theorem, z_1 is a differentiable function of a on A and

$$w'(z_1,a)\, dz_1/da + \delta(z_1,a) = 0.$$

Since $w'(z_1,a)<0$ it follows from *i)* of Lemma 4.2 that $dz_1/da<0$ on A; therefore z_1 moves monotonically to the left as a increases. Thus if A is non-empty it is a semi-infinite interval. We next show that if (4.5) and (4.6) hold for $a=a_1$, then A is non-empty and a_1 is the left endpoint of A. This will clearly imply Theorem 4.1. Let a_1 be as above and let $a_2>a_1$; we shall show that if a_2-a_1 is sufficiently small, then the assumption

$$w(r,a_2)>0 \quad \text{on } (0,\infty) \tag{4.8}$$

leads to a contradiction. We put $w_i = w_i(r, a)$, $i = 1, 2$. By our assumption on a_1 and *ii)* of Lemma 4.2, we can choose $r_0 > 0$ so that

(4.9) $$3r^{-2} w_1^2 < 1/2, \qquad r \geqq r_0$$

and

(4.10) $$\delta(r_0, a_1) < 0, \qquad \delta'(r_0, a_1) < 0.$$

From Lemma 4.1 and (4.10) it follows that if $a_2 > a_1$, and $a_2 - a_1$ is sufficiently small, then

(4.11) $$w_2(r_0) < w_1(r_0), \qquad w_2'(r_0) < w_1'(r_0).$$

We put $v = w_1 - w_2$, so that v satisfies

(4.12) $$v'' - v + r^{-2}(w_1^2 + w_1 w_2 + w_1^2) v = 0.$$

We suppose that $w_2 = w(r, a_2)$ satisfies (4.8) and that

(4.13) $$0 < w_2 < w_1$$

on $[r_0, r_1)$ for some $r_1 > r_0$ (there exists such an r_1 by (4.11)). From (4.11), (4.12) and (4.9) it follows that v is positive and convex on $[r_0, r_1)$; moreover, from (4.11), $v'(r_0) > 0$ so that v is increasing on $[r_0, r_1)$. Thus (4.13) holds at $r = r_1$; hence by a standard argument we conclude that (4.13) holds on $[r_0, \infty)$ and that v is increasing there. The inequality (4.13) on $[r_0, \infty)$ implies that

$$\lim_{r \to \infty} (w_1^2 + w_1 w_2 + w_2^2) = 0.$$

Using this fact and the monotone character of v, we conclude from asymptotic integration of (4.12) [12, Corollary 9.2, p. 381], that v grows exponentially as $r \to \infty$. From the definition of v, this is clearly a contradiction of (4.6) for $a = a_1$ and (4.8). We conclude therefore that (4.8) cannot hold for $a_2 > a_1$ and $a_2 - a_1$ arbitrarily small; therefore $a_2 \in A$ for all $a_2 > a_1$ with $a_2 - a_1$ sufficiently small. This completes the proof of Theorem 4.1.

The proof of Lemma 4.2 will be based on the sequence of lemmas to follow. We shall assume that $w = w(r, a)$ satisfies the assumptions either of *i)* or of *ii)* of Lemma 4.2, and we shall let z_1 denote the least zero in $(0, \infty)$ if the first of these assumptions holds and put $z_1 = \infty$ if the second holds.

Lemma 4.3. *Let* $a > 0$ *and let* $w = w(r, a)$ *either vanish at least once in* $(0, \infty)$ *or satisfy* (4.6); *then* $a > \sqrt{2}$, $w(r, a) = r$ *for precisely one value* $r = r_0$ *in* $(0, z_1)$, *and* $w'(r_0, a) < 0$.

Proof. The function $v(r, a) = r^{-1} w(r, a)$ satisfies

(4.14) $$v'' + 2r^{-1} v' - v + v^3 = 0$$

and

(4.15) $$\lim_{r \to 0} v(r, a) = a, \qquad \lim_{r \to 0} v'(r, a) = 0.$$

Upon differentiating the function

$$\Phi(r) = v'^2 + \tfrac{1}{2} v^4 - v^2 \qquad (v = v(r, a))$$

and using (4.14), we conclude that, for $a > 0$, $\Phi(r)$ is a strictly decreasing function of r. Since $-v^2 < \Phi(r)$, it follows from the monotone character of Φ that if $\Phi(r_0)$ is non-positive for some r_0 in $[0, \infty)$, then v does not vanish in (r_0, ∞) and $\liminf_{r\to\infty} v^2(r) > 0$. If $0 < a \leqq \sqrt{2}$, then from (4.15) it follows that $\Phi(0) \leqq 0$ and that w neither vanishes on $(0, \infty)$ nor satisfies (4.6) (since (4.6) clearly implies $\lim_{r\to\infty} v(r) = 0$).

