The Second Law of Thermodynamics and Stability

C. M. DAFERMOS

Dedicated to Clifford Truesdell on the occasion of his sixtieth birthday

Εὖ ἂν ἔχοι εἰ τοιοῦτον εἴη ἡ σοφία, ὥστ' ἐκ τοῦ πληρεστέρου εἰς τὸν κενώτερον ῥεῖν ἡμῶν, ἐὰν ἁπτώμεθα ἀλλήλων.

PLATO

1. Introduction

There are several indications pointing to an intimate relationship between the second law of thermodynamics and "stability". Most notably, the work of ERICKSEN [1], COLEMAN & DILL [2] and GURTIN [3] has revealed that the Clausius-Duhem inequality induces Liapunov stability of equilibrium processes in a variety of materials. Here I attempt to establish a different connection between stability and the second law, in that "stability" will be interpreted as continuous dependence of thermodynamic processes upon initial state and supply terms.

The ideas will be presented within the context of thermoelasticity theory, without heat conduction. The hyperbolic character of the field equations causes the breakdown of smooth solutions and the development of shock waves, so the class of smooth functions is far too narrow to encompass all processes of physical interest. The natural framework is the class of *proper processes*, characterized by the property that velocity, deformation gradient, specific entropy, stress, temperature and internal energy are functions of bounded variation in the sense of Tonelli-Cesari.* The balance laws, in integral as well as in local form, are meaningful in the class of proper processes, and the classical geometric theory of wave propagation can be transplanted into this class. One expects that existence of solutions to the field equations will eventually be established within the class of proper processes, but so far this has only been accomplished in the one-dimensional case [5].

* that is, measurable functions whose first derivatives are locally Borel measures. This is essentially the broadest function class in which the Gauss-Green theorem holds. There is an analogy between the geometric structures of functions of bounded variation and functions that are piecewise smooth. For a survey see [4].

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The interpretation of the role of the second law of thermodynamics in the theory of proper processes poses an interesting problem. The practice of restricting constitutive relations so that the Clausius-Duhem inequality be automatically satisfied by smooth processes originated in the work of COLEMAN & NOLL [6] and has by now become standard in continuum thermomechanics. On the other hand, within the class of processes with shock waves, the Clausius-Duhem inequality has traditionally been viewed as an admissibility criterion. It turns out that this criterion generally rules out some but not all extraneous processes, so that one must either impose shock admissibility restrictions [7] or else strengthen the second law [8], in order to single out those processes that are physically relevant.

Despite the above remarks, it will be shown here that, whenever they exist, smooth processes are stable within the class of (not necessarily smooth) proper processes that satisfy the Clausius-Duhem inequality. In other words, so long as one is dealing with smooth processes, the second law in its traditional form is a satisfactory admissibility criterion.

The above result is established under certain assumptions on material response,* relating to convexity of internal energy. With the exception of one-dimensional bodies, global convexity of the internal energy function is contrary to experience (it is incompatible, in particular, with the principle of frame indifference). Even so, internal energy will in general be locally convex on a certain region in state space. In Section 4 we prove uniqueness and stability for smooth processes residing in the convexity region of internal energy. For the case where the body is smooth and bounded and the motion of its boundary is prescribed, stability is established in Section 5 under the weaker assumption that the smooth process resides in the strong ellipticity region.

The general strategy of the proof was inspired by the important paper [10] of DIPERNA on uniqueness of solutions to the initial value problem for quasilinear hyperbolic systems. The heart of the proof is an inequality, derived in Section 3, which estimates the evolution in time of the "distance" between the states of two processes originating at neighboring states. The time rate of increase of this distance can be controlled, with the help of the Clausius-Duhem inequality, provided that at least one of the processes is smooth and resides in the convexity or the strong ellipticity region of internal energy.

2. Adiabatic Processes in Thermoelastic Materials

We consider a thermoelastic body with reference configuration $\mathscr{B}$ in R^n ($n=1,2$ or 3). The reference mass density $\rho(X)$, defined on $\bar{\mathscr{B}}$, is smooth and strictly positive,

$$\rho(X) \geqq \rho_0 > 0, \qquad X \in \bar{\mathscr{B}}. \tag{2.1}$$

A motion $x = x(X, t)$ determines the velocity field $v = \dot{x}$ and the deformation gradient field $F = \operatorname{grad}_X x$. The internal energy ε, the Piola-Kirchhoff stress T and

* It is not generally to be expected that the second law will of itself induce stability, unless it is supported by appropriate restrictions on constitutive relations. In this connection, see the illuminating remarks of SERRIN [9].

the temperature θ are determined by the deformation gradient $\boldsymbol{F}$ and the specific entropy η via constitutive relations

(2.2) $$\varepsilon=\varepsilon^*(\boldsymbol{F},\eta;\ \boldsymbol{X}),\ \ \boldsymbol{T}=\boldsymbol{T}^*(\boldsymbol{F},\eta;\ \boldsymbol{X}),\ \ \theta=\theta^*(\boldsymbol{F},\eta;\ \boldsymbol{X}),$$

where $\varepsilon^*, \boldsymbol{T}^*$ and θ^* are smooth functions defined for $\boldsymbol{F}$ in the set M^+ of $n\times n$ matrices with positive determinant, η in R and $\boldsymbol{X}$ in $\bar{\mathscr{B}}$.* We also require that

(2.3) $$\theta^*(\boldsymbol{F},\eta;\ \boldsymbol{X})>0, \qquad \boldsymbol{F}\in M^+,\ \eta\in R,\ \boldsymbol{X}\in\bar{\mathscr{B}}.$$

We shall be assuming that the material is a non-conductor of heat so that heat flux and entropy flux vanish.

