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EXTENSIONALITY IN SENTENCE POSITION

I. TWO PHENOMENA DISTINGUISHED

Given a notion of *extension* for linguistic expressions in general, an *extensional context* is one in which replacement of a contained expression by a co-extensive expression yields as its result an expression again having the same extension as the original. Thus some contexts are extensional in name position, or 'referentially transparent' as it is more usually (after Quine) put, in that replacement of co-extensive names within, e.g., a sentence, results in a sentence materially equivalent to that in which the replacement is made. In what follows we shall be concerned with sentential contexts in which the substitution of co-extensive (*sc.* materially equivalent) subsentences preserves truth-value. A casual perusal of the philosophical literature in this area reveals a tendency to describe such contexts as truth-functional. Such a terminology is, while generally harmless, at least *misleading*, in that reasonable precisification of the concepts involved reveals that connectives which are not truth-functional may yet yield contexts extensional in sentence position.[1] The present section of this paper illustrates that possibility, and the next provides some perspective on the relation between truth-functionality and sentence-position extensionality; the remaining sections explore some parts of logical theory opened up by the discussion of I and II. (To reduce the length of the paper for publication in its present form, I have chosen to omit, *inter alia*, the proofs – none of them presenting any essential difficulty – of the various claims and theorems.)

We turn to the proposed precisification. By the *language based on* some set of connectives we mean the set of all formulae compounded in the usual inductively specified manner from a countable stock of propositional variables by means of the zero-place connective $\bot$ and binary connective $\rightarrow$ and any connectives from the set in question. (Thus if κ is n-ary and $A_1, \ldots, A_n$ are formulae in the language, so is $\kappa(A_1, \ldots, A_n)$; 'infix' notation will be used for binary connectives, such as $\rightarrow$.) A *language* is a language based on some set of connectives. Thus the smallest language to be countenanced here (until Section IV) may be viewed as the language of classical truth-functional logic with primitive connectives $\perp$ and $\rightarrow$. A *logic* in a language is a collection of formulae of that language which includes all the classical tautologies in those primitives and is closed under truth-functional consequences (equivalently, *modus ponens*) and uniform substitution. The (unique) language which a logic is a logic in will be called the language of that logic. An n-ary connective κ in the language of a logic L will be said to be *truth-functional in* L iff there is some formula A containing at most the propositional variables $p_1, \ldots, p_n$ and no connectives apart from $\perp$, $\rightarrow$ such that the formulae $A \rightarrow \kappa(p_1, \ldots, p_n)$ and $\kappa(p_1, \ldots, p_n) \rightarrow A$ belong to L. Henceforth we discuss all languages with the abbreviative aid of the usual defined connectives of truth-functional logic (notated T, $\wedge$, $\vee$, $\sim$, and $\leftrightarrow$); this, given the restricted sense in which we are taking the term 'logic' enables us to re-phrase the *definiens* just given as requiring the membership in L of the formula $A \leftrightarrow \kappa(p_1, \ldots, p_n)$.[2]

On the other hand, we define an n-ary connective κ in the language of logic L to be *extensional in sentence position in* L iff L contains each instance (in its language) of the schema of 'replacement of material equivalents':

$$(RME_n) \quad ((A_1 \leftrightarrow B_1) \wedge \ldots \wedge (A_n \leftrightarrow B_n)) \rightarrow (\kappa(A_1, \ldots, A_n) \leftrightarrow \kappa(B_1, \ldots, B_n)).$$

The point, then, is that not every connective with the latter property (in some logic) has the former property (in that logic). Since our concern in this note is exclusively with sentential logic, no confusion wil be risked by our generally saying 'extensional (in L)' in place of 'extensional in sentence position (in L)'. The definition just offered of this concept trades heavily on the presence of the truth-functional connectives in the language, and on the inclusion of all tautologous formulae involving them in the logics considered. In Section IV we shall take a brief glance over the same terrain in which this dependence on the logical machinery governing truth-functional connectives is removed.

We begin with the straightforward example of a connective from modal logic. Think of the modal system K as developed in the language based on the one-place connective '$\Box$' (in discussing which we use '$\Diamond$' as abbreviating '$\sim\Box\sim$'). Now extend this language by a new singularly connective 'O', and consider the smallest logic in this language extending K, closed under the rule of necessitation, and containing all instances of the schema:

$$OA \leftrightarrow ((\Diamond T \wedge A) \vee (\Box\bot \wedge \sim A)).$$

We may think of the semantics of this logic, which we call K^+, in terms of the apparatus of Kripke models. The schema exhibited tells us that OA has the same truth-value as A at any point in such a model $\langle \boldsymbol{W}, \boldsymbol{R}, \boldsymbol{V}\rangle$, if that point bears $\boldsymbol{R}$ to at least one point (in $\boldsymbol{W}$), but the opposite truth-value to A at any point which lacks $\boldsymbol{R}$-successors. It is immediate that no point in such a model can falsify any instance of the schema of Replacement of Material Equivalents appropriate for a singularly connective:

$$(RME_1) \quad (A \leftrightarrow B) \rightarrow (OA \leftrightarrow OB)$$

so that the connective 'O' is indeed extensional in sentence position. To see this quickly, note that at a point with successor(s) (points $\boldsymbol{R}$-related to it, that is), OA and OB have the same truth-values as A and B and so agree in truth-value if A and B do, while at a point without successors, they again agree in having truth-values opposite to those of A and B. But it is equally clear that the right-hand side of the biconditional schema above with 'OA' on its left could not be replaced by a formula equivalent (in K) to that given yet containing only our truth-functional primitives '$\rightarrow$' and '$\bot$' for its connectives. Equivalently, no truth-table for 'O' can be given:

OA	A
?	T
?	F

with the property that when truth at a world in a model is constrained by the table, K^+ consists of precisely the formulae false at no world in any model. So here we have a simple illustration of the possibility of a connective's being extensional in sentence position without being a truth-functional connective.

In connexion with this particular illustration of the possibility, it seems appropriate to make a further remark. One might say, adapting a terminology popularized by Kripke's discussion of proper names in [11] that 'O' is a non-rigid expression for a truth-function, rather than saying that it is simply non-truth-functional *tout court*. For while '$\sim$', for example, as a rigid truth-functor, expresses negation at every world, our 'O' expresses negation at the successorless worlds and the identity truth-function at the rest. I shall continue to speak of 'O' and other such connectives as non-truth-functional in the logic in question, though, because it is this very variability that deprives that logic of any equivalence of the form that could count as a definition in terms of the truth-functional connectives standardly construed. Though the idea of non rigidity for connectives seems one worth developing, I shall say no more about it other than that we have already in some semantic treatments of the weaker non-normal modal logics such as Lewis' S2 and S3, a somewhat similar situation as regards the clause for '$\Box$' in the definition of truth at a point in a model.[3] For this operator, according to such treatments, in effect expresses necessity (truth at all R-related points) at so-called 'normal worlds', and the constant false truth-function at the rest.

