On Stability and Uniqueness in Finite Elasticity

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Dedicated to Clifford Truesdell on the occasion of his 60th birthday

Introduction

In finite elasticity one neither expects nor desires unqualified uniqueness. Indeed, counterexamples exhibiting non-uniqueness are known for all of the standard boundary-value problems of the equilibrium theory.[1] This, however, does not rule out the possibility of having uniqueness in certain subsets of the solution space, and in this paper we present our initial attempts to delineate these subsets. In particular, we consider the displacement and mixed problems[2] and show that *uniqueness holds in any convex, stable set of deformations.* Further, using this result as a basis, we establish uniqueness:[3]

(a) in a neighborhood of a uniformly stable deformation;[4]

(b) in a neighborhood of a positive, natural configuration;[5]

(c) (for the displacement problem) in a neighborhood of a homogeneous, strongly-elliptic configuration.[6]

[1] For the displacement problem cf. JOHN [1964]. For the mixed problem cf. BALL [1977] and GURTIN [1977]. For the traction problem cf. ARMANNI [1915] and ANTMAN [1979] (eversion of a sphere), ALMANSI [1916] (eversion of a hollow cylinder), and ERICKSEN (WANG & TRUESDELL [1973]). See also TRUESDELL & NOLL [1965], pp. 128–129 and TRUESDELL [1977].

[2] The traction problem, which is far more difficult, will be the subject of a future paper.

[3] For (a) and (b) we need the assumption of dead loads. The extension to more general loadings is given by Spector [1979].

[4] ERICKSEN & TOUPIN [1956] and HILL [1957] have established an analogous result within the linear theory of *infinitesimal* deformations superimposed on a stressed state. See also TRUESDELL & NOLL [1965], p. 255.

[5] i.e., a natural configuration whose elasticity tensor is positive-definite.

[6] This particular result was first established by JOHN [1972] using completely different methods. We became aware of JOHN's work only after we had completed our analysis.

Our results (b) and (c) are similar in nature to results established previously by STOPELLI [1954] and VAN BUREN [1968].[1] However, in contrast to these authors, our goal is uniqueness, rather than uniqueness *and* existence, and for this reason our proofs are far simpler.

Many of the counterexamples displaying lack of uniqueness arise because the path of loading is not specified. But specification of this path does not rule out non-uniqueness due to instabilities such as buckling. Here results concerning local uniqueness play an important role. For example, (a) tells us that a solution cannot bifurcate in a continuous manner at a uniformly stable deformation.

Our definition of stability is quite general: roughly speaking, a deformation f is stable if the incremental power required to move the body from f is strictly positive.

Finally, we remark that none of our results require the assumption of hyperelasticity.

1. Notation

We use lower case Greek letters for scalars (elements of $\mathbb{R}$), lower case Latin letters for vectors (elements of $\mathbb{R}^3$), and upper case Latin letters for *tensors*[2] (linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$). We use the standard inner product $a \cdot b$ on $\mathbb{R}^3$, while on

$$\mathrm{Lin} = \text{the space of all tensors}$$

we use the inner product

$$G \cdot H = \mathrm{tr}(GH^T)$$

with H^T the transpose of H and tr the trace. We write

$$\mathrm{Lin}^+ = \{H \in \mathrm{Lin}\colon \det H > 0\},$$
$$\mathrm{Orth}^+ = \{Q \in \mathrm{Lin}^+\colon Q^T Q = I\},$$

where det is the determinant and I the identity. Any tensor H admits the unique decomposition

$$H = E + W$$

into a symmetric tensor E and a skew tensor W; in fact,

$$E = \tfrac{1}{2}(H + H^T), \qquad W = \tfrac{1}{2}(H - H^T).$$

We call E and W, respectively, the *symmetric* and *skew* parts of H. Finally, the tensor product $a \otimes b$ of $a, b \in \mathbb{R}^3$ is the tensor defined by $(a \otimes b)x = (b \cdot x)a$ for every $x \in \mathbb{R}^3$.

[1] See also TRUESDELL & NOLL [1965], § 46 and WANG & TRUESDELL [1973], pp. 494-509.

[2] As is customary, we will use the term elasticity *tensor* for a certain linear map A: Lin → Lin. Of course, A is not a tensor in the above sense.

