Nonlinear Scalar Field Equations, I
Existence of a Ground State

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Introduction

This paper as well as a subsequent one (part II of this study, immediately following in this journal) is concerned with the existence of nontrivial solutions for some semi-linear elliptic equations in $\mathbb{R}^N$. Such problems are motivated in particular by the search for certain kinds of solitary waves (stationary states) in nonlinear equations of the Klein-Gordon or Schrödinger type. To be more precise, consider the following nonlinear Klein-Gordon equation

(1) $$\Phi_{tt} - \Delta\Phi + a^2\Phi = f(\Phi).$$

where $\Phi = \Phi(t, x)$ is a complex function defined on $t \in \mathbb{R}$, and $x \in \mathbb{R}^N$, $\Delta\Phi = \sum_{i=1}^{N} \partial^2\Phi/\partial x_i^2$, and a is a real constant. Suppose that

(2) $$f(\varrho e^{i\theta}) = f(\varrho)\, e^{i\theta}, \qquad \forall \varrho, \theta \in \mathbb{R}.$$

Hence, we may assume that $f: \mathbb{R} \to \mathbb{R}$ is a real continuous function which is odd, and $f(0) = 0$. Equation (1) corresponds to the Lagrangian density

$$\mathscr{L}_\Phi = -\frac{1}{2}|\Phi_t|^2 + \frac{1}{2}|\nabla\Phi|^2 + \frac{a^2}{2}|\Phi|^2 - F(|\Phi|),$$

where $\nabla\Phi = (\partial\Phi/\partial x_1, \ldots, \partial\Phi/\partial x_n)$ and

$$F(\varrho) = \int_0^{\varrho} f(s)\, ds, \qquad \varrho \in \mathbb{R}.$$

Then, looking for a solitary wave in (1) of "standing wave" type, that is, Φ of the form $\Phi(t, x) = e^{i\omega t} u(x)$, $\omega \in \mathbb{R}$, and $u : \mathbb{R}^N \to \mathbb{R}$, one is led to the equation

$$-\Delta u + mu = f(u) \text{ in } \mathbb{R}^N, \tag{3}$$

where $m = a^2 - \omega^2$. Notice that $u \equiv 0$ is always a trivial solution of (3), while of course one is interested in nontrivial solutions, that is $u \not\equiv 0$.

In terms of u, the Lagrangian $S(u)$ has the expression

$$S(u) = \int_{\mathbb{R}^N} \mathscr{L}_\Phi\, dx = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{m}{2} \int_{\mathbb{R}^N} u^2\, dx - \int_{\mathbb{R}^N} F(u)\, dx. \tag{4}$$

For physical reasons, one wants the Lagrangian to be finite, and hence one requires u to vanish at infinity. This plays the role of a boundary condition for (3). Therefore, we impose here the condition $u \in H^1(\mathbb{R}^N)$. Actually, this condition more than suffices for our purpose since, as will be seen later, in most cases the solutions will have exponential decay at infinity. A weaker condition than $u \in H^1(\mathbb{R}^N)$ is considered in section 5 (namely, $|\nabla u| \in L^2(\mathbb{R}^N)$).

Another classical type of solitary waves is that of travelling waves. Consider a real Klein-Gordon equation (1), that is $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Then, looking for a travelling wave solution of (1), that is Φ of the form $\Phi(t, x) = u(x - ct)$ where $u : \mathbb{R}^N \to \mathbb{R}$, and $c \in \mathbb{R}^N$ is a fixed vector such that $|c| < 1$, one obtains the following equation for u:

$$-\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} + a^2 u = f(u) \quad \text{in } \mathbb{R}^N. \tag{5}$$

Here $a_{ij} = \delta_{ij} - c_i c_j$, c_i being the coordinates of c. It is easily checked, using the fact that $|c| < 1$, that the constant coefficient operator in the left hand side of (5) is elliptic. Thus, after a change of coordinates, (5) can be converted into an equation of type (3).

Stationary states of nonlinear Schrödinger equations lead to similar problems. Indeed, consider the equation

$$i\Phi_t - \Delta\Phi = f(\Phi), \tag{6}$$

where $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ and f satisfies the symmetry property (2). Then, looking for standing wave solutions, that is $\Phi(t, x) = e^{-imt} u(x)$, one is again led to problem (3).

To sum up, we consider the following semi-linear elliptic problem

$$(*) \qquad \begin{aligned} &-\Delta u = g(u) \text{ in } \mathbb{R}^N, \\ &u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0, \end{aligned}$$

where we always assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function which is odd, and thus $g(0) = 0$.

Equations of type (*) arise in various other contexts of physics (for example, the classical approximation in statistical mechanics, constructive field theory, false vacuum in cosmology, nonlinear optics, laser propagations, etc.) They are also called nonlinear Euclidean scalar field equations (cf. e.g. [6, 28, 34, 35]).[1]

The Lagrangian associated with (*), $S(u)$ is defined by

(7) $$S(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx,$$

where $G(s) = \int_0^s g(t) \, dt$. The functional $S(u)$ is also called the "action" associated with (*) (when (*) is thought of as a Euclidean field equation; cf. e.g. [28, 29]). Moreover, by analogy with nonlinear elliptic problems in bounded domains, $S(u)$ is sometimes (unprecisely) called the energy associated with (*). In the following, we use indifferently the terms action or Lagrangian to designate S.

In a totally different context, a solution of (*) can also be interpreted as a non-trivial stationary state for a nonlinear heat equation

(8) $$\frac{\partial \psi}{\partial t} - \Delta \psi = g(\psi),$$

$\psi = \psi(t, x)$, $t \geqq 0$, $x \in \mathbb{R}^N$. Such problems arise in biology, especially in population dynamics theory (cf. e.g. [7, 33, 60] and the survey article of FIFE [32]).

There is an important and well known litterature about semi-linear elliptic boundary value problems in a bounded domain of $\mathbb{R}^N$. We refer the reader for instance to [2, 5, 20, 48, 53] for the existence of positive solutions and to [4, 5, 22, 23, 25, 26, 39, 53] for the existence of an infinite number of distinct solutions. Evidently, a striking contrast between semi-linear elliptic boundary value problems on a bounded domain and on $\mathbb{R}^N$ is the apparent lack of compactness in treating the latter. Therefore, a first natural approach to (*) would be to use the above works and to approximate a solution of (*) by a solution of an analogous problem on the ball $B_R = \{x \in \mathbb{R}^N, |x| < \mathbb{R}\}$, that is, first solve $-\Delta u_R = g(u_R)$ in B_R, $u_R \mid_{\partial B_R} = 0$, and then let $R \to +\infty$. One of the difficulties to overcome in such an approach is the absence of uniform a priori bounds (i.e. independent of R) in the works mentioned above. This method is nevertheless developed in [9], though it requires some restrictions of a technical nature on the nonlinear term g. Here and in part II, we study (*) by using variational methods, working with an appropriate constraint in order to have some compactness. This constraint can be made transparent because of the "autonomous" character of (*) and the fact that one can use scale changes in $\mathbb{R}^N$. The fact that g is "autonomous" (that is, depends only on u), and the operator is the Laplacian (or a constant coefficient elliptic operator), constitute the main restrictions on the method presented in this study.

A special feature of (*) is its invariance under the group of displacements. That is, if $\mathcal{R}$ is a rotation in $\mathbb{R}^N$ and $C \in \mathbb{R}^N$ is a fixed vector, then, for any solution u of (*), the function v defined by $v(x) = u(\mathcal{R}x + C)$ is also a solution of

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[1] References are gathered at the end of part II of this study, following in this journal.

(*). Such an indeterminacy will not be present in what follows, since we will be seeking "radial" solutions of (*), that is, solutions u with spherical symmetry: u depends only on $|x|$. Such solutions are sometimes called "particle-like" (cf. e.g. [6]). In this case, u, as a function of $r = |x|$, satisfies the ordinary differential equation

$$-\frac{d^2u}{dr^2} - \frac{N-1}{r}\frac{du}{dr} = g(u), \qquad r \in (0, +\infty). \tag{9}$$

One can also obtain certain existence results by analysing (9) using a "shooting" type argument (cf. [15]). This approach, as well as the local one evoked above, also provides results for the "nonautonomous" case, that is when g is allowed to depend also on r. However, the optimal results for the autonomous case which we present here are only obtained by working directly with (*).

In section 4.3 we show, following an argument of COLEMAN, GLAZER and MARTIN [29], that the solution u_0 of (*) which we derive from our variational problem has the property of having the least action among all possible solutions of (*), namely $0 < S(u_0) \leqq S(u)$, for any solution u of (*). Such a solution u_0 is called a "ground state" for (*). It can be shown that a ground state is necessarily a positive and radial solution of (*).

It is conjectured (and not known except for a very special case, see [27]) that, at least for certain classes of nonlinearities g, the positive radial solution of (*) is unique. Therefore, it is customary (though not quite correct) to call a positive radial solution of (*) a ground state. Solutions u of (*) such that $S(u) > S(u_0)$ are called "bound states". In this paper (part I), we prove the existence of a ground state. In the second part of this study, we show the existence of infinitely many distinct bound states u_k of (*), $k \in \mathbb{N}$, with furthermore $S(u_k) \uparrow +\infty$. The proof there rests on two results in critical point theory (cf. section 8 in part II), the main value of these theorems being to allow a greater flexibility in the choice of the manifold defining the constraint.

Several of the results presented here were announced in [8, 11, 12]. Some results concerning the existence of solutions for certain semi-linear weakly coupled elliptic systems in $\mathbb{R}^N$, as well as a discussion of certain bifurcation questions in (*), can be found in [12, 13].

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1. The Main Result; Examples

Throughout this paper, with the exceptions of sections 5 and 6, it will be assumed that the dimension of the space N is at least 3. We recall that $g: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$. We also assume that g is odd.

We consider the problem

$$(*)\qquad \begin{aligned} -\Delta u &= g(u) \text{ in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), &\quad u \not\equiv 0. \end{aligned}$$

The function g is required to satisfy the following conditions:

$$(1.1)\qquad -\infty < \varliminf_{s\to 0^+} g(s)/s \leqq \varlimsup_{s\to 0^+} g(s)/s = -m < 0.$$

$$(1.2)\qquad -\infty \leqq \varlimsup_{s\to +\infty} g(s)/s^l \leqq 0, \quad \text{where } l = \frac{N+2}{N-2}.$$

$$(1.3)\qquad \text{There exists } \zeta > 0 \text{ such that } \quad G(\zeta) = \int_0^\zeta g(s)\,ds > 0.$$

The following theorem concerns the existence of a ground state of (*).

Theorem 1. *Suppose* $N \geqq 3$ *and that g satisfies* (1.1)–(1.3). *Then* (*) *possesses a solution u such that*

i) $u > 0$ *on* $\mathbb{R}^N$.
ii) u is spherically symmetric: $u(x) = u(r)$, *where* $r = |x|$, *and u decreases with respect to r.*
iii) $u \in C^2(\mathbb{R}^N)$.
iv) u together with its derivatives up to order 2 *have exponential decay at infinity:*

$$|D^\alpha u(x)| \leqq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^N,$$

for some $C, \ \delta > 0$ *and for* $|\alpha| \leqq 2$.

This theorem will be proved in section 3 by means of a constrained minimization method. Let us first illustrate the result by giving a few simple examples. We will also derive some necessary conditions in section 2.