The schema (RME_1) for 'O' says that the *immediate* scope of one of its occurrences was extensional in sentence position rather than that any substitution on the basis of material equivalence within its scope preserved truth-value, because of course such substitutions within the scope of an occurrence of '$\Box$' itself within the scope of 'O' cannot be expected to do so, the operator '$\Box$' (in K) not being likewise extensional in sentence position. This makes the example of K^+ less than ideal for illustrating certain subsequent points, for which reason we now present a modification of the example, using the language based on just the connective 'O'. The logic involved we call 'de-modalized K^+' and define semantically with the aid – somewhat extravagantly since at most one element of W will be pertinent to the evaluation of any formula – of models of the form $\langle W, X, V\rangle$ where W is a non-empty set and $X \subseteq W$. The definition of truth is standard for variables, $\perp$, and $\rightarrow$; we have the following clause for 'O':

$$\langle W, X, V\rangle \models_X OA \quad \text{iff} \quad (A \text{ iff } x \in X).$$

In other words X corresponds to what in terms of the relational models was $\{x\colon \exists x \in W.Rxy\}$. Then the logic we call de-modalized K^+ is characterized semantically, as the set of formulae valid (*sc.* unfalsifiable at any point in any such model) according to the above semantic account. A matching syntactic characterization may be given as: the smallest logic (in the present language) containing all instances of the schema got by strengthening (RME_1) stated for 'O' to a biconditional. (A straightforward application of the canonical model technique justifies the completeness assertion involved in this remark.)

I want to close this section with a remark about the definitions of truth-functionality (in L) and extensionality (in L) with which we are working. To begin with the latter term, it is worth mentioning that sometimes the principle asserting the substitutivity of *provable* equivalents within the (immediate) scope of a connective is referred to as a rule of extensionality; it seems preferable (following, e.g., [25]) to use the term congruentiality here. More precisely, the definition of what is for a connective κ belonging to the language of a logic L to be *congruential in* L is got by modifying the definition given above for extensionality in L by turning the requirement that a conditional be provable in L into the corresponding metalinguistic conditional requirement to the effect that if the components involved are provably equivalent in L then so are the corresponding κ-compounds. (For an example of a discussion of modal logic employing 'extensional' in this sense, see [7]; obviously this usage results from thinking of the extension of a sentence as being a set of possible worlds (the truth-set of the sentence) rather than a truth-value. [14] provides a discussion of some of the issues that come up in adjudicating between rival accounts of the intension/extension distinction for a language. The points I have been making obviously survive, rephrased, any re-drawing of this distinction.

As to the notion of truth-functionality in play in the present paper, we should draw attention to the fact that it is rather more liberal than might be wanted for some purposes. For an example, consider the modal logic of actuality, formulated in a language which adds to the primitive truth-functional connectives, not only the operator $\Box$ but also singularly connective, '$\mathscr{A}$' (for 'actually'). Models, for simplicity, are taken as triples $\langle W, w^*, V\rangle$, in which W and V are as usual for modal logic and $w^* \in W$, truth being defined for $\Box$-formulae by universal quantification over W (so that the $\mathscr{A}$-free formulae have an S5-logic), and for $\mathscr{A}$-formulae by: $\langle W, w^*, V\rangle \models_x \mathscr{A}A$ iff $\langle W, w^*, V\rangle \models_{w^*} A$. Two alternatives present themselves for the definition of validity: unfalsifiability at any point in any model (the option pursued in [3]), and unfalsifiability at the distinguished point w^* in any model (the option taken by Hazen in [8]; cf. also Kamp [9], Kaplan [10]). On the latter approach – and in fact the present example was suggested to me by Hazen – we have, if we denote the class of valid formulae by $L_{\mathscr{A}}$, $\mathscr{A}p \leftrightarrow p \in L_{\mathscr{A}}$, making the connective $\mathscr{A}$ truth-functional in L_A according to our definition, somewhat counterintuitively since, this not being a congruential logic (in the sense defined above; e.g., $\Box\mathscr{A}p \leftrightarrow \Box p \notin L_{\mathscr{A}}$), we cannot find a truth-functional formula in the variable p to substitute for $\mathscr{A}p$ wherever the latter occurs. Accordingly, for some purposes (though not ours here), one might prefer the following stronger conception of truth-functionality: the n-place connective κ is truth-functional in L iff there is a formula A containing precisely the variables $p_1, \ldots, p_n$ and no connectives other than '$\rightarrow$' and '$\bot$', such that for any formula B containing $\kappa(p_1, \ldots, p_n)$ the formula $B \leftrightarrow B' \in L$, where B' is like B save in having every occurrence of $\kappa(p_1, \ldots, p_n)$ replaced by the formula A. (Even the revised definition is only suitable, like the original, with the general assumption of closure under uniform substitution.)

II. PSEUDO-TRUTH-FUNCTIONAL CONNECTIVES

We ought to try to place the example of 'O' in a more general setting. The non-modal logic of 'O' described at the end of Section I (and which will be the subject of some remarks later in this section also) has as one of its theorems the following disjunction:

$$(Op \leftrightarrow p) \vee (Oq \leftrightarrow \sim q).$$

In fact, instead of axiomatizing the set of formulae valid according to the semantics offered by adding (RME_1) for 'O' and its converse to a basis for truth-functional logic, we could simply have taken this disjunction as the sole new axiom. Since neither of its disjuncts is provable in the system (valid on the semantics) and they have no propositional variables in common we have an example of Halldén-incompleteness, a point to which we return below (Observation 1). The presence of such variable-disjoint disjunctions turns out to be a fruitful feature of the given logic from which to abstract, and accordingly we say that an n-ary connective κ in the language of a logic L is *pseudo-truth-functional in L* iff L contains for some m, the disjunction:

$$(\kappa(p_1^1, \ldots, p_n^1) \leftrightarrow A_1(p_1^1, \ldots, p_n^1) \vee \ldots \vee$$
$$\vee\ \kappa(p_1^m, \ldots, p_n^m) \leftrightarrow A_m(p_1^m, \ldots, p_n^m)),$$

where each A_j is a formula containing no connectives other than $\rightarrow$ and $\bot$ and no variables other than those exhibited, which are to be taken as all distinct.[4] Note that we can be more specific than this and require that $m = 2^{2^n}$; for example, it is necessary and sufficient for a one-place connective κ to be pseudo-truth-functional in a logic, that the logic should contain:

$$(\kappa p \leftrightarrow T) \vee (\kappa q \leftrightarrow \bot) \vee (\kappa r \leftrightarrow r) \vee (\kappa s \leftrightarrow {\sim} s).$$

Thus for example taking κ as the 'O' of the previous section and L as K^+ (demodalized or otherwise) we have the disjunction in virtue of having its subdisjunction consisting of the final two disjuncts. Obviously, any connective truth-functional in a logic is psuedo-truth-functional in that logic, by similar weakening moves (or, to put it in terms of the definition above, by taking the case $m = 1$). The main interest of the concept for our purpose is the 'handle' it gives on the notion of extensionality in a logic, *via* the

THEOREM. For any logic L and n-ary connective κ in the language of L, κ is extensional in L iff κ is pseudo-truth-functional in L.