We write ∇ and div for the gradient and divergence operators[1] in $\mathbb{R}^3$: for a vector field u, ∇u is the tensor field with components $(\nabla u)_{ij} = \partial u_i/\partial x_j$; for a tensor field S, $\operatorname{div} S$ is the vector field with components $\sum_j \partial S_{ij}/\partial x_j$.

Given a function $\Phi(a, b, \ldots, c)$ with vector or tensor arguments, we write, e.g., $\partial_a \Phi(a, b, \ldots, c)$ for the partial Frechet derivative with respect to a holding the remaining arguments fixed.

Throughout this paper $\mathscr{B}$ will denote a *properly regular*[2] region in $\mathbb{R}^3$; thus, in particular, $\mathscr{B}$ is compact and has piecewise smooth[3] boundary $\partial\mathscr{B}$. Further, $\mathscr{D}$ will always designate a subsurface of $\partial\mathscr{B}$ with

$$\mathscr{D} \textit{ non-empty and relatively open,} \tag{1.1}$$

and

$$\mathrm{Var} = \{u \in C^1(\mathscr{B}, \mathbb{R}^3) \colon u \not\equiv 0,\ u = 0 \text{ on } \mathscr{D}\}. \tag{1.2}$$

Lemma (Korn Inequality). *There exists a* $\kappa > 0$ *such that*

$$\|E\| \geqq \kappa \, \|\nabla u\| \qquad (1.3)^4$$

for all $u \in \mathrm{Var}$, *where* E *is the symmetric part of* ∇u.

Here and in what follows $\|\cdot\|$ denotes the $L^2(\mathscr{B})$ norm; thus, e.g.,

$$\|E\|^2 = \int_{\mathscr{B}} |E|^2.$$

2. Kinematics

For convenience, we identify the body with the (properly regular) region of $\mathbb{R}^3$ it occupies in a fixed reference configuration.

Definition. A *deformation* f (of $\mathscr{B}$) is a member of the space

$$\mathrm{Def} = \{f \in C^1(\mathscr{B}, \mathbb{R}^3) \colon \det \nabla f > 0\}.$$

A *process* g is a one-parameter family g_σ $(0 \leqq \sigma \leqq \beta)$ of *deformations* such that:

(a) the derivatives[5]

$$\dot{g}_\sigma(x), \qquad G_\sigma(x) = \nabla g_\sigma(x), \qquad \dot{G}_\sigma(x), \qquad \ddot{G}_\sigma(x)$$

exist and are jointly continuous in (x, σ) on $\mathscr{B} \times [0, \beta]$;

(b) for all $\sigma \in [0, \beta]$,

$$\dot{g}_\sigma = 0 \quad \text{on } \mathscr{D}; \tag{2.1}$$

(c) $\dot{g}_0 \not\equiv 0$.

We say that g *starts from* f if $g_0 = f$.

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[1] For our applications these operations will be with respect to the *material point* x.

[2] Cf. FICHERA [1972], p. 351.

[3] We use smooth as a synonym for C^1.

[4] Cf., e.g., FICHERA [1972], p. 384, whose proof, with minor modifications, applies in the present circumstances.

[5] A superposed dot denotes the partial derivative with respect to σ; ∇ is the gradient with respect to x.

When we discuss the mixed problem $\mathscr{D}$ will be the portion of the boundary over which the deformation is prescribed. Condition *(b)* preserves this type of boundary condition; it asserts that if g starts from f, then $g_\sigma = f$ on $\mathscr{D}$ for all σ. Note also that *(b)*, *(c)*, and (1.1) rule out the possibility of a rigid process, while *(a)*, *(b)*, and *(c)* insure that

$$\dot{g}_0 \in \mathrm{Var}. \tag{2.2}$$

Proposition 1. *Given a deformation f and a field $u \in \mathrm{Var}$, there is a $\beta > 0$ such that $g_\sigma (0 \leqq \sigma \leqq \beta)$ defined by*

$$g_\sigma(x) = f(x) + \sigma u(x) \tag{2.3}$$

is a process. Moreover,

$$\dot{g}_\sigma = u, \qquad \dot{G}_\sigma = \nabla u. \tag{2.4}$$

Proof. The non-trivial portion of the proof consists in showing that for some $\beta > 0$, $g_\sigma \in \mathrm{Def}$ for $0 \leqq \sigma \leqq \beta$, or equivalently, $\det \nabla g_\sigma > 0$ for $0 \leqq \sigma \leqq \beta$. But this follows from the relation $\nabla g_\sigma = \nabla f + \sigma \nabla u$, since $\det \nabla f > 0$, f and u are smooth, and $\mathscr{B}$ is compact. □

We call a process of the form (2.3) *straight*, since it represents a straight line in the space $C^1(\mathscr{B}, \mathbb{R}^3)$.