Suppose now that $w(r, a)$ were to satisfy the hypothesis of Lemma 4.3 but that for $r_0 \in (0, z_1)$, $w(r_0) = r_0$ while $w'(r_0) \geqq 0$. Since w is convex as long as $0 < w < r$, this assumption would imply the existence of an r_1 with $r_0 < r_1 < z_1$ such that $w(r_1) = r_1$, so that the assertion concerning the sign of the slope where w crosses the 45° line reduces to the assertion that there is a single such crossing in $(0, z_1)$. Suppose that there were two such crossings, r_0, r_1. We would then have $v(r_0) = v(r_1) = 1$, and we can assume $0 < v < 1$ on (r_0, r_1). There is then an $r_3 \in (r_0, r_1)$ with $v'(r_3) = 0$, but then $\Phi(r_3) < 0$ (with Φ defined as above). This implies, as indicated above, that w cannot vanish on (r_3, ∞) nor satisfy (4.6). Thus the assumption that $w(r) - r$ can vanish twice in $(0, z_1)$ has led to a contradiction, and the proof of the lemma is complete.

For $w = w(r, a)$ as in Lemma 4.3, it follows from that result that there will exist positive numbers α, β, γ which are, respectively, the least positive values of r for which

$$w'(r) = 1, \qquad w'(r) = 0, \qquad w(r) = r.$$

Moreover, by Lemma 4.3, $0 < \alpha < \beta < \gamma < z_1$, $r < w(r)$ on $(0, \gamma)$ and $0 < w(r) < r$ on (γ, z_1); finally w is concave on $(0, \gamma)$ and convex on (γ, z_1).

We shall require the following identities which are valid for $w = w(r, a)$, $\delta = \delta(r, a)$:

(4.16) $$(w'\delta - \delta' w)' = 2r^{-2} w^3 \delta,$$

(4.17) $$(w'\delta' - w''\delta)' = -2r^{-3} w^3 \delta,$$

(4.18) $$\big(r(w'\delta' - w''\delta) - w'\delta\big)' = -2w\delta,$$

(4.19) $$\big((w'-1)\delta' - w''\delta\big)' = -r^{-3}\delta(w-r)^2(2w+r),$$

(4.20) $$\big[r\big((w'-1)\delta' - w''\delta\big) - (w'-1)\delta\big]' = r^{-1}\delta(w-r)(3w+r).$$

Let y_1 denote the least positive zero of $\delta = \delta(r, a)$.

Lemma 4.4. $\alpha < y_1 < \beta$.

Proof. Suppose first that

(4.21) $$y_1 \leqq \alpha$$

and integrate (4.20) between 0 and y_1. The expression $\delta(w-r)(3w+r)$ is positive on $(0, y_1)$, which implies

$$y_1(w'(y_1) - 1)\delta'(y_1) > 0.$$

Because of the definition of α the assumption (4.21) implies that $w'(y_1) \geqq 1$, while clearly $\delta'(y_1) < 0$, so that the assumption (4.21) has led to a contradiction. Suppose next that

(4.22) $$y_1 \geqq \beta$$

and integrate (4.18) between 0 and β. This gives

$$-w''(\beta)\,\delta(\beta) < 0;$$

but (4.22) implies $\delta(\beta) \geqq 0$, and clearly $w''(\beta) < 0$, so (4.22) has also led to a contradiction and the lemma is proved.

Since w is concave on $(0, \gamma)$ we have $w'(r) < 1$ on (α, γ). In particular, by Lemma 4.1

(4.23) $$w'(y_1) < 1.$$

We now complete the proof of Lemma 4.2. Suppose that δ has a zero, say y_2, in $(y_1, z_1]$, and integrate (4.19) between y_1 and y_2. This gives

(4.24) $$\big(w'(y_2) - 1\big)\,\delta'(y_2) = \big(w'(y_1) - 1\big)\,\delta'(y_1) - \int_{y_1}^{y_2} r^{-3}\,\delta(w - r)^2 (2w + r)\,dr.$$

We assume (as we obviously can) that $\delta < 0$ on (y_1, y_2). Then $\delta'(y_1) < 0$ and by (4.23), $\big(w'(y_1) - 1\big) < 0$, so that the right side of (4.24) is positive. Since $\delta'(y_2) > 0$, (4.24) implies that $w'(y_2) > 1$. This is clearly a contradiction since $w' < 1$ on (α, γ) and, since w is convex on $(\gamma, z_1]$, $w' < 0$ on that interval. Thus $\delta < 0$ on $(y_1, z_2]$, and *i)* of Lemma 4.2 is proved.