Some interesting features of the general situation with non-truth-functional but sentence-position extensional connectives may be illustrated by looking a little more closely at de-modalized K^+, which for brevity will be denoted by 'L' until the end of Observation 3 below. I should like to say something about a possible reaction to the provability in L of the disjunction in the absence of the provability of either disjunct. The thought might be that still 'O' is a truth-functional connective, though L doesn't tell us which one. The reply is that we are operating with the notion of truth-functionality in L, and this cannot be something about which there is a truth not told to us by L. Thinking of L as given syntactically, no semantics w.r.t. which L is sound and complete can both respect the classical meanings of '$\rightarrow$' and '$\perp$' and render valid any equivalence of 'Op' with a formula in which only they figure as connectives. It seems to be pertinent at this point to remark that in spite of the impossibility of *reducing* the question of OA's truth-value to that of A, we do have, and always will have, when (RME_1) is satisfied, a case of *supervenience*: for what (RME_1) for 'O' says is precisely that the truth-values of OA and OB cannot differ without a difference between those of A and B.

This brings us to the illustrative features of L to which I thought it worthwhile drawing attention:

OBSERVATION 1. Since L extends truth-functional logic with a disjunction which is 'Halldén-unreasonable', it is the intersection of two of its proper extensions, neither of which is included in the other (see [12] or [13]); call them L_{Id} and L_{Neg}: these are the extensions of L by new axioms $Op \leftrightarrow p$ and $Oq \leftrightarrow \sim q$, respectively. In each of these logics 'O' is a redundant ornament being respectively, the identity (or 'empty') connective in disguise, and a notational variant on '$\sim$' (i.e., on '$\ldots \rightarrow \perp$').

OBSERVATION 2. For any formula A, let A^{Id} be the formula A with all occurrences of 'O' omitted, and A^{Neg} be the formula A with all occurrences of 'O' replaced by '$\sim$'. Then since $A \in L_{\mathrm{Id}}$ iff $A^{\mathrm{Id}} \in L$ and $A \in L_{\mathrm{Neg}}$ iff $A^{\mathrm{Neg}} \in L$, we may deduce the following from Observation 1:

$$A \in L \quad \text{iff} \quad \text{both } A^{\mathrm{Id}} \in L \quad \text{and} \quad A^{\mathrm{Neg}} \in L.$$

OBSERVATION 3. If the four-valued matrices of '$\rightarrow$' and '$\bot$' which result from taking the product of each of their respective 2-valued matrices with itself are used to evaluate formulae of L (see [21], pp. 96ff), in conjunction with the following table for 'O':

OA	A
⟨T, F⟩	⟨T, T⟩
⟨T, T⟩	⟨T, F⟩
⟨F, F⟩	⟨F, T⟩
⟨F, T⟩	⟨F, F⟩

then let us say that a formula is *4-valid* when no assignment of these four values to its constituent variables can yield a consequential assignment other than that of ⟨T, T⟩ to the formula itself. We may now observe that L contains all and only the 4-valid formulae. (*Cf.* [27].)

Faced with this last fact, someone might say that the connective is after all truth-functional: for has not a truth-table, admittedly in four values, been provided? Of course what we had in mind in denying this was: truth-functionality in T and F. This is why we took as our logic-relative conception of truth-functionality for systems extending classical propositional logic (i.e., the notion 'is truth-functional in L') the availability of a defining equivalence with '$\rightarrow$' and '$\bot$' on the right-hand side; these connectives are of course no longer functionally complete in the four-valued setting and a different criterion would need to be chosen if one wished to take the four values seriously as truth-values. However, it is a well-known fact that for any reasonable logic a semantic account in terms of generalized truth-values and associated matrices can be found, provided one is prepared to use enough truth-values, so that the most that could be claimed of interest in this direction for the product tables is that we can keep the number finite. In any case, the device that yielded non-truth-functional connectives extensional in sentence position in two-valued logic can be re-applied in the four-valued (or any other many-valued) framework, to yield connectives obeying an appropriate form of (*RME*) while resisting a truth-table characterization in that framework.

We next consider, as being of interest in the area of modal logic – and readers with no special interest in this area may safely omit the present paragraph – a certain strengthening of the concept of pseudo-truth-functionality. We define an n-place connective κ of an extension of the language of modal logic (as in I, with '$\Box$' alongside our truth-functional primitives) to be strongly pseudo-truth-functional in L just as we defined pseudo-truth-functionality in L, except that instead of requiring that for some m, an m-termed disjunction of biconditionals of a certain form belong to L, we here require that every result of necessitating on the various disjuncts should yield a disjunction belonging to L. That is, where the disjuncts of the original definition are $D_1, \ldots, D_m$, what we now require is not simply that L should contain $D_1 \vee \ldots \vee D_m$, but that L should contain each formula:

$$S_1 D_1 \vee \ldots \vee S_m D_m,$$

where each S_i is a (possibly zero-termed) sequence of occurrences of '$\Box$'. The interest of this concept arises with the normal modal logics. Recall that the idea of non-rigidity seemed appropriate in connexion with the logic K^+ in Section I because the truth-function associated with 'O' was apt to vary from world to world within any given model for the logic. It may be, however, that such intra-model variations are not crucial in the following sense: we could have a logic determined by a class of models throughout each of which the connective in question was associated with a truth-function, (*sc.* the same truth-function at each point within a model) though the connective was not truth-functional in the logic because the truth-function in question varied form model to model. When this is the case, the connective in question will be – unlike our 'O' – strongly pseudo-truth-functional in the above sense.[5] In fact, taking L to be the smallest normal modal logic to contain all instances of the schema $A \rightarrow \Box A$, indeed the intersection of the two Post-complete normal modal logics ($K + \Box\bot$, and $K + \Box A \leftrightarrow A$, in each of which $\Box$ is truth-functional), gives a choice of L (in the language based on $\{\Box\}$) for which we can say that $\Box$ is itself not only pseudo-truth-functional – without being truth-functional – L, but is strongly so. Incidentally, a point that is perhaps not generally noticed is that this system is in fact precisely the same (save in notation) as Łukasiewicz's Ł-modal system, a logic not usually discussed under that description, though [28] provides something of an exception, when the range of normal modal logics is considered, but set aside as something of a curiosity. (No doubt because of the 4-valued presentation of the semantics, as well as the fact that, however far-fetched the axioms may be, one is still inclined to read '$\Box$' as 'necessarily', while the symbol Łukasiewicz used that behaves as '$\Box$' does in this logic he read as 'possibly'.)