3. The Constitutive Relation

We assume that the body is elastic with *smooth* response function S: $\mathrm{Lin}^+ \times \mathscr{B} \to \mathrm{Lin}$. S gives the (Piola-Kirchhoff) *stress*

$$S(\nabla f(x), x)$$

at any point $x \in \mathscr{B}$ when the body is deformed by f. Writing F for $\nabla f(x)$, we assume that S obeys the following hypotheses [1] at each (F, x) in its domain:

$$\begin{aligned} QS(F,x) &= S(QF,x) \quad \text{for all } Q \in \mathrm{Orth}^+, \\ S(F,x)F^T &= FS(F,x)^T. \end{aligned} \tag{3.1}$$

The restriction $(3.1)_1$ is a consequence of frame-indifference, while $(3.1)_2$ follows from balance of moments.

The linear transformation $A(F,x)$: $\mathrm{Lin} \to \mathrm{Lin}$ defined by

$$A(F,x) = \partial_F S(F,x) \tag{3.2}$$

is called the *elasticity tensor*.

Definition. The reference configuration is:

(a) natural if

$$S(I,x) = 0 \tag{3.3}$$

for all $x \in \mathscr{B}$;

[1] These hypotheses and the remaining results of this section are not needed until Section 6.

(b) homogeneous if $A(I,x)$ is independent of x;
(c) positive if, for each $x \in \mathscr{B}$,

$$E \cdot A(I,x) E > 0 \tag{3.4}$$

for all symmetric $E \neq 0$;
(d) strongly-elliptic if, for each $x \in \mathscr{B}$,

$$H \cdot A(I,x) H > 0$$

whenever $H = a \otimes e$ with $a \neq 0$, $e \neq 0$.

Proposition 2. *Assume that the reference configuration is natural. Then, for all $x \in \mathscr{B}$ and $H \in \mathrm{Lin}$,*

$$H \cdot A(I,x) H = E \cdot A(I,x) E, \tag{3.5}$$

where E is the symmetric part of H.

Proof. If we take $F = I$ and $Q = Q(t) = e^{Wt}$ in $(3.1)_1$, where W is skew, and differentiate with respect to t at $t = 0$, we conclude, with the aid of (3.3), that

$$A(I,x) W = 0. \tag{3.6}$$

Next, if we take the directional derivative of $(3.1)_2$ with respect to F in the direction $H \in \mathrm{Lin}$, we find that

$$A(I,x) H = [A(I,x) H]^T.$$

Thus $A(I,x)$ has symmetric values and hence

$$W \cdot A(I,x) H = 0 \tag{3.7}$$

whenever W is skew. Now let E and W, respectively, denote the symmetric and skew parts of H. Then, by (3.6) and (3.7),

$$H \cdot A(I,x) H = (E + W) \cdot A(I,x)(E + W) = E \cdot A(I,x) E. \qquad \square$$

For convenience, we write $S(\nabla f)$ and $A(\nabla f)$ for the fields on $\mathscr{B}$ with values $S(\nabla f(x), x)$ and $A(\nabla f(x), x)$.

4. Stability

We assume that in each deformation f the environment exerts a *body force* $b_f(x)$ on the points x of $\mathscr{B}$ and a surface force $s_f(x)$ on the points x of

$$\mathscr{S} = \partial \mathscr{B} - \mathscr{D}$$

(cf. (1.1), (2.1)). We assume further that, for each deformation f, $s_f \in L^1(\mathscr{S}, \mathbb{R}^3)$ and $b_f \in L^1(\mathscr{B}, \mathbb{R}^3)$.