By a *proper thermodynamic process* we mean fields $(\boldsymbol{x},\eta)(\boldsymbol{X},t)$ such that $(\boldsymbol{v},\boldsymbol{F},\eta)(\boldsymbol{X},t)$ are functions of locally bounded variation, in the sense of Tonelli-Cesari, satisfying the balance laws of momentum and energy, *viz.*,

(2.4) $$\rho\,\dot{v}_i=T_{i\alpha,\alpha}+\rho\, b_i,$$

(2.5) $$\rho\overline{(\varepsilon+\tfrac{1}{2}v_i v_i)}^{\boldsymbol{\cdot}}=(T_{i\alpha}v_i)_{,\alpha}+\rho\, b_i v_i+\rho\, r,$$

where $\boldsymbol{b}(\boldsymbol{X},t)$ is the body force and $r(\boldsymbol{X},t)$ is the energy supply. The process will be called *admissible* if it also satisfies the Clausius-Duhem inequality

(2.6) $$\dot{\eta}-\frac{r}{\theta}\geqq 0.$$

The reader should bear in mind that for the class of proper processes (2.4)–(2.6) only hold in the sense of measures (or distributions) and thus one cannot simplify (2.5) in the standard fashion, since the usual product differentiation rule does not apply to products of functions of bounded variation.

A process $(\bar{\boldsymbol{x}},\bar{\eta})(\boldsymbol{X},t)$ will be called *smooth* if the functions $(\bar{\boldsymbol{v}},\bar{\boldsymbol{F}},\bar{\eta})(\boldsymbol{X},t)$ are Lipschitz continuous, uniformly on bounded subsets of their domain. Thus a smooth process may contain weak waves but not shock waves. For smooth processes, one may write the balance laws in reduced form

(2.7) $$\rho\,\dot{\bar{v}}_i=\bar{T}_{i\alpha,\alpha}+\rho\,\bar{b}_i$$

(2.8) $$\rho\,\dot{\bar{\varepsilon}}=\bar{T}_{i\alpha}\bar{v}_{i,\alpha}+\rho\,\bar{r}.$$

As is known, the standard requirement that every smooth process be admissible will be satisfied if and only if

(2.9) $$\boldsymbol{T}^*=\rho\frac{\partial\varepsilon^*}{\partial\boldsymbol{F}}, \qquad \theta^*=\frac{\partial\varepsilon^*}{\partial\eta}.$$

Indeed, upon using (2.9), (2.8) yields

(2.10) $$\dot{\bar{\eta}}-\frac{\bar{r}}{\bar{\theta}}=0,$$

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* In particular, we assume that the partial derivatives of ε^*, θ^* and $\boldsymbol{T}^*$, at any fixed $\boldsymbol{F}\in M^+$ and $\eta\in R$, are bounded functions of $\boldsymbol{X}$ on $\bar{\mathscr{B}}$.

which shows that the Clausius-Duhem inequality is automatically satisfied by smooth processes, as an equality.

3. The Evolutionary Inequality

We establish here the estimate that will be the basis of our stability analysis. We define

$$\begin{aligned} H^*(\boldsymbol{v},\boldsymbol{F},\eta;\,\bar{\boldsymbol{v}},\bar{\boldsymbol{F}},\bar{\eta};\,\boldsymbol{X}) &= \tfrac{1}{2}\rho(v_i-\bar{v}_i)(v_i-\bar{v}_i)+\rho\,\varepsilon^*(\boldsymbol{F},\eta;\,\boldsymbol{X}) \\ &\quad -\rho\,\varepsilon^*(\bar{\boldsymbol{F}},\bar{\eta};\boldsymbol{X})-T^*_{i\alpha}(\bar{\boldsymbol{F}},\bar{\eta};\,\boldsymbol{X})(F_{i\alpha}-\bar{F}_{i\alpha})-\rho\,\theta^*(\bar{\boldsymbol{F}},\bar{\eta};\,\boldsymbol{X})(\eta-\bar{\eta}), \end{aligned} \tag{3.1}$$

$$G^*_\alpha(\boldsymbol{v},\boldsymbol{F},\eta;\,\bar{\boldsymbol{v}},\bar{\boldsymbol{F}},\bar{\eta};\,\boldsymbol{X}) = -(T^*_{i\alpha}(\boldsymbol{F},\eta;\,\boldsymbol{X})-T^*_{i\alpha}(\bar{\boldsymbol{F}},\bar{\eta};\,\boldsymbol{X}))(v_i-\bar{v}_i). \tag{3.2}$$

On account of (2.9), H^* and $\boldsymbol{G}^*$ are of quadratic order in $(\boldsymbol{v}-\bar{\boldsymbol{v}},\boldsymbol{F}-\bar{\boldsymbol{F}},\eta-\bar{\eta})$.

Consider now a smooth process $(\bar{\boldsymbol{x}},\bar{\eta})(\boldsymbol{X},t)$, with supply terms $(\bar{\boldsymbol{b}},\bar{r})(\boldsymbol{X},t)$, and let $(\boldsymbol{x},\eta)(\boldsymbol{X},t)$ be any admissible process with supply terms $(\boldsymbol{b},r)(\boldsymbol{X},t)$. We set

$$H(\boldsymbol{X},t)=H^*\big(\boldsymbol{v}(\boldsymbol{X},t),\boldsymbol{F}(\boldsymbol{X},t),\eta(\boldsymbol{X},t);\,\bar{\boldsymbol{v}}(\boldsymbol{X},t),\bar{\boldsymbol{F}}(\boldsymbol{X},t),\bar{\eta}(\boldsymbol{X},t);\,\boldsymbol{X}\big), \tag{3.3}$$

$$\boldsymbol{G}(\boldsymbol{X},t)=\boldsymbol{G}^*\big(\boldsymbol{v}(\boldsymbol{X},t),\boldsymbol{F}(\boldsymbol{X},t),\eta(\boldsymbol{X},t);\,\bar{\boldsymbol{v}}(\boldsymbol{X},t),\bar{\boldsymbol{F}}(\boldsymbol{X},t),\bar{\eta}(\boldsymbol{X},t);\,\boldsymbol{X}\big). \tag{3.4}$$