III. THE LATTICE OF *n*-ARY EXTENSIONAL CONNECTIVES

The commonest use of the term 'connective' in logical writings is perhaps to denote a certain symbol which attaches to a given number of formulae to form a new formula; more abstractly one could regard an *n*-ary connective as any operation taking n formulae to a formula. This linguistic interpretation of the notion is the one we have been working with up to this point. But often connectives are individuated by reference to their logical powers. Thus, one speaks, e.g., of relevant implication, intuitionistic negation, or the '$\Box$' of S4, as a connective. If κ is a connective in the earlier sense and L is a logic in a language based on a set including κ, we could identify a connective in the newer sense with the pair $\langle \kappa, L \rangle$: the connective κ as it behaves in L. Thus instead of saying κ is extensional (or truth-functional, or whatever) in L, one would describe the 'connective' $\langle \kappa, L \rangle$ as extensional (truth-functional, . . .). We shall have this understanding in mind generally in what follows. Our concern will be only with logics which, like demodalized K^+, are cast in languages based on $\{\kappa\}$ for some connective κ-languages, that is, with at most a single connective in addition to $\bot$, $\rightarrow$. In fact, to represent the 'logical powers' notion of a connective, it would be more appropriate to disregard altogether the linguistic guise in which the connective appears in a logic. We do not care, in other words, whether, to cite one of the earlier examples, the necessity operator of S4 is written as 'L' or as '$\Box$'. Thus we should identify connectives $\langle \kappa, L \rangle$ and $\langle \kappa', L' \rangle$ when for each formula $A(\kappa)$ in the language of L and corresponding formula $A(\kappa')$ in the language of L':

$$(*) \qquad A(\kappa) \in L \quad \text{iff} \quad A(\kappa') \in L'.$$

Here the formulae $A(\kappa)$ and $A(\kappa')$ differ in that all the first has the connective κ precisely where the second has κ'. (This stipulation requires that κ and κ' have the same arity, and makes sense without further qualification only with the 'single additional connective' restriction on L, L' in force. This will be sufficient for what follows.) More precisely, we may define a connective to be not an ordered pair $\langle \kappa, L \rangle$ but to be the equivalence class of such a pair under the equivalence relation given above (by (*)). Notice that this legitimizes talk of *the* connective O in de-modalized K^+, in spite of indifference as to how this connective is written (this being simply the equivalence class of $\langle O$, de-modalized $K^+ \rangle$). I make this point explicitly because there is another conception of uniqueness of connectives (in the $\langle \kappa, L \rangle$ sense), due to Belnap [1], which the extensional but non-truth-functional connectives spectacularly *fail* to display. Adapted to the present setting, what this conception would require would be that (for the case of O just mentioned) the smallest logic whose language was based on two connectives of the same arity as O, call them O_1 and O_2, and which contained $A(O_1)$ and $A(O_2)$ for each formula $A(0)$ of demodalized K^+ should also contain: $O_1 p \leftrightarrow O_2 p$.[7] This obviously does not hold in our case because we could interpret, e.g., O_1 as expressing the identity truth-function and O_2 as negation making all theorems of the 'combined' logic tautologous but not the biconditional exhibited. (The extensional connectives do meet a certain generalized version of the Belnap uniqueness criterion, but limitations of space prevent me from going into that here.) To clarify what is at issue here it may help to remark that for a connective to be uniquely characterized in a given logic (in the Belnap-derived sense) is for that logic to give it such logical powers that no two connectives could survive as non-equivalent in a logic giving those powers to each of them. But this does not prevent us here from speaking of *the* connective O of de-modalized K^+, for example, since this is just a way of alluding to the set of logical powers in question.

Replacing the 'iff' in (*) above by 'only if' gives a relation with the aid of which we can chart the various extensional connectives conveniently. We could call it the *subconnective* relation. It holds between a connective of one logic and a connective of another when whatever the first logic says about the first connective, the second says about the second: we have, to use the informal vocabulary of the preceeding paragraph, inclusion of logical powers. The n-ary connectives for fixed n are partially ordered by the subconnective relation. We shall use this relation to throw light on the relations between the extensional connectives, for which case we may infer from the Theorem of Section II that the partially ordered set in question is a boolean lattice, since each n-place connective is determined by a selection from amongst the 2^n available truth-function-equating disjuncts of the Theorem. This presumes we can make sense of a selection of zero of these disjuncts, of course, and I propose to make the usual identification of an empty disjunction with $\bot$. We look at the case of $n = 1$, for simplicity. Begin by baptizing the four truth-functional connectives (connectives in the *new* sense): '**V**' (for 'Verum': the constant-true truth-function), '**F**' ('falsum': the constant-false), '**I**' (identity) and '**N**' (negation). In terms of the official definition of connectives we are working with: let L be the smallest logic in the language based on some singularly connective κ, which contains the formula $\kappa p \leftrightarrow (p \rightarrow p)$; then **V** is the equivalence class under the relation described above at (*) of $\langle \kappa, L \rangle$. And similarly, for **I**, **F**, and **N**, *mutatis mutandis*. The lattice meet is an operation, symbolized below by concatenation, which we might usefully call the *product* operation: the product of two connectives has the logical powers of any connective having all the logical powers of either of its 'factors'. Thus the O of demodalized K^+ is, in the present notation, **IN**, the product of **I** and **N**. Many other such products are independently interesting; for example we note that a simple approach to the logic of contrariety, simple in the sense of being extensional unlike the treatment *via* modal logic in [17], consists in the study of the logic of the connective **FN**. This connective forms a compound when attached to a formula A, whose logical powers are those that follow merely from its being inconsistent with A.[6] We have already has occasion (in Observation 3 of Section II) to recall the use of the term 'product' in connexion with many-valued truth-tables, for connectives semantically specified by co-ordinatewise computation in accordance with the truth-tables for the factors. I prefer to think of this as one way of approaching products in the present sense, a way not always available (e.g., when the factor connectives are not themselves amenable to a many-valued truth-functional treatment), and even when available, arguably not the most illuminating mode of description for the phenomenon. (For examples of this way of proceeding, see [20], [26]). Here, then, is a picture of the lattice of the 16 1-place extensional connectives:

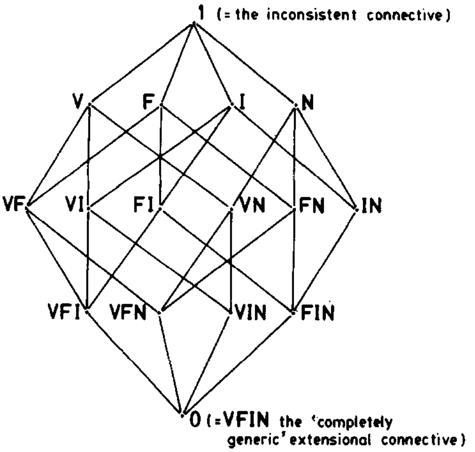

Fig. 1

The **O** (zero) of this lattice is a connective whose logic is axiomatized simply by the schema (RME_1) by itself: it might be called the completely generic one-place extensional connective. Equivalently, we may just cite the four-termed disjunction mentioned before the statement of the Theorem in Section II: if κ has that disjunction provable for it, but no proper subdisjunction, in a given logic, then it has the logical powers represented by this zero element. If the fourth disjunct is omitted, then it is **VFI** we are dealing with, and so on up the lattice. At the top, we might reach what might be regarded as an embarrassment for the present way of describing the lattice. For what, you may ask, is an inconsistent *connective*? The oddity may come from continuing to think of connective as symbols, identified by their physical appearance rather than by their logical powers. And these powers must, for a connective to have more than one truth-functional connective as a subconnective, be such as to render inconsistent any logic in which a connective is given those powers. But if you find the embarrassment overwhelming, it is always possible to stick with the 'connectives as symbols' way of talking, and redescribe the lattice of Figure 1 as a lattice of logics in some fixed language (with a single additional connective), with 'sublogic' for 'subconnective' as the $\leqslant$-relation involved. In fact, so viewed, since the connective is singulary, what we have is just a particularly simple sublattice of the lattice of all modal logics.

Whichever of the two ways just distinguished one chooses to describe the lattices illustrated by Figure 1 it is important that they not be confused with the Lindenbaum algebras of some more inclusive logic (in a language based on a set of up to 16 additional connectives). The subconnective relation does not correspond to any kind of consequence relation, in the way that we could explain our saying that κ was 'at least as strong as' λ in a logic involving both when that logic contained every instance of the schema (suppose these to be n-ary connectives) $\kappa(A_1, \ldots, A_n) \to \lambda(A_1, \ldots, A_n)$. The latter 'comparative strength' relation has indeed been confused in the past with what I am calling the subconnective relation. For example, [29] cites numerous logic texts as arguing that since everyone agrees that a necessary condition for the truth of an indicative conditional in English is the truth of the corresponding material conditional (whether or not they think this is sufficient), any argument involving the former conditional will, if valid, convert into a valid argument when it is replaced by the latter. As Staines pointed out, this argument about arguments is simply fallacious. It by no means follows from the fact that in a language containing both conditionals, the material and the 'natural' conditional, the material conditional forms compounds implied by the corresponding compounds formed by the natural conditional, that any statement of what follows from what will continue to hold when references to the natural conditional are replaced by references to the material conditional. This point could naturally be described as involving a confusion between comparative deductive strength and the subconnective relation, especially when these notions are given the obvious redefinition they would require in a sequent-based rather than formula-based approach to logic, a change of perspective we consider for quite different reasons in the following section. Amongst the one-place extensional connectives, a simpler, though less interesting example of this confusion would be to think that because in a logic containing symbols for both **V** and **F**, say κ and λ, we should have $\lambda \rightarrow \kappa$, any formula it contained with λ would have to remain in the logic when this was replaced by κ.

IV. A SEQUENT-LOGICAL APPROACH TO EXTENSIONAL CONNECTIVES

The labelling of the lattice elements in Figure 1 as well as the surrounding discussion in Section III highlighted the rôle of the product operation. The dual operation we call *sum* ('+'). The sum of two connectives is a connective having all the logical powers *either* of them has. This is not a very interesting operation on truth-functional connectives such as the dual atoms of Figure 1, since as we saw in Section III, inconsistency results from summing any two of them. But if we start from the atoms of the lattice, clearly we can build up all the extensional connectives by successive summing. This affords an 'anatomical' perspective on the truth-functions themselves, the interest of which is presumably at its greatest when they are not already (as on the conception of a logic in Sections I – III they are) taken to be all definable. Thus we work in the present section with an alternative logical framework, the sequent-logical approach of Scott (e.g. [23]), taking a language to be as before, except now with no requirement (like the requirement about $\rightarrow$ and $\bot$) that any connectives be present. A logic in such a language will be a collection of ordered pairs $\langle \Gamma, \Delta \rangle$ of finite (possibly empty) sets of formulae of the language. For familiarity we write $\Gamma \vdash \Delta$ for such a pair, and call it a sequent. To qualify as a logic, we demand that the set of sequents be closed under Scott's rules (R), (M) and (T). An n-ary κ connective in the language of such a logic L is *extensional* in L iff L is closed under the rule:

(RME_n sequent form)

$$\frac{\Gamma, A_1 \vdash B_1, \Delta \quad \Gamma, B_1 \vdash A_1, \Delta \quad \ldots \quad \Gamma, A_n \vdash B_n, \Delta \quad \Gamma, B_n \vdash A_n, \Delta}{\Gamma, \kappa(A_1, \ldots, A_n) \vdash \kappa(B_1, \ldots, B_n), \Delta}.$$

This is the appropriate re-casting of the definition in Section I in that if the truth-functional connectives of that section are assumed available in the language and governed by the usual rules, closure under the above rule is equivalent to the provability of all instances of the sequent-schema with nothing to the left of the '$\vdash$' and the (RME_n) of that section to the right. Note that the condition of congruentiality is similarly re-expressible in terms of a variation on the requirement of closure under (RME_n sequent form) got by adding the proviso that Γ and Δ be empty.

A *valuation* for a language is a function from formulae of the language to {T, F}, and a valuation V is *consistent with* a logic L when for all $\Gamma \vdash \Delta \in L$, $V(C) = T$ for each $C \in \Gamma$ only if $V(D) = T$ for some $D \in \Delta$. Denote by $\mathscr{V}_L$ the set of all valuations consistent with the logic L. Then, as Scott shows in [23] with the aid of an elegant Lindenbaum argument, the set $\mathscr{V}_L$ can be seen as providing a semantics for L in the sense that for all finite sets Γ, Δ of formulae of the language of L:

$$\Gamma \vdash \Delta \in L \text{ iff } \forall V \in \mathscr{V}_L, \quad V(C) = T$$
$$\text{for each } C \in \Gamma \text{ implies } V(D) = T \text{ for some } D \in \Delta.$$

(The 'only if' is trivial; the 'if' is Scott's abstract completeness theorem.) We may use the concept here recalled to throw light on extensionality with the following

THEOREM. If κ is an n-ary connective in the language of a logic L, then κ is extensional in L iff $\forall V \in \mathscr{V}_L, \exists f \forall A_1, \ldots, A_n\ V(\kappa(A_1, \ldots, A_n) = f(V(A_1), \ldots, V(A_n))$.

Here the 'f' ranges over functions from {T, F} $\times \ldots \times$ {T, F} (n times) to {T, F}. Note that although we do not presume available the connectives $\bot$ and $\rightarrow$ governed by the principles of classical logic, and so cannot directly mimick the definition in Section I of truth-functionality, a reasonable definition in the present setting would be to identify the n-ary connectives of L as truth-functional when the right-hand side of the biconditional in this Theorem held with '$\exists f$' and '$\forall V$' interchanged. (A more direct, proof-theoretic, characterization equivalent to this would use Segerberg's concept of type-determination [25].) Putting this Theorem together with Scott's result above yields a semantic account of the various extensional connectives in the sequent framework: each is interpretable by the specification of a *range* of truth-functions (*cf.* the $\langle V, f \rangle$ interpretations of the following section).