Central to our definition of stability is the functional

$$P_\sigma(g)=\int_{\mathscr{B}}(S_\sigma-S_0)\cdot\dot{G}_\sigma-\int_{\mathscr{S}}(s_\sigma-s_0)\cdot\dot{g}_\sigma-\int_{\mathscr{B}}(b_\sigma-b_0)\cdot\dot{g}_\sigma, \tag{4.1}$$

where

$$G_\sigma=\nabla g_\sigma,\quad S_\sigma=S(G_\sigma),\quad s_\sigma=s_{g_\sigma},\quad b_\sigma=b_{g_\sigma}.$$

To interpret (4.1) physically assume that the process g is sufficiently smooth and use the divergence theorem and (2.1) to rewrite the first integral as

$$\int_{\mathscr{S}}(\hat{s}_\sigma-\hat{s}_0)\cdot\dot{g}_\sigma+\int_{\mathscr{B}}(\hat{b}_\sigma-\hat{b}_0)\cdot\dot{g}_\sigma,$$

where

$$\hat{s}_\sigma=S_\sigma n,\qquad \hat{b}_\sigma=-\operatorname{div}S_\sigma$$

with n the outward unit normal to $\partial\mathscr{B}$. The fields $\hat{s}_\sigma$ and $\hat{b}_\sigma$ represent the surface force and body force necessary to equilibrate S_σ. Thus *$P_\sigma(g)$ is the total incremental*[1] *power needed to sustain g minus the incremental power available from the environment.*

A reasonable definition for the stability of a deformation f is that $P_\sigma(g)$ be strictly positive[2] near $\sigma=0$ in any process g starting from f, so that such processes actually absorb power.

Definition. A deformation f is *stable* if given any process g_σ $(0\leqq\sigma\leqq\beta)$ starting from f there is a $\lambda\in(0,\beta)$ such that

$$P_\sigma(g)>0$$

for all $\sigma\in(0,\lambda)$. A set $\Omega\subset\mathrm{Def}$ is stable if every $f\in\Omega$ is stable.

The next result is the essential ingredient in our proof of uniqueness.

Theorem 1. *Let $\Omega\subset\mathrm{Def}$ be stable. Then given any straight process g_σ $(0\leqq\sigma\leqq\beta)$ lying in Ω, the mapping $\sigma\mapsto P_\sigma(g)$ is strictly increasing on $[0,\beta]$.*

Proof. Let g be as above. Choose $\alpha\in[0,\beta)$ and consider the straight process g^*_σ $(0\leqq\sigma\leqq\beta-\alpha)$ defined by $g^*_\sigma=g_{\alpha+\sigma}$. Then, since $\dot{g}_\sigma$ and $\dot{g}^*_\sigma$ are independent of σ and equal, as are $\dot{G}_\sigma$ and $\dot{G}^*_\sigma$ (cf. (2.4)), a simple calculation based on (4.1) shows that

$$P_{\alpha+\sigma}(g)=P_\alpha(g)+P_\sigma(g^*).$$

But g^* starts from $g_\alpha\in\Omega$ which is stable. Thus there exists a $\lambda(\alpha)>0$ such that $P_\sigma(g^*)>0$ for $0<\sigma<\lambda(\alpha)$, and

$$P_{\alpha+\sigma}(g)>P_\alpha(g)$$

for all such σ. Therefore $\sigma\mapsto P_\sigma(g)$ is strictly increasing on $[0,\beta)$. To show that this monotonicity is preserved at the end point, consider the process $^*g_\sigma$

[1] We use the term "incremental" because the underlying forces are the actual loads minus their values at $\sigma=0$.

[2] Recall that, because of (1.1) and (2.1), processes are necessarily non-rigid.

$(0 \leqq \sigma \leqq \beta)$ defined by $^*g_\sigma = g_{\beta-\sigma}$. Then $\dot{g}_\sigma$ and $^*\dot{g}_\sigma$ are independent of σ with $\dot{g}_\sigma = -^*\dot{g}_\sigma$, and a similar assertion applies to $\dot{G}_\sigma$ and $^*\dot{G}_\sigma$. Thus (4.1) implies

$$P_{\beta-\sigma}(g) = P_\beta(g) - P_\sigma(^*g),$$

and, since $P_\sigma(^*g) > 0$ for all sufficiently small $\sigma > 0$, we have

$$P_\beta(g) > P_{\beta-\sigma}(g). \qquad \square$$

5. The Mixed Problem. Uniqueness

The *mixed problem* consists in finding a deformation f that satisfies the *equation of equilibrium*

$$\operatorname{div} S(\nabla f) + b_f = 0 \quad \text{in } \mathscr{B} \tag{5.1}$$

and the *boundary conditions*

$$f = d \text{ on } \mathscr{D}, \qquad S(\nabla f)\,n = s_f \text{ on } \mathscr{S}, \tag{5.2}$$

where $d \in C^0(\mathscr{D}, \mathbb{R}^3)$ is the *prescribed deformation* of $\mathscr{D}$, while n is the outward unit normal to $\partial\mathscr{B}$.