We shall be viewing H as a measure of the "distance" between the two processes. In order to see how H evolves in time, we compute below $\dot{H}+\operatorname{div}_{\boldsymbol{X}}\boldsymbol{G}$, bearing in mind that the usual product differentiation rule applies (in the sense of measures) to the product of a Lipschitz continuous function with a function of bounded variation. We have

$$\begin{aligned} \dot{H}+G_{\alpha,\alpha} &= \rho\,\overline{(\varepsilon+\tfrac{1}{2}v_i v_i)}^{\boldsymbol{\cdot}}-\rho\,v_i\dot{\bar{v}}_i-\rho\,\bar{v}_i\dot{v}_i+\rho\,\bar{v}_i\dot{\bar{v}}_i-\rho\,\dot{\bar{\varepsilon}} \\ &\quad -\dot{\bar{T}}_{i\alpha}(F_{i\alpha}-\bar{F}_{i\alpha})-\bar{T}_{i\alpha}\dot{F}_{i\alpha}+\bar{T}_{i\alpha}\dot{\bar{F}}_{i\alpha}-\rho\,\dot{\bar{\theta}}(\eta-\bar{\eta}) \\ &\quad -\rho\,\bar{\theta}(\dot{\eta}-\dot{\bar{\eta}})-(T_{i\alpha}v_i)_{,\alpha}+(T_{i\alpha}-\bar{T}_{i\alpha})\dot{\bar{F}}_{i\alpha}+\bar{T}_{i\alpha}\dot{F}_{i\alpha} \\ &\quad +v_i\,\bar{T}_{i\alpha,\alpha}+\bar{v}_i\,T_{i\alpha,\alpha}-\bar{v}_i\,\bar{T}_{i\alpha,\alpha}. \end{aligned} \tag{3.5}$$

Observing that $v_{i,\alpha}=\dot{F}_{i\alpha}$, $\bar{v}_{i,\alpha}=\dot{\bar{F}}_{i\alpha}$ and using the balance laws (2.4), (2.5), (2.7), (2.8), we may rewrite (3.5) in the form

$$\begin{aligned} \dot{H}+G_{\alpha,\alpha} &= \rho(v_i-\bar{v}_i)(b_i-\bar{b}_i)+\rho(r-\bar{r})-\dot{\bar{T}}_{j\beta}(F_{j\beta}-\bar{F}_{j\beta}) \\ &\quad +\dot{\bar{F}}_{i\alpha}(T_{i\alpha}-\bar{T}_{i\alpha})-\rho\,\dot{\bar{\theta}}(\eta-\bar{\eta})-\rho\,\bar{\theta}(\dot{\eta}-\dot{\bar{\eta}}) \\ &= \rho(v_i-\bar{v}_i)(b_i-\bar{b}_i)+\rho(r-\bar{r})-\overline{\frac{\partial T^*_{j\beta}}{\partial F_{i\alpha}}}\,\dot{\bar{F}}_{i\alpha}(F_{j\beta}-\bar{F}_{j\beta}) \\ &\quad -\overline{\frac{\partial T^*_{j\beta}}{\partial \eta}}\,\dot{\bar{\eta}}(F_{j\beta}-\bar{F}_{j\beta})+\dot{\bar{F}}_{i\alpha}(T_{i\alpha}-\bar{T}_{i\alpha})-\rho\,\overline{\frac{\partial \theta^*}{\partial F_{i\alpha}}}\,\dot{\bar{F}}_{i\alpha}(\eta-\bar{\eta}) \\ &\quad -\rho\,\overline{\frac{\partial \theta^*}{\partial \eta}}\,\dot{\bar{\eta}}(\eta-\bar{\eta})-\rho\,\bar{\theta}(\dot{\eta}-\dot{\bar{\eta}}). \end{aligned} \tag{3.6}$$

By virtue of (2.9),

$$\rho\frac{\partial\theta^*}{\partial F_{i\alpha}}=\frac{\partial T^*_{i\alpha}}{\partial\eta},\qquad \frac{\partial T^*_{j\beta}}{\partial F_{i\alpha}}=\frac{\partial T^*_{i\alpha}}{\partial F_{j\beta}},\tag{3.7}$$

so that (3.6) takes the form

$$\begin{aligned}\dot H+G_{\alpha,\alpha}&=\rho(v_i-\bar v_i)(b_i-\bar b_i)+\rho(r-\bar r)\\&+\dot{\bar F}_{i\alpha}\left\{T_{i\alpha}-\bar T_{i\alpha}-\overline{\frac{\partial T^*_{i\alpha}}{\partial F_{j\beta}}}(F_{j\beta}-\bar F_{j\beta})-\overline{\frac{\partial T^*_{i\alpha}}{\partial\eta}}(\eta-\bar\eta)\right\}\\&+\rho\dot{\bar\eta}\left\{\theta-\bar\theta-\overline{\frac{\partial\theta^*}{\partial F_{j\beta}}}(F_{j\beta}-\bar F_{j\beta})-\overline{\frac{\partial\theta^*}{\partial\eta}}(\eta-\bar\eta)\right\}\\&-\rho\dot{\bar\eta}(\theta-\bar\theta)-\rho\bar\theta(\dot\eta-\dot{\bar\eta}).\end{aligned}\tag{3.8}$$

From the Clausius-Duhem inequality (2.6) and (2.10), we obtain

$$\rho(r-\bar r)-\rho\dot{\bar\eta}(\theta-\bar\theta)-\rho\bar\theta(\dot\eta-\dot{\bar\eta})\leqq\frac{\rho}{\theta}(\theta-\bar\theta)(r-\bar r)-\frac{\rho\bar r}{\theta\bar\theta}(\theta-\bar\theta)^2;\tag{3.9}$$

hence (3.8) yields

$$\begin{aligned}\dot H+G_{\alpha,\alpha}&\leqq\rho(v_i-\bar v_i)(b_i-\bar b_i)+\frac{\rho}{\theta}(\theta-\bar\theta)(r-\bar r)-\frac{\rho\bar r}{\theta\bar\theta}(\theta-\bar\theta)^2\\&+\dot{\bar F}_{i\alpha}\left\{T_{i\alpha}-\bar T_{i\alpha}-\overline{\frac{\partial T^*_{i\alpha}}{\partial F_{j\beta}}}(F_{j\beta}-\bar F_{j\beta})-\overline{\frac{\partial T^*_{i\alpha}}{\partial\eta}}(\eta-\bar\eta)\right\}\\&+\frac{\rho\bar r}{\bar\theta}\left\{\theta-\bar\theta-\overline{\frac{\partial\theta^*}{\partial F_{j\beta}}}(F_{j\beta}-\bar F_{j\beta})-\overline{\frac{\partial\theta^*}{\partial\eta}}(\eta-\bar\eta)\right\}.\end{aligned}\tag{3.10}$$