If w satisfies (4.5) and (4.6), then $w'(r) < 0$ on (β, ∞) and $\delta < 0$ in (y_1, ∞). Integration of (4.19) from y_1 to γ gives $\delta'(\gamma) < 0$, and integration of (4.17) from γ then gives

$$\lim_{r \to \infty} \big(w'(r)\,\delta'(r) - w''(r)\,\delta(r)\big) > 0,$$

and this implies that $-\delta$ grows exponentially. This completes the proof of Lemma 4.2.

Remark. Once again we consider the more general equation

$$\Delta u - u + |u|^{p-2} u = 0$$

in R^3 and remark that with regard to the uniqueness theorem the generalization from the case $p = 4$ is not straightforward. In fact, if one writes the comparison formulas (4.16)–(4.20) for the general case, then the argument presented above will fail except for the case $p = 4$. Whether or not a uniqueness proof can be obtained in the general case using other comparison formulas, we do not know. In particular, the question of uniqueness remains open for the equation

$$\Delta u - u + u^2 = 0$$

which was studied by SYNGE in [22].

5. Numerical Values for λ_1

The inequality (2.4) gives the lower estimate

$$\lambda_1 > 2.$$

Numerical values have been computed by TESHIMA [23] and RYDER [21]; the results of the former are quoted in [2]. RYDER gives the value

$$\tfrac{1}{2}\int_0^\infty (w_1'^2 + w_1^2)\,dr = 3.00787$$

where w_1 is the positive solution of (4.1), (4.2). TESHIMA gives

$$(16\pi)^{-1}\int \varphi_1^4(x)\,dx = 1.503.$$

For λ_1 these two values give

$$(4\pi)^{-1}\lambda_1 = 6.01574$$

and

$$(4\pi)^{-1}\lambda_1 = 6.012.$$

6. Non-Positive Solutions

For the purposes of this section it will be convenient to consider the Rayleigh quotient associated with the problem (4.1), (4.2); this will also be denoted by J: thus

$$J(w) = \left(\int_0^\infty (w'^2 + w^2)\,dr\right)^2 \Big/ \int_0^\infty w^4 r^{-2}\,dr.$$

We shall let W denote the space of functions w on $[0,\infty)$ which are absolutely continuous and satisfy the conditions

$$w(0) = 0, \qquad \int_0^\infty (w'^2 + w^2)\,dr < \infty.$$

Finally, we let U denote the space of measurable functions w on $[0,\infty)$ such that

$$\int_0^\infty w^4 r^{-2}\,dr < \infty.$$

If these two spaces are normed in the obvious way, then they are isometrically equivalent, respectively, to the space V defined in §2 and to the subspace of $L^4(R^3)$ consisting of radially symmetric functions; except for a factor these equivalences are given by

$$w \to u$$

where $u(x) = |x|^{-1} w(|x|)$. In particular it follows that $J(w)$ is defined whenever $w \in W$.

We are interested here in "characteristic values" or stationary values of $J(w)$ greater than the minimum. Geometrically the corresponding stationary points are those points in $W \setminus \{0\}$ where the gradients of the W norm and the U norm are colinear.

We shall first quote RYDER's characterization [21] of the stationary values of J. For practical purposes this is probably the most useful characterization since it lends itself most readily to estimation. Indeed estimates as well as asymptotic formulas are given in [21]. RYDER actually defines "characteristic values" for (4.1), (4.2) in such a way that they differ by a factor of $\frac{1}{2}$ from the stationary values of $J(w)$; we merely omit this factor in quoting his definition. Let $0 = r_0 < r_1 < \cdots < r_{n-1} < r_n = \infty$ and let $w \in W$ satisfy

$$w(r_\nu) = 0, \qquad \nu = 1, 2, \ldots, n-1,$$

and not vanish identically in any of the intervals $(r_{\nu-1}, r_\nu)$, $\nu = 1, \ldots, n$. The n^{th} stationary value of J is

