Stability of Cosserat Fluid Motions

M. SHAHINPOOR & G. AHMADI

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Abstract

We obtain criteria of stability of the unsteady motion of an incompressible Cosserat fluid in an arbitrary time-dependent domain, employing a general energy method due to J. SERRIN. It is shown that the original motion is stable if $R_e^2 \leqq 80 + 12800\ C_0$ or if $\lambda R_e \leqq 80 + 6400\ C_0$. The quantities R_e and C_0 are the Reynolds number and Cosserat number, respectively, and $-\lambda$ is the lower bound for the eigenvalues of the strain rate tensor D_{ij}.

The theorems established for the stability criteria are universal in the sense that they do not depend either on the shape of the domain or on the distribution of the basic field variables. Finally an experimental scheme is proposed to determine the upper bound of the Cosserat number and consequently the characteristic length of a Cosserat fluid.

Introduction

JAMES SERRIN [1], in a remarkable and now classic paper, examined the exact stability of motion for viscous fluids, employing a general energy technique. He obtained conditions sufficient to ensure the stability of motions for arbitrary disturbances in bounded domains of arbitrary shape. He showed that his results were applicable also to unbounded domains if the disturbances were periodic. He applied his theory also to Couette flows. Following the work of SERRIN [1], JOSEPH [2] extended the investigation to the stability of the Boussinesq equations. Recently RAO [3] employed SERRIN's energy method to investigate the stability of the flow of an incompressible micropolar fluid motions. The model of fluids employed by RAO [3], as presented by ERINGEN [4, 5], consists in a coupled system of differential equations for the two basic vectors, namely velocity and microrotation. In the present paper we consider another model of a viscous fluid with couple stresses. This model is based essentially on the theory of couple stresses proposed by TRUESDELL & TOUPIN [6], TOUPIN [7], and MINDLIN & TIERSTEN [8]. We choose to call the fluid under discussion a *Cosserat fluid.*

Stability criteria for an incompressible Cosserat fluid in unsteady motion in an arbitrary time-dependent domain are obtained. The method employs a positive definite Liapunoff function in the form of the kinetic energy of the difference motion. Then it is shown that the decay of the above function in time will lead to stability of the original flow.

1. Governing Equations of a Cosserat Fluid

We consider the motion of an isotropic material continuum of volume V, bounded by a surface S with outward normal $\boldsymbol{n}$. Upon S act stress and couple stress vectors $\boldsymbol{t}_n$ and $\boldsymbol{m}_n$, and within V act body force and body couple vectors $\boldsymbol{f}$ and $\boldsymbol{C}$. Then the governing equations are

mass:

$$\frac{d}{dt}\int_V \rho\, dV = 0, \tag{1.1}$$

momentum:

$$\frac{d}{dt}\int_V \rho \boldsymbol{v}\, dV = \int_S \boldsymbol{t}_n\, dS + \int_V \rho \boldsymbol{f}\, dV, \tag{1.2}$$

moment of momentum:

$$\frac{d}{dt}\int_V (\boldsymbol{r}\times\boldsymbol{v} + j\,\boldsymbol{\omega})\,\rho\, dV = \int_S (\boldsymbol{r}\times\boldsymbol{t}_n + \boldsymbol{m}_n)\, dS + \int_V (\boldsymbol{r}\times\boldsymbol{f} + \boldsymbol{C})\,\rho\, dV, \tag{1.3}$$

energy:

$$\begin{aligned}\frac{d}{dt}\int_V \left(\frac{v^2}{2} + U + \frac{j\,\omega^2}{2}\right)\rho\, dV \\ = \int_S \left(\boldsymbol{t}_n\cdot\boldsymbol{v} + \tfrac{1}{2}\boldsymbol{m}_n\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})\right) dS - \int_V \boldsymbol{\nabla}\cdot\boldsymbol{q}\, dV + \int_V \left(\boldsymbol{f}\cdot\boldsymbol{v} + \tfrac{1}{2}\boldsymbol{C}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})\right) dV,\end{aligned} \tag{1.4}$$

