On the Response Functions of Isotropic Elastic Shells

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1. Introduction

In a mathematical model proposed in [1] for elastic shells we define a *surface manifold* to be an orientable 2-dimensional differentiable manifold $\mathscr{S}$ which can be imbedded globally into the physical space $\mathscr{R}^3$. Let X denote a typical point in $\mathscr{S}$. We define a *local state* of X to be a triple $(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e})$ consisting in a pair of linear injections $\boldsymbol{v}_X$ and $\hat{\boldsymbol{v}}_X$ of the tangent space $\mathscr{S}_X$ into $\mathscr{R}^3$ and a vector $\boldsymbol{e}$ in $\mathscr{R}^3$. $\boldsymbol{e}$ is called the *director* of X in the local state. It is required that $\boldsymbol{e}$ be non-zero, not belonging to $\boldsymbol{v}_X(\mathscr{S}_X)$ and $\hat{\boldsymbol{v}}_X(\mathscr{S}_X)$, and having the same orientation relative to $\boldsymbol{v}_X(\mathscr{S}_X)$ and $\hat{\boldsymbol{v}}_X(\mathscr{S}_X)$. That is, if $\{\boldsymbol{E}_1, \boldsymbol{E}_2\}$ is a basis for $\mathscr{S}_X$, we define

$$\boldsymbol{e}_\Gamma \equiv \boldsymbol{v}_X(\boldsymbol{E}_\Gamma), \qquad \hat{\boldsymbol{e}}_\Gamma \equiv \hat{\boldsymbol{v}}_X(\boldsymbol{E}_\Gamma), \qquad \Gamma = 1, 2; \tag{1.1}$$

then it is required that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}\}$ and $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \boldsymbol{e}\}$ be bases having the same orientation in $\mathscr{R}^3$.

As usual, we may define an *orientation* for X by designating a particular equivalence class of bases which have the same orientation in $\mathscr{S}_X$ to be the *positive class*. When X has been oriented in this way, a local state $(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e})$ is called *positive* if the bases $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}\}$ and $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \boldsymbol{e}\}$ defined by (1.1) are positive in $\mathscr{R}^3$ for any positive basis $\{\boldsymbol{E}_1, \boldsymbol{E}_2\}$ in $\mathscr{S}_X$. For definiteness, we shall now assume that X is oriented and, naturally, only positive local states of X will be considered.

To characterize a local state, we can introduce a *local reference configuration* $\boldsymbol{\mu}_X$ for X. Let $\boldsymbol{n}$ be the positive unit normal of $\boldsymbol{\mu}_X(\mathscr{S}_X)$ in $\mathscr{R}^3$. We define the *deformation gradients* $(\boldsymbol{F}, \hat{\boldsymbol{F}})$ from $(\boldsymbol{\mu}_X, \boldsymbol{n})$ to $(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e})$ by the conditions

$$\boldsymbol{F}\boldsymbol{n} = \hat{\boldsymbol{F}}\boldsymbol{n} = \boldsymbol{e}, \tag{1.2}$$

and

$$\boldsymbol{F}\boldsymbol{v} = \boldsymbol{v}_X \circ \boldsymbol{\mu}_X^{-1}(\boldsymbol{v}), \qquad \hat{\boldsymbol{F}}\boldsymbol{v} = \hat{\boldsymbol{v}}_X \circ \boldsymbol{\mu}_X^{-1}(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{\mu}_X(\mathscr{S}_X). \tag{1.3}$$

Since $(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e})$ is positive, and since $\boldsymbol{n}$ is the positive unit normal of $\mu_X(\mathscr{S}_X)$, (1.2) and (1.3) imply that

$$\det \boldsymbol{F} > 0, \quad \det \hat{\boldsymbol{F}} > 0. \tag{1.4}$$

As in [1], we denote the set of all pairs $(\boldsymbol{F}, \hat{\boldsymbol{F}})$ which satisfy the conditions $(1.2)_1$ and (1.4) by $\mathscr{D}_{\boldsymbol{n}}$. This set constitutes a representation for the set of positive local states of X. The set $\mathscr{D}_{\boldsymbol{n}}$ is a submanifold of dimension 15 in $\mathscr{GL}(3)^+ \times \mathscr{GL}(3)^+$. We define *left-multiplication* of $\mathscr{GL}(3)^+$ on $\mathscr{D}_{\boldsymbol{n}}$ by

$$\boldsymbol{K}(\boldsymbol{F}, \hat{\boldsymbol{F}}) \equiv (\boldsymbol{K}\boldsymbol{F}, \boldsymbol{K}\hat{\boldsymbol{F}}) \tag{1.5}$$

and *right-multiplication* similarly by

$$(\boldsymbol{F}, \hat{\boldsymbol{F}})\boldsymbol{K} \equiv (\boldsymbol{F}\boldsymbol{K}, \hat{\boldsymbol{F}}\boldsymbol{K}) \tag{1.6}$$

for any $\boldsymbol{K} \in \mathscr{GL}(3)^+$ and any $(\boldsymbol{F}, \hat{\boldsymbol{F}}) \in \mathscr{D}_{\boldsymbol{n}}$. Then it can be verified easily that

$$\boldsymbol{K}(\mathscr{D}_{\boldsymbol{n}}) = \mathscr{D}_{\boldsymbol{n}}, \tag{1.7}$$

and

$$(\mathscr{D}_{\boldsymbol{n}})\boldsymbol{K} = \mathscr{D}_{\boldsymbol{m}}, \tag{1.8}$$

where $\boldsymbol{m}$ is the unit vector in the direction of $\boldsymbol{K}^{-1}\boldsymbol{n}$, *viz*,

$$\boldsymbol{m} = \frac{\boldsymbol{K}^{-1}\boldsymbol{n}}{\|\boldsymbol{K}^{-1}\boldsymbol{n}\|}. \tag{1.9}$$