$$\lambda_n = \inf \sum_{\nu=1}^{n} \left(\int_{r_{\nu-1}}^{r_\nu} (w'^2 + w^2)\, dr \right)^2 \Big/ \int_{r_{\nu-1}}^{r_\nu} w^4 r^{-2}\, dr \tag{6.1}$$

where the infimum is taken over all such functions w and over all such finite sequences $r_1, \ldots, r_{n-1}$. RYDER proves that for each n there is a solution w_n of (4.1), (4.2) with exactly $n-1$ zeros in $(0, \infty)$ and with

$$J(w_n) = \lambda_n. \tag{6.1'}$$

We shall describe two other ways of characterizing the stationary values of $J(w)$. The first of these is the exact analogue of the Poincaré minimum-maximum principle. In what follows, M will always denote a finite dimensional subspace of W. We then define

$$J(M) = \max_{M \setminus \{0\}} J(w)$$

and

$$\lambda_n' = \inf \{ J(M) \colon \dim M \geqq n \}. \tag{6.2}$$

It was mentioned earlier that the Lyusternik-Schnirelman theory has been applied to (4.1), (4.2) by BERGER [1] to obtain existence assertions similar to RYDER'S. We remark also that, because of *h)* of § 2, the general results of [4] are applicable to the integral equation (2.2) in the space Y defined in § 2. To characterize the stationary values of J in accordance with the Lyusternik-Schnirelman theory, we use the notion of *genus*, due to KRASNOSELSKII (the genus is 1 greater than the *co-index* defined in [8]). We shall call a subset S of W *admissible* if $S \subseteq W \setminus \{0\}$ and S is symmetric, compact, and non-empty. In what follows S will always denote an admissible subset of W. The genus of S, $\rho(S)$, is the least integer n such that there exists an odd continuous mapping of S into $R^n \setminus \{0\}$. We put

$$J(S) = \max_{S} J(w)$$

and define

$$\lambda_n'' = \inf \{ J(S) \colon \rho(S) \geqq n \}. \tag{6.3}$$

It readily follows from the results of [4] that for each n there is a solution w of (4.1), (4.2) with $J(w) = \lambda_n''$; here however we shall rely only on the existence theorem of RYDER.

If M is a finite dimensional subspace of W and S is the unit sphere in M, then S is admissible,

$$J(S) = J(M)$$

and by the Borsuk Antipodal Theorem [9], $\rho(S)=\dim M$. It follows that

$$\lambda_n'' \leqq \lambda_n'.$$

Let w_n be a solution of (4.1), (4.2) with $n-1$ zeros in $(0,\infty)$ and which satisfies (6.1′). Then w_n is the n^{th} eigenfunction of the boundary value problem

$$\theta'' - \theta + \mu r^{-2} w_n^2 \theta = 0, \tag{6.4}$$

$$\theta(0)=0, \quad \lim_{r\to\infty} \theta(r)=0, \tag{6.5}$$

and the corresponding eigenvalue μ_n is 1. Therefore, by the theory of the Sturm-Liouville problem, if M denotes the space spanned by the first n eigenfunctions of (6.4), (6.5), then for $\theta \in M \setminus \{0\}$

$$\int_0^\infty (\theta'^2+\theta^2)\,dr \Big/ \int_0^\infty r^{-2} w_n^2 \theta^2\,dr \leqq 1. \tag{6.6}$$

By the Schwarz inequality

$$\int_0^\infty r^{-2} w_n^2 \theta^2\,dr \leqq \left(\int_0^\infty r^{-2} w_n^4\,dr\right)^{\frac{1}{2}} \left(\int_0^\infty r^{-2} \theta^4\,dr\right)^{\frac{1}{2}},$$

and therefore, from (6.6)

$$\int_0^\infty (\theta'^2+\theta^2)\,dr \Big/ \left(\int_0^\infty r^{-2} \theta^4\,dr\right)^{\frac{1}{2}} \leqq \left(\int_0^\infty r^{-2} w_n^4\,dr\right)^{\frac{1}{2}}. \tag{6.7}$$