Inequality (3.10) will be the starting point of our stability analysis. The crucial observation is that its right-hand side is of quadratic order in ($\boldsymbol{v}-\bar{\boldsymbol{v}}$, $\boldsymbol{F}-\bar{\boldsymbol{F}}$, $\eta-\bar\eta$, $\boldsymbol{b}-\bar{\boldsymbol{b}}$, $r-\bar r$), so that one may establish continuous dependence of processes upon initial data and supply terms by applying Gronwall type inequalities, provided that H be positive definite in an appropriate sense. This program will be implemented in the following two sections.

4. Stability of Smooth Processes in the Convexity Region of Internal Energy

In this section we shall be dealing with pairs of processes that render $H(\boldsymbol{X},t)$ pointwise positive definite. On account of (3.1) and (2.9), the sign of H^* is dependent upon the convexity properties of the function ε^*.

For fixed $\boldsymbol{X}\in\mathscr{B}$, we let $J^*(\boldsymbol{F},\eta;\boldsymbol{X})$ denote the Hessian matrix of $\varepsilon^*(\boldsymbol{F},\eta;\boldsymbol{X})$. We will say that a process $(\bar{\boldsymbol{x}},\bar\eta)(\boldsymbol{X},t)$ *resides in the convexity region of internal energy* if $J^*(\bar{\boldsymbol{F}}(\boldsymbol{X},t),\bar\eta(\boldsymbol{X},t);\boldsymbol{X})$ is uniformly positive definite on the closure of the domain of the process. By virtue of (2.9), J^* is the Jacobian matrix of the transformation $(\boldsymbol{F},\eta)\mapsto(\rho^{-1}\boldsymbol{T},\theta)$. Thus, by Sylvester's theorem, $J^*(\bar{\boldsymbol{F}}(\boldsymbol{X},t),\bar{\eta}(\boldsymbol{X},t);\boldsymbol{X})$ will be positive definite if and only if

$$\left[\frac{\partial \boldsymbol{T}^*(\bar{\boldsymbol{F}}(\boldsymbol{X},t),\bar{\eta}(\boldsymbol{X},t);\ \boldsymbol{X})}{\partial \boldsymbol{F}}\right] \quad \text{is positive definite,} \tag{4.1}$$

and

$$\det J^*(\bar{\boldsymbol{F}}(\boldsymbol{X},t),\bar{\eta}(\boldsymbol{X},t);\ \boldsymbol{X})>0. \tag{4.2}$$

In order to see the thermodynamic interpretation of (4.2), we observe that when J^* is nonsingular at a point one may invert locally the transformation $(\boldsymbol{F},\eta)\mapsto(\rho^{-1}\boldsymbol{T},\theta)$ to get $\boldsymbol{F}=\boldsymbol{F}^*(\rho^{-1}\boldsymbol{T},\theta;\ \boldsymbol{X})$, $\eta=\eta^*(\rho^{-1}\boldsymbol{T},\eta;\ \boldsymbol{X})$ and

$$\frac{\partial \eta^*}{\partial \theta}=\rho^{-n}(\det J^*)^{-1}\det\left(\frac{\partial \boldsymbol{T}^*}{\partial \boldsymbol{F}}\right). \tag{4.3}$$

It follows that (4.1), (4.2) are equivalent to the following two conditions (in the notation of classical thermodynamics):

$$\left(\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{F}}\right)_\eta \quad \text{is positive definite,} \tag{4.4}$$

$$\left(\frac{\partial \eta}{\partial \theta}\right)_{\boldsymbol{T}}>0. \tag{4.5}$$

It is easily seen that (4.4), (4.5) are in turn equivalent to

$$\left(\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{F}}\right)_\theta \quad \text{is positive definite,} \tag{4.6}$$

$$\left(\frac{\partial \eta}{\partial \theta}\right)_{\boldsymbol{F}}>0. \tag{4.7}$$

The relevance of (4.5), (4.7) to thermodynamic stability was first pointed out by GIBBS [11]. It would be appropriate to impose these conditions on any region in state space that does not contain critical points at which the material undergoes phase transitions. On the contrary, (4.4) and (4.6) are quite restrictive. These conditions may be globally satisfied for one dimensional bodies but are incompatible, in the multidimensional case, with the principle of material frame indifference on a large portion of state space that includes, in particular, natural states. The theory of elastic stability (e.g. [12]) elucidates the role of (4.4), (4.6) and makes it plausible that these conditions will be satisfied on some region in state space where strains are moderate and stresses are predominately tensile.

We now state two stability results, the first one for the mixed initial-boundary value problem and the second for the pure initial value problem.