We call X an *elastic point* if it has a list of constitutive relations of the form

$$\boldsymbol{Z} = \boldsymbol{\Phi}(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e}), \tag{1.10}$$

where $\boldsymbol{\Phi}$ is called the *response function*. The value $\boldsymbol{Z}$ of $\boldsymbol{\Phi}$ may be a scalar, a vector, or a tensor on $\mathscr{R}^3$. For example, in the case where X is a *hyperelastic point*, the response function is a scalar-valued function, which can be interpreted as the *stored-energy function* of X. More generally, the value $\boldsymbol{Z}$ may be the stress tensor, the director stress tensor, or the internal director force, *etc.*, in any local state $(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e})$ of X. Using a local reference configuration $(\mu_X, \boldsymbol{n})$, we can represent (1.10) by

$$\boldsymbol{Z} = \boldsymbol{\Phi}_{\mu_X}(\boldsymbol{F}, \hat{\boldsymbol{F}}), \tag{1.11}$$

where $\boldsymbol{\Phi}_{\mu_X}$ is called the *relative response function*. The domain of $\boldsymbol{\Phi}$ is, of course, the set of positive local states of X, while the domain of $\boldsymbol{\Phi}_{\mu_X}$ is the set $\mathscr{D}_{\boldsymbol{n}}$.

The response function $\boldsymbol{\Phi}$ and the relative response function $\boldsymbol{\Phi}_{\mu_X}$ are both restricted by the principle of material frame-indifference. Let f be scalar-valued, $\boldsymbol{g}$ be vector-valued, and $\boldsymbol{H}$ be tensor-valued response functions, respectively. Then they must obey the following transformation rules:

$$f(\boldsymbol{Q}\boldsymbol{v}_X, \boldsymbol{Q}\hat{\boldsymbol{v}}_X, \boldsymbol{Q}\boldsymbol{e}) = f(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e}), \tag{1.12}$$

$$\boldsymbol{g}(\boldsymbol{Q}\boldsymbol{v}_X, \boldsymbol{Q}\hat{\boldsymbol{v}}_X, \boldsymbol{Q}\boldsymbol{e}) = \boldsymbol{Q}\boldsymbol{g}(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e}), \tag{1.13}$$

and

$$\boldsymbol{H}(\boldsymbol{Q}\boldsymbol{v}_X, \boldsymbol{Q}\hat{\boldsymbol{v}}_X, \boldsymbol{Q}\boldsymbol{e}) = \boldsymbol{Q}\boldsymbol{H}(\boldsymbol{v}_X, \hat{\boldsymbol{v}}_X, \boldsymbol{e})\boldsymbol{Q}^T, \tag{1.14}$$

relative to an arbitrary rotation Q of $\mathscr{R}^3$ and for an arbitrary local state $(v_X, \hat{v}_X, e)$. Here Q^T denotes the transpose of Q. In terms of the corresponding relative response functions, the rules (1.12)–(1.14) become

$$f_{\mu_X}(QF, Q\hat{F}) = f_{\mu_X}(F, \hat{F}), \tag{1.15}$$

$$g_{\mu_X}(QF, Q\hat{F}) = Q g_{\mu_X}(F, \hat{F}), \tag{1.16}$$

and

$$H_{\mu_X}(QF, Q\hat{F}) = Q H_{\mu_X}(F, \hat{F}) Q^T, \tag{1.17}$$

respectively, for all $(F, \hat{F})$ in $\mathscr{D}_n$. Here we have used the condition (1.7), which implies that the left-hand side of (1.15)–(1.17) are well defined.

The response function Φ and the relative response function Φ_{μ_X} are also restricted by the condition of material symmetry. Specifically, an orientation preserving linear isomorphism

$$\Gamma\colon \mathscr{S}_X \to \mathscr{S}_X \tag{1.18}$$

is a material automorphism of X if

$$\Phi(v_X \Gamma, \hat{v}_X \Gamma, e) = \Phi(v_X, \hat{v}_X, e) \tag{1.19}$$

for all local states $(v_X, \hat{v}_X, e)$ of X. The set of all material automorphisms of X form a group g_X, called the (*abstract*) *symmetry group* of X. Thus (1.19) holds for all $\Gamma \in g_X$. In terms of the relative response function Φ_{μ_X}, we say that K is a *material automorphism relative to* μ_X if it satisfies the conditions

$$K n = n, \qquad K n^{\perp} = n^{\perp}, \qquad \det K > 0 \tag{1.20}$$

and the identity

$$\Phi_{\mu_X}(FK, \hat{F}K) = \Phi_{\mu_X}(F, \hat{F}) \tag{1.21}$$

for all $(F, \hat{F})$ in $\mathscr{D}_n$. Here we have used the condition (1.8), which implies that the left-hand side of (1.21) is well defined. The set of all material automorphisms relative to μ_X also form a group g_{μ_X}, called the *relative symmetry group* of X. Then (1.21) holds for all $K \in g_{\mu_X}$.

In accordance with NOLL's general rule we call X an *isotropic solid* point if there exists an inner product m_X on $\mathscr{S}_X$ such that g_X coincides with the rotational group $\mathscr{SO}(\mathscr{S}_X)$ of $\mathscr{S}_X$ relative to m_X. In terms of the relative symmetry group, isotropy requires the existence of a local reference configuration μ_X, called an *undistorted local configuration*, relative to which g_{μ_X} coincides with the rotational group $\mathscr{SO}_n$, which is the subgroup of $\mathscr{SO}(3)$ subject to the condition $(1.20)_1$.