Example 1. Consider the equation

$$(1.4)\qquad \begin{aligned} -\Delta u + mu &= \lambda\, |u|^{p-1} u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), &\quad u \not\equiv 0, \end{aligned}$$

where λ and m are positive constants and $p > 1$. This equation was treated by S. Pohozaev [51]; he showed that (1.4) possesses a solution if and only if $1 < p < \frac{N+2}{N-2}$. Notice that in this case the hypotheses of Theorem 1 reduce to the same condition. The fact that for $p \geqq \frac{N+2}{N-2}$, (1.4) does not have any solution follows from a well known identity of Pohozaev [51] which we recall in section 2. The method of Pohozaev consisted in maximizing

$$\frac{\lambda}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\,dx$$

over the set

$$\left\{u \in H^1(R^N); \quad \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{m}{2}\int_{\mathbb{R}^N} u^2\,dx = 1\right\}.$$

The constraint causes a Lagrange multiplier to appear, and one obtains a positive solution of $-\Delta u + mu = \lambda\theta u^p$. The Lagrange multiplier θ, shown to be positive, can then be removed by looking for a solution $v = \sigma u$, $\sigma > 0$, and using the special homogeneity feature of (1.4). Equation (1.4) was also studied by BERGER [16, 17] and by COFFMAN [27] who showed (1.4) to possess infinitely many distinct solutions, using the same special feature of homogeneity of (1.4).

Example 2.

$$\text{(1.5)} \qquad \begin{gathered} -\Delta u + mu = \lambda\,|u|^{p-1}\,u - \mu\,|u|^{q-1}\,u \quad \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \qquad u \not\equiv 0, \end{gathered}$$

where λ, μ, m are positive constants and $p \neq q$, $1 < p, q$. Here the hypotheses of theorem 1 reduce to $1 < p < \max\left(\frac{N+2}{N-2}, q\right)$ and to the existence of $\zeta > 0$ such that

$$G(\zeta) = \frac{\lambda}{p+1}\zeta^{p+1} - \frac{\mu}{q+1}\zeta^{q+1} - \frac{m}{2}\zeta^2 > 0.$$

The second condition is automatically fulfilled if $q < p$. Thus Theorem 1 applies for $1 < q < p < \frac{N+2}{N-2}$. This case was treated by STRAUSS [55], who also showed the existence of an infinite number of solutions. If $q \leqq \frac{N+2}{N-2} \leqq p$, it follows from the Pohozaev identity (see Section 2 below) that there cannot exist a non-trivial solution of (1.5). The case $\frac{N+2}{N-2} < q < p$ remains open: it is not known whether or when there exists a solution of (1.5). Lastly, when $p < q$, Theorem 1 applies if there is a $\zeta > 0$ such that $G(\zeta) > 0$. Actually, again from the Pohozaev identity, it can be seen that (1.5) has no solution if $G(\zeta) \leqq 0$ for all $\zeta > 0$. Hence the condition (1.3) is in this case both necessary and sufficient. This condition means, in particular, that for given $\mu, m > 0$, there exists $\lambda^* > 0$ such that (1.5) has no solution for $0 < \lambda \leqq \lambda^*$, while Theorem 1 applies for $\lambda > \lambda^*$. For $p < q$ a weaker result was obtained in [55], namely the existence of at least one $\lambda > 0$ for which (1.5) possesses a positive solution). In fact λ^* has the expression

$$\lambda^* = \left(\frac{m}{2}\right)^a \mu^b (q-1)\,(p+1)\,(q-p)^{-a}\,(p-1)^{-b}\,(q+1)^{-b},$$

where $a = \frac{q-p}{q-1}$ and $b = \frac{p-1}{q-1}$. In the particular case of (1.5) with $m = 1$, $\lambda = 1$, $p = 3$, $q = 5$, which was studied by ANDERSON [6], the requirement

(1.3) yields exactly $\mu < 3/16$. This explains why the latter condition appears in [6].

The proof of Theorem 1 in the particular framework of equation (1.5). which serves as a model and for which technicalities are somewhat simpler, is developed in [11].

Equations of type (*) have been considered in a number of works in addition to the ones already mentioned. The one dimensional problem ($N = 1$) was studied by STUART [56] and DANCER [30], while in higher dimensions, existence results were obtained by NEHARI [47], SYNGE [58], RYDER [54]. The first general study of this type of equations is due to STRAUSS [55]. A general result for the existence of a ground state is given in COLEMAN, GLAZER and MARTIN [29].

2. Necessary Conditions

2.1. *Pohozaev's identity*. Several necessary conditions for the existence of a solution of problem (*) can be derived from an identity which seems to be due to S. I. POHOZAEV [51][1]. It asserts that a solution of (*) which, together with its first derivatives, is sufficiently small at infinity, necessarily satisfies

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx = N\int_{\mathbb{R}^N} G(u)\,dx \tag{2.1}$$

(G always denotes the function $G(z) = \int_0^z g(s)\,ds$). Before being more precise, let us give a formal argument explaining (2.1). Define the two functionals

$$T(u) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx, \qquad V(u) = \int_{\mathbb{R}^N} G(u)\,dx.$$

(By analogy, $\frac{1}{2}T$ corresponds to kinetic energy while V corresponds to potential energy. Thus $S = \frac{1}{2}T - V$.) Consider a scale change in $\mathbb{R}^N$: for $\sigma > 0$ define $u_\sigma(x) = u(x/\sigma)$. One readily checks that

$$T(u_\sigma) = \sigma^{N-2}T(u), \qquad V(u_\sigma) = \sigma^N V(u).$$

Hence $S(u_\sigma) = \frac{\sigma^{N-2}}{2}T(u) - \sigma^N V(u)$. Now, if u is a solution of (*) then at least formally it can be interpreted as a critical point of S. Therefore, one has $\frac{d}{d\sigma}S(u_\sigma)_{|\sigma=1} = 0$, which is precisely (2.1).

The preceding argument is not rigorous, however, for two reasons at least. Firstly, one needs to know that S is a C^1 functional on the space where it is defined (see the Appendix of Part I, for such a result). Then, one also needs to show that $\frac{d}{d\sigma}u_\sigma(x)|_{\sigma=1} = -\nabla u(x)\cdot x$ lies in the right integration space.

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[1] This identity is also known under other names. In particular it is also associated with the names of ROSEN and DERRICK. Notice moreover that it is just the "Virial theorem".

We now proceed to give a rigorous proof of the fact that *any* solution of (*) satisfies Pohozaev's identity (2.1). This will be derived as a corollary of the following more general statement. Let us remark that this level of generality—even in the case of the corollary 1 below—seems to be new for the identity.

Proposition 1. *Suppose g is a continuous function*: $\mathbb{R} \to \mathbb{R}$ *such that* $g(0) = 0$, *and let* $G(t) = \int_0^t g(s)\, ds$. *Let u satisfy*

$$-\Delta u = g(u) \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

Assume furthermore that

$$u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N), \quad \nabla u \in L^2(\mathbb{R}^N), \quad G(u) \in L^1(\mathbb{R}^N).$$

Then u satisfies

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u)\, dx. \tag{2.1}$$

Remark 2.1. The condition $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ can still be weakened. As for the conditions $\nabla u \in L^2(\mathbb{R}^N)$, $G(u) \in L^1(\mathbb{R}^N)$, they are needed for the integrals in (2.1) to make sense at all.

Proof of Proposition 1. In this proof we write $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i\, \partial x_j}$ and we adopt the summation convention on repeated indices. Observe first that because of $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, standard regularity theory (see e.g. section 4.1 below) shows $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for any q, $1 \leqq q < +\infty$. In the first part of the proof we use the device of POHOZAEV [51], multiplying the equation by $x_i u_i$ and integrating by parts to get the Pohozaev identity on a ball $B_R = \{x \in \mathbb{R}^N, |x| < R\}$. We then show the boundary term (on ∂B_R) to converge to 0 as $R \to +\infty$. Indeed, integration by parts yields

$$\int_{B_R} g(u)\, u_i x_i\, dx = \int_{B_R} \frac{\partial}{\partial x_i}(G(u))\, x_i\, dx = -N \int_{B_R} G(u)\, dx + \int_{\partial B_R} G(u)\, x_i n_i\, dS.$$

But $-u_{jj} = g(u)$ and we have

$$-\int_{B_R} u_{jj} u_i x_i\, dx = \int_{B_R} u_j(\delta_{ij} u_i + x_i u_{ij})\, dx - \int_{\partial B_R} u_j n_j x_i u_i\, dS$$

$$= \int_{B_R} |\nabla u|^2\, dx - \frac{N}{2} \int_{B_R} |\nabla u|^2\, dx - \frac{1}{2} \int_{\partial B_R} \left|\frac{\partial u}{\partial n}\right|^2 R\, dS.$$

Thus,

$$\int_{B_R} |\nabla u|^2\, dx - \frac{2N}{N-2} \int_{B_R} G(u)\, dx = \frac{-2R}{N-2}\left[\frac{1}{2} \int_{\partial B_R} \left|\frac{\partial u}{\partial n}\right|^2 dS + \int_{\partial B_R} G(u)\, dS\right]. \tag{2.2}$$

We will now show that the right hand side in (2.2) converges to 0 for at least one suitably chosen sequence $R_n \to +\infty$. We have

$$(2.3) \qquad \int_{\mathbb{R}^N} \{|G(u)| + |\nabla u|^2\}\, dx = \int_0^{+\infty} \left\{ \int_{\partial B_R} [|G(u)| + |\nabla u|^2]\, dS \right\} dR < +\infty.$$

Hence, there exists a sequence $R_n \to +\infty$ such that

$$R_n \int_{\partial B_{R_n}} \{|G(u)| + |\nabla u|^2\}\, dS \to 0 \quad \text{as } n \to +\infty.$$

Indeed, if

$$\varliminf_{R \to +\infty} R \int_{\partial B_R} \{|G(u)| + |\nabla u|^2\}\, dS = \alpha > 0,$$

then

$$\int_{\partial B_R} \{|G(u)| + |\nabla u|^2\}\, dS$$

would not be in $L^1(0, +\infty)$, which contradicts (2.3). Therefore, since

$$\int_{B_{R_n}} |\nabla u|^2\, dx \to \int_{\mathbb{R}^N} |\nabla u|^2\, dx, \qquad \int_{B_{R_n}} G(u)\, dx \to \int_{\mathbb{R}^N} G(u)\, dx$$

as $n \to +\infty$, we derive the identity (2.1) from (2.2) (with the choice $R = R_n$, and $n \to +\infty$).

Corollary 1. *Assume g satisfies* (1.1) *and* (1.2). *Then any solution of* (*) *satisfies the Pohozaev identity* (2.1).

Proof. To be more precise, we mean any solution of (*) corresponding to a truncated function $\tilde{g}$ (see section 3 below), where (i) $\tilde{g}(s) = g(s)$ if $g(s) \geq 0$ for $s \geq \zeta$ and (ii) if $\exists s_0 > \zeta$ such that $g(s_0) \leqq 0$, then $\tilde{g}(s) = g(s)$ on $[0, s_0]$ and $\tilde{g}(s) = g(s_0)$ on $[s_0, +\infty)$ (for simplicity we take g and $\tilde{g}$ to be odd). As will be seen later, solutions of (*) corresponding to $\tilde{g}$ are also solutions of (*) with g. (If $\varliminf_{s \to +\infty} g(s) > 0$ or $\varlimsup_{s \to +\infty} g(s) < 0$, we can actually take $\tilde{g}$ in such a way that the two problems have the same solutions). The corollary follows immediately from proposition 1, for if $u \in H^1(\mathbb{R}^N)$ solves (*), then by a standard bootstrap argument (see Section 4.1 below) $u \in L^\infty(\mathbb{R}^N)$, while by Theorem A.VI in the Appendix, $G(u) \in L^1(\mathbb{R}^N)$.

Remark 2.2. In the case of radial solutions of (*), one can use the exponential decay of u and ∇u (see Section 3.3) to obtain a simpler proof of the identity, by making the preceeding scale-change argument rigorous (an alternative way would be to multiply the ordinary differential equation satisfied by such a solution by $r^N u'(r)$ and to integrate by parts). We remark that by a recent result of Gidas, Ni, and Nirenberg [37], if $g'(0) < 0$ and if $\hat{g}(s) = g(s) - g'(0)\, s$ is an increasing function of s, then any *positive* solution of (*) is spherically symmetric (and decreases with r).