where d/dt is the material time-derivative, ρ is the mass density, $\boldsymbol{r}$ is the spatial position vector from a fixed origin, $\boldsymbol{v}$ is the material velocity $d\boldsymbol{r}/dt$, $\boldsymbol{\omega}$ is the local angular spin vector, j is the local micro-inertia, U is the internal energy per unit mass, $\boldsymbol{\nabla}$ is the spatial gradient $\partial/\partial\boldsymbol{r}$, $\boldsymbol{q}$ is the heat flux vector. In writing (1.4) we have assumed that the work of the couple stress vector $\boldsymbol{m}_n$, and the body couple vector $\boldsymbol{C}$ on the translation of rotation $\boldsymbol{\omega}$ can be neglected.

The model of the continuum considered in (1.1)–(1.4) is different from that of [6], [7] and [8] due to the presence of independent spin vector $\boldsymbol{\omega}$ and local micromoment of inertia j and heat flux vector $\boldsymbol{q}$. Note also that

$$\boldsymbol{t}_n = \boldsymbol{n}\cdot\boldsymbol{\tau}, \qquad \boldsymbol{m}_n = \boldsymbol{n}\cdot\boldsymbol{\mu}, \tag{1.5}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ are the stress tensor and couple stress tensor, respectively.

Definition of a Cosserat Fluid. We choose to define a Cosserat fluid as one for which

$$\boldsymbol{\omega} = \tfrac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{v}, \qquad j\approx 0. \tag{1.6}$$

For a Cosserat fluid the governing equations, which are based on (1.6) and (1.1)–(1.5) reduce to

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\boldsymbol{v} = 0, \tag{1.7}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}^s + \tfrac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot\boldsymbol{\mu}^d + \rho\boldsymbol{f} + \tfrac{1}{2}\boldsymbol{\nabla}\times\rho\boldsymbol{C} = \rho\mathring{\boldsymbol{v}}, \tag{1.8}$$

$$\rho\frac{dU}{dt} = \boldsymbol{\tau}\colon \boldsymbol{\nabla}\boldsymbol{v} - \tfrac{1}{2}(\boldsymbol{\tau}\times\boldsymbol{I})\cdot\boldsymbol{\nabla}\times\boldsymbol{v} + \tfrac{1}{2}\boldsymbol{\mu}\colon \boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v} - \boldsymbol{\nabla}q, \tag{1.9}$$

where the superscripts s and d denote the symmetric and deviatoric parts, respectively.

Constitutive Equations for a Cosserat Fluid: For our fluid we choose to consider linear constitutive equations analogous to those of AERO & KUVSHINSKII [9] and the linearized equations of TOUPIN [7]. In addition a hydrostatic pressure term appears in the constitutive equations, thus:

$$\boldsymbol{\tau}^s = -p\boldsymbol{I} + \lambda \nabla \cdot \boldsymbol{v}\boldsymbol{I} + \mu(\nabla \boldsymbol{v} + \boldsymbol{v}\nabla), \tag{1.10}$$

$$\boldsymbol{\mu}^d = 2\eta \nabla\nabla \times \boldsymbol{v} + 2\eta' \nabla \times \boldsymbol{v}\nabla, \tag{1.11}$$

where $\lambda, \mu, \eta, \eta'$, are material constants. As in [5], it may be shown that the Clausius-Duhem Inequality imposes the following restrictions on the constants:

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad -1 < \eta'/\eta < 1, \quad \eta > 0. \tag{1.12}$$

Furthermore, as shown in [8], the restrictions (1.12) also ensure the uniqueness of solutions for this problem.

Finally the displacement equation of motion based on (1.10), (1.11), (1.8) with $\boldsymbol{f}$ and $\boldsymbol{C}$ set equal to 0 is

$$-\nabla p + \mu \nabla^2 \boldsymbol{v} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{v} + \eta \nabla^2 \nabla \times \nabla \times \boldsymbol{v} = \rho \dot{\boldsymbol{v}}. \tag{1.13}$$

Boundary Conditions: We assume that on $S(t)$ the velocity and angular velocity are prescribed arbitrarily:

$$\boldsymbol{v}(x, t) = \boldsymbol{U}(x, t) \qquad \text{on } S(t), \tag{1.14}$$

$$\boldsymbol{\omega}(x, t) = \tfrac{1}{2}\nabla \times \boldsymbol{v}(x, t) = \boldsymbol{\Omega}(x, t) \qquad \text{on } S(t). \tag{1.15}$$

These conditions reflect a sort of super adherence [1], [3], [10]. According to KOITER [11] the conditions (1.14) and (1.15) are kinematically admissible.