Remark. Note that the displacement problem $(\mathscr{D} = \partial\mathscr{B},\ \mathscr{S} = \emptyset)$ is a special case of the mixed problem. In view of our original hypothesis that $\mathscr{D}$ be non-empty, the traction problem $(\mathscr{S} = \partial\mathscr{B},\ \mathscr{D} = \emptyset)$ is excluded from our considerations.

Let f be a class C^2 solution of the mixed problem and let u be a variation (i.e., $u \in \mathrm{Var}$). Then (1.2), (5.1), (5.2), and the divergence theorem imply that

$$\begin{aligned}\int_{\mathscr{S}} s_f \cdot u = \int_{\partial\mathscr{B}} S(\nabla f)\,n \cdot u &= \int_{\mathscr{B}} [S(\nabla f) \cdot \nabla u + u \cdot \operatorname{div} S(\nabla f)] \\ &= \int_{\mathscr{B}} [S(\nabla f) \cdot \nabla u - b_f \cdot u],\end{aligned}$$

and we have the identity

$$\int_{\mathscr{S}} s_f \cdot u + \int_{\mathscr{B}} b_f \cdot u = \int_{\mathscr{B}} S(\nabla f) \cdot \nabla u. \tag{5.3}$$

The converse is also true. Indeed, let

$$\mathrm{Kin} = \{f \in \mathrm{Def} : f = d \text{ on } \mathscr{D}\};$$

then a class C^2 deformation $f \in \mathrm{Kin}$ that satisfies (5.3) for every $u \in \mathrm{Var}$ will automatically satisfy (5.1) and (5.2). This motivates the following *weak* statement of the mixed problem: Find an $f \in \mathrm{Kin}$ that satisfies (5.3) for every $u \in \mathrm{Var}$. A deformation $f \in \mathrm{Kin}$ with this property will be called a *solution*.

Let Ω be a set of deformations. We say that *uniqueness holds in* Ω provided the mixed problem has at most one solution in Ω. The next theorem shows that stability implies uniqueness, at least in convex[1] subsets of Def.

Theorem 2. *For the mixed problem uniqueness holds in any convex, stable set of deformations.*

Proof. Let $\Omega \subset \mathrm{Def}$ be convex and stable. Let $f, h \in \Omega$ be solutions with $f \not\equiv h$, and let $u = h - f$. Then $h = f$ on $\mathscr{D}$, so that $u = 0$ on $\mathscr{D}$. Thus $u \in \mathrm{Var}$. Consider the *straight* path

$$g_\sigma = f + \sigma u \qquad (0 \leqq \sigma \leqq 1)$$

from f to h. Since Ω is convex, g lies in Ω and hence represents a process. Thus we may conclude from Theorem 1 and (4.1) that

$$P_1(g) > P_0(g) = 0.$$

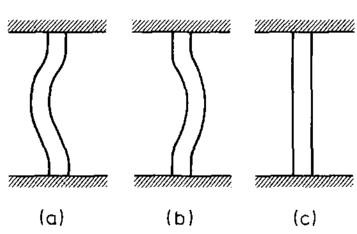
On the other hand, (2.4), (4.1), (5.3), and the fact that both f and h are solutions imply that

$$P_1(g) = \int_{\mathscr{B}} [S(\nabla h) - S(\nabla f)] \cdot \nabla u - \int_{\mathscr{S}} (s_h - s_f) \cdot u - \int_{\mathscr{B}} (b_h - b_f) \cdot u = 0,$$

and we have a contradiction. Thus $f \equiv h$. $\square$

Corollary. *Let f and h be two solutions of the mixed problem. Then the straight path from f to h (provided it lies in* Def*) cannot be stable.*

Remark. Consider a straight rod placed between two parallel rigid plates which are moved toward each other until the rod buckles ((a) and (b) denote two possible buckled states). If the buckling is not too severe, the straight path connecting these states will lie in Def. The corollary asserts that at least one deformation on this path is not stable; a strong candidate for such a deformation is the intermediate state (c).