Theorem 4.1. *Assume that $\mathscr{B}$ is bounded and has finite perimeter* [4]. *Let $(\bar{\boldsymbol{x}},\bar{\eta})(\boldsymbol{X},t)$ be a smooth process defined on $\mathscr{B}\times[0,t_0]$, residing in the convexity region of internal energy, with supply terms $(\bar{\boldsymbol{b}},\bar{r})(\boldsymbol{X},t)\in L^\infty(\mathscr{B}\times[0,t_0])$. Then there are positive constants δ,α,M,N with the following property:*

If $(\boldsymbol{x},\eta)(X,t)$ *is any admissible process defined on* $\mathscr{B}\times[0,t_0]$, *with supply terms* $(\boldsymbol{b},r)(\cdot,t)\in L^1([0,t_0];\,L^2(\mathscr{B}))$, *and such that*★

$$|F(X,t)-\bar{F}(X,t)|+|\eta(X,t)-\bar{\eta}(X,t)|<\delta,\qquad (X,t)\in\mathscr{B}\times[0,t_0], \tag{4.8}$$

$$(\boldsymbol{v}-\bar{\boldsymbol{v}})\cdot(\boldsymbol{\tau}-\bar{\boldsymbol{\tau}})\leqq 0,\qquad \text{on } \partial\mathscr{B}, \tag{4.9}$$

then we have, for any $s\in[0,t_0]$,

$$\begin{aligned}&\|(\boldsymbol{v}-\bar{\boldsymbol{v}},\boldsymbol{F}-\bar{\boldsymbol{F}},\eta-\bar{\eta})(\cdot,s)\|_{L^2(\mathscr{B})}\\ &\leqq M e^{\alpha s}\|(\boldsymbol{v}-\bar{\boldsymbol{v}},\boldsymbol{F}-\bar{\boldsymbol{F}},\eta-\bar{\eta})(\cdot,0)\|_{L^2(\mathscr{B})}\\ &\quad+N e^{\alpha s}\int_0^s\|(\boldsymbol{b}-\bar{\boldsymbol{b}},r-\bar{r})(\cdot,t)\|_{L^2(\mathscr{B})}\,dt.\end{aligned} \tag{4.10}$$

Theorem 4.2. *Assume that* $\mathscr{B}=R^n$. *Let* $(\bar{\boldsymbol{x}},\bar{\eta})(X,t)$ *be a smooth process defined on* $\mathscr{B}\times[0,t_0]$, *residing in the convexity region of internal energy, with supply terms* $(\bar{\boldsymbol{b}},\bar{r})(X,t)\in L^\infty_{\mathrm{loc}}(\mathscr{B}\times[0,t_0])$ *and such that* $(\bar{\boldsymbol{F}},\bar{\eta})(X,t)$ *is bounded on* $\mathscr{B}\times[0,t_0]$. *Then there are positive constants* δ,k,α,M,N *with the following property:*

If $(\boldsymbol{x},\eta)(X,t)$ *is any admissible process defined on* $\mathscr{B}\times[0,t_0]$, *with supply terms* $(\boldsymbol{b},r)(\cdot,t)\in L^1([0,t_0];\,L^2_{\mathrm{loc}}(\mathscr{B}))$, and such that

$$|F(X,t)-\bar{F}(X,t)|+|\eta(X,t)-\bar{\eta}(X,t)|<\delta,\qquad (X,t)\in\mathscr{B}\times[0,t_0], \tag{4.11}$$

then we have for every $a>0,\ s\in[0,t_0]$,

$$\begin{aligned}&\|(\boldsymbol{v}-\bar{\boldsymbol{v}},\boldsymbol{F}-\bar{\boldsymbol{F}},\eta-\bar{\eta})(\cdot,s)\|_{L^2(|X|<a)}\\ &\leqq M e^{\alpha s}\|(\boldsymbol{v}-\bar{\boldsymbol{v}},\boldsymbol{F}-\bar{\boldsymbol{F}},\eta-\bar{\eta})(\cdot,0)\|_{L^2(|X|<a+ks)}\\ &\quad+N e^{\alpha s}\int_0^s\|(\boldsymbol{b}-\bar{\boldsymbol{b}},r-\bar{r})(\cdot,t)\|_{L^2(|X|<a+k(s-t))}\,dt.\end{aligned} \tag{4.12}$$

From the above propositions one draws immediately the following

Corollary 4.1. *Let* $\mathscr{B}$, $(\bar{\boldsymbol{x}},\bar{\eta})(X,t)$ *and* $(\boldsymbol{x},\eta)(X,t)$ *be as in Theorem* 4.1 *or as in Theorem* 4.2. *Assume that the corresponding supply terms* $(\bar{\boldsymbol{b}},\bar{r})(X,t)$ *and* $(\boldsymbol{b},r)(X,t)$ *coincide on* $\mathscr{B}\times[0,t_0]$ *and that both processes originate from the same state, that is,*

$$\boldsymbol{x}(X,0)=\bar{\boldsymbol{x}}(X,0),\qquad \boldsymbol{v}(X,0)=\bar{\boldsymbol{v}}(X,0),\qquad \eta(X,0)=\bar{\eta}(X,0),\qquad X\in\mathscr{B}. \tag{4.13}$$

Then $(\bar{\boldsymbol{x}},\bar{\eta})(X,t)$ *and* $(\boldsymbol{x},\eta)(X,t)$ *coincide on* $\mathscr{B}\times[0,t_0]$.

The above uniqueness result is only local since the two processes are restricted a priori by (4.8) or (4.11). Roughly speaking, the corollary states that the only way that an admissible process may bifurcate out of a smooth process, at an instant t_1, is by the spontaneous generation at t_1 of a shock wave of large ($\geqq\delta$) amplitude. In the one dimensional case it is known [13] that this is

★ In Equation (4.9) τ, $\bar{\tau}$ denote the stress vectors. For the meaning of the traces of v and τ on $\partial\mathscr{B}$, see [4].

impossible, i.e., shock waves originating at a smooth state start out with "zero" amplitude and are then gradually amplified. One would need a similar property for the multidimensional case in order to deduce global uniqueness from Corollary 4.1, but a proof does not seem feasible at the present stage of development of the theory.

For the proof of Theorems 4.1 and 4.2 we shall need the following Gronwall type inequality.

Lemma 4.1. *Assume that the nonnegative functions* $y(t)\in L^\infty[0,s]$ *and* $g(t)\in L^1[0,s]$ *satisfy the inequality*

$$y^2(\sigma)\leqq M^2 y^2(0)+\int_0^\sigma [2\alpha y^2(t)+2N g(t)\, y(t)]\, dt, \qquad \sigma\in[0,s], \tag{4.14}$$

where α, M, N *are nonnegative constants. Then*

$$y(s)\leqq M e^{\alpha s} y(0)+N e^{\alpha s}\int_0^s g(t)\, dt. \tag{4.15}$$

The proof of the above lemma is straightforward and in any case a more general result (Lemma 5.1) will be established in Section 5.