2.2. *Some consequences of Pohozaev's identity, and some necessary conditions.* We now show that conditions (1.1)–(1.3) are "almost" necessary for the existence of a solution of problem (*).

(a) Hypothesis (1.3) is necessary, for if u is a solution of (*), then by (2.1)

$$\int_{\mathbb{R}^N} G(u)\,dx > 0.$$

(b) Hypothesis (1.2) is justified by considering the pure power case, that is equation (1.4) (Example 1 in Section 1), where $g(u) = \lambda\,|u|^{p-1}\,u - mu$, λ, $m > 0$. If u satisfies (1.4), then multiplying (1.4) by u and using (2.1) yields

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx = \int_{\mathbb{R}^N} g(u)\,u\,dx = \frac{2N}{N-2}\int_{\mathbb{R}} G(u)\,dx,$$

whence

$$\lambda\left(\frac{1}{p+1} - \frac{N-2}{2N}\right)\int_{\mathbb{R}^N} |u|^{p+1}\,dx = \frac{m}{N}\int_{\mathbb{R}^N} u^2\,dx > 0.$$

This implies

$$\frac{1}{p+1} > \frac{N-2}{2N}, \quad \text{that is} \quad p < \frac{N+2}{N-2} = l.$$

Thus (1.4) has no solution when $p \geqq l$. On the other hand, it is known [51, 17, 55] that when $p < l$ (1.4) admits (infinitely many distinct) radial solutions. From this example, we see that a growth hypothesis like (1.2) is needed and that $l = \frac{N+2}{N-2}$ (the "critical exponent") is indeed the cut-off.

(c) Consider now hypothesis (1.1). We claim it is "almost" necessary in the sense that if $g'(0) > 0$, then (*) has no radial solution. Indeed, if $u \in H^1(\mathbb{R}^N)$ is spherically symmetric, then by a result of STRAUSS [55] (see Appendix, Radial Lemma A II) there exists a constant $C\,(= C(N)) > 0$ such that

$$|u(x)| \leqq C\frac{\|u\|_{H^1(\mathbb{R}^N)}}{|x|^{(N-1)/2}},$$

and actually $|u(x)| = o(|x|^{-(N-1)/2})$ as $|x| \to +\infty$. Let $m = g'(0)$ and $q(r) = m - g(u(r))/u(r)$. Then, considering the case $N = 3$ and assuming g is C^2 in the neighborhood of 0, one has $q(r) = o(r^{-1})$ as $r \to +\infty$ while u satisfies the linear equation

$$-\Delta u + q(r)\,u = mu \text{ in } \mathbb{R}^3.$$

But this is impossible, since it violates a result of KATO [40] which states that the linear Schrödinger operator $-\Delta + q(r)$ has no positive eigenvalues associated with eigenfunctions in $L^2(\mathbb{R}^3)$ under the condition $q(r) = o(r^{-1})$. A careful discussion of equation (9) (in the Introduction) in the case $g'(0) > 0$ would lead to the same conclusion. Thus hypothesis (1.1) is "almost" necessary.

Notice, however, that $g'(0) > 0$ is not exactly the negation of (1.1). The only remaining case, essentially, is the limiting "zero mass" case where $g'(0) = 0$.

Then the existence question becomes much more complex and many different phenomena may take place, depending on the structure of g. We will study this case in Section 5 below, where we prove an existence theorem (which we believe to be very nearly optimal) for a ground state solution.

(d) Another consequence of Pohozaev's identity is the following

Corollary 2. *If u is any solution of* (*), *then* $S(u) = \frac{1}{N} T(u) > 0$.

Proof. Just use (2.1). Then

$$S(u) = \frac{1}{2} T(u) - V(u) = \frac{1}{2}\left[1 - \frac{N-2}{N}\right] T(u) = \frac{1}{N} T(u) > 0.$$

3. The Constrained Minimization Method

A natural method to solve (*) would be to look directly for critical points of the action S on the space $H^1(\mathbb{R}^N)$. Indeed by Theorem A.V in the Appendix, after a suitable modification of g (see below), S is a C^1 functional on $H^1(\mathbb{R}^N)$.

Actually this method was used in [55] for some particular cases and in [31] for some existence results for dimension $N = 2$. However, a first difficulty encountered in this approach is the fact that S is neither bounded from above nor from below on $H^1(\mathbb{R}^N)$. That S is not bounded above is well known (due to the presence of the gradient term). On the other hand, under hypothesis (1.3) there exists $w \in H^1(\mathbb{R}^N)$ such that

$$V(w) = \int_{\mathbb{R}^N} G(w)\, dx > 0$$

(see below, Section 3, proof of Theorem 2). By the scale change of section 2.1 one has $S(w_\sigma) = \frac{\sigma^{N-2}}{2} T(w) - \sigma^N V(w)$. It follows from $V(w) > 0$ that $S(w_\sigma) \to -\infty$ as $\sigma \to +\infty$, as claimed. Another difficulty in this approach lies in the fact that S does not satisfy conditions of the type (PS^+) or (PS^-) in an obvious way.

Therefore, rather than looking for critical points of S, we will consider a constrained minimization problem. First, however, we need to modify the function g in order to make V of class C^1 and a meaningful functional on $H^1(\mathbb{R}^N)$.

Define a new function $\tilde{g}: \mathbb{R} \to \mathbb{R}$ as follows:

(i) if $g(s) \geqq 0$ for all $s \geqq \zeta$, put $g = \tilde{g}$;

and

(ii) if $\exists\, s_0 \geqq \zeta$ such that $g(s_0) = 0$, put

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0] \\ 0 & \text{for } s \geqq s_0. \end{cases}$$

For $s \leqq 0$, $\tilde{g}$ is defined (as g) by $\tilde{g}(s) = -g(-s)$. Observe that $\tilde{g}$ satisfies the same conditions as g. Furthermore, by the maximum principle, solutions of problem (*) with $\tilde{g}$ are also solutions of (*) with g. (Indeed, in case (ii) above, a solution u of (*) with $\tilde{g}$ satisfies $|u| < s_0$, whence $\tilde{g}(u) = g(u)$). Hence there is no loss in generality in replacing g by $\tilde{g}$ in the following discussion. *Henceforth we will always adopt the convention that g has been replaced by $\tilde{g}$; we keep however the same notation g.* With this modification, g satisfies the stronger condition

$$\text{(1.2 bis)} \qquad \lim_{s\to\pm\infty} \frac{|g(s)|}{|s|^l} = 0 \quad \text{with} \quad l = \frac{N+2}{N-2}.$$

Theorem A.V of the Appendix then applies, and thus

$$V(w) = \int_{\mathbb{R}^N} G(w)\,dx$$

is a meaningful C^1 functional on $H^1(\mathbb{R}^N)$.

Consider the following constrained minimization problem:

$$\text{(3.1)} \qquad \text{minimize}\,\{T(w);\, w \in H^1(\mathbb{R}^N),\ V(w) = 1\},$$

introduced by COLEMAN, GLAZER and MARTIN [29]. The problem (3.1) leads to a solution of (*). Indeed if u solves (3.1) then, since T and V are of class C^1 on $H^1(\mathbb{R}^N)$, there exists a Lagrange multiplier θ such that $T'(u) = \theta V'(u)$; that is (at least in the distribution sense)

$$\text{(3.2)} \qquad -\Delta u = \theta g(u) \ \text{ in } \mathbb{R}^N.$$

We will show below that necessarily $\theta > 0$. Thus, letting $u_\sigma(x) = u(x/\sigma)$, $\sigma > 0$, one has

$$-\Delta u_\sigma = \frac{\theta}{\sigma^2} g(u_\sigma) \quad \text{in } \mathbb{R}^N.$$

Therefore, choosing $\sigma = \sqrt{\theta}$, one obtains a solution of (*).

Theorem 2. *Under the hypotheses of Theorem* 1 *the minimization problem* (3.1) *has a solution* $u \in H^1(\mathbb{R}^N)$ *which is positive, spherically symmetric, and decreases with* $r = |x|$. *Furthermore, there exists a Lagrange multiplier* $\theta > 0$ *such that* u *satisfies* (3.2). *Hence* u_σ, *for* $\sigma = \sqrt{\theta}$, *is a solution of* (*).

Proof of Theorem 2. This will be divided into several steps:

1. The set $\{w \in H^1(\mathbb{R}^N),\ V(w) = 1\}$ is not empty.
2. Selection of an adequate minimizing sequence.
3. A priori estimates.
4. Passage to the limit.
5. Conclusion.

Step 1. *The set* $\{w \in H^1(\mathbb{R}^N);\ V(w) = 1\}$ *is not empty.* This is the only place where hypothesis (1.3) is used. Let $\zeta > 0$ be such that $G(\zeta) > 0$. For $R > 1$, define

$$w_R(x) = \begin{cases} \zeta & \text{for } |x| \leqq R \\ \zeta(R+1-r) & \text{for } r = |x| \in [R, R+1] \\ 0 & \text{for } |x| \geqq R+1. \end{cases}$$

Thus $w_R \in H^1(\mathbb{R}^N)$. Letting $|\cdot|$ denote Lebesgue measure, it is easily checked that one has

$$V(w_R) \geqq G(\zeta)\,|B_R| - |B_{R+1} - B_R| \left(\max_{s\in[0,\zeta]} |G(s)|\right).$$

Hence there exist constants $C, C' > 0$ such that

$$V(w_R) \geqq CR^N - C'R^{N-1}.$$

For $R > 0$ large enough, this shows that $V(w_R) > 0$. Then, introducing a scale change on w_R, $w_{R,\sigma}(x) = w_R(x/\sigma)$, we have $V(w_{R,\sigma}) = \sigma^N V(w_R)$. Thus for an appropriate choice of $\sigma > 0$, we have $V(w_{R,\sigma}) = 1$.

Step 2. *Selection of an adequate minimizing sequence.* There exists a sequence $(u_n) \subset H^1(\mathbb{R}^N)$ such that $V(u_n) = 1$ and $\lim_{n\to+\infty} T(u_n) = I \equiv \inf\{T(w);\, w \in H^1(\mathbb{R}^N)$ $V(w) = 1\} \geqq 0$. Let u_n^* denote the Schwarz spherical rearrangement of $|u_n|$. (The definition and some properties of the Schwarz symmetrization are recalled in the Appendix A3). One has $u_n^* \in H^1(\mathbb{R}^N)$, $V(u_n^*) = 1$, and $I \leqq T(u_n^*) \leqq T(u_n)$. This means that (u_n^*) is also a minimizing sequence. Replacing (u_n) by (u_n^*), we will assume henceforth that, for all n, u_n is nonnegative, spherically symmetric and nonincreasing with $r = |x|$.

Step 3. *Estimates for* u_n. We will show that $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded. For $s \geqq 0$, define $g_1(s) = (g(s) + ms)^+$ and $g_2(s) = g_1(s) - g(s)$. (Here $a^+ = \max(a, 0)$ is the positive part of a). Extend g_1 and g_2 as odd functions for $s \leqq 0$. Then one has $g = g_1 - g_2$ with $g_1, g_2 \geqq 0$ on $\mathbb{R}^+$, and

$$\text{(3.3)} \qquad g_1(s) = o(s) \quad \text{as} \quad s \to 0; \qquad \lim_{s\to\infty} \frac{g_1(s)}{s^l} = 0, \quad \text{where} \quad l = \frac{N+2}{N-2},$$

and

$$\text{(3.4)} \qquad g_2(s) \geqq ms, \qquad \forall\, s \geqq 0.$$

Let $G_i(z) = \int_0^z g_i(s)\,ds$, $i = 1, 2$. From (3.3) and (3.4) we see that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\text{(3.5)} \qquad G_1(s) \leqq C_\varepsilon |s|^{l+1} + \varepsilon G_2(s), \qquad \forall\, s \in \mathbb{R}.$$

(Indeed, one has $g_1(s) \leqq C_\varepsilon s^l + \varepsilon g_2(s)$, $\forall\, s \geqq 0$.) Now, since $T(u_n) \downarrow I$, $\|\nabla u_n\|_{L^2(\mathbb{R}^N)}$ is bounded, which implies by the Sobolev embedding Theorem* that $\|u_n\|_{L^{2^*}(\mathbb{R}^N)}$

* $\mathscr{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ (cf. e.g. [44]).