Remark. To place Theorems 1 and 2 in a slightly different context, let $\mathscr{L}$ denote the non-linear operator that carries $f \in \mathrm{Kin}$ into the linear functional $\mathscr{L}_f$: $\mathrm{Var} \to \mathbb{R}$ defined by

$$\mathscr{L}_f(u) = \int_{\mathscr{B}} S(\nabla f) \cdot \nabla u - \int_{\mathscr{S}} s_f \cdot u - \int_{\mathscr{B}} b_f \cdot u.$$

[1] Convexity here is with respect to the linear structure in $C^1(\mathscr{B}, \mathbb{R}^3)$.

By definition $\mathscr{L}$ is *strictly monotone* on $\Omega \subset \text{Kin}$ if

$$(\mathscr{L}_f - \mathscr{L}_h)(f-h) > 0$$

whenever $f, h \in \Omega$ with $f \neq h$. For Ω convex this is equivalent to $P_1(g) > 0$ with $g_\sigma = f + \sigma(h-f)$; hence Theorem 1 asserts that $\mathscr{L}$ is strictly monotone on Ω if Ω is convex and stable. Theorem 2 is therefore a corollary of the well-known result that strictly monotone operators are one-to-one.

6. Local Stability and Uniqueness for Dead Loading

We now confine our attention to dead loading, so that both $s_f (=s)$ and b_f $(=b)$ are independent of the deformation f. The incremental power (4.1) then takes the simple form

$$P_\sigma(g) = \int_{\mathscr{B}} (S_\sigma - S_0) \cdot \dot{G}_\sigma. \tag{6.1}$$

When discussing local uniqueness it is often more convenient to work with the following slightly stronger definition of stability.

Definition. A deformation f {respectively, set $\Omega \subset \text{Def}$} is *Hadamard-stable*,[1] or simply *H-stable*, if

$$\dot{P}_0(g) = \frac{d}{d\sigma} P_\sigma(g)|_{\sigma=0} > 0 \tag{6.2}$$

for all processes g which start from f {respectively, in Ω}. The stability is *uniform* if for some $\kappa > 0$

$$\dot{P}_0(g) \geqq \kappa \|\nabla \dot{g}_0\|^2$$

for all processes g which start from f {respectively, in Ω}.

Proposition 3. *For a deformation f or a set* $\Omega \subset \text{Def}$

$$H\text{-}stability \Rightarrow stability.$$

Proof. By (4.1), $P_0(g) = 0$. Thus if $\dot{P}_0(g) > 0$, then $P_\sigma(g)$ must be strictly positive in some interval $(0, \lambda)$. □

Our next result shows that H-stability is equivalent to stability under infinitesmal perturbations.

[1] Cf. HADAMARD [1903], p. 252, who uses the phrase "stabilité de l'equilibre interne". Actually, HADAMARD's definition is based on the equivalent condition (6.3). See also PEARSON [1956], HILL [1957], GREEN & ADKINS [1960], TRUESDELL & NOLL [1965], BEATTY [1965]. In the mechanics literature a condition equivalent to (6.2), but formulated in terms of work, is usually referred to as DRUCKER's postulate (DRUCKER [1964]).

Theorem 3. *A deformation f is H-stable if and only if*

$$\int_{\mathscr{B}} \nabla u \cdot A(\nabla f)\, \nabla u > 0 \tag{6.3}$$

for all $u \in \mathrm{Var}$. A set $\Omega \subset \mathrm{Def}$ is uniformly H-stable if and only if for some $\kappa > 0$

$$\int_{\mathscr{B}} \nabla u \cdot A(\nabla f)\, \nabla u \geqq \kappa \,\|\nabla u\|^2 \tag{6.4}$$

for all $f \in \Omega$ and $u \in \mathrm{Var}$.

Proof. By (6.1) and (3.2),

$$\dot{P}_0(g) = \int_{\mathscr{B}} \dot{S}_0 \cdot \dot{G}_0 = \int_{\mathscr{B}} \dot{G}_0 \cdot A(\nabla f)\, \dot{G}_0 \qquad (f = g_0) \tag{6.5}$$

for any process g. Thus, by (2.2), (6.3) implies the H-stability of f. Conversely, assume that f is H-stable, and choose $u \in \mathrm{Var}$. Then, by Proposition 1, there is a process g which starts from f and has $\dot{g}_0 = u$; for this process (6.5) and (6.2) yield (6.3). The remainder of the proof is equally simple. □

We now endow Def with the topology generated by the semi-norm

$$\sup_{\mathscr{B}} |\nabla f|. \tag{6.6}$$

Then, as is clear from the next result, the set of all uniformly H-stable deformations is an open subset of Def.