We shall, for illustrative purposes, make a few observations about the extensional logics in a language containing a single (note: we do not now have to say 'single additional') singularly connective-symbol. Write it as '$*$'. The weakest such logic is of course the smallest logic in this language closed under the rule (RME_1 sequent form) for '$*$'. An interesting deriveable rule is the following:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, **A \vdash *B, \Delta}$$

which corresponds to a principle 'known as a curiosity to Lesniewski'[8] in the form of an axiom(-schema) $**\bot \rightarrow *B$, in the sense that if connectives $\rightarrow$ and $\bot$ are added to the language, subject to the usual sequent-logical rules the condition that a system be closed under the rule and that it contain all instances of the sequent-schema with nothing on the left of the '$\vdash$' and the above principle on the right are equivalent. A few sequents of special interest belonging to this smallest (RME_1)-closed logic are also worth listing, though proofs will not be given. We have all instances of the following schemata:

(i)	$A, *A \vdash **A$	(ii)	$**B \vdash *B, B$
(iii)	$*A \dashv\vdash ***A$	(iv)	$A, **B \vdash **A, B$
(*LE*)	$A, B, *A \vdash *B$	(*RE*)	$*A \vdash *B, A, B$

Instead of defining extensionality for $*$ in terms of the rule (RME_1) we could equivalently (given (*R*), (*M*), (*T*)) have simply required that a logic contain all instances of the last two schemata. The first of these schemata, (*LE*) – for 'left-extensionality' – imposes the condition on valuations for any logic containing all its instances: $*$-formulae cannot differ in truth-value when their immediate subformulae are alike in being true. For (*RE*) the condition is the same of course except that we say '. . . their subformulae are alike in being false'. We turn now to some extensions of this minimal extensional logic.

In the logic just considered, the connective-symbol $*$ represents the completely generic one-place extensional connective (in the 'logical powers' sense of connective: we assume the vocabulary introduced in this connexion Section III to be re-defined in the obvious way to apply to the sequent framework). If we, for example, wish to isolate the minimal logic of connectives which are *veridical*, meaning by this that the consistent valuations for the logic satisfy the condition: $V(*) = \mathrm{T}$ only if $V(A) = \mathrm{T}$, then adding to L the schema $*A \vdash A$ will do the job, as is easily verified. This is of course none other than the logic of **FI**, in the terminology of Section III. If one is interested in economy of axiomatization, one may take this sequent schema alongside (LE) and consider this as the smallest logic containing all instances of both schema (since the schema in question yields (RE) by (M)). Similar economies, not further commented on, are available for many of the cases presented below.

It is worth seeing what happens in the present framework to the other entries in Figure 1. There remain, apart from the generic case (the lattice **O**), the inconsistent case (on which more below), and the case of **FI** just dealt with, thirteen logics. Schemata axiomatizing these logics (where '$\dashv\vdash$' is used to refer to a sequent and its converse) are the following:

for **F**: $*A \vdash$ for **V**: $\vdash *B$ for **FN**: $A, *A \vdash$
for **VI**: $B \vdash *B$ for **VF**: $*A \vdash *B$ for **VN**: $\vdash *B, B$
for **IN** (= **VIN** + **FIN**): $A \dashv\vdash **A$ for **I**: $A \dashv\vdash *A$
for **N**: $A, *A \vdash$ alongside $\vdash A, *A$ for **VIN**: $A \vdash **A$
for **VFN**: $A, *A \vdash *B$ or alternatively $*A \vdash *B, B$ for **FIN**: $**A \vdash A$
for **VFI**: $B, *A \vdash *B$ or alternatively $*A \vdash *B, A$

There is more to the story in the sequent framework than there was to tell in Section III however. An intriguing complication is that whereas in that section we had a boolean lattice, the lattice of extensions of the generic logic in the present setting is not even distributive. The sequent logic usually referred to as inconsistent in a given language is that containing all sequents $\Gamma \vdash \Delta$, where Γ and Δ are sets of formulae of the language. Equivalently: iff the logic contains the 'empty' sequent ⊢ (i.e., $\varnothing \vdash \varnothing$). Now this logic has some pathological neighbours which are distinct from it. I will call the logic in a given language (our $*$-language for present purposes) 'Yes' if it contains every sequent $\vdash A$ (for A a formula of the language) and 'No' if it contains every sequent $A \vdash$. Note that $*$ is extensional in these logics. Their lattice sum is the inconsistent logic. Their product is the smallest logic to contain every sequent $A \vdash B$ (A, B, arbitrary formulae). It thus consists precisely of the sequents $\Gamma \vdash \Delta$ satisfying the condition that neither Γ nor Δ is empty. Semantically, we may describe it as the system of Constant-Valued Logic, in that its consistent valuations are those which never assign different truth-values to different formulae. (The set of valuations consistent with '**Yes**' ('**No**') consists of precisely those that assign T (F) to every formula.) These logics interact with those we have been discussing in ways which distort the lattice relations depicted in Figure 1. For example, whereas there we have **VF** + **I** = **1** (the inconsistent logic), here we have **VF** + **I** = **YesNo**. Our present concern is not with mapping out the whole of the new lattice, but with showing its non-distributivity. For this purpose, it suffices to examine the following sublattice:

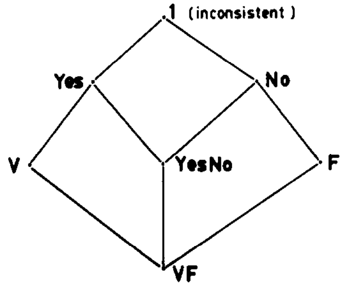

Fig. 2

Here notice that **No(V** + **F)** = **No1** = **No**, while **(NoV)** + **(NoF)** = **VF** + **F** = **F**; so **No(V** + **F)** ≠ **(NoV)** + **(NoF)**.

Such oddities aside, the + operation really comes into its own in the setting of the present approach to senential logic. In Section III, we worked the non-truth-functional but extensional connectives into a language already boasting the full range of truth-functional connectives (as primitive or definable), and so it was natural that we should think of the novelties as products of those antecedently available connectives. But the approach of the present section allows us a reversal of perspective. Beginning with a language containing no connectives, we can introduce first the extensional connectives which are not truth-functional, and arrive at the truth-functional connectives themselves by taking sums. We could, for example, begin with the 'veridical' connective $*$ satisfying ($*A \vdash A$) mentioned above, and the 'weakening' connective $*'$ axiomatized by the converse sequent-schema (*alias* **FI** and **VI** respectively) and think of **I** as $* + *'$. The interest of such a reorientation may be illustrated by considering the case of the binary connectives. For example, we can think of the usual conjunction connective as the sum of two connectives, call them $\wedge_I$ and $\wedge_E$, the logic of the former being axiomatized by adding to a basis for the logic of the completely generic extensional binary connective (i.e., the smallest (RME_2)-closed logic) the schema $A, B \vdash A \wedge_I B$, that of the latter by $A \wedge_E B \vdash A$ and $A \wedge_E B \vdash B$. This decomposition, $\wedge = \wedge_I + \wedge_E$ is suggested by a certain proof-theoretic approach, namely, natural deduction (the I and E are intended to recall Introduction and Elimination; with $\vee_I$ similarly understood, recall that Prior [19] had examined $\vee_I + \wedge_E$ and observed that this gave the Constant-Valued logic of a binary connective.)