2. Serrin Energy Criterion

Let us consider a basic motion ($\boldsymbol{v}$) of an incompressible Cosserat fluid in a time-dependent domain $V(t)$ bounded by the surface $S(t)$. Further, we assume that on $S(t)$ we have the conditions (1.14) and (1.15). Suppose that the motion is disturbed from the basic flow velocity $\boldsymbol{v}$ to a disturbed flow velocity $\boldsymbol{v}^*$. Now to determine whether or not the disturbed flow approaches the basic flow asymptotically and in the mean as $t \to \infty$ we employ a positive-definite Liapunoff function in the form of the kinetic energy of the difference motion $\boldsymbol{\mu} = \boldsymbol{v}^* - \boldsymbol{v}$. The kinetic energy of the difference motion is*:

$$T = \tfrac{1}{2}\int_V \rho \mu^2 \, dV + \tfrac{1}{2}\int_V \rho j \omega^2 \, dV. \tag{2.1}$$

Note that in the model considered it was assumed that $j \simeq 0$ so that the second term in the right-hand side of (2.1) can be dropped. Thus we consider only a

* Note that this is not the difference in kinetic energies of the starred and the basic flow.

Liapunoff measure of stability in the form

$$T=\tfrac{1}{2}\int_V \rho \mu^2 \, dV, \qquad \boldsymbol{\mu}=\nabla\times\boldsymbol{\mu}=0 \quad \text{on } S(t). \tag{2.2}$$

we note that from (2.2) and the Leibnitz rule that

$$\frac{dT}{dt}=\int_V \rho \boldsymbol{\mu}\cdot\frac{\partial \boldsymbol{\mu}}{\partial t}\, dV. \tag{2.3}$$

Further,

$$\rho \dot{\boldsymbol{v}}^* - \rho \dot{\boldsymbol{v}} = \rho\left(\frac{\partial \boldsymbol{\mu}}{\partial t}+\boldsymbol{\mu}\cdot\nabla\boldsymbol{v}+\boldsymbol{v}\cdot\nabla\boldsymbol{\mu}+\boldsymbol{\mu}\cdot\nabla\boldsymbol{\mu}\right), \tag{2.4}$$

so that from (2.3) and (2.4)

$$\frac{dT}{dt}=\int_V \boldsymbol{\mu}\cdot(\rho\dot{\boldsymbol{v}}^*-\rho\dot{\boldsymbol{v}})\,dV-\int_V(\boldsymbol{\mu}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\mu}+\boldsymbol{v}\cdot\nabla\boldsymbol{\mu}\cdot\boldsymbol{\mu}-\boldsymbol{\mu}\cdot\nabla\boldsymbol{\mu}\cdot\boldsymbol{\mu})\,dV. \tag{2.5}$$

Since the conditions (1.14) and (1.15) ensure that $\boldsymbol{\mu}=\nabla\times\boldsymbol{\mu}=0$ on $S(t)$, and since $\nabla\cdot\boldsymbol{v}^*=\nabla\cdot\boldsymbol{v}=0$ because the fluid is incompressible,

$$\int_V \boldsymbol{v}\cdot\nabla\boldsymbol{\mu}\cdot\boldsymbol{\mu}=\int_V \tfrac{1}{2}\nabla\cdot(\boldsymbol{v}\mu^2)\,dv=\int_S \tfrac{1}{2}\boldsymbol{v}\mu^2\cdot d\boldsymbol{S}=0, \tag{2.6}$$