$\leqq C$, where $2^* = l + 1 = 2N/(N-2)$. (Here and in the sequel C designates various positive constants independent of n). Writing $V(u_n) = 1$ in the form

$$\int_{\mathbb{R}^N} G_1(u_n)\,dx = \int_{\mathbb{R}^N} G_2(u_n)\,dx + 1 \tag{3.6}$$

and using (3.5), we derive (with $\varepsilon = 1/2$ in (3.5))

$$C + \frac{1}{2}\int_{\mathbb{R}^N} G_2(u_n)\,dx \geqq \int_{\mathbb{R}^N} G_2(u_n)\,dx + 1.$$

Hence $\int_{\mathbb{R}^N} G_2(u_n)\,dx \leqq C$, and by (3.4),

$$\frac{m}{2}\int u_n^2\,dx \leqq G_2(u_n)\,dx \leqq C.$$

Thus $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded. This implies by Hölder's inequality that $\|u_n\|_{L^p(\mathbb{R}^N)} \leqq C$ for any p, $2 \leqq p \leqq 2^*$.

Step 4. *Passage to the limit.* First, observe that $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n. Indeed, since u_n is radial and nondecreasing and u_n is bounded in $L^2(\mathbb{R}^N)$, it is easily seen that $|u_n(x)| \leqq C|x|^{-N/2}$, $x \in \mathbb{R}^N$, with C independent of n (see Radial Lemma A IV in the Appendix). Now, since u_n is bounded in $H^1(\mathbb{R}^N)$, one can extract a subsequence of u_n, again denoted by u_n, such that u_n converges weakly in $H^1(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$ to a function u. Observe that $u \in H^1(\mathbb{R}^N)$ is spherically symmetric and non-increasing with r.

Now, let $Q(s) = s^2 + |s|^l$. From (3.3) and (3.4), we derive

$$\frac{G_1(s)}{Q(s)} \to 0 \quad \text{as} \quad s \to +\infty \quad \text{and as} \quad s \to 0. \tag{3.7}$$

We also know that

$$\sup_n \int_{\mathbb{R}^N} Q(u_n)\,dx < +\infty, \tag{3.8}$$

$$G_1(u_n) \to G(u) \text{ a.e. in } \mathbb{R}^N, \tag{3.9}$$

$$u_n(x) \to 0 \text{ as } |x| \to +\infty, \text{ uniformly in } n. \tag{3.10}$$

Therefore the compactness lemma of STRAUSS [55] (see Appendix, Theorem A1) applies. Thus,

$$\int_{\mathbb{R}^N} G_1(u_n)\,dx \to \int_{\mathbb{R}^N} G_1(u)\,dx \quad \text{as} \quad n \to +\infty.$$

Using Fatou's Lemma in (3.6) we deduce that

$$\int_{\mathbb{R}^N} G_1(u)\,dx \geqq \int_{\mathbb{R}^N} G_2(u)\,dx + 1,$$

that is, $V(u) \geqq 1$. On the other hand, we also know that

$$T(u) \leqq \varliminf_{n\to+\infty} T(u_n) = I.$$

Now suppose for contradiction that $V(u) > 1$. Then, by the scale change $u_\sigma(x) = u(x/\sigma)$ we have $V(u_\sigma) = \sigma^N V(u) = 1$ for some σ, $0 < \sigma < 1$. Also, $T(u_\sigma) = \sigma^{N-2} T(u) \leqq \sigma^{N-2} I$. But by the very definition of I, $T(u_\sigma) \geqq I$. But this would imply $I = 0$, whence $T(u) = 0$, i.e. $u = 0$, contradicting $V(u) > 0$. This is impossible and therefore $V(u) = 1$ and $T(u) = I > 0$; u is a solution of the minimization problem (3.1).

Step 5. *Conclusion.* Since V and T are C^1 functionals on $H^1(\mathbb{R}^N)$ (see Appendix), there exists a Lagrange multiplier θ such that $\frac{1}{2} T'(u) = \theta V'(u)$. We remark first that $\theta \neq 0$, since if $\theta = 0$ one would have $u = 0$ which is impossible. Let us show that $\theta > 0$. Suppose for contradiction that $\theta < 0$. Observe that $V'(u) \neq 0$ ($V'(u) = 0$ gives $g(u) \equiv 0$, which implies $u \equiv 0$ since $g(s) \neq 0$ for $s > 0$ small, a contradiction to $V(u) = 1$). Consider a function $w \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\langle V'(u), w\rangle = \int_{\mathbb{R}^N} g(u)\, w\, dx > 0.$$

Since $V(u + \varepsilon w) \simeq V(u) + \varepsilon\langle V'(u), w\rangle$ and

$$T(u + \varepsilon w) \simeq T(u) + 2\varepsilon\theta \langle V'(u), w\rangle \text{ for } \varepsilon \to 0 \text{ and } \theta > 0,$$

one can find $\varepsilon > 0$ small enough so that $v = u + \varepsilon w$ satisfies $V(v) > V(u) = 1$ and $T(v) < T(u) = I$. Again by a scale change, there exists σ, $0 < \sigma < 1$ such that $V(v_\sigma) = 1$ and $T(v_\sigma) < I$, which is absurd. Hence $\theta > 0$.

Thus u satisfies, at least in the H^1 sense, the equation

$$-\Delta u = \theta g(u) \text{ in } \mathbb{R}^N$$

and so $u(\cdot/\sqrt{\theta}) = u_{\sqrt{\theta}}$ is a solution of problem (*).

Remark 3.1. In dimension $N = 1$ and $N = 2$, the method used in Step 3 for obtaining bounds on the sequence (u_n) fails. The reason for this failure is that in those dimensions a bound on $|\nabla u|$ in $L^2(\mathbb{R}^N)$ alone does not yield a bound on u in an $L^{l+1}(\mathbb{R}^N)$ space ($l < +\infty$). Actually, the constrained minimization problem (3.1) has no solutions when $N = 1$ or $N = 2$. Indeed, let us examine separately the cases $N = 1$ and $N = 2$.

Case (*i*): $N = 2$. Under a scale change, one has the following relations

$$T(u_\sigma) = T(u), \qquad V(u_\sigma) = \sigma^2 V(u).$$

Thus,

$$\inf_{\{V(u)=1\}} T(u) = \inf_{\{V(u)>0\}} T(u).$$

Now if it is supposed for contradiction that u_0 is a solution of (3.1), then one has $V(u_0) = 1$ and $T(u_0) = \min_{\{V(u)>0\}} T(u)$. Hence u_0 is an "interior minimum" for $T(u)$; thus $T'(u_0) = 0$, whence $u_0 = 0$, a contradiction to $V(u_0) = 1$.

Case (ii): $N = 1$. The scaling relations in this case read

$$T(u_\sigma) = \sigma^{-1} T(u), \qquad V(u_\sigma) = \sigma V(u).$$

Choose a $w \in H^1(\mathbb{R})$ such that $V(w) = 1$. Recalling that $\overline{\lim}_{s \to 0^+} g(s)/s = -m < 0$, we see that there exists $\theta_0 \in (0, 1)$ such that $V(\theta_0 w) = 0$ and $V(\theta w) > 0$ for $\theta_0 < \theta < 1$. Clearly, $V(\theta w) \to 0^+$ as $\theta \to \theta_0^+$. Let $\sigma(\theta) = V(\theta w)^{-1}$; thus $V(\theta w_{\sigma(\theta)}) = 1$. Now, $T(\theta w_{\sigma(\theta)}) = \sigma(\theta)^{-1} T(\theta w) = \theta^2 V(\theta w) T(w)$.
Letting $\theta \downarrow \theta_0$, this shows that $\inf_{\{V(u)=1\}} T(u) = 0$.

The case $N = 1$ for problem (*) is treated in Section 6 below. A very general existence theorem, involving conditions which are both necessary and sufficient, is proved there by means of simple methods of ordinary differential equations. For the case $N = 2$ and under more restrictive assumptions, one can obtain existence results by a "local approach" (see [9]) or by "shooting methods" (see [15]). Other results for the case $N = 2$ were obtained by M. J. ESTEBAN [31] using critical point theorems due to AMBROSETTI & RABINOWITZ [5] for the action S.

Remark 3.2. There is a "dual" variational method to (3.1). Consider the problem

$$\text{maximize } \{V(u);\ u \in H^1(\mathbb{R}^N),\ T(u) = 1\}. \tag{3.11}$$

One can check that the proof of Theorem 2 easily adapts to problem (3.11) and derive from it a solution of problem (*). This observation will be used in Part II.

4. Further Properties of the Solution

Let u denote the solution to problem (*) which we have obtained in the preceding section. We shall consider the regularity and exponential decay of u, and thus complete the proof of Theorem 1. Lastly, we show that u has minimum action among all possible solutions of (*). We assume throughout this section that g is odd.

4.1. *Regularity.* We show that $u \in C^2(\mathbb{R}^N)$, using the following more general lemma (also applicable for the regularity of the solutions obtained in Part II).

Lemma 1. *Under conditions* (1.1), (1.2 *bis*), *if u is a spherically symmetric solution of* (*) *then* $u \in C^2(\mathbb{R}^N)$.

Proof of Lemma 1. u satisfies the equation

$$-\Delta u = q(x)\, u \text{ in } \mathbb{R}^N \tag{4.1}$$

where $q(x) = g(u(x))/u(x)$. By (1.2 bis), one has

$$\left|\frac{g(u)}{u}\right| \leq C + |u|^{\frac{4}{N-2}}.$$

Since $u \in H^1(\mathbb{R}^N)$, we also have $u \in L^{2^*}(\mathbb{R}^N)$. Noticing that $2^* = \frac{4}{N-2} \cdot \frac{N}{2}$, we see that $q \in L^{N/2}(\mathbb{R}^N)$. Now, using a result of BREZIS & KATO [19], we obtain $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ for $1 \leqq p < \infty$. A classical bootstrap argument (on balls B_R) then shows that $u \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. Thus by the L^p estimate [1] we know that $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ for any $p < +\infty$. Hence $u \in C^{1,\alpha}(\mathbb{R}^N)$, $\alpha \in (0, 1)$.

Since u satisfies the equation

$$-u_{rr} - \frac{N-1}{r} u_r = g(u), \quad r \in (0, +\infty), \tag{4.2}$$

we already know that u_{rr} is continuous, except possibly at 0. Let us put $v(r) = g(u(r))$; v is continuous on $[0, +\infty)$. Rewriting (4.2) as $-\frac{d}{dr}(r^{N-1}u_r) = r^{N-1}v(r)$ and integrating from 0 to r yields

$$r^{N-1}u_r = -\int_0^r s^{N-1}v(s)\,ds.$$

With a change of variable, we have

$$u_r = -r\int_0^1 t^{N-1}v(rt)\,dt$$

or

$$\frac{u_r}{r} = -\int_0^1 t^{N-1}v(rt)\,dt.$$

Since

$$\int_0^1 t^{N-1}v(rt)\,dt \to \frac{v(0)}{N} \quad \text{as } r \to 0,$$

we deduce that $u_{rr}(0)$ exists and $u_{rr}(0) = -v(0)/N$. Furthermore, from equation (4.2) we then see that $u_{rr} \to -v(0)/N$ as $r \to 0$. Thus, $u \in C^2(\mathbb{R}^N)$.

Hence the solution of (*) obtained in Section 3 satisfies $u \in C^2(\mathbb{R}^N)$. We also observe that $u > 0$ on $\mathbb{R}^N$, by the maximum principle, and that u is a decreasing function of r, since by the strong maximum principle, $u'(r) < 0$ for any $r > 0$.