Since w_n satisfies (4.1), (4.2)

$$\int_0^\infty (w_n'^2+w_n^2)\,dr = \int_0^\infty r^{-2} w_n^4\,dr,$$

and thus

$$\int_0^\infty r^{-2} w_n^4\,dr = J(w_n) = \lambda_n.$$

Therefore (6.7) implies

$$J(\theta) \leqq \lambda_n$$

for $\theta \in M \setminus \{0\}$. Since $\dim M = n$ it follows that

$$\lambda_n' \leqq \lambda_n.$$

Having established that

$$\lambda_n'' \leqq \lambda_n' \leqq \lambda_n,$$

we shall now show that $\lambda_n \leqq \lambda_n''$, that is, the three characterizations of the stationary values of J are in fact equivalent. For a given $w \in W \setminus \{0\}$ we let $\mu_k = \mu_k(w)$ denote the k^{th} eigenvalue of

$$\theta'' - \theta + \mu r^{-2} w^2 \theta = 0 \tag{6.8}$$

$$\theta(0)=0, \quad \lim_{r\to\infty} \theta(r)=0; \tag{6.9}$$

similarly, let $\theta_k = \theta_k(r, w)$ be the k^{th} eigenfunction normalized by

$$\theta'(0) > 0, \qquad \int_0^\infty r^{-2} w^2 \theta^2 \, dr = 1. \tag{6.10}$$

The Poincaré minimum-maximum principle shows that μ_k is a continuous function of w on $W \setminus \{0\}$, and it then follows that $w \to \theta_k(\cdot, w)$ is a continuous mapping of $W \setminus \{0\}$ into W. Let

$$a_k = a_k(w) = \int_0^\infty r^{-2} w^3 \theta_k \, dr;$$

then a_k is an odd continuous function of w and

$$A_n \colon w \to (a_1(w), \ldots, a_{n-1}(w))$$

is an odd mapping of $W \setminus \{0\}$ into R^{n-1}. If S is any admissible set of genus n then by definition of the genus there is a $\tilde{w} \in S$ such that $A_n(\tilde{w}) = 0$. From the definition of A_n and the theory of the Sturm-Liouville problem, this implies

$$\mu_n(\tilde{w}) \leqq \int_0^\infty (\tilde{w}'^2 + \tilde{w}^2) \, dr \Big/ \int_0^\infty \tilde{w}^4 r^{-2} \, dr. \tag{6.11}$$

If $0 < r_1 < r_2 < \cdots < r_{n-1} < \infty$ are the zeros of θ_n then

$$\lambda_n \leqq \sum_{\nu=1}^{n} \left(\int_{r_{\nu-1}}^{r_\nu} (\theta_n'^2 + \theta_n^2) \, dr \right)^2 \Big/ \int_{r_{\nu-1}}^{r_\nu} r^{-2} \theta_n^4 \, dr. \tag{6.12}$$

By the Schwarz inequality

$$\left(\int_{r_{\nu-1}}^{r_\nu} r^{-2} \tilde{w}^2 \theta_n^2 \, dr \right)^2 \leqq \left(\int_{r_{\nu-1}}^{r_\nu} r^{-2} \theta_n^4 \, dr \right) \left(\int_{r_{\nu-1}}^{r_\nu} r^{-2} \tilde{w}^4 \, dr \right),$$

and using this together with the relation

$$\mu_n \int_{r_{\nu-1}}^{r} r^{-2} \tilde{w}^2 \theta_n^2 \, dr = \int_{r_{\nu-1}}^{r_\nu} (\theta_n'^2 + \theta_n^2) \, dr$$

we obtain

$$\lambda_n \leqq \mu_n^2 \int_0^\infty r^{-2} \tilde{w}^4 \, dr.$$

Upon combining this last inequality with (6.11), we find

$$\lambda_n \leqq J(\tilde{w}).$$

Since S was an arbitrary admissible set of genus $\geqq n$, it follows that

$$\lambda_n \leqq \lambda_n''.$$

We have proved the following result.

Theorem 6.1. *The formulas* (6.1) (RYDER), (6.2) *(Poincaré minimum-maximum principle) and* (6.3) *(Lyusternik-Schnirelman minimum-maximum principle) determine the same sequence of numbers.*

Remarks. 1. We have referred to the numbers λ_n as the stationary values of J. We do not know whether the nodal properties of w_n uniquely determine (to within sign) a solution of (4.1), (4.2); therefore neither do we know if all of the stationary values of J are among the $\{\lambda_n\}$.

2. The Lyusternik-Schnirelman principle would more conventionally be formulated in terms of the category of sets in the identification space of $W\backslash\{0\}$ under the equivalence $u \equiv -u$. A result of WEISS [24] shows that such a formulation is equivalent to that in terms of the genus.