Proof of Theorem 4.1. Since $(\bar{x},\bar{\eta})(X,t)$ resides in the convexity region of internal energy, it follows from (2.9) that there exist positive constants δ,λ such that, whenever (4.8) holds,

$$\begin{aligned} &\varepsilon^*(F(X,t),\eta(X,t);\, X)-\varepsilon^*(\bar{F}(X,t),\bar{\eta}(X,t);\, X)\\ &\quad -\rho^{-1}\, T^*_{i\alpha}(\bar{F}(X,t),\bar{\eta}(X,t);\, X)\,[F_{i\alpha}(X,t)-\bar{F}_{i\alpha}(X,t)]\\ &\quad -\theta^*(\bar{F}(X,t),\bar{\eta}(X,t);\, X)\,[\eta(X,t)-\bar{\eta}(X,t)]\\ &\geqq \lambda(|F(X,t)-\bar{F}(X,t)|^2+|\eta(X,t)-\bar{\eta}(X,t)|^2), \end{aligned} \tag{4.16}$$

for every $(X,t)\in\mathscr{B}\times[0,t_0]$.

We now fix $s\in[0,t_0]$, integrate (3.10) over $\mathscr{B}\times[0,\sigma]$, $\sigma\in[0,s]$, apply the Gauss-Green theorem, and then use (4.9), (4.16) and the observation that the right-hand side of (3.10) is of quadratic order in $(v-\bar{v}, F-\bar{F}, \eta-\bar{\eta}, b-\bar{b}, r-\bar{r})$. Thus we arrive at an estimate of the form (4.14) with

$$y(t)=\|(v-\bar{v}, F-\bar{F}, \eta-\bar{\eta})(\cdot\,,t)\|_{L^2(\mathscr{B})}, \tag{4.17}$$

$$g(t)=\|(b-\bar{b}, r-\bar{r})(\cdot\,,t)\|_{L^2(\mathscr{B})}. \tag{4.18}$$

An application of Lemma 4.1 completes the proof.

Proof of Theorem 4.2. As in the proof of Theorem 4.1 there are positive constants δ,λ such that (4.16) is satisfied for all $(X,t)\in\mathscr{B}\times[0,t_0]$, whenever (4.11) holds. Consequently, since G^* is of quadratic order in $(v-\bar{v}, F-\bar{F}, \eta-\bar{\eta})$, one may determine a sufficiently large positive constant k with the property that

$$kH(X,t)-G(X,t)\cdot\frac{X}{|X|}\geqq 0, \qquad (X,t)\in\mathscr{B}\times[0,t_0]. \tag{4.19}$$

We now fix $s \in [0, t_0]$, $a > 0$, and for each $\sigma \in [0, s]$ integrate (3.10) over the frustum $\{(X, t) | t \in [0, \sigma], |X| < a + k(s-t)\}$. Applying the Gauss-Green theorem and using (4.19), (4.16), we obtain (as in the proof of Theorem 4.1) an estimate of the form (4.14) with

$$y(t) = \|(\boldsymbol{v} - \bar{\boldsymbol{v}}, \boldsymbol{F} - \bar{\boldsymbol{F}}, \eta - \bar{\eta})(\cdot, t)\|_{L^2(|X| < a + k(s-t))}, \tag{4.20}$$

$$g(t) = \|(\boldsymbol{b} - \bar{\boldsymbol{b}}, r - \bar{r})(\cdot, t)\|_{L^2(|X| < a + k(s-t))}. \tag{4.21}$$

Hence Lemma 4.1 yields (4.12). The proof of the theorem is complete.

From the proof of Theorems 4.1 and 4.2 it becomes clear that in the one-dimensional case, where the internal energy is globally convex, one may choose δ in (4.8) or (4.11) to be arbitrarily large. Thus the stability and uniqueness results become global. As a matter of fact, in the one-dimensional case and under some additional assumptions (most notably, genuine nonlinearity), DiPerna [10] has established uniqueness of piecewise smooth processes in the class of proper admissible processes. DiPerna's analysis relies on a number of novel ideas and the problem of extending it to the multidimensional case is still open.

5. Stability of Smooth Processes in the Strong Ellipticity Region

The object of the restrictive convexity assumption in Section 4 was to ensure that $H(X, t)$ be pointwise positive definite. In reviewing the proof of Theorem 4.1, however, it becomes clear that definiteness of $\int_{\mathscr{B}} H(X, t)\, dX$ would suffice for stability. We prove here that, whenever the body is smooth and the motion of its boundary is prescribed, stability is induced by a mere strong ellipticity condition.

We will say that a process $(\bar{x}, \bar{\eta})(X, t)$ *resides in the strong ellipticity region* if there is a positive constant ν with the property that, for any vectors $\xi, \zeta \in R^n$, every $\mu \in R$ and all (X, t) in the domain of the process,

$$\begin{aligned} &\frac{\partial^2 \varepsilon^*(\bar{F}(X,t), \bar{\eta}(X,t); X)}{\partial F_{i\alpha} \partial F_{j\beta}} \xi_i \xi_j \zeta_\alpha \zeta_\beta + \frac{\partial^2 \varepsilon^*(\bar{F}(X,t), \bar{\eta}(X,t); X)}{\partial F_{i\alpha} \partial \eta} \xi_i \zeta_\alpha \mu \\ &\quad + \frac{\partial^2 \varepsilon^*(\bar{F}(X,t), \bar{\eta}(X,t); X)}{\partial \eta^2} \mu^2 \geqq \nu(|\xi|^2 |\zeta|^2 + \mu^2). \end{aligned} \tag{5.1}$$

Strong ellipticity in nonlinear elasticity has been studied in connection with wave propagation (e.g. [14]) as well as with existence of equilibrium configurations [15]. As an assumption, it is weaker than convexity of the internal energy and, in particular, it is not incompatible with frame indifference at a natural state. For a discussion of the extent of the strong ellipticity region in two-dimensional, homogeneous, isotropic, hyperelastic bodies, see [16]. See also [17, 18] for relevant illuminating remarks.