$$\int_V \boldsymbol{\mu}\cdot\nabla\boldsymbol{\mu}\cdot\boldsymbol{\mu}=\int_V \tfrac{1}{2}\nabla\cdot(\boldsymbol{\mu}\mu^2)\,dV=\int_S \tfrac{1}{2}\boldsymbol{\mu}\mu^2\cdot d\boldsymbol{S}=0, \tag{2.7}$$

and furthermore from (1.13) and certain vector identities

$$\begin{aligned}\boldsymbol{\mu}\cdot(\rho\mathring{\boldsymbol{v}}^*-\rho\mathring{\boldsymbol{v}})&=-\nabla\cdot[\boldsymbol{\mu}(p^*-p)]\\&\quad-\mu[(\nabla\times\boldsymbol{\mu})^2-\nabla\cdot(\boldsymbol{\mu}\times\nabla\times\boldsymbol{\mu})]-\eta[\nabla\cdot(\boldsymbol{\mu}\times\nabla\times\nabla\times\nabla\times\boldsymbol{\mu})\\&\quad+\nabla\cdot(\nabla\times\boldsymbol{\mu}\times\nabla\times\nabla\times\boldsymbol{\mu})+(\nabla\times\nabla\times\boldsymbol{\mu})^2].\end{aligned} \tag{2.8}$$

Since all surface integrals involving $\boldsymbol{\mu}$ or $\nabla\times\boldsymbol{\mu}$ vanish,

$$\frac{dT}{dt}=-\int_V[\mu(\nabla\times\boldsymbol{\mu})^2+\eta(\nabla\times\nabla\times\boldsymbol{\mu})^2+\rho\boldsymbol{\mu}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\mu}]\,dV. \tag{2.9}$$

We eliminate dimensions by putting

$$\boldsymbol{\mu}=V_0\boldsymbol{\mu}', \quad \nabla=\frac{1}{L_0}\nabla', \quad t=t'\frac{L_0}{V_0}, \quad \boldsymbol{v}=V_0\boldsymbol{v}', \quad T'=\frac{1}{\rho V_0^2}T. \tag{2.10}$$

Then from (2.9) and (2.10)

$$\frac{dT'}{dt'}=-\int_V\left(\frac{1}{R_e}(\nabla'\times\boldsymbol{\mu}')^2+\frac{C_0}{R_e}(\nabla'\times\nabla'\times\boldsymbol{\mu}')^2+\boldsymbol{\mu}'\cdot\nabla'\boldsymbol{v}'\cdot\boldsymbol{\mu}'\right)dV, \tag{2.11}$$

where

$$\begin{aligned}&R_e\equiv\text{Reynolds number}=\frac{\rho V_0 L_0}{\mu},\\&C_0\equiv\text{Cosserat number}=\frac{\eta}{\mu L_0^2},\\&V_0\equiv\text{Maximum velocity},\\&L_0\equiv\text{Reference dimension of the domain.}\end{aligned} \tag{2.12}$$

SERRIN [1], [10] has developed certain vector inequalities for spherical domains. Let us now assume that our region $V(t)$ can be included in a sphere of diameter L_0. Then according to SERRIN [1] we have the following inequalities*:

$$\int_V (\nabla' \boldsymbol{\mu}')^2 \, dV = \int_V (\nabla' \times \boldsymbol{\mu}')^2 \, dV \geqq 80 \int_V \mu'^2 \, dV,$$

$$\int_V (\nabla' \times \nabla' \times \boldsymbol{\mu}')^2 \geqq 80 \int_V (\nabla' \times \mu')^2 \, dV \geqq 6400 \int_V \mu'^2 \, dV. \tag{2.13}$$

Also note that

$$\int_V (\boldsymbol{\mu}' \cdot \nabla' \boldsymbol{v}' \cdot \boldsymbol{\mu}') \, dV = -\int_V (\boldsymbol{\mu}' \cdot \nabla' \boldsymbol{\mu}' \cdot \boldsymbol{v}') \, dV, \tag{2.14}$$

and, as shown by SERRIN [11],

$$\boldsymbol{\mu}' \cdot \nabla' \mu' \cdot \boldsymbol{v}' \leqq \frac{1}{2R_e} (\nabla' \boldsymbol{\mu}')^2 + \frac{R_e}{2} \boldsymbol{\mu}'^2 \boldsymbol{v}'^2. \tag{2.15}$$