Having explained the conditions of material frame-indifference and material symmetry, we see that a scalar-valued response function f relative to an undistorted local reference configuration of an isotropic solid point X must satisfy the following identities:

$$f_{\mu_X}(QF, Q\hat{F}) = f_{\mu_X}(F, \hat{F}) \quad \forall Q \in \mathscr{SO}(3) \tag{1.15}$$

and

$$f_{\mu_X}(FQ', \hat{F}Q') = f_{\mu_X}(F, \hat{F}) \quad \forall Q' \in \mathscr{SO}_n, \tag{1.22}$$

for all $(F, \hat{F})$ in $\mathscr{D}_n$. Likewise, when the response function g is vector-valued, the restrictions are

$$g_{\mu_X}(QF, Q\hat{F}) = Q g_{\mu_X}(F, \hat{F}) \quad \forall Q \in \mathscr{SO}(3) \tag{1.16}$$

and

$$g_{\mu_X}(FQ', \hat{F}Q') = g_{\mu_X}(F, \hat{F}) \qquad \forall Q' \in \mathscr{S}\mathscr{O}_n, \tag{1.23}$$

and when the response function H is tensor-valued, the restrictions become

$$H_{\mu_X}(QF, Q\hat{F}) = QH_{\mu_X}(F, \hat{F})Q^T \qquad \forall Q \in \mathscr{S}\mathscr{O}(3) \tag{1.17}$$

and

$$H_{\mu_X}(FQ', \hat{F}Q') = H_{\mu_X}(F, \hat{F}) \qquad \forall Q' \in \mathscr{S}\mathscr{O}_n. \tag{1.24}$$

The aim of this paper is to solve the identity-pairs (1.15), (1.22), (1.16), (1.23), and (1.17), (1.24). We shall follow the general procedure developed in [2] to obtain general solutions for the identities.

2. Representations for Scalar-Valued Response Functions

For brevity we suppress the notation μ_X, so we write the conditions (1.15) and (1.22) as

$$f(QF, Q\hat{F}) = f(F, \hat{F}) \qquad \forall Q \in \mathscr{S}\mathscr{O}(3) \tag{2.1}$$

and

$$f(FQ', FQ') = f(F, \hat{F}) \qquad \forall Q' \in \mathscr{S}\mathscr{O}_n, \tag{2.2}$$

where $(F, \hat{F}) \in \mathscr{D}_n$. As usual, we denote the plane $\mu_X(\mathscr{S}_X)$ whose positive unit normal is n by $n^\perp$.

Given any deformation gradient F, we define its *normalized gradient* G relative to n by

$$Gn \equiv h, \qquad Gv \equiv Fv \qquad \forall v \in n^\perp, \tag{2.3}$$

where h is the positive unit normal of $F(n^\perp)$. Clearly, F can be determined by G and the vector

$$e = Fn, \tag{2.4}$$

and *vice versa*. Further, if we choose any positive basis $\{n_1, n_2\}$ in $n^\perp$, then $(2.3)_1$ can be represented by

$$Gn = \frac{Fn_1 \times Fn_2}{\|Fn_1 \times Fn_2\|} = h. \tag{2.5}$$

As usual G admits the polar decompositions:

$$G = RU = VR, \tag{2.6}$$

where R is the rotation tensor, and U and V are the right stretch tensor and the left stretch tensor, respectively. In view of (2.3) or (2.5), we have

$$Rn = h, \qquad Un = n, \qquad Vh = h \tag{2.7}$$

and

$$U(n^\perp) = n^\perp, \qquad V(h^\perp) = h^\perp. \tag{2.8}$$

We denote a positive principal basis for U in $n^\perp$ by $\{u_1, u_2\}$. Then a positive principal basis for V in $h^\perp$ is given by

$$\{v_1, v_2\} = R\{u_1, u_2\}. \tag{2.9}$$

The principal stretches $\{\alpha_1, \alpha_2\}$ and the principal invariants of $\boldsymbol{U}$ and $\boldsymbol{V}$ are, of course, related by

$$\alpha_1+\alpha_2=\operatorname{tr}\boldsymbol{U}-1=\operatorname{tr}\boldsymbol{V}-1, \qquad \alpha_1\alpha_2=\det\boldsymbol{U}=\det\boldsymbol{V}, \tag{2.10}$$

since the proper number of $\boldsymbol{U}$ and $\boldsymbol{V}$ in the direction of the proper vectors $\boldsymbol{n}$ and $\boldsymbol{h}$, respectively, is 1. We call $\boldsymbol{G}$ *degenerate* if $\alpha_1=\alpha_2$; otherwise, $\boldsymbol{G}$ is called *non-degenerate*. For a non-degenerate $\boldsymbol{G}$, α_1 or α_2 need not be different from 1.

From (2.7), (2.8), and (2.9) the tensors $\boldsymbol{R}$, $\boldsymbol{U}$, and $\boldsymbol{V}$ can be characterized by

$$\boldsymbol{R}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n}\}=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{h}\},$$
$$\boldsymbol{U}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n}\}=\{\alpha_1\boldsymbol{u}_1, \alpha_2\boldsymbol{u}_2, \boldsymbol{n}\}, \tag{2.11}$$

and

$$\boldsymbol{V}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{h}\}=\{\alpha_1\boldsymbol{v}_1, \alpha_2\boldsymbol{v}_2, \boldsymbol{h}\}.$$

Then from (2.6) and (2.3) the tensors $\boldsymbol{G}$ and $\boldsymbol{F}$ can be characterized by

$$\boldsymbol{G}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n}\}=\{\alpha_1\boldsymbol{v}_1, \alpha_2\boldsymbol{v}_2, \boldsymbol{h}\},$$

and

$$\boldsymbol{F}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n}\}=\{\alpha_1\boldsymbol{v}_1, \alpha_2\boldsymbol{v}_2, \boldsymbol{e}\}. \tag{2.12}$$