We now state our stability result.

Theorem 5.1. *Assume that $\mathscr{B}$ is smooth and bounded. Let $(\bar{x}, \bar{\eta})(X, t)$ be a smooth process defined on $\mathscr{B} \times [0, t_0]$, residing in the strong ellipticity region, with* *supply terms* $(\bar{b},\bar{r})(X,t)\in L^\infty(\mathscr{B}\times[0,t_0])$. *Then there are positive constants* δ, α, β, M, N *with the following property:*

If $(x,\eta)(X,t)$ *is any admissible process defined on* $\mathscr{B}\times[0,t_0]$, *with supply terms* $(b,r)(X,t)\in L^1([0,t_0];L^2(\mathscr{B}))$ *and such that*

$$|F(X,t)-\bar{F}(X,t)|+|\eta(X,t)-\bar{\eta}(X,t)|<\delta,\qquad (X,t)\in\mathscr{B}\times[0,t_0], \tag{5.2}$$

$$x(X,t)=\bar{x}(X,t),\qquad X\in\partial\mathscr{B},\ t\in[0,t_0], \tag{5.3}$$

then we have, for any $s\in[0,t_0]$,

$$\begin{aligned}&\|(v-\bar{v},F-\bar{F},\eta-\bar{\eta})(\cdot,s)\|_{L^2(\mathscr{B})}\\ &\quad\leqq M\exp(\alpha s+\beta s^2)\|(v-\bar{v},F-\bar{F},\eta-\bar{\eta})(\cdot,0)\|_{L^2(\mathscr{B})}\\ &\quad\quad+N\exp(\alpha s+\beta s^2)\int_0^s\|(b-\bar{b},r-\bar{r})(\cdot,t)\|_{L^2(\mathscr{B})}\,dt.\end{aligned} \tag{5.4}$$

From the above proposition we get the following uniqueness result.

Corollary 5.1. *Let* $\mathscr{B}$, $(\bar{x},\bar{\eta})(X,t)$ *and* $(x,\eta)(X,t)$ *be as in Theorem* 5.1. *Assume that the corresponding supply terms* $(\bar{b},\bar{r})(X,t)$ *and* $(b,r)(X,t)$ *coincide on* $\mathscr{B}\times[0,t_0]$ *and that both processes originate from the same state, that is*

$$x(X,0)=\bar{x}(X,0),\ v(X,0)=\bar{v}(X,0),\qquad \eta(X,0)=\bar{\eta}(X,0),\qquad X\in\mathscr{B}. \tag{5.5}$$

Then $(\bar{x},\bar{\eta})(X,t)$ *and* $(x,\eta)(X,t)$ *coincide on* $\mathscr{B}\times[0,t_0]$.

For the proof of Theorem 5.1 we will employ the following Gronwall type inequality.

Lemma 5.1. *Assume that the nonnegative functions* $y(t)\in L^\infty[0,s]$ *and* $g(t)\in L^1[0,s]$ *satisfy the inequality*

$$y^2(\sigma)\leqq M^2y^2(0)+\int_0^\sigma[(2\gamma+4\beta\sigma)y^2(t)+2N\,g(t)\,y(t)]\,dt,\qquad \sigma\in[0,s], \tag{5.6}$$

with β,γ,M,N *nonnegative constants. Then*

$$y(s)\leqq M\exp(\alpha s+\beta s^2)\,y(0)+N\exp(\alpha s+\beta s^2)\int_0^s g(t)\,dt, \tag{5.7}$$

where $\alpha=\gamma+\beta/\gamma$.

Proof. We define a nonnegative function $z(\sigma)$ by

$$z^2(\sigma)=M^2y^2(0)+\int_0^\sigma[(2\gamma+4\beta\sigma)y^2(t)+2Ng(t)\,y(t)]\,dt,\qquad \sigma\in[0,s], \tag{5.8}$$

and we note that

$$\begin{aligned}2z(\sigma)\,\dot{z}(\sigma)&=(2\gamma+4\beta\sigma)\,y^2(\sigma)+2N\,g(\sigma)\,y(\sigma)+4\beta\int_0^\sigma y^2(t)\,dt\\ &\leqq(2\alpha+4\beta\sigma)\,z^2(\sigma)+2N\,g(\sigma)\,z(\sigma).\end{aligned} \tag{5.9}$$

Hence

$$\dot{z}(\sigma) \leqq (\alpha + 2\beta\sigma) z(\sigma) + N g(\sigma). \tag{5.10}$$

Integrating the differential inequality (5.10) under the initial condition $z(0) = M y(0)$ we arrive at (5.7). The proof is complete.

We note that Lemma 5.1 reduces to Lemma 4.1 when $\beta = 0$.

Lemma 5.2. *Let $\mathscr{B}$, $(\bar{x}, \bar{\eta})(X, t)$ and $(x, \eta)(X, t)$ be as in Theorem 5.1. Then there are constants $\lambda > 0$ and κ with the property that, for any $\sigma \in [0, t_0]$,*

$$\begin{aligned}
&\int_{\mathscr{B}} \rho(X) \left\{ \frac{\partial^2 \varepsilon^*(\bar{F}(X,\sigma), \bar{\eta}(X,\sigma); X)}{\partial F_{i\alpha} \partial F_{j\beta}} [F_{i\alpha}(X,\sigma) - \bar{F}_{i\alpha}(X,\sigma)][F_{j\beta}(X,\sigma) - \bar{F}_{j\beta}(X,\sigma)] \right. \\
&\quad + \frac{\partial^2 \varepsilon^*(\bar{F}(X,\sigma), \bar{\eta}(X,\sigma); X)}{\partial F_{i\alpha} \partial \eta} [F_{i\alpha}(X,\sigma) - \bar{F}_{i\alpha}(X,\sigma)] [\eta(X,\sigma) - \bar{\eta}(X,\sigma)] \\
&\quad \left. + \frac{\partial^2 \varepsilon^*(\bar{F}(X,\sigma), \bar{\eta}(X,\sigma); X)}{\partial \eta^2} [\eta(X,\sigma) - \bar{\eta}(X,\sigma)]^2 \right\} dX \\
&\geqq 2\lambda \int_{\mathscr{B}} \{ |F(X,\sigma) - \bar{F}(X,\sigma)|^2 + |\eta(X,\sigma) - \bar{\eta}(X,\sigma)|^2 \} \, dX \\
&\quad - \kappa \int_{\mathscr{B}} |x(X,\sigma) - \bar{x}(X,\sigma)|^2 \, dX.
\end{aligned} \tag{5.11}$$