4.2. *Exponential decay*. The decay of u, $|D^\alpha u|$ ($|\alpha| \leqq 2$) at infinity is shown in the next lemma (this result also applies to the (non-positive) bound states obtained in Part II).

Lemma 2. *Under conditions* (1.1), (1.2 *bis*), *if u is a spherically symmetric solution of* (*) *then*

$$|D^\alpha u(x)| \leqq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^N$$

for some C, $\delta > 0$ and for $|\alpha| \leqq 2$.

Proof. The exponential decay of u at infinity follows from a standard argument of ordinary differential equations (see e.g. [55]). For the sake of completeness, we repeat it here.

By Lemma 1 u is of class $C^2(\mathbb{R}^N)$; accordingly it satisfies equation (4.2). Set $v = r^{(N-1)/2}u$; then v satisfies

$$v_{rr} = \left[q(r) + \frac{b}{r^2}\right] v$$

where $q(r) = -g(u(r))/u(r)$ and $b = (N-1)(N-3)/4$. For r large enough, say $r \geqq r_0$, one has

$$q(r) + \frac{b}{r^2} \geqq \frac{m}{2}$$

(recall that $u(r) \to 0$ as $r \to 0$ by the Radial Lemma A II in the Appendix).

Let $w = v^2$; then w verifies

$$\frac{1}{2} w_{rr} = v_r^2 + \left[q(r) + \frac{b}{r^2}\right] w.$$

Thus for $r \geqq r_0$ one has $w_{rr} \geqq mw$, and $w \geqq 0$.

Now let $z = e^{-\sqrt{m}r}(w_r + \sqrt{m}\, w)$. We have $z_r = e^{-\sqrt{m}r}(w_{rr} - mw) \geqq 0$; hence z is nondecreasing on $(r_0, +\infty)$. If there exists $r_1 > r_0$ such that $z(r_1) > 0$, then $z(r) \geqq z(r_1) > 0$ for all $r \geqq r_1$. This implies that

$$w_r + \sqrt{m}\, w \geqq (z(r_1))\, e^{\sqrt{m}r},$$

whence $w_r + \sqrt{m}\, w$ is not integrable on $(r_1 + \infty)$. But v^2 and $v\, v_r$ are integrable near ∞ (for $u \in H^1(\mathbb{R}^N)$), so that w_r and w are also integrable, a contradiction. Hence $z(r) \leqq 0$ for $r \geqq r_1$. This implies that

$$\left(e^{\sqrt{m}r} w\right)_r = e^{2\sqrt{m}r} z \leqq 0 \quad \text{for} \quad r \geqq r_1.$$

Hence $w(r) \leqq Ce^{-\sqrt{m}r}$ and in turn

$$\text{(4.3)} \qquad |u(r)| \leqq C\, r^{-\frac{N-1}{2}} e^{-\frac{\sqrt{m}}{2}r} \quad \text{for} \quad r \geqq r_1,$$

for certain positive constants C and r_1.

To obtain the exponential decay of u_r, observe that u_r satisfies

$$\text{(4.4)} \qquad (r^{N-1}u_r)_r = -r^{N-1}g(u).$$

Hence using (1.1) and the exponential decay of u it is easily seen that for r large enough, say $r \geqq r_0$, one has $m_1 |u| \leqq |g(u)| \leqq m_2 |u|$, where $m_2 \geqq m_1 > 0$. Hence, integrating (4.4) on (r, R), using (4.3), and letting r, $R \to +\infty$ shows that $r^{N-1} u_r$ has a limit as $r \to \infty$; this limit can only be zero by (4.3). Integrating (4.3) on $(r, +\infty)$ then implies that u_r has exponential decay. Lastly, the exponential decay of u_{rr} (and thus of $|D^\alpha u(x)|$ for $|\alpha| \leqq 2$) follows immediately from equation (4.2).

4.3. *Minimum of the action among solutions of* (*). By a result of COLEMAN, GLAZER & MARTIN [29] the solution of (*) obtained by the constrained minimization method of Section 3 has the important property of minimizing the action among all solutions of (*). The proof of this fact which we present now relies essentially on the Pohozaev identity (2.1). Therefore it is crucial, from this viewpoint, to know that *any* solution of (*) satisfies this identity (cf. Proposition 1 above).

Theorem 3. *Let u denote the solution of (*) obtained in Theorem 2. Then for any solution v of (*) one has*

$$0 < S(u) \leqq S(v).$$

Let us recall again that by any solution v of (*) we mean a solution corresponding to the truncated g, that is $\tilde{g}$ (see above). We follow the argument of [29]. Let $\bar{u}$ be the solution of (3.1) obtained in Theorem 2, so that

$$V(\bar{u}) = 1 \text{ and } T(\bar{u}) = \min\{T(w);\, w \in H^1(\mathbb{R}^N),\, V(w) = 1\}.$$

Then, as we have seen, there exists $\theta > 0$ such that $-\Delta\bar{u} = \theta g(\bar{u})$ in $\mathbb{R}^N$, and u is defined by $u = \bar{u}_{\sqrt{\theta}}$. By (2.1), one has

$$T(u) = \frac{2N}{N-2} V(u). \tag{2.1}$$

The scale change relations yield

$$T(u) = \theta^{\frac{N-2}{2}} T(\bar{u}), \qquad V(u) = \theta^{N/2} V(\bar{u}) = \theta^{N/2}.$$

Hence from (2.1) we derive

$$\theta = \frac{N-2}{2N} T(\bar{u}).$$

By Corollary 2, the action for a solution of (*) has the expression $S(u) = \frac{1}{N} T(u)$. Thus,

$$S(u) = \frac{1}{N}\left(\frac{N-2}{2N}\right)^{(N-2)/2} [T(\bar{u})]^{N/2}. \tag{4.5}$$

Now, let v denote another solution of (*), so by (2.1)

$$T(v) = \frac{2N}{N-2} V(v).$$

Let $\sigma > 0$ be such that $V(v_\sigma) = 1$, that is $\sigma = V(v)^{-(1/N)}$, or using (2.1),

$$\sigma = \left(\frac{N-2}{2N}\right)^{-(1/N)} [T(v)]^{-(1/N)}.$$

Let us express $S(v)$ in terms of $T(v_\sigma)$. We know (Corollary 2) that $S(v) = T(v)/N$. On the other hand, $T(v_\sigma) = \sigma^{N-2}\, T(v)$, so using the preceding expression of σ,

$$T(v_\sigma) = \left(\frac{N-2}{2N}\right)^{-(N-2)/N} [T(v)]^{2/N}.$$

Hence

$$S(v) = \frac{1}{N} T(v) = \frac{1}{N} \left(\frac{N-2}{2N}\right)^{(N-2)/2} [T(v_\sigma)]^{N/2}. \tag{4.6}$$

Since $\bar{u}$ solves the minimization problem (3.1) and $V(v_\sigma) = 1$, we have $T(v_\sigma) \geqq T(\bar{u})$. Using this inequality together with (4.5) and (4.6) yields $S(v) \geqq S(u)$.

5. The "Zero Mass" Case

As we have seen in Section 2, the situation where $g'(0) = 0$ is a limiting case from the viewpoint of existence results. Indeed, we have seen that when $g'(0) > 0$ there are no solutions of (*) while when $g'(0) < 0$, Theorem 1 applies. We call the case $g'(0) = 0$, the "zero mass" case. This situation arises in certain problems related to the Yang-Mills equations; see e.g. [35, 36]). In this section, we prove an existence result which is more general than Theorem 1, as it also includes situations where $g'(0) = 0$. We believe this result to be a nearly optimal one.

We assume here that $g: \mathbb{R}^+ \to \mathbb{R}$ is continuous and satisfies:

$$g(0) = 0 \quad \text{and} \quad \overline{\lim_{s \to 0^+}} \frac{g(s)}{s^l} \leqq 0, \quad \text{where} \quad l = \frac{N+2}{N-2}. \tag{5.1}$$

$$\text{There exists } \zeta > 0 \text{ such that } G(\zeta) > 0. \tag{5.2}$$

(5.3) Let $\zeta_0 = \inf\{\zeta > 0;\ G(\zeta) > 0\}$. If $g(s) > 0$ for all $s > \zeta_0$, then

$$\lim_{s \to +\infty} \frac{g(s)}{s^l} = 0.$$

Theorem 4. *Under hypotheses* (5.1)–(5.3) *there exists a positive, spherically symmetric, and decreasing* (*with* r) *solution* u *of the equation*

$$-\Delta u = g(u) \ \text{in} \ \mathbb{R}^N$$

such that $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. *Furthermore,* u *is a classical solution* (*i.e.* $u \in C^2(\mathbb{R}^N)$).

Proof. As in Section 3, we modify the function g by letting $\tilde{g}(s) = g(s \wedge s_0)$ if there exists $s_0 > \zeta_0$ such that $g(s_0) \leqq 0$. We again denote by g the truncated function $\tilde{g}$. The proof of Theorem 4 rests on the same constrained minimization method as in Section 3. Consider the problem

$$\text{minimize } \{T(w);\ w \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ |G(w)| \in L^1(\mathbb{R}^N),\ V(w) = 1\} \tag{5.4}$$

where again

$$T(w) = \int_{\mathbb{R}^N} |\nabla w|^2 \, dx, \qquad V(w) = \int_{\mathbb{R}^N} G(w) \, dx.$$

For the definition of $\mathcal{D}^{1,2}$ the reader is referred to [44]; $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is just the Hilbert space obtained by taking the completion of $\mathcal{D}(\mathbb{R}^N)$ for the norm $\|w\|_{\mathcal{D}^{1,2}} = \sqrt{T(w)}$.

(Equivalently, using Sobolev's embedding theorem, $\mathscr{D}^{1,2}(\mathbb{R}^N)$ is the space of functions in $L^{2^*}(\mathbb{R}^N)$ such that $\nabla u \in (L^2(\mathbb{R}^N))^N$).

The proof will be divided into the following steps:

5a. Existence of a solution to the minimization problem (5.4).
5b. Existence of a Lagrange multiplier $\theta > 0$.
5c. Regularity of the solution of (*).

5a. *Existence of a solution to the minimization problem.* By taking the same w_R as in Step 1 of the Proof of Theorem 2, we see that the set

$$A = \{w \in \mathscr{D}^{1,2}(\mathbb{R}^N); \quad |G(w)| \in L^1(\mathbb{R}^N), \quad V(w) = 1\}$$

is not empty. Let (u_n) be a minimizing sequence for (5.4), i.e. $u_n \in A$ and

$$T(u_n) \downarrow I = \inf\{T(w); w \in A\} \text{ as } n \uparrow +\infty.$$

We may always assume, as in Step 2 of the Proof of Theorem 2, that u_n is nonnegative, sperhically symmetric and nonincreasing. (Indeed if (u_n) is a minimizing sequence, then so is (u_n^*), where u_n^* is the Schwarz symmetrization of u_n; note that there is no difficulty in defining the Schwarz symmetrization on $\mathscr{D}^{1,2}(\mathbb{R}^N)$; see Appendix A3).

Thus $\|u_n\|_{\mathscr{D}^{1,2}(\mathbb{R}^N)}$ and hence $\|u_n\|_{L^{2^*}(\mathbb{R}^N)}$ remain bounded.