It should be noted that the expressions (2.3)–(2.12) can be applied to any deformation gradient. In particular, when the deformation gradient is $\hat{\boldsymbol{F}}$, we shall designate the corresponding quantities by the superimposed $\wedge$, *viz*,

$$\hat{\boldsymbol{G}}\{\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \boldsymbol{n}\}=\{\hat{\alpha}_1\hat{\boldsymbol{v}}_1, \hat{\alpha}_2\hat{\boldsymbol{v}}_2, \hat{\boldsymbol{h}}\},$$
$$\hat{\boldsymbol{F}}\{\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \boldsymbol{n}\}=\{\hat{\alpha}_1\hat{\boldsymbol{v}}_1, \hat{\alpha}_2\hat{\boldsymbol{v}}_2, \boldsymbol{e}\}, \tag{2.13}$$

etc. However, from (1.2), we have

$$\boldsymbol{e}=\boldsymbol{F}\boldsymbol{n}=\hat{\boldsymbol{F}}\boldsymbol{n}=\hat{\boldsymbol{e}}. \tag{2.14}$$

Next we consider the transformation rules for the quantities introduced in (2.3)–(2.13) under a change of configuration of the forms (2.1) and (2.2). First, when the reference configuration is rotated by $\boldsymbol{Q}'$, the deformation gradient $\boldsymbol{F}$ is transformed to $\boldsymbol{F}\boldsymbol{Q}'$. In this case we have the following transformation rule:

$$\boldsymbol{F}\mapsto\boldsymbol{F}\boldsymbol{Q}' \Rightarrow \begin{cases} \boldsymbol{n}\mapsto\boldsymbol{n},\ \boldsymbol{e}\mapsto\boldsymbol{e},\ \boldsymbol{h}\mapsto\boldsymbol{h}, \\ \boldsymbol{G}\mapsto\boldsymbol{G}\boldsymbol{Q}',\ \boldsymbol{R}\mapsto\boldsymbol{R}\boldsymbol{Q}',\ \boldsymbol{U}\mapsto\boldsymbol{Q}'^T\boldsymbol{U}\boldsymbol{Q}',\ \boldsymbol{V}\mapsto\boldsymbol{V}, \\ \alpha_\Gamma\mapsto\alpha_\Gamma,\ \boldsymbol{u}_\Gamma\mapsto\boldsymbol{Q}'^T\boldsymbol{u}_\Gamma,\ \boldsymbol{v}_\Gamma\mapsto\boldsymbol{v}_\Gamma \quad \Gamma=1,2, \end{cases} \tag{2.15}$$

for any $\boldsymbol{Q}'\in\mathscr{SO}_{\boldsymbol{n}}$. Similarly, when the deformed configuration is rotated by $\boldsymbol{Q}$, the deformation gradient $\boldsymbol{F}$ is transformed to $\boldsymbol{Q}\boldsymbol{F}$. In this case we have the following transformation rule:

$$\boldsymbol{F}\mapsto\boldsymbol{Q}\boldsymbol{F} \Rightarrow \begin{cases} \boldsymbol{n}\mapsto\boldsymbol{n},\ \boldsymbol{e}\mapsto\boldsymbol{Q}\boldsymbol{e},\ \boldsymbol{h}\mapsto\boldsymbol{Q}\boldsymbol{h}, \\ \boldsymbol{G}\mapsto\boldsymbol{Q}\boldsymbol{G},\ \boldsymbol{R}\mapsto\boldsymbol{Q}\boldsymbol{R},\ \boldsymbol{U}\mapsto\boldsymbol{U},\ \boldsymbol{V}\mapsto\boldsymbol{Q}\boldsymbol{V}\boldsymbol{Q}^T, \\ \alpha_\Gamma\mapsto\alpha_\Gamma,\ \boldsymbol{u}_\Gamma\mapsto\boldsymbol{u}_\Gamma,\ \boldsymbol{v}_\Gamma\mapsto\boldsymbol{Q}\boldsymbol{v}_\Gamma \quad \Gamma=1,2, \end{cases} \tag{2.16}$$

for any $\boldsymbol{Q}\in\mathscr{SO}(3)$. Of course, a similar set of transformation rules can be stated for the deformation gradient $\hat{\boldsymbol{F}}$ and its corresponding quantities, such as those in (2.13). In particular, in a joint transformation of the form (2.1) and (2.2), we have the transformation rule:

$$(\boldsymbol{F},\hat{\boldsymbol{F}})\mapsto \boldsymbol{Q}(\boldsymbol{F},\hat{\boldsymbol{F}})\boldsymbol{Q}' \Rightarrow \begin{cases} \boldsymbol{n}\mapsto \boldsymbol{n},\ (\boldsymbol{e},\boldsymbol{h},\hat{\boldsymbol{h}})\mapsto \boldsymbol{Q}(\boldsymbol{e},\boldsymbol{h},\hat{\boldsymbol{h}}),\\ (\boldsymbol{G},\hat{\boldsymbol{G}},\boldsymbol{R},\hat{\boldsymbol{R}})\mapsto \boldsymbol{Q}(\boldsymbol{G},\hat{\boldsymbol{G}},\boldsymbol{R},\hat{\boldsymbol{R}})\boldsymbol{Q}',\\ (\boldsymbol{V},\hat{\boldsymbol{V}})\mapsto \boldsymbol{Q}(\boldsymbol{V},\hat{\boldsymbol{V}})\boldsymbol{Q}^T,\ (\boldsymbol{U},\hat{\boldsymbol{U}})\mapsto \boldsymbol{Q}'^T(\boldsymbol{U},\hat{\boldsymbol{U}})\boldsymbol{Q}'. \end{cases} \tag{2.17}$$

The preceding transformation rule suggests that we define a *relative rotation tensor* $\boldsymbol{A}$ for any pair $(\boldsymbol{F},\hat{\boldsymbol{F}})\in\mathscr{D}_n$ by

$$\boldsymbol{A}\equiv\hat{\boldsymbol{R}}\boldsymbol{R}^T. \tag{2.18}$$

Then under the same joint transformation as in (2.17) we have also the transformation rule:

$$(\boldsymbol{F},\hat{\boldsymbol{F}})\mapsto \boldsymbol{Q}(\boldsymbol{F},\hat{\boldsymbol{F}})\boldsymbol{Q}' \Rightarrow \boldsymbol{A}\mapsto\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T. \tag{2.19}$$