More semantically inspired building blocks out of which to construct the truth-functional connective of conjunction would be the sequent-schemata which force, separately, each of the various values (four, in our case, and in general 2^n for an n-ary connective) for the compound. Taking a terminological cue from Segerberg's theory of type-determination ([25], p. 59), we could call these schemata (or zero-premiss rules) *determinants*. For conjunction they are the following:

$$A, B \vdash A\ \kappa_{ttt}\ B \qquad\qquad A, A\ \kappa_{tff}\ B \vdash B$$

$$B, A\ \kappa_{ftf}\ B \vdash B \qquad\qquad A\ \kappa_{fff}\ B \vdash A, B$$

The rather cumbersome notation is intended to suggest the semantic effect of the governing schema: thus valuations consistent with the logic of κ_{tff}, for example, cannot assign the value T to the first component of a compound formed with the aid of this connective, and

F to the second, without also assigning F to the compound; and similarly for the other cases. Conjunction, then, is the sum of the four connectives listed, and the first and fourth on the list figure likewise amongst the four determinants whose sum is (inclusive) disjunction, the remaining two being the connectives we should, on the present pattern, refer to κ_{tft} and κ_{ftt}.

We have used the symbol '+' in our labels for various connectives, without supposing it to figure as a symbol of the object languages containing those connectives. That would be a possibility worth exploring. Then '+' would be a connective-modifier, and the language would contain simple and also complex connectives, formed by application of this modifier, which links two *n*-ary connectives to make a new *n*-ary connective. But a systematic investigation of this line would require that we considered languages containing more than one connective (or more than one additional connective, if pursued as in Section III) at a time and so lies beyond the scope of the present paper.

V. CLOSING COMMENTS

In this paper, I have drawn attention to the existence of extensional connectives which are not truth-functional, and said something about some of their more obvious properties. I want to close by considering the question of whether the symbols we have been discussing really are connectives. An alternative view might be that what we are really discussing is classes of connectives, and that we are conducting the discussion in schematic terms, abstracting from certain differences between the various connectives involved. The view has two forms. The first charge would be that the symbols allegedly denoting extensional but non-truth-functional connectives in the object-language of this or that system are really meta-linguistic variables over ordinary truth-functional connectives. This first form of the objection is definitely wrong: we can put whatever symbols we like into our object-languages, and subject them to whatever axioms or rules we care to. A second version of this 'schematic symbol' objection would concede that no meta-language/object-language confusion has been committed, but still our discussion has ignored the fact that the various symbols we have counted as connectives differ from the ordinary connectives usually considered in something like the way that individual variables differ from individual constants.[9] Just as a variable may stand in a formula in the same position that a name might, so, e.g., our completely generic singularly connective can stand precisely where a connective like $\sim$ stands, but the force of the provability of a sequent (or a formula, in the framework of earlier sections) involving the former symbol is to be understood as prefixed by a universal quantifier, invisible in our notation. This is an interpretation in the style of Lesniewski's protothetic, and it gives very much the spirit of [15], in which Łukasiewicz uses the symbol 'δ' as what he calls a 'variable functor', to be thought of as ranging over the one-place truth-functions, so that the formulae containing it which are deemed valid are those valid for, as the terminology of the present paper would have it, the completely generic singularly extensional connective. But Łukasiewicz could not himself insist on the 'variable functor' description for any connective which is a product (as this one is: of **V**, **F**, **I** and **N**) of the two-valued singularly truth-functional connectives, since he thinks of his own modal operators, themselves capable of being presented in this light, as *bona fide* connectives in their own right, while these are themselves viewable as such products. Prior ([18], p. 145), on the other hand, explicitly takes the line that the 'higher' products such as the Łukasiewicz modal operators are 'variable functors with a limited range'.

The cost of ceasing to describe the extensional connectives as genuine connectives unless they happen to be truth-functional would be considerable, however. We have already seen that this description fits the modal operator '$\Box$' in certain modal logics, not normally something one regards as a schematic symbol: further this happens to logics sandwiched between logics in which the description does not apply (e.g., to $K + A \rightarrow \Box A$, between K, in which $\Box$ is intensional and so presumably immune from the argument, and the Trivial system, in which again, now because it is not only extensional but actually truth-functional, $\Box$ is immune). I take it that the oscillations in whether or not we regard '$\Box$' as a connective or a schematic symbol in the position of a connective which would thus result are hardly to be welcomed.[10]

Actually, the analogy with modal logic is rather more general than the above considerations may suggest, when attention is turned to the semantics of the extensional connectives. For simplicity, we revert to the assumption that we are dealing (in the sequent-logical framework) with a language containing a single connective, and that (call it $*$) singulary. Then we may recast our semantic scheme in simulation of the (relational) frames-and-models approach to the semantics of normal modal systems, by now restricting a *valuation* to being an assignment of truth-values to the atomic formulae of the language, and defining an *interpretation* to be a pair $\langle f, V\rangle$ where V is such a valuation and f (as before) is a one-place truth-function. Truth of a formula on an interpretation is defined inductively in the following way:

$$\begin{array}{lll} \langle f, V\rangle \models A & \text{iff} & V(A) = T, \text{ for } A \text{ atomic} \\ \langle f, V\rangle \models *A & \text{iff} & \langle f, V\rangle \models A \text{ and } f(T) \models T \text{ or} \\ & & \langle f, V\rangle \models A \text{ and } f(F) \models T. \end{array}$$

Let us now say that a sequent $\Gamma \vdash \Delta$ is *valid* over a class $\mathscr{C}$ of interpretations if for no $\langle f, V\rangle \in \mathscr{C}$ do we have $\langle f, V\rangle \models A$ for each $A \in \Gamma$ unless we also have $\langle f, V\rangle \models B$ for some $B \in \Delta$. Then following the lead of the customary presentations of modal logic we describe a logic as *sound* (*complete*) with respect to a class of interpretations if the sequents provable in the logic are all (only those) sequents valid over the class. Then our remarks in Section IV can be rephrased to: the smallest (RME_1)-closed logic is sound and complete w.r.t. the class of *all* interpretations, and, to recall another case considered in that section, defining an interpretation $\langle f, V\rangle$ to be *veridical* when $f(F) = F$, we may describe the extension of that logic by the schema $*A \vdash A$ as sound and complete w.r.t. the class of all veridical interpretations.