Theorem 4. *Every uniformly H-stable deformation has a neighborhood which is uniformly H-stable.*

Proof. Let

$$\varphi(f, u) = \int_{\mathscr{B}} \nabla u \cdot A(\nabla f)\, \nabla u.$$

Then

$$\varphi(h, u) = \varphi(f, u) - \int_{\mathscr{B}} \nabla u \cdot [A(\nabla f) - A(\nabla h)]\, \nabla u \tag{6.7}$$

for all $f, h \in \mathrm{Def}$ and $u \in \mathrm{Var}$. Assume that f is uniformly H-stable. Then, if we let

$$\varepsilon_h = \sup_{\mathscr{B}} |A(\nabla f) - A(\nabla h)|,$$

(6.4) (with $\Omega = \{f\}$) and (6.7) imply

$$\varphi(h, u) \geqq (\kappa - \varepsilon_h)\, \|\nabla u\|^2$$

for all $u \in \mathrm{Var}$. Moreover, by the continuity of A and the choice of topology (6.6) on Def, there exists a neighborhood Ω of f such that

$$\kappa - \varepsilon_h \geqq \kappa_1 > 0$$

for all $h \in \Omega$; hence

$$\int_{\mathscr{B}} \nabla u \cdot A(\nabla h)\, \nabla u \geqq \kappa_1 \,\|\nabla u\|^2$$

for all $h \in \Omega$ and $u \in \mathrm{Var}$. Thus, by Theorem 3, Ω is uniformly H-stable. □

By a *neighborhood of the reference configuration* we mean a neighborhood of the identity deformation ($f(x)=x$ for all $x\in\mathscr{B}$).

Theorem 5. *Assume that either:*

(a) the reference configuration is positive and natural; or

(b) $\mathscr{D}=\partial\mathscr{B}$ and the reference configuration is homogeneous and strongly-elliptic.

Then the reference configuration has a neighborhood which is uniformly H-stable.

Proof. Assume first that *(a)* holds. By (3.4), (3.5), and the continuity of A, there is a $\kappa_0>0$ such that

$$H\cdot A(I,x)\,H\geqq\kappa_0|E|^2$$

for all $x\in\mathscr{B}$ and $H\in\mathrm{Lin}$, where E is the symmetric part of H. Thus if we take $H=\nabla u(x)$, integrate over $\mathscr{B}$, and use Korn's inequality (1.3), we infer the existence of a $\kappa_1>0$ such that

$$\int_{\mathscr{B}}\nabla u\cdot A(I)\,\nabla u\geqq\kappa_1\,\|\nabla u\|^2 \tag{6.8}$$

for all $u\in\mathrm{Var}$. Further, (6.8) also holds in case *(b)*; indeed, in this instance (6.8) is simply Gärding's inequality.[1] In any event, (6.8) and Theorem 3 tell us that the reference configuration is uniformly H-stable, and the desired conclusion follows from Theorem 4. □

We now return to the mixed problem (5.1), (5.2), but now, of course, with dead loads. For convenience, we use the term *data*[2] for the triplet (d,s,b).

Remark. It is important to note that the neighborhoods established in Theorems 4 and 5 are independent of the data.

Trivial examples of convex sets are sufficiently small open balls in Def. Thus Theorems 2, 4, and 5, Proposition 3, and the preceding Remark have the following immediate consequences.

Theorem 6. *Every uniformly H-stable deformation has a neighborhood in which uniqueness holds, and this neighborhood is independent of the data.*

Theorem 7. *Assume that either:*

(a) the reference configuration is natural and positive; or

(b) $\mathscr{D}=\partial\mathscr{B}$ and the reference configuration is homogeneous and strongly-elliptic.

Then the reference configuration has a neighborhood in which uniqueness holds, and this neighborhood is independent of the data.

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[1] GÄRDING [1953].

[2] Note that we do not consider the response function S to be part of the data.

Thus, as one would expect, the usual hypotheses of the infinitesimal theory of elasticity yield local uniqueness in the finite theory.

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