Now as in [2] we define an *equivalence set* in $\mathscr{D}_n$ by the equivalence relation

$$(\boldsymbol{F},\hat{\boldsymbol{F}})\sim(\boldsymbol{F}',\hat{\boldsymbol{F}}')\Leftrightarrow \exists\,\boldsymbol{Q}\in\mathscr{SO}(3),\,\boldsymbol{Q}'\in\mathscr{SO}_n\colon\ (\boldsymbol{F}',\hat{\boldsymbol{F}}')=\boldsymbol{Q}(\boldsymbol{F},\hat{\boldsymbol{F}})\boldsymbol{Q}'. \tag{2.20}$$

From the general representation theorem in [2] we know that a general solution for the scalar-valued response function f subject to the condition (2.1) and (2.2) can be characterized by a set of invariants, which are constant on each equivalence set in $\mathscr{D}_n$ but which take on distinct constant values on different equivalence sets. We shall determine such a set of invariants by using the following

Theorem 2.1. *An equivalence set in* $\mathscr{D}_n$ *can be characterized by the rotational invariants of the set*

$$\{\boldsymbol{e},\boldsymbol{h},\hat{\boldsymbol{h}},\boldsymbol{V},\hat{\boldsymbol{V}},\boldsymbol{A}\}. \tag{2.21}$$

That is to say, the condition (2.20) *is equivalent to the condition*

$$(\boldsymbol{F},\hat{\boldsymbol{F}})\sim(\boldsymbol{F}',\hat{\boldsymbol{F}}')\Leftrightarrow \exists\,\boldsymbol{Q}\in\mathscr{SO}(3)\colon \begin{cases}(\boldsymbol{e}',\boldsymbol{h}',\hat{\boldsymbol{h}}')=\boldsymbol{Q}(\boldsymbol{e},\boldsymbol{h},\hat{\boldsymbol{h}}),\\ (\boldsymbol{V}',\hat{\boldsymbol{V}}',\boldsymbol{A}')=\boldsymbol{Q}(\boldsymbol{V},\hat{\boldsymbol{V}},\boldsymbol{A})\boldsymbol{Q}^T.\end{cases} \tag{2.22}$$

It should be noted that the quantities in the set (2.21) are not entirely arbitrary; besides the obvious restrictions that $\boldsymbol{h}$ and $\hat{\boldsymbol{h}}$ be unit vectors, $\boldsymbol{V}$ and $\hat{\boldsymbol{V}}$ be positive-definite and symmetric, and $\boldsymbol{A}$ be a rotation, they must obey also the conditions

$$\boldsymbol{h}\cdot\boldsymbol{e}>0,\qquad \hat{\boldsymbol{h}}\cdot\boldsymbol{e}>0 \tag{2.23}$$

and

$$\boldsymbol{V}\boldsymbol{h}=\boldsymbol{h},\qquad \hat{\boldsymbol{V}}\hat{\boldsymbol{h}}=\hat{\boldsymbol{h}},\qquad \boldsymbol{A}\boldsymbol{h}=\hat{\boldsymbol{h}}. \tag{2.24}$$

Necessity of the condition (2.22) follows readily from the transformation rules (2.17) and (2.19). Conversely, when the right-hand side of (2.22) holds, we may choose any rotation $\boldsymbol{R}$ such that

$$\boldsymbol{R}\boldsymbol{n}=\boldsymbol{h}. \tag{2.25}$$

Then $\hat{\boldsymbol{R}}$ is given by

$$\hat{\boldsymbol{R}}=\boldsymbol{A}\boldsymbol{R}, \tag{2.26}$$

and from (2.24) we have

$$\hat{\boldsymbol{R}}\boldsymbol{n}=\hat{\boldsymbol{h}}. \tag{2.27}$$

Similarly, we may choose any rotation $\boldsymbol{R}'$ such that

$$\boldsymbol{R}'\boldsymbol{n}=\boldsymbol{h}'. \tag{2.28}$$

Then we have also

$$\hat{R}' = A' R' \tag{2.29}$$

and

$$\hat{R}' n = \hat{h}'. \tag{2.30}$$

From (2.25)–(2.30) and the right-hand side of (2.22), we obtain

$$R^T Q^T R' n = n \tag{2.31}$$

or, equivalently,

$$Q' \equiv R^T Q^T R' \in \mathcal{SO}_n. \tag{2.32}$$

We now show that $(F, \hat{F})$ and $(F', \hat{F}')$ are related by the right-hand side of (2.20). Indeed, from (2.25)–(2.31) we have

$$QFQ' n = QFR^T Q^T R' n = QFR^T Q^T h' = QFR^T h = QF n = Q e = e' = F' n, \tag{2.33}$$

and

$$QGQ' = QVRR^T Q^T R' = QVQ^T R' = V' R' = G'. \tag{2.34}$$

As explained before, these two conditions imply that

$$F' = QFQ'. \tag{2.35}$$

Exactly the same argument yields also

$$\hat{F}' = Q\hat{F}Q'. \tag{2.36}$$

Thus the proof of Theorem 2.1 is complete.

Note. The preceding theorem can be stated in terms of the response function f as follows:

Theorem 2.2. *A representation for the scalar-valued response function f subject to the conditions* (2.1) *and* (2.2) *is given by*

$$f(F, \hat{F}) = \hat{f}(e, h, \hat{h}, V, \hat{V}, A), \tag{2.37}$$

where $\hat{f}$ is a hemitropic function; i.e.,

$$\hat{f}(Qe, Qh, Q\hat{h}, QVQ^T, Q\hat{V}Q^T, QAQ^T) = \hat{f}(e, h, \hat{h}, V, \hat{V}, A) \tag{2.38}$$

for all $Q \in \mathcal{SO}(3)$.