We can now state the schematic-symbol objection as: although each interpretation for the present language does live up to its name and interpret the '$*$' as some truth-function, the logics in which this symbol behaves as (in my terminology) a non-truth-functional though extensional connective are sound and complete only w.r.t. classes of interpretations with *different* f-components: thus the symbol '$*$' is being treated *schematically* by these logics, in the sense of being given no fixed interpretation. However, we can see now also that a parallel objection could be made in connexion with normal modal logics since '$\Box$' is interpreted *via* the relation $\boldsymbol{R}$ in the models $\langle \boldsymbol{W}, \boldsymbol{R}, \boldsymbol{V} \rangle$, and this (as well as the $\boldsymbol{V}$) is allowed to vary from model to model, even (typically) for $\boldsymbol{W}$ fixed. One can, if one likes, reverse the usual perspective on the semantics of modal logic and abstract from models not (as is generally done) frames (pairs $\langle \boldsymbol{W}, \boldsymbol{R} \rangle$) but instead what I might call *cradles*: pairs $\langle \boldsymbol{W}, \boldsymbol{V} \rangle$. To get a model 'on' such a cradle one has to supply some $\boldsymbol{R} \subseteq \boldsymbol{W} \times \boldsymbol{W}$, and we can call a sequent $\Gamma \vdash \Delta$ *secure* on a cradle when at no point in any model on the cradle are all formulae in Γ true and all formulae in Δ false, just as we should normally say that such a sequent was valid on a frame when at no point in any model on the frame are all formulae in Γ true and all formulae in Δ false. Then the fact normally expressed by saying that the (sequent-logically formulated) system *K* consists of precisely the sequents valid on every frame may be reformulated as: *K* consists of the sequents secure on every cradle. The equivalence of these two formulations is clear, the difference in suggestiveness (in particular as regards certain consequence relations among sequents) between the two terminologies notwithstanding. Neither is the 'right' way to conceptualize the given subject-matter, though the 'cradle' approach certainly makes '$\Box$' appear more in the light of a schematic or variable-like symbol in that it focusses at some level of description on structures making no fixed assignment (of an accessibility relation) to that symbol. I cannot believe much of any significance to hang on a preference for either approach, in the absence of quantication into the position occupied by the symbol in question.[11] With these remarks I conclude my rejection of the sentiment that our extensional yet non-truth-functional connectives are not really connectives at all, but abstractions therefrom symbolized by schematic letters. The sharp contrast between symbols which are and those which are not intrinsically schematic or 'variable-like' that is presumed when such a sentiment is expressed does not exist.

NOTES

[1] The description of contexts as truth-functional when they are (in the present terminology) extensional in sentence position is common in philosophical discussions, as is

well illustrated by the voluminous literature on the 'Frege Argument', from which the examples [4] and [16] of (what I regard as) this misdescription are chosen at random. However, I do not dwell on this since nothing of substance hangs on the use of the term *truth-functional* in these discussions in place of *extensional in sentence position*, though that use of terms strikes me as inviting a confusion between the two concepts I use this pair of terms to distinguish.

[2] The 'p_i' notation here may be taken as either object-linguistic or meta-linguistic; in the latter case we assume $p_i \neq p_j$ if $i \neq j$. Other truth-functional primitives could be chosen in place of $\bot$, $\rightarrow$, though some of our formulations exploit the presence of $\bot$, making available, at it does, formulae devoid of propositional variables.

[3] See [12]. A similar rigidity concept for connectives was isolated by Cresswell at p. 369 of [2] under the name 'truth-functionally constant functor (in a set of worlds)'.

[4] I would have preferred to exploit the 'almost' connotations of 'quasi' and called these connectives *quasi*-truth-functional, but this term has been appropriated by Rescher with a different meaning (e.g., in [21]). The need to change variables, in going from disjnct to disjunct will be clear from what follows; without it even the $\Box$ of the modal system *K* would emerge as psuedo-truth-functional (since $(\Box p \leftrightarrow p) \vee (\Box p \leftrightarrow \sim p)$ is provable therein). For the moment, it may help to think of the variable-changing as tantamount to what would be expressed in a language with propositional quantifiers by prefixing each disjunct with a universal quantifier.

[5] There is an analogy with the concept of piecewise definability from the model theory of standard logic here. (See for example [22].)

[6] Geach [6] complained of McCall's attempt to examine the logic of a contrary-forming sentential operator on the grounds that there is no such thing as *the* contrary of a given statement (unlike the case of contradictories), but this observation does not prevent one from exploring the logical properties of an arbitrarily selected contrary for a given statement. (*Cf.* the discussion of uniqueness below.)

[7] For the record: I believe that justice can only be done to the idea of a logic's uniquely characterizing a connective if a richer conception of logic is in play from that of Sections I – III or Section IV. A sensitivity is required to the rules of proof (primitive or derived) of logical systems. We can safely ignore this complication for present purposes, however.

[8] This remark is from [15], p. 324; the sentence in which it occurs runs in full: "It is a thesis which was already known as a curiosity to Lesniewski, and was brought to Dublin by Sobocinski in 1947." (One wonders how he got it through Customs.)

[9] It is perhaps not necessary to mention here that the contrast between variable/ schematic symbols for connectives and constant symbols for connectives is not to be confused with any contrast between symbols which are not, and symbols which are, *logical constants*.

[10] Since we can use the product terminology to describe relations between connectives, there is no need to deny 'genuine' connective status to all but the dual atoms of Figure 1, any more that one should deny the status of 'genuine' logic to a logic that is not Post-complete. It is not as though one is deprived of making observations with the aid of schematic symbols ranging over a variety of connectives, or of putting the present apparatus to work in the description of logics not themselves possessing the products in terms of which the description is cast. In this spirit, for example, we may say that the logic (in the style of Section IV) of the completely generic *n*-place connective consists of

the sequents involving an n-ary truth-functional connective which are 'structurally valid' in the sense of Gareth Evans (see [5]).

[11] The idea that schematicity is a feature of the way connectives (in the logical powers sense) are *treated* rather than of the connectives themselves, may be further illustrated with the case of $\bot$, as it behaves in minimal logic. This zero-place connective has precisely the same logical powers as any other atomic formula, but is not treated, e.g., in [24] as equally variable-like, since the logic is studied semantically with the aid of frames which assign a truth-set to this formula which remains fixed while the changes are rung (as we change from model to model on a given frame) on the other atomic formulae. It is certainly convenient to present the semantics in this way (for ease of comparison with various extensions of the system in which $\bot$ comes to have more distinctive logical powers) but obviously for the sake of just describing minimal logic, we could have had the models rather than the frames provide the assignment to it. This would be to remove a level of description at which $\bot$ had a fixed semantic value while other atomic formulae did not, and so constitute a way of *treating* it as more schematic or variable-like.

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