From (2.11), (2.13), (2.15)

$$\frac{dT'}{dt'} \leqq \left[-\int_V \mu'^2 \, dV\right] \left[\frac{4_0}{R_e} + 6400 \frac{C_0}{R_e} - \frac{R_e}{2}\right], \tag{2.16}$$

3. First Condition of Universal Stability

Theorem 1. *If the Reynolds number R_e and the Cosserat number C_0 of a Cosserat fluid in a bounded domain $V(t)$ of space are such that*

$$R_e^2 \leqq 80 + 12800\, C_0, \tag{3.1}$$

the Liapunoff measure of stability T' tends to zero as $t \to \infty$, and the basic flow is stable.

We note that if (3.1) holds, then from (2.16)

$$T'(t) \leqq T'(0) \exp[-a t'], \tag{3.2}$$

$$T'(t) \to 0$$

$$t' \to \infty$$

where

$$a = \frac{40}{R_e} + 6400 \frac{C_0}{R_e} - \frac{R_e}{2} \geqq 0. \tag{3.3}$$

4. Second Condition of Universal Stability

We may, alternatively, write

$$\int_V (\boldsymbol{\mu}' \cdot \nabla' \boldsymbol{v}' \cdot \boldsymbol{\mu}') \, dV = \int_V (\boldsymbol{\mu}' \cdot \boldsymbol{D}' \cdot \boldsymbol{\mu}') \, dV, \tag{4.1}$$

where $\boldsymbol{D}'$ is the non-dimensional strain-rate matrix in the form

$$\boldsymbol{D}' = \tfrac{1}{2}(\nabla' \boldsymbol{v}' + \boldsymbol{v}' \nabla'). \tag{4.2}$$

* The constants employed here are the improved ones obtained by L. E. PAYNE & H. F. WEINBERGER.

Now we have the inequality

$$\boldsymbol{\mu}' \cdot \boldsymbol{D}' \cdot \boldsymbol{\mu}' \geqq -\lambda \mu'^2, \tag{4.3}$$

where $-\lambda$ is a lower bound for the eigenvalues of $\boldsymbol{D}'$. Now from (2.11), (2.13), (4.1), (4.2), (4.3),

$$\frac{dT'}{dt'} \leqq [-\int \mu'^2 \, dV] \left[\frac{80}{R_e} + 6400 \frac{C_0}{R_e} - \lambda\right]. \tag{4.4}$$

Theorem 2. *If the Reynolds number R_e and the Cosserat number C_0 of a Cosserat fluid in a bounded domain $V(t)$ of space are such that*

$$\lambda R_e \leqq 80 + 6400\, C_0, \tag{4.5}$$

the Liapunoff measure of stability T' tends to zero as $t \to \infty$, and the basic flow is stable.

Note that if (4.5) holds, then from (4.4)

$$T'(t) \leqq T'(0) \exp[-b t'], \tag{4.6}$$

$$T'(t) \to 0$$

$$t' \to \infty$$

where

$$b = \frac{8_0}{R_e} + 6400 \frac{C_0}{R_e} - \lambda \geqq 0. \tag{4.7}$$

As (3.1) and (4.5) indicate, disturbances in the Cosserat fluid flows are damped more rapidly than in the corresponding viscous Navier-Stokes fluid flows. This should be no surprise, for a Cosserat fluid has an additional dissipative mechanism.

From the theorems above the following corollaries are established once and for all:

Corollary 1. (Uniqueness of unsteady Cosserat fluid flows.) *If $\boldsymbol{v}$ and $\boldsymbol{v}^*$ are two flows in a bounded region $V(t)$ having the same velocity distribution at $t=0$ and on the boundary of V, then they must be identical.*

The proof of the corollary is simply seen by putting $\boldsymbol{\mu} = \boldsymbol{v}^* - \boldsymbol{v}$ and noting that μ must satisfy equations (3.2) or (4.6). But since at $t=0$, $\mu=0$, we find that $T'(t)=0$ for all t, and therefore $\boldsymbol{\mu}=0$, and hence $\boldsymbol{v}$ must be equal to $\boldsymbol{v}^*$ at all time $t>0$.