From this theorem we see that the conditions (2.1) and (2.2) are equivalent to the condition (2.38), which we shall now proceed to solve. First, among the six variables of $\hat{f}$ the last one, A, may be disposed of first. Indeed, since A is a rotation and must satisfy the condition (2.24), there are two possibilities:

1. h and $\hat{h}$ are parallel. In this case A is a rotation about h and $\hat{h}$. We can characterize A by the angle of rotation $\theta(A)$ about h and $\hat{h}$.
2. h and $\hat{h}$ are not parallel. In this case the vector $h \times \hat{h}$ is orthogonal to both h and $\hat{h}$. We can characterize A by the angle[1] $\theta(A)$ from $h \times \hat{h}$ to $A(h \times \hat{h})$.

[1] The sign of θ is determined as usual by the right-hand rule relative to the common normal $\hat{h}$ for $h \times \hat{h}$ and $A(h \times \hat{h})$.

Clearly, $\theta(A)$ is a scalar invariant of A, *viz*,

$$\theta(QAQ^T) = \theta(A) \quad \forall Q \in \mathscr{S}\mathscr{O}(3). \tag{2.39}$$

Hence when we replace A by $\theta(A)$ in (2.38), that condition becomes

$$\hat{f}(Qe, Qh, Q\hat{h}, QVQ^T, Q\hat{V}Q^T, \theta(QAQ^T)) = \hat{f}(e, h, \hat{h}, V, \hat{V}, \theta(A)) \tag{2.40}$$

for all rotations Q. For the remaining five variables of $\hat{f}$ we know that they can be characterized by a set of fundamental invariants and a set of relative invariants. The fundamental invariants are

$$\{\|e\|, \operatorname{tr} V, \operatorname{tr} \hat{V}, \det V, \det \hat{V}\}, \tag{2.41}$$

which characterize separately the five variables to within a rotation.

Since that rotation must be the same for all five variables, there are also some relative invariants, which characterize the directions of the vectors $\{e, h, \hat{h}\}$ and the principal axis of the symmetric tensors $\{V, \hat{V}\}$ relative to one another. Naturally, the number of these relative invariants depend on the degeneracy of the tensor variables. There are three possibilities:

1. V and $\hat{V}$ are both degenerate. The relative invariants are those among the oriented directions of the vectors $\{e, h, \hat{h}\}$.
2. Only one of the two tensors, V and $\hat{V}$, is degenerate. The relative invariants are those among the oriented directions of the vectors $\{e, h, \hat{h}\}$ and the non-oriented direction of a particular principal axis of the non-degenerate tensor V or $\hat{V}$.
3. V and $\hat{V}$ are both non-degenerate. The relative invariants are those among the oriented directions of the vectors $\{e, h, \hat{h}\}$ and the non-oriented directions of a particular principal axis of each V and $\hat{V}$.

For definiteness, we choose the particular principal axes in Cases 2 and 3 above to be the axes v_1 and $\hat{v}_1$ for V and $\hat{V}$, respectively. Then from the celebrated Cauchy-Weyl theorem it is well known that the relative invariants are

$$\left\{\begin{matrix} e\cdot h, e\cdot\hat{h}, h\cdot\hat{h}, v_1\cdot e, v_1\cdot\hat{h}, \hat{v}_1\cdot h \\ \hat{v}_1\cdot e, v_1\cdot\hat{v}_1, \Delta(e, h, \hat{h}, v_1, \hat{v}_1) \end{matrix}\right\} \tag{2.42}$$

(*cf.* [3, §11]), where Δ denotes the *determinant product* (*cf.* [4, II.9]). Since v_1 and $\hat{v}_1$ are non-oriented, the relative invariants (2.42) are subject to an equivalence relation defined by the mutually independent transformations

$$v_1 \mapsto -v_1, \quad \hat{v}_1 \mapsto -\hat{v}_1. \tag{2.43}$$

Having determined the basic invariants among the variables of the function $\hat{f}$ on the right-hand side of (2.40), we can now summarize our results in the following

Representation Theorem 2.3. *A scalar-valued response function f obeys the identities* (2.1) *and* (2.2) *if and only if it can be expressed as a function of the fundamental invariants* (2.41) *and* $\theta(A)$ *and the relative invariants* (2.42) *subject to the equivalence relation* (2.43).

It should be noted that, unlike the usual representation theorems for isotropic or hemitropic functions, the theorem above has not reduced the relative invariants among the variables explicitly to some particular hemitropic functions. While it is, of course, possible to determine such hemitropic functions, which characterize the relative invariants among the variables $\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}\}$, we choose not to do so here. Since from (2.24) $\boldsymbol{V}$ and $\hat{\boldsymbol{V}}$ are essentially 2-dimensional tensors, it is actually easier to calculate $\boldsymbol{v}_1$ and $\hat{\boldsymbol{v}}_1$ directly than to make use of a long list of hemitropic functions.

A special case of the representation theorem is a representation theorem for scalar-valued response function for an isotropic elastic membrane; *cf.* [5]. In the membrane theory the only pertinent variable is $\boldsymbol{V}$. As a result, $\hat{f}$ can be represented by a function of $\operatorname{tr}\boldsymbol{V}$ and $\det\boldsymbol{V}$.