After truncation of G, from (5.3) it follows that

$$|g(s)| \leqq C + |s|^l, \quad s \in \mathbb{R}; \tag{5.3 bis}$$

hence for any finite R, there exists a constant $C(R)$ such that

$$\int_{B_R} |G(u_n)|\,dx \leqq C(R). \tag{5.5}$$

Furthermore, $|u_n(r)| \leqq \beta(r)$, where $\beta(r)$ is independent of n and $\lim_{r\to+\infty} \beta(r) = 0$. (Indeed by the Radial Lemma A III of the Appendix,

$$|u_n(r)| \leqq Cr^{-\alpha},$$

where $\alpha > 0$ depends only on N and C depends only on $\|\nabla u_n\|_{L^2(\mathbb{R}^N)}$, which is bounded). Using (5.5) and $V(u_n) = 1$, we see that, for any $R > 0$ fixed, there exists a constant $C(R)$ such that

$$\int_{\mathbb{R}^N - B_R} G(u_n)\,dx \geqq -C(R). \tag{5.6}$$

Put $g^+ = \max(g, 0)\, g^- = (-g)^+$, so that $g = g^+ - g^-$ and $g^+, g^- \geqq 0$. Also, set

$$G_1(z) = \int_0^z g^+(s)\,ds, \quad G_2(z) = \int_0^z g^-(s)\,ds.$$

From (5.6) we deduce that

$$\int_{\mathbb{R}^N - B_R} G_2(u_n)\,dx \leqq C(R) + \int_{\mathbb{R}^N - B_R} G_1(u_n)\,dx. \tag{5.7}$$

We know that $0 \leqq u_n(r) \leqq \beta(R)$ for $r \geqq R, n \in N$, where $\beta(R) \to 0$ as $R \to +\infty$. By (5.1), for R large enough, there exists a constant $\varepsilon(R) > 0$ such that

$$0 \leqq G_1(u_n(r)) \leqq \varepsilon(R)\,|u_n(r)|^{l+1}, \qquad r \geqq R, \qquad n \in N.$$

Futhermore (by (5.1)) we can suppose $\varepsilon(R) \to 0$ as $R \to +\infty$. Hence

$$\int_{\mathbb{R}^N - B_R} |G_1(u_n)|\,dx \leqq \varepsilon(R) \int_{\mathbb{R}^N - B_R} |u_n|^{l+1}\,dx \leqq C\varepsilon(R). \tag{5.8}$$

This together with (5.5) shows that $|G_1(u_n)|$ is bounded in $L^1(\mathbb{R}^N)$.

We can extract a subsequence of (u_n), again denoted by (u_n), such that $u_n \rightharpoonup u$ weakly in $\mathscr{D}^{1,2}(\mathbb{R}^N)$ and $u_n \to u$ almost everywhere in $\mathbb{R}^N$. Note that u is nonnegative, radial and nonincreasing. Now, for any R, since u_n is bounded in $H^1(B_R)$ we have

$$\int_{B_R} G_1(u_n)\,dx \to \int_{B_R} G_1(u)\,dx \quad \text{as} \quad n \to +\infty,$$

using (5.3 bis). Since by (5.8),

$$\int_{\mathbb{R}^N - B_R} |G_1(u_n)|\,dx \to 0 \quad \text{as} \quad R \to +\infty,$$

uniformly with respect to n, we then derive that

$$\int_{\mathbb{R}^N} G_1(u_n)\,dx \to \int_{\mathbb{R}^N} G_1(u)\,dx.$$

From $V(u_n) = 1$, that is

$$\int_{\mathbb{R}^N} G_1(u_n)\,dx = 1 + \int_{\mathbb{R}^N} G_2(u_n)\,dx,$$

it follows (using Fatou's Lemma) that $G_1(u), G_2(u) \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} G_1(u)\,dx \geqq 1 + \int_{\mathbb{R}^N} G_2(u)\,dx,$$

that is $V(u) \geqq 1$. On the other hand, we know that $T(u) \leqq \varliminf_{n\to+\infty} T(u_n) = I$. Thus we conclude as in the Proof of Theorem 2 that $V(u) = 1$, i.e. $u \in A$ and $T(u) = I$. Hence u is a solution of the minimization problem (5.4).

5b. *Existence of a Lagrange multiplier* $\theta > 0$. We now prove that there exists $\theta \neq 0$ such that $-\Delta u = \theta g(u)$ in $\mathbb{R}^N$. Observe that $u \in H^1(C_\varepsilon)$ on any region

$$C_\varepsilon = \{x \in \mathbb{R}^N, \varepsilon < |x| < 1/\varepsilon\} \text{ for } 0 < \varepsilon < 1.$$

Denoting by $\mathscr{D}_r^{1,2}$ the space of radial functions in $\mathscr{D}^{1,2}$, we remark that u is also a solution of the minimization problem

$$\text{minimize } \{T(w);\, w \in \mathscr{D}_r^{1,2},\, w = u \text{ in } \mathbb{R}^N - C_\varepsilon,\, V(w) = 1\}.$$

This is a classical problem in the calculus of variations, as T and V are C^1 functionals on $H^1(C_\varepsilon)$. Therefore there exists a constant θ such that $-\Delta u = \theta g(u)$ in the sense of $\mathscr{D}'(C_\varepsilon)$, for all $\varepsilon > 0$. That is, $-\Delta u = \theta g(u)$ in $\mathscr{D}'(\mathbb{R}^N - \{0\})$.

To see that this equation is also satisfied at the origin, we use a result of [21] about the singularities of solutions of semi-linear elliptic problems. Indeed, if $u \in H^1(B_1)$ satisfies

$$-\Delta u = \theta g(u) \text{ in } \mathscr{D}'(B_1 - \{0\})$$

and g verifies condition (5.3 bis), then the equation also holds at the origin: $-\Delta u = \theta g(u)$ is satisfied in $\mathscr{D}'(\mathbb{R}^N)$.

The possibility $\theta = 0$ is ruled out by $V(u) = 1$. We can eliminate the case $\theta < 0$ too, by using the same argument as in Section 3. Indeed, if $w \in \mathscr{D}(\mathbb{R}^N)$ is such that $\int_{\mathbb{R}^N} g(u)\, w\, dx > 0$, then for $\theta < 0$ and $\varepsilon > 0$ small enough the function $v = u + \varepsilon w$ would satisfy $V(v) > V(u) = 1$ and $T(v) < T(u)$. But as we have seen by a simple scale change argument this is impossible. Hence $u \in \mathscr{D}^{1,2}(\mathbb{R}^N) \cap H^1_{\text{loc}}(\mathbb{R}^N)$ satisfies $-\Delta u = \theta g(u)$ in $\mathbb{R}^N$, with $\theta > 0$.

5c. *Regularity of the solution of* (*). Using a scale change, we find that $u_{\sqrt{\theta}}$ which we again denote by u in this section, is a positive, spherically symmetric, and non-increasing solution of (*). Write (*) as

$$-\Delta u = q(x)\, u \text{ in } \mathbb{R}^N,$$

where $q(x) = g(u(x))/u(x)$. By (5.3 bis), $q \in L^{N/2}_{\text{loc}}(\mathbb{R}^N)$. Since $u \in H^1_{\text{loc}}(\mathbb{R}^N)$, we find from a result of Brézis & Kato [19] that $u \in L^p_{\text{loc}}(R^N)$ for $1 \leqq p < \infty$. A standard bootstrap argument then shows that $u \in C^2(\mathbb{R}^N)$ as before.

6. The Case of Dimension N = 1 (Necessary and Sufficient Conditions)

In this section we will show, with very few restrictions, that there exists a necessary and sufficient condition for the solvability of problems like (*) in one variable. In this case, furthermore, the solution of (*) is unique when it exists. The proofs rely on simple arguments adapted from H. Berestycki & P. L. Lions [11].

Let $f \in C(\mathbb{R}, \mathbb{R})$ be a locally Lipschitz continuous function with $f(0) = 0$. Let $F(z) = \int_0^z f(s)\, ds$. Consider the problem

$$\begin{gathered} -u'' = f(u), \quad u \in C^2(\mathbb{R}), \\ \lim_{x \to \pm\infty} u(x) = 0, \quad u(x_0) > 0 \text{ for some } x_0 \in \mathbb{R}. \end{gathered} \tag{6.1}$$

Theorem 5. *A necessary and sufficient condition for the existence of a solution u of problem* (6.1) *is that*

$$\begin{gathered} \zeta_0 = \inf\{\zeta > 0;\, F(\zeta) = 0\} \text{ exists}, \\ \zeta_0 > 0, \quad f(\zeta_0) > 0. \end{gathered} \tag{6.2}$$

Furthermore, if (6.2) *is satisfied, then* (6.1) *has a unique solution up to translations of the origin, and this solution satisfies* (*after a suitable translation of the origin*):

(*i*) $u(x) = u(-x)$, $x \in \mathbb{R}$ ("*u is radial*"),
(*ii*) $u(x) > 0$, $x \in \mathbb{R}$,
(*iii*) $u(0) = \zeta_0$,
(*iv*) $u'(x) < 0$, $x > 0$.

Remark 6.1. Under the assumptions of Theorem 5, the solution u of (6.1) can be obtained as the solution of the initial value problem

$$-u'' = f(u) \text{ in } \mathbb{R}, \quad u(0) = \zeta_0, u'(0) = 0. \tag{6.3}$$

The other solutions are obtained by translations; $v(x) = u(x + C)$, $C \in \mathbb{R}$ being a constant.

Remark 6.2. Observe that condition (6.2) implies that

i) There exists $\zeta > 0$ such that $F(\zeta) > 0$.
ii) If f is differentiable at 0, then $f'(0) \leqq 0$. We thus have the same type of assumptions that were made for the case $N \geqq 3$. However, in dimension 1 (as is well known) no growth restriction needs to be imposed on f.

Remark 6.3. If one assumes in addition to (6.2) that $\lim\limits_{s \to 0} \frac{f(s)}{s} \leqq -m < 0$, then u, u', u'' have exponential decay at infinity: There exist C, $\delta > 0$ such that

$$0 \leqq u(x), |u'(x)|, |u''(x)| \leqq Ce^{-\delta|x|}, \qquad x \in \mathbb{R}$$

(see Section 4 above).

Proof of Theorem 5. (*a*) *Condition* (6.2) *is sufficient.* Let u denote the solution of (6.3). This solution exists and is unique on a certain maximal interval $(-\bar{x}, \bar{x})$. Multiplying the equation by u' yields

$$-\frac{1}{2} u'(x)^2 = F(u(x)), \quad |x| < \bar{x}. \tag{6.4}$$

Now observe that:

(i) $u(-x) = u(x)$, $|x| < \bar{x}$.
(ii) $u(x) > 0$, $|x| < \bar{x}$. Indeed, if not, there exists x_0 with $u(x_0) = 0$. But then by (6.4), $u'(x_0) = 0$, and by the uniqueness of solutions of the initial value problem, $u \equiv 0$, which is impossible.
(iii) $u'(x) < 0$, $0 < x < \bar{x}$. Indeed, since $u'(0) = 0$ and $u''(0) = -f(\zeta_0) < 0$, it is clear that $u'(x) < 0$ is negative for small $x > 0$. Now, suppose there exists $x_0 > 0$ such that $u'(x_0) = 0$ and $0 < u(x_0) < \zeta_0$. This implies by (6.4) that $F(u(x_0)) = 0$, which is impossible by the very definition of ζ_0. Hence $u'(x) < 0$, $x < \bar{x}$.

From (ii) and (iii) above we derive that u is bounded: $0 < u(x) < \zeta_0$ for $x > 0$. Applying a standard continuation argument we then show that u is defined on all of $\mathbb{R}: \bar{x} = +\infty$.

(iv) $\lim_{x \to +\infty} u(x) = 0$. Indeed, let $L = \lim_{x \uparrow +\infty} \downarrow u(x)$; thus $0 \leqq L < \zeta_0$. From (6.4) we see that $-\frac{1}{2}(u'(x))^2 \to F(L)$ as $x \to +\infty$, which obviously implies $F(L) = 0$ and $L = 0$.