Corollary 2. (Uniqueness of steady Cosserat fluid flows.) *If $\boldsymbol{v}$ and $\boldsymbol{v}^*$ are two steady motions of a Cosserat fluid in a bounded domain $V(t)$ subject to the adherence boundary conditions, and if the relations* (3.1) *and* (4.5) *hold for both of them, then the two motions are identical.*

The proof of the corollary is seen easily by putting $\boldsymbol{\mu} = \boldsymbol{v}^* - \boldsymbol{v}$ and noting that $\boldsymbol{\mu}$ must also be steady. On the other hand (3.2) and (4.6) must also hold. This is possible only if $T'(t)=0$ for all t, and therefore $\boldsymbol{\mu}=0 \rightleftarrows \boldsymbol{v}^* = \boldsymbol{v}$.

5. Program for Experimental Determination of the Upper Bound of the Cosserat Number

One of the consequences of the above universal stability analysis will come to light in experimental determination of an upper bound for the Cosserat number C_0. As mentioned by MINDLIN & TIERSTEN [8], the ratio

$$L=\sqrt{\eta/\mu} \tag{5.1}$$

is actually a material length, the characteristic length of the Cosserat fluid. Inasmuch as the common theory of elasticity has been verified experimentally in great detail, L is probably very small in comparison with bodily dimensions and wave-lengths that are commonly encountered. It is now clear that the Cosserat number C_0 is in fact the square of a non-dimensional characteristic length of the Cosserat medium. We propose here a scheme for determining experimentally an upper bound for C_0 and consequently an upper bound for the material length L. Suppose that in the laboratory a certain flow of a Cosserat fluid be produced, the flow velocity $\boldsymbol{v}$ of which can always be determined. Suppose also that the experiment be such that we can increase $\boldsymbol{v}$ as desired. Upon observation of the system, we may predict the instant of instability, *i.e.*, when any increase in the flow velocity $\boldsymbol{v}$ will lead to a drastic change of the flow field, a radically different flow velocity $\boldsymbol{v}^*$. This is the instant of instability. According to Theorem 1 this happens when

$$R_e^2 \geqq 80+12800\, C_0, \tag{5.2}$$

so that the characteristic length will satisfy the relation

$$\sqrt{\frac{\eta}{\mu}}=L \leqq \frac{\sqrt{2}}{160} L_0 (R_e^2-80)^{\frac{1}{2}}. \tag{5.3}$$

References

1. SERRIN, J., Arch. Rational Mech. Anal. **3**, 1 (1959).
2. JOSEPH, D. D., Arch. Rational Mech. Anal. **20**, 59 (1965).
3. RAO, S. K. L., Int. J. Eng'g. Sci. **8** (1970).
4. ERINGEN, A. C., Int. J. Eng'g. Sci. **2**, 205 (1964).
5. ERINGEN, A. C., J. Math. Mech. **16**, 1 (1966).
6. TRUESDELL, C., & R. A. TOUPIN, The Classical Field Theories. Encyclopedia of Physics, Vol. III/1, Ed. FLÜGGE 1960.
7. TOUPIN, R. A., Arch. Rational Mech. Anal. **11**, 385 (1962).
8. MINDLIN, R. D., & H. F. TIERSTEN, Arch. Rational Mech. Anal. **11**, 415 (1962).
9. AERO, E. L., & E. V. KUVSHINSKII, Fiziha Tverdogo Tela **2** (1960) (Trans. Soviet Physics Solid State **2** (1961)).
10. SERRIN, J., Mathematical Principles of Classical Fluid Mechanics. Handbuch der Physik, VIII/1, Ed. FLÜGGE & TRUESDELL. Berlin-Göttingen-Heidelberg: Springer 1959.
11. KOITER, W. T., Proc. Roy. Netherlands Academy of Sciences, Series B, LXVII, 1, 1964.

Department of Mechanical Engineering
Pahlavi University
Shiraz
Iran

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