3. Representations for Vector-Valued Response Functions

For brevity we write the conditions (1.16) and (1.23) as

$$\boldsymbol{g}(\boldsymbol{Q}\boldsymbol{F}, \boldsymbol{Q}\hat{\boldsymbol{F}}) = \boldsymbol{Q}\boldsymbol{g}(\boldsymbol{F}, \hat{\boldsymbol{F}}) \qquad \forall \boldsymbol{Q} \in \mathscr{SO}(3) \tag{3.1}$$

and

$$\boldsymbol{g}(\boldsymbol{F}\boldsymbol{Q}', \hat{\boldsymbol{F}}\boldsymbol{Q}') = \boldsymbol{g}(\boldsymbol{F}, \hat{\boldsymbol{F}}) \qquad \forall \boldsymbol{Q}' \in \mathscr{SO}_{\boldsymbol{n}}. \tag{3.2}$$

By exactly the same argument as that of Theorems 2.1 and 2.2 we obtain

Theorem 3.1. *A vector-valued response function g subject to the conditions* (3.1) *and* (3.2) *can be represented by*

$$\boldsymbol{g}(\boldsymbol{F}, \hat{\boldsymbol{F}}) = \hat{\boldsymbol{g}}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}), \tag{3.3}$$

where $\hat{\boldsymbol{g}}$ is a hemitropic function, i.e.,

$$\hat{\boldsymbol{g}}(\boldsymbol{Q}\boldsymbol{e}, \boldsymbol{Q}\boldsymbol{h}, \boldsymbol{Q}\hat{\boldsymbol{h}}, \boldsymbol{Q}\boldsymbol{V}\boldsymbol{Q}^T, \boldsymbol{Q}\hat{\boldsymbol{V}}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T) = \boldsymbol{Q}\hat{\boldsymbol{g}}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}) \tag{3.4}$$

for all $\boldsymbol{Q} \in \mathscr{SO}(3)$.

As explained in [6], a solution for (3.4) can be found in the following way: First, we define the *invariance group* $\mathscr{SO}_{\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}}$ for each particular $\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}$ by the condition

$$\boldsymbol{Q} \in \mathscr{SO}_{\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}} \Leftrightarrow \boldsymbol{Q} \in \mathscr{SO}(3){:}\; \begin{cases} \{\boldsymbol{Q}\boldsymbol{e}, \boldsymbol{Q}\boldsymbol{h}, \boldsymbol{Q}\hat{\boldsymbol{h}}, \boldsymbol{Q}\boldsymbol{V}\boldsymbol{Q}^T, \boldsymbol{Q}\hat{\boldsymbol{V}}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T\} \\ = \{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}. \end{cases} \tag{3.5}$$

From (3.4) we see that

$$\boldsymbol{Q}\hat{\boldsymbol{g}}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}) = \hat{\boldsymbol{g}}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}) \qquad \forall \boldsymbol{Q} \in \mathscr{SO}_{\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}}, \tag{3.6}$$

so we define the *admissible space* $\mathscr{V}_{\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}}$ by the condition

$$\boldsymbol{v} \in \mathscr{V}_{\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}} \Leftrightarrow \boldsymbol{v} \in \mathscr{R}^3{:}\;\; \boldsymbol{Q}\boldsymbol{v} = \boldsymbol{v} \qquad \forall \boldsymbol{Q} \in \mathscr{SO}_{\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}}. \tag{3.7}$$

Then from (3.6) we have

$$\hat{\boldsymbol{g}}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}) \in \mathscr{V}_{\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}} \qquad \forall \{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, \boldsymbol{V}, \hat{\boldsymbol{V}}, \boldsymbol{A}\}. \tag{3.8}$$

The theory developed in [6] now implies that a representation for $\hat{\boldsymbol{g}}$ is given by a linear combination of vector-valued hemitropic functions which generate the admissible space $\mathscr{V}_{\{e,h,\hat{h},V,\hat{V},A\}}$ at each $\{e, h, \hat{h}, V, \hat{V}, A\}$, the coefficients in the linear combination of the generators being scalar-valued hemitropic functions which we have already considered in general in the preceding section.

In view of the restrictions (2.23) and (2.24) on $\{e, h, \hat{h}, V, \hat{V}, A\}$, we see that there are only two possibilities for $\mathscr{V}$:

1. e, h, and $\hat{h}$, are all parallel. In this case the invariance group contains at least a rotation π about e, h, and $\hat{h}$. The admissible space is the 1-dimensional subspace spanned by e, namely $(e^{\perp})^{\perp}$.

2. e, h, and $\hat{h}$ are not all parallel. In this case the invariance group contains the identity tensor I only, so that the admissible space is the whole space $\mathscr{R}^3$. There are two possibilities in this case:

2a. e and h are not parallel. A generating set is $\{e, h, e\times h\}$.

2b. e and $\hat{h}$ are not parallel. A generating set is $\{e, \hat{h}, e\times \hat{h}\}$.

Having exhausted the admissible spaces for all $\{e, h, \hat{h}, V, \hat{V}, A\}$, we can now state the following

Representation Theorem 3.2. *A vector-valued response function g obeys the identities* (3.1) *and* (3.2) *if and only if it can be expressed as a linear combination*

$$g=\hat{g}=\hat{f}_1 e+\hat{f}_2 h+\hat{f}_3 \hat{h}+\hat{f}_4 e\times h+\hat{f}_5 e\times \hat{h}, \tag{3.9}$$

where $\hat{f}_1, \ldots, \hat{f}_5$ *are scalar-valued response functions satisfying* (2.1) *and* (2.2).

As usual, although there are five generators in (3.9) to express a vector g in $\mathscr{R}^3$, none of the generators can be deleted in (3.9), since the remaining four are not sufficient to span the admissible space at every $\{e, h, \hat{h}, V, \hat{V}, A\}$. For instance, if $\hat{h}$ is deleted from (3.9), then $\{e, h, e\times h, e\times \hat{h}\}$ do not span $\mathscr{V}_{\{e,h,\hat{h},V,\hat{V},A\}}$ when e and h are parallel but not parallel to $\hat{h}$. However, since a set of five vectors is always linearly dependent in $\mathscr{R}^3$, the coefficients $\hat{f}_1, \ldots, \hat{f}_5$ in (3.9) are not unique. This property is typical in most representations for vector-valued or tensor-valued isotropic or hemitropic functions; *cf.* [6].