(v) *The solution of* (6.1) *is unique up to translations.* Let v be another solution. By translating the point where v reaches its maximum to the origin, we can assume that $v'(0) = 0$. By uniqueness of the initial value problem v must be symmetric $(v(-x) = v(x))$. Multiplying (6.1) by v' yields

$$-\frac{1}{2} v'(x)^2 = F(v(x)) - F(v(0)), \quad x \in \mathbb{R}.$$

Thus, $\frac{1}{2} v'(x)^2 \to F(v(0))$ as $x \to +\infty$, implying $F(v(0)) = 0$. Suppose $v(0) > \zeta_0$. Then there exists x_0 such that $v(x) > \zeta_0$ for $0 \leqq x < x_0$, and $v(x_0) = \zeta_0$. Since v also satisfies (6.4) we have $u'(x_0) = 0$. But then, $v''(x_0) = -f(\zeta_0) < 0$, contradicting $v(x) > \zeta_0$ for $x < x_0$. Hence, it must be the case that $v(0) = \zeta_0$, which means that v is a solution of (6.3). Thus $v = u$ by uniqueness of this problem.

(*b*) *Condition* (6.2) *is necessary.* Suppose (6.2) is violated and that there exists a solution v of (6.1) such that $v(0) > 0$ and $v'(0) = 0$ (indeed, v is even modulo a translation). By the same argument as above, $F(v(0)) = 0$. Hence $\zeta_0 < +\infty$ (i.e. ζ_0 exists). Hence (6.2) can only fail to be satisfied in two ways:

(i) *Case* 1. $\zeta_0 > 0$ but $f(\zeta_0) \leqq 0$. Then $v(0) \geqq \zeta_0$, and then exists $x_0 \geqq 0$ such that $v(x_0) = \zeta_0$. Then also $v'(x_0) = 0$ and $v''(x_0) = -f(\zeta_0) \geqq 0$. If $f(\zeta_0) = 0$, then the conditions $v(x_0) = \zeta_0$, $v'(x_0) = v''(x_0) = 0$ imply $v \equiv \zeta_0$, which is impossible. On the other hand if $f(\zeta_0) < 0$, then since whenever $v(x_0) = \zeta_0$ one also has $v'(x_0) = 0$ and $v''(x_0) > 0$, v can never go below ζ_0, which is impossible too. Hence, this case is ruled out.

(ii) *Case* 2. $\zeta_0 = 0$. Let $v(0) = \zeta_1 > 0$. Then $F(\zeta_1) = 0$ and $f(\zeta_1) > 0$ (if not, as we have seen in Case 1, v could not go below ζ_1). There exists ζ_2, $0 < \zeta_2 < \zeta_1$, such that $F(\zeta_2) = 0$. Again, $f(\zeta_2) > 0$. But there exists $\hat{x} > 0$ such that $v(\hat{x}) = \zeta_2$ and $v(x) > \zeta_2$ for any x, $0 < x < \hat{x}$. Then $v'(\hat{x}) = 0$ and $v''(\hat{x}) = -f(\zeta_2) < 0$, which is a contradiction.

Remark 6.4. The sharp difference between the case $N = 1$ and the dimensions $N \geqq 3$ should be stressed. Indeed, for the case $N = 1$, when there is existence for problem (*), one also has uniqueness. On the other hand for dimensions $N \geqq 3$, roughly speaking, when there is existence, then, in the odd case, there also exist ifinitely many distinct solutions. The optimal conditions for the existence of a ground state in dimension $N = 2$ are not entirely clear for the moment (partial results are however available for dimension 2 in [9], [15], [31]).

In a general manner, the results concerning existence for (*), strongly depend on

1) the dimension N, and

2) the function spaces where solutions are being sought. This shows that one has to be quite careful in defining with precision the function spaces or the variational problems used to solve (*). One also has to be careful in approaching (*) via the ordinary differential equation (9), since solvability depends on N.

Appendix

A.I. A compactness lemma of Strauss. We recall here a compactness result (used in section 3) due to STRAUSS [55], and present a very simple proof of it.

Theorem A.I. *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(s)}{Q(s)} \to 0 \quad \text{as} \quad |s| \to +\infty. \tag{I.1}$$

Let (u_n) be a sequence of measurable functions: $\mathbb{R}^N \to \mathbb{R}$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| \, dx < +\infty \tag{I.2}$$

and

$$P(u_n(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N, \text{ as } n \to +\infty. \tag{I.3}$$

Then for any bounded Borel set B one has

$$\int_B |P(u_n(x)) - v(x)| \, dx \to 0 \quad \text{as} \quad n \to +\infty.$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \to 0 \quad \text{as} \quad s \to 0, \tag{I.4}$$

and

$$u_n(x) \to 0 \text{ as } |x| \to +\infty, \text{ uniformly with respect to } n, \tag{I.5}$$

then $P(u_n)$ converges to v in $L^1(\mathbb{R}^N)$ as $n \to +\infty$.

Example. Suppose (w_n) is a sequence of functions $\mathbb{R}^N \to \mathbb{R}$ which is included in a bounded set of $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, $1 < p < q < +\infty$. Suppose furthermore that w_n converges to some w a.e. in $\mathbb{R}^N$, and that w_n as well as w satisfies (I.5). Then w_n converges strongly to w in $L^r(\mathbb{R}^N)$ for any $r \in (p, q)$. Indeed, we remark that by Fatou's Lemma, $w \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, and apply Theorem A.I. with

$$P(s) = |s|^r, \quad Q(s) = |s|^p + |s|^q, \quad u_n = w_n - w.$$

Proof of Theorem A.I. (1) To prove the first part of the theorem, we need to show that $P(u_n)$ is uniformly integrable on B. But by condition (I.1), there exists $C > 0$ such that

$$|P(u_n(x))| \leqq C + C\,|Q(u_n(x))|, \quad x \in \mathbb{R}^N.$$

Thus $P(u_n)$ and v (by Fatou's lemma) are in $L^1(B)$. Next, we see that

$$\int_{B \cap \{|P(u_n(x))| \geqq K\}} P(u_n(x))\,dx \leqq \int_{\{|u_n(x)| \geqq \varphi(K)\} \cap B} |P(u_n(x))|\,dx,$$

for some function φ such that $\varphi(K) \to +\infty$ as $K \to +\infty$.

Applying condition (I.1), we then have

$$\int_{\{|P(u_n(x))| \geqq K\} \cap B} |P(u_n(x))|\,dx \leqq \varepsilon(K) \int_B |Q(u_n(x))|\,dx \leqq C\varepsilon(K),$$

where $\varepsilon(K) \to 0$ as $K \to +\infty$. This shows the uniform integrability.

(2) Now let $\varepsilon > 0$; by conditions (I.4) and (I.5), there exists $R_0 > 0$ such that

$$|x| \geqq R_0 \text{ implies } |P(u_n(x))| \leqq \varepsilon\,|Q(u_n(x))|, \quad n \in N.$$

Therefore, by Fatou's Lemma, $v \in L^1(\mathbb{R}^N)$, and

$$\int_{\{|x| \geqq R_0\}} |v(x)|\,dx \leqq \varepsilon C.$$

Now, from the first part of the theorem, there exists n_0 such that for any $n \geqq n_0$,

$$\int_{\{|x| < R_0\}} |P(u_n(x)) - v(x)|\,dx \leqq \varepsilon.$$

To sum up, we have for $n \geqq n_0$ $(n_0 = n_0(\varepsilon))$,

$$\int_{\mathbb{R}^N} |P(u_n(x)) - v(x)|\,dx \leqq 2\varepsilon C + \varepsilon.$$

A.II. Some radial lemmas. We prove here some useful radial lemmas concerning the uniform decay at infinity of certain radial functions. The first one is due again to STRAUSS [55].

Radial Lemma A.II. *Let $N \geqq 2$; every radial function $u \in H^1(\mathbb{R}^N)$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$ and such that*

(II.1) $$|U(x)| \leqq C_N\,|x|^{(1-N)/2}\,\|u\|_{H^1(\mathbb{R}^N)} \quad \textit{for } |x| \geqq \alpha_N$$

where C_N and α_N depend only on the dimension N.

Proof. Since the first part is classical, we just indicate the way to show (II.1). By a standard density argument, it suffices to consider the case $u \in \mathcal{D}(\mathbb{R}^N)$. Let $m = \frac{N-1}{2}$; we have

$$\frac{d}{dr}(r^{2m}u^2) = 2\frac{d}{dr}(r^m u) \cdot r^m u \leqq \left[\frac{d}{dr}(r^m u)\right]^2 + (r^m u)^2.$$

Now if $N \geqq 3$, integrating over $[0, r]$, we obtain

$$r^{N-1}u^2(r) \leqq \int_0^r \left\{\left(\frac{du}{dr}\right)^2 + u^2\right\} \varrho^{N-1}\, d\varrho + mr^{N-2}\, u^2(r),$$

or

$$\left(1 - \frac{m}{r}\right) r^{N-1}u^2(r) \leqq C_N \|u\|_{H^1(\mathbb{R}^N)}.$$

We next prove a radial lemma (in the same spirit as the preceding one) for the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Recall that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the closure of $\mathcal{D}(\mathbb{R}^N)$ for the norm

$$\|\varphi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx.$$

Then (see [44]) by Sobolev's inequality, $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, for $N \geqq 3$, with $2^* = 2N/(N-2)$.

Radial Lemma A.III. *Let $N \geqq 3$. Every radial function u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$, such that*

(II.2) $$|U(x)| \leqq C_N |x|^{(2-N)/2} \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \qquad |x| \geqq 1,$$

where C_N only depends on N.

Proof. As above, it suffices to consider the case when $u \in \mathcal{D}(\mathbb{R}^N)$. Now, following a device of COLEMAN, GLAZER and MARTIN [29], we set $r = e^y$ and define

$$v(y) = u(r) \exp\left\{\frac{1}{2}(N-2)\, y\right\}.$$

An elementary calculation then shows that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^N)} = |S^{N-1}| \left\{ \int_{-\infty}^{+\infty} (v'(y))^2\, dy + \int_{-\infty}^{+\infty} \frac{(N-2)^2}{4} (v(y))^2\, dy \right\}$$

($|S^{N-1}|$ is the area of the unit sphere in $\mathbb{R}^N$). Since for any $v \in H^1(\mathbb{R})$ one has

$$|v(y)| \leqq 2\, \|v\|_{L^2(\mathbb{R})} \left\|\frac{dv}{dy}\right\|_{L^2(\mathbb{R})},$$

we obtain

$$|u(r)\, r^{(N-2)/2}| \leqq C_N \|\nabla u\|_{L^2(\mathbb{R}^N)}$$

and the desired inequality follows at once.

We next prove an easy lemma, used in the proof of the existence of a ground state solution.

Radial Lemma A.IV. *If* $u \in L^p(\mathbb{R}^N)$, $1 \leqq p < +\infty$, *is a radial nonincreasing function (i.e.* $0 \leqq u(x) \leqq u(y)$ *if* $|x| \geqq |y|$), *then one has*

$$|u(x)| \leqq |x|^{-N/p} \left(\frac{N}{|S^{N-1}|}\right)^{1/p} \|u\|_{L^p(\mathbb{R}^N)}, \quad x \neq 0. \tag{II.3}$$

Proof. For all $r > 0$, we have (setting $r = |x|$)

$$\|u\|^p_{L^p(\mathbb{R}^N)} \geqq |S^{N-1}| \int_0^r [u(s)]^p s^{N-1} ds \geqq |S^{N-1}| \, [u(r)]^p \frac{r^N}{N}.$$

Denote by $H^1_r(\mathbb{R}^N)$ the subspace of $H^1(\mathbb{R}^N)$ formed by the radial functions. An important corollary of Lemma A.II is

Theorem A.I'. *The injection* $H^1_r(\mathbb{R}^N) \hookrightarrow L^p(R^N)$ *is compact, for* $2 < p < 2N/(N-2)$.