3. A mapping corresponding to A_n was used for similar purposes in [5]. A related construction was used in conjunction with fixed point (rather than variational) techniques by WOLKOWISKY [25].

The author wishes to acknowledge several very helpful conversations with Dr. C. BANDLE and Prof. Z. NEHARI, and in particular is grateful to the former for suggesting the use of the Schwarz symmetrization.

This research was supported by National Science Foundation Grant GP-21512.

References

1. BERGER, M. S., Stationary states for a nonlinear wave equation. J. Math. Physics **11**, 2906–2912 (1970).
2. BETTS, D. D., H. SCHIFF, & W. B. STRICKFADEN, Approximate solution of a nonlinear field equation. J. Math. Physics **4**, 334–338 (1963).
3. COFFMAN, C. V., On the equation $\Delta u - u + |u|^{p-2} u = 0$. Department of Mathematics. Carnegie-Mellon University.
4. COFFMAN, C. V., Spectral theory of monotone Hammerstein operators. Pac. J. Math. **36**, 303–322 (1971).
5. COFFMAN, C. V., A minimum-maximum principle for a class of non-linear integral equations. J. d'Analyse Math. **22**, 391–419 (1969).
6. COFFMAN, C. V., On the positive solutions of boundary-value problems for a class of nonlinear differential equations. J. Diff. Eq. **3**, 92–111 (1967).
7. COFFMAN, C. V., & A. J. DAS, A class of eigenvalues of the fine structure constant and internal energy obtained from a class of exact solutions of the combined Klein-Gordon-Maxwell-Einstein field equations. J. Math. Physics **8**, 1720–1735 (1967).
8. CONNER, P. E., & E. E. FLOYD, Fixed point free involutions and equivariant maps. Bull. Amer. Math. Soc. **66**, 416–441 (1960).
9. DUGUNDJI, J., Topology. Boston: Allyn and Bacon 1966.
10. FINKLESTEIN, R., R. LELEVIER, & M. RUDERMAN, Nonlinear spinor fields. Phys. Review **83**, 326–332 (1951).
11. FRIEDMAN, A., Partial Differential Equations. New York: Holt. Rinehart and Winston 1969.
12. HARTMAN, P., Ordinary Differential Equations. New York: Wiley 1964.
13. KOLODNER, I. I., Heavy rotating string—a nonlinear eigenvalue problem. Comm. Pure and Appl. Math. **8**, 395–408 (1955).
14. LADYZHENSKAYA, O. A., The Mathematical Theory of Viscous Incompressible Flow. New York: Gordon and Breach 1963.
15. MOSER, J., A sharp form of an inequality by N. Trudinger. Indiana University Mathematics Journal **20**, 1077–1092 (1971).
16. MOSTOW, G. D., Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. Inst. des Hautes Études Scient., Publ. Math., No **34** (1968).
17. NEHARI, Z., On a nonlinear differential equation arising in nuclear physics. Proc. Royal Irish Academy, **62**, 117–135 (1963).
18. NIRENBERG, L., Remarks on strongly elliptic partial differential equations, Comm. Pure and Appl. Math. **8**, 648–674 (1955).

19. PÓLYA, G., & G. SZEGÖ, Isoperimetric Inequalities in Mathematical Physics. Princeton: Princeton University Press 1951.
20. ROBINSON, P. D., Extremum principles for the equation $\nabla^2 \phi = \phi - \phi^3$. J. Math. Physics **12**, 23–28 (1971).
21. RYDER, G. H., Boundary value problems for a class of nonlinear differential equations. Pac. J. Math. **22**, 477–503 (1967).
22. SYNGE, J. L., On a certain nonlinear differential equation. Proc. Royal Irish Academy **62**, 17–42 (1961).
23. TESHIMA, R. K., M. Sc. Thesis. University of Alberta, Edmonton, Alberta, Canada, 1960.
24. WEISS, S., Nonlinear eigenvalue problems. Ph. D. Dissertation, University of Chicago, August 1969.
25. WOLKOWISKY, J. H., Existence of buckled states of circular plates. Comm. Pure and Appl. Math. **20**, 549–560 (1967).
26. SANSONE, G., Su un'equazione differenziale non lineare della fisica nucleare, Symposia Mathematica, vol. VI. Istituto Nazionale di Alta Matematica, Bologna 1970.

Department of Mathematics
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

(Received October 9, 1971)