Proof. We recall that $(\bar{x}, \bar{\eta})(X, t)$ resides in the strong ellipticity region and that $x - \bar{x}$ vanishes on $\partial \mathscr{B}$. In the case where $\eta(X,t) - \bar{\eta}(X,t)$ vanishes identically, (5.11) reduces to the classical Gårding inequality. In general, (5.11) is established by imitating the proof of Gårding's inequality, as described, for example, in [19].

Proof of Theorem 5.1. On account of (2.9),

$$\begin{aligned}
&\varepsilon^*(F, \eta; X) - \varepsilon^*(\bar{F}, \bar{\eta}; X) - \frac{1}{\rho(X)} T^*_{i\alpha}(\bar{F}, \bar{\eta}; X)(F_{i\alpha} - \bar{F}_{i\alpha}) - \theta^*(\bar{F}, \bar{\eta}; X)(\eta - \bar{\eta}) \\
&\quad = \frac{\partial^2 \varepsilon^*(\bar{F}, \bar{\eta}; X)}{\partial F_{i\alpha} \partial F_{j\beta}} (F_{i\alpha} - \bar{F}_{i\alpha})(F_{j\beta} - \bar{F}_{j\beta}) + \frac{\partial^2 \varepsilon^*(\bar{F}, \bar{\eta}; X)}{\partial F_{i\alpha} \partial \eta} (F_{i\alpha} - \bar{F}_{i\alpha})(\eta - \bar{\eta}) \\
&\quad + \frac{\partial^2 \varepsilon^*(\bar{F}, \bar{\eta}; X)}{\partial \eta^2} (\eta - \bar{\eta})^2 + o(|F - \bar{F}|^2 + |\eta - \bar{\eta}|^2).
\end{aligned} \tag{5.12}$$

Combining (5.11) with (5.12), we conclude that there is a positive constant δ with the property that, when (5.2) is satisfied,

$$\begin{aligned}
&\int_{\mathscr{B}} \{ \rho(X) \varepsilon^*(F(X,\sigma), \eta(X,\sigma); X) - \rho(X) \varepsilon^*(\bar{F}(X,\sigma), \bar{\eta}(X,\sigma); X) \\
&\quad - T^*_{i\alpha}(\bar{F}(X,\sigma), \bar{\eta}(X,\sigma); X)[F_{i\alpha}(X,\sigma) - \bar{F}_{i\alpha}(X,\sigma)] \\
&\quad - \rho(X) \theta^*(\bar{F}(X,\sigma), \bar{\eta}(X,\sigma); X)[\eta(X,\sigma) - \bar{\eta}(X,\sigma)] \} \, dX \\
&\geqq \lambda \int_{\mathscr{B}} \{ |F(X,\sigma) - \bar{F}(X,\sigma)|^2 + |\eta(X,\sigma) - \bar{\eta}(X,\sigma)|^2 \} \, dX \\
&\quad - \kappa \int_{\mathscr{B}} |x(X,\sigma) - \bar{x}(X,\sigma)|^2 \, dX.
\end{aligned} \tag{5.13}$$

Furthermore, upon using Schwarz's inequality and the Poincaré inequality, we find that

$$\begin{aligned}
&\int_{\mathscr{B}} |x(X,\sigma)-\bar{x}(X,\sigma)|^2\,dX \\
(5.14)\qquad &\quad = \int_{\mathscr{B}} \left| x(X,0)-\bar{x}(X,0)+\int_0^\sigma \{v(X,t)-\bar{v}(X,t)\}\,dt\right|^2 dX \\
&\quad \leqq 2c\int_{\mathscr{B}} |F(X,0)-\bar{F}(X,0)|^2\,dX \\
&\qquad + 2\sigma\int_0^\sigma\int_{\mathscr{B}} |v(X,t)-\bar{v}(X,t)|^2\,dX\,dt.
\end{aligned}$$

We now proceed as in the proof of Theorem 4.1. We fix $s\in[0,t_0]$, integrate (3.10) over $\mathscr{B}\times[0,\sigma]$, $\sigma\in[0,s]$, and apply the Gauss-Green theorem. We then use the boundary conditions (5.3), (5.13), (5.14) and the observation that the right-hand side of (3.10) is of quadratic order in ($v-\bar{v}$, $F-\bar{F}, \eta-\bar{\eta}$, $b-\bar{b}$, $r-\bar{r}$), thus arriving at an estimate of the form (5.6) with

$$(5.15)\qquad y(t)=\|(v-\bar{v}, F-\bar{F}, \eta-\bar{\eta})(\cdot,t)\|_{L^2(\mathscr{B})},$$

$$(5.16)\qquad g(t)=\|(b-\bar{b}, r-\bar{r})(\cdot,t)\|_{L^2(\mathscr{B})}.$$

An application of Lemma 5.1 yields (5.4) and thus completes the proof.

Acknowledgment. I am indebted to RONALD DIPERNA for many stimulating discussions. This work was supported in part by the National Science Foundation under grant MCS-76-07247-A03, in part by the U.S. Army under contract ARO-DAAG27-76-G-0294 and in part by the Office of Naval Research under contract N0014-76-C-0278-P0002.

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Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island

(Received December 28, 1978)