4. Representations for Tensor-Valued Response Functions

As explained in [1] we are interested in tensor-valued response functions

$$Z=H(F,\hat{F}) \tag{4.1}$$

which can be identified as a linear transformation of the form

$$Z\colon\ h^{\perp}\rightarrow \mathscr{R}^3, \tag{4.2}$$

or of the form

$$\hat{Z}\colon\ \hat{h}^{\perp}\rightarrow \mathscr{R}^3. \tag{4.3}$$

For definiteness, we consider tensors of the form (4.2) first, and we write the conditions (1.17) and (1.24) as

$$H(QF, Q\hat{F})=QH(F,\hat{F})Q^T \qquad \forall Q\in\mathscr{SO}(3) \tag{4.4}$$

and

$$H(FQ', \hat{F}Q') = H(F, \hat{F}) \qquad \forall Q \in \mathscr{SO}_n, \tag{4.5}$$

where the right-hand side of (4.4) corresponds to a linear transformation of the form

$$QZQ^T:\ Qh^{\perp} \to \mathscr{R}^3. \tag{4.6}$$

As before, we can reduce (4.4) and (4.5) to a single identity by using the following

Theorem 4.1. *A tensor-valued response function* H *of the forms* (4.2) *or* (4.3) *and subject to the conditions* (4.4) *and* (4.5) *can be represented by*

$$H(F, \hat{F}) = \hat{H}(e, h, \hat{h}, V, \hat{V}, A), \tag{4.7}$$

where $\hat{H}$ *is a hemitropic function, i.e.,*

$$\hat{H}(Qe, Qh, Q\hat{h}, QVQ^T, Q\hat{V}Q^T, QAQ^T) = Q\hat{H}(e, h, \hat{h}, V, \hat{V}, A)\,Q^T \tag{4.8}$$

for all $Q \in \mathscr{SO}(3)$.

We can find a representation for (4.8) in exactly the same way as before. However, before solving (4.8) directly, we introduce first a canonical decomposition for tensors of the form (4.2). Let Z be an arbitrary tensor of that form. We define the *tangential projection* $Z_{h^{\perp}}$ and the *normal projection* Z_h of Z with respect to h by

$$Z_{h^{\perp}}v \equiv Zv - (h \cdot Zv)h, \qquad Z_h v \equiv (h \cdot Zv)h \qquad \forall v \in h^{\perp}. \tag{4.9}$$

Then $Z_{h^{\perp}}$ is a linear transformation of $h^{\perp}$, *viz,*

$$Z_{h^{\perp}}:\ h^{\perp} \to h^{\perp}, \tag{4.10}$$

while Z_h is a linear transformation from $h^{\perp}$ to $(h^{\perp})^{\perp}$ *viz.*

$$Z_h:\ h^{\perp} \to (h^{\perp})^{\perp}. \tag{4.11}$$

Clearly $Z_{h^{\perp}}$ and Z_h are uniquely determined by Z, and *vice versa*, since from (4.9) we have

$$Zv = (Z_{h^{\perp}} + Z_h)v \qquad \forall v \in h^{\perp}. \tag{4.12}$$

The transformation rule for $Z_{h^{\perp}}$ and Z_h is

$$Z \mapsto QZQ^T \Rightarrow \begin{cases} h \mapsto Qh,\ h^{\perp} \mapsto Qh^{\perp}, \\ Z_{h^{\perp}} \mapsto QZ_{h^{\perp}}Q^T,\ Z_h \mapsto QZ_hQ^T, \end{cases} \tag{4.13}$$

for all $Q \in \mathscr{SO}(3)$.

From (4.9), if we define a vector $z \in h^{\perp}$ by

$$z \equiv Z^T h - (h \cdot Z^T h)h, \tag{4.14}$$

then z characterizes Z_h in such a way that

$$Z_h v = (z \cdot v)h \qquad \forall v \in h^{\perp}. \tag{4.15}$$

Clearly, the vector z obeys the transformation rule:

$$Z \mapsto QZQ^T \Rightarrow z \mapsto Qz \tag{4.16}$$

for all $Q \in \mathscr{SO}(3)$.

As usual, we can decompose the tensor $Z_{h^\perp}$ into a symmetric part X and a skew symmetric part Y, *viz*,

$$X = \tfrac{1}{2}(Z_{h^\perp} + Z^T_{h^\perp}), \qquad Y = \tfrac{1}{2}(Z_{h^\perp} - Z^T_{h^\perp}). \tag{4.17}$$

Since $h^\perp$ is 2-dimensional, the action of Y on $h^\perp$ can be characterized by a scalar y such that

$$Yv = y(h \times v) \qquad \forall v \in h^\perp. \tag{4.18}$$

Moreover, X and y obey the transformation rule

$$Z \mapsto QZQ^T \Rightarrow X \mapsto QXQ^T,\ y \mapsto y \tag{4.19}$$

for all $Q \in \mathscr{SO}(3)$.

In view of the decompositions (4.17) and (4.9) and the transformation rules (4.19), (4.16), and (4.13), we see that a tensor-valued response function H of the form (4.2) can be characterized by a hemitropic scalar-valued response function, a hemitropic vector-valued response function, and a hemitropic symmetric tensor-valued response function. Since in the preceding sections we have already obtained representations for hemitropic scalar-valued and vector-valued response functions, it suffices to consider hemitropic symmetric tensor-valued response functions only. Hence we shall now assume that $\hat{H}$ is a symmetric tensor on $h^\perp$ and obeys the identity (4.8).

As before we define the admissible space $\mathscr{H}_{\{e,h,\hat{h},V,\hat{V},A\}}$ by the condition

$$X \in \mathscr{H}_{\{e,h,\hat{h},V,\hat{V},A\}} \Leftrightarrow QXQ^T = X \qquad \forall Q \in \mathscr{SO}_{\{e,h,\hat{h},V,\hat{V},A\}}, \tag{4.20}