Proof. The injection $H^1(R^N) \hookrightarrow L^p(\mathbb{R}^N)$, when $2 < p < 2N/(N-2)$, follows from Sobolev's Theorem and Hölder's inequality. Now let $(u_n) \subset H^1_r(\mathbb{R}^N)$ be a sequence of radial functions such that $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded. From Lemma A. II we deduce that (u_n) has uniform behaviour at infinity with respect to n, i.e., $\lim_{|x|\to\infty} |u_n(x)| = 0$ uniformly with respect to n. We can extract a subsequence (u_{n_k}) which converges almost everywhere in $\mathbb{R}^N$, and weakly in $H^1_r(\mathbb{R}^N)$ to a radial function u. Applying Theorem A.I above (with the choice $P(s) = |s|^p$, $Q(s) = s^2 + |s|^{2^*}$) we then find that u_{n_k} converges strongly to u in $L^p(\mathbb{R}^N)$.

A.III. Some results about Schwarz symmetrization. We recall here, without proofs, the basic properties of Schwarz symmetrization. First, let us recall the definition of the sperhical rearrangement (or symmetrization) of a function. Let $f \in L^1(\mathbb{R}^N)$; then f^*, the Schwarz symmetrized function of f, is a radial, nonincreasing (in r), measurable function such that for any $\alpha > 0$,

$$m\{f^* \geqq \alpha\} = m\{|f| \geqq \alpha\},$$

where m is the Lebesgue measure. It is obvious that

$$\int_{\mathbb{R}^N} F(f)\,dx = \int_{\mathbb{R}^N} F(f^*)\,dx$$

for every continuous function F such that $F(f)$ is integrable.

A fundamental property of the mapping $f \to f^*$ is the following

Riesz Inequality. Let f, g *be in* $L^2(\mathbb{R}^N)$; *then*

$$\int_{\mathbb{R}^N} f(x)\,g(x)\,dx \leqq \int_{\mathbb{R}^N} f^*(x)\,g^*(x)\,dx. \tag{III.1}$$

From this inequality, we see that

$$\|f^* - g^*\|_{L^2(\mathbb{R}^N)} \leqq \|f - g\|_{L^2(\mathbb{R}^N)}, \quad f, g \in L^2(\mathbb{R}^N). \tag{III.2}$$

Another important consequence of the Riesz inequality is the following result.

Let u be in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ if $N \geqq 3$ (respectively, in $H^1(\mathbb{R}^N)$ for any N). Then u^* belongs to $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (respectively, to $H^1(\mathbb{R}^N)$), and we have

$$\int_{\mathbb{R}^N} |\nabla u^*(x)|^2 \, dx \leqq \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx. \tag{III.3}$$

This result is essentially well known (see [38]), but with stronger regularity requirements and with a somewhat delicate proof; recently LIEB gave a quite simple and general proof, using only the Riesz inequality and the symmetry property of the fundamental solution of the heat equation (see [43]). Though the result was established in the case of H^1 functions only, the case $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ follows easily from a simple density argument.

A.IV. Some functionals of class C^1 on $H^1(\mathbb{R}^N)$. We prove here some assertions about the C^1 character of certain integral functionals defined on $H^1(\mathbb{R}^N)$. We believe these are standard results (they are certainly well-known and used in the case of a bounded domain), but since we would not find any precise reference, we include the proofs. We start with the bounded domain case.

Theorem A.V. *Let Ω be a bounded, regular domain in $\mathbb{R}^N$, with $N \geqq 3$. Let $g \in C(\mathbb{R})$ satisfy $g(0) = 0$ and*

$$\overline{\lim_{|s| \to +\infty}} \frac{|g(s)|}{|s|^l} < +\infty \text{ with } l = \frac{N+2}{N-2}. \tag{IV.1}$$

Then the functional

$$V(u) = \int_\Omega G(u(x)) \, dx, \text{ where } G(t) = \int_0^t g(s) \, ds,$$

is well-defined and of class C^1 on the space $H^1(\Omega)$. Moreover one has

$$\langle V'(u), v \rangle = \int_\Omega g(u(x)) \, v(x) \, dx, \quad u, v \in H^1(\Omega). \tag{IV.2}$$

Theorem A.VI. *Let $N \geqq 3$ and let g be a continuous function on $\mathbb{R}$ satisfying $g(0) = 0$, condition* (IV.1), *and*

$$\overline{\lim_{\substack{s \to 0 \\ s \neq 0}}} \frac{|g(s)|}{|s|} < +\infty. \tag{IV.3}$$

Then, the functional $V(u) = \int_{\mathbb{R}^N} G(u(x)) \, dx$ is well-defined and of class C^1 on the space $H^1(\mathbb{R}^N)$. Moreover

$$\langle V'(u), v \rangle_{H^{-1}, H^1} = \int_{\mathbb{R}^N} g(u(x)) \, v(x) \, dx, \quad u, v \in H^1(\mathbb{R}^N). \tag{IV.2'}$$

Remark. The above results remain true (with the same proofs) if g also depends on $x: g = g(x, s)$, and satisfies the Carathéordory condition. Conditions (IV.1) to (IV.3) must then, however, be satisfied in a uniform sense with respect to $x \in \bar{\Omega}$ or $x \in \mathbb{R}^N$.

Proof of Theorem A.V. The fact that V is meaningful on $H^1(\Omega)$ is immediate. To show that it is of class C^1 (see VAINBERG [59]), we only need to show:

(i) For $u, v \in H^1(\Omega)$,

$$\left|\frac{1}{t}\left\{V(u+tv) - V(u) - t\int_\Omega g(u)\, v\, dx\right\}\right| \to 0 \text{ as } t \to 0, t > 0.$$

(ii) If $u_n \to u$ in $H^1(\Omega)$ (strongly), then

$$\sup_{v \in H^1(\Omega), \|v\|_{H^1(\Omega)} \leqq 1} \left|\int_\Omega \{g(u_n) - g(u)\}\, v\, dx\right| \to 0 \text{ as } n \to +\infty.$$

Proof of (i). We have

$$\left|\frac{1}{t}\left\{V(u+tv) - V(u) - t\int_\Omega g(u)\, v\, dx\right\}\right| \leqq \int_\Omega \left|\{G(u+tv) - G(u) - tg(u)\, v\}\frac{1}{t}\right| dx.$$

Now, almost everywhere in Ω, one has

$$\left|\frac{1}{t}\{G(u+tv) - G(u) - tg(u)\, v\}\right| \leqq \left\{\operatorname*{Sup}_{t \in [0,1]} |g(u+tv)| + |g(u)|\right\} |v|$$
$$\leqq \{C + C|u|^l + C|v|^l\}|v|$$

(here, we use the fact that $|g(s)| \leqq C + C|s|^l$ for $s \in \mathbb{R}$). Thus, letting $h = \{C + C|u|^l + C|v|^l\}|v|$, h lies in $L^1_+(\Omega)$ by the Sobolev imbedding theorem $(H^1(\Omega) \hookrightarrow L^{l+1}(\Omega))$. We have next

$$\frac{1}{t}\{G(u+tv) - G(u) - tg(u)\, v\} \to 0 \text{ a.e. } x \in \Omega, \text{ as } t \to 0$$

and

$$\left|\frac{1}{t}\{G(u+tv) - G(u) - tg(u)\, v\}\right| \leqq h, \text{ a.e. } x \in \Omega, h \in L^1_+(\Omega).$$

The conclusion now follows by applying Lebesgue's dominated convergence theorem.

Proof of (ii). We also know that $u_n \to u$ in $L^{l+1}(\Omega)$ (note that $l + 1 = 2^*$). Hence by a standard result in integration theory (cf. [18]), there exists $\bar{u} \in L^{l+1}_+(\Omega)$ such that (taking subsequences, if necessary)

$$|u|, |u_n| \leqq \bar{u} \text{ a.e. in } \Omega, \quad n \in \mathbb{N}.$$

Therefore we have

$$|g(u_n) - g(u)|^{\frac{l+1}{l}} \leqq C + C(\bar{u})^{l+1}.$$

This shows that $g(u_n) \to g(u)$ in $L^{\frac{l+1}{l}}(\Omega)$, and in turn

$$\sup_{\|v\|_{H^1(\Omega)} \leqq 1} \left| \int_\Omega \{g(u_n) - g(u)\} v \, dx \right|$$

$$\leqq \left\{ \int_\Omega |g(u_n) - g(u)|^{\frac{l+1}{l}} dx \right\}^{\frac{l}{l+1}} \sup_{\|v\|_{H^1(\Omega)} \leqq 1} \left\{ \int_\Omega |v|^{l+1} dx \right\}^{\frac{l}{l+1}}.$$

This concludes the proof of (ii) and hence of Theorem A.7.

Proof of Theorem A.VI. We follow the proof above.

(*i*) For any $u, v \in H^1(\mathbb{R}^N)$, one has

$$\left| \frac{1}{t} \left\{ V(u + tv) - V(u) - t \int_{\mathbb{R}^N} g(u) \, v \, dx \right\} \right| \to 0 \text{ as } t \to 0, \ t > 0.$$

We now use the inequality

$$|g(s)| \leqq C|s| + C|s|^l, \quad s \in \mathbb{R}$$

for some positive constant C. With h now defined by

$$h = \{C|u| + C|v| + C|u|^l + C|v|^l\} |v|$$

we have $h \in L^1_+(\mathbb{R}^N)$. This enables us to repeat the previous argument.

(*ii*) We show that

$$\sup_{v \in H^1(\mathbb{R}^N), \|v\|_{H^1(R^N)} \leqq 1} \left| \int_{\mathbb{R}^N} \{g(u_n) - g(u)\} v \, dx \right| \to 0 \text{ as } n \to +\infty.$$

Let $\varepsilon > 0$; we assert that there exists $R_0 > 0$ such that

$$\sup_{\|v\|_{H^1(\mathbb{R}^N)} \leqq 1} \left| \int_{|x| \geqq R_0} \{g(u_n) - g(u)\} v \, dx \right| \leqq \varepsilon.$$

In view of Theorem A.V and its proof above, we will then be finished). Since $u_n \to u$ in $H^1(\mathbb{R}^N)$, and so also in $L^{l+1}(\mathbb{R}^N)$, taking subsequences if needed, there exist $\bar{u} \in L^{l+1}_+(\mathbb{R}^N)$, $\tilde{u} \in L^2_+(\mathbb{R}^N)$ such that (cf. [18])

$$|u|, |u_n| \leqq \bar{u} \text{ a.e. in } \mathbb{R}^N, \quad |u|, |u_n| \leqq \tilde{u} \text{ a.e. in } \mathbb{R}^N.$$

Then, for any $R > 0$, we have

$$\sup_{\|v\|_{H^1(\mathbb{R}^N)} \leqq 1} \left| \int_{|x \geqq| R} \{g(u_n) - g(u)\} v \, dx \right| \leqq C \|\tilde{u}\|_{L^2(|x| \geqq R)} \left\{ \sup_{\|v\|_{H^1(\mathbb{R}^N)} \leqq 1} \|v\|_{L^2(|x| \geqq R)} \right\}$$

$$+ C \|\bar{u}\|^l_{L^{l+1}(|x| \geqq R)} \left\{ \sup_{\|v\|_{H^1(\mathbb{R}^N)} \leqq 1} \|v\|_{L^{l+1}(|x| \geqq R)} \right\}.$$

Hence,

$$\sup_{v \in H^1(\mathbb{R}^N), \|v\|_{H^1(\mathbb{R}^N)} \leqq 1} \left| \int_{|x| \geqq R} \{g(u_n) - g(u)\} v \, dx \right| \leqq C \|\tilde{u}\|_{L^2(|x| \geqq R)} + C \|\bar{u}\|^l_{L^{l+1}(|x| \geqq R)}.$$

Since $\tilde{u} \in L^2(\mathbb{R}^N)$ and $\bar{u} \in L^{l+1}(\mathbb{R}^N)$, we derive the existence of $R_0 > 0$ such that

$$\sup_{\|v\|_{H^1(\mathbb{R}^N)} \leqq 1} \left| \int_{|x| \geqq R_0} \{g(u_n) - g(u)\} v \, dx \right| \leqq \varepsilon.$$

This concludes the proof of Theorem A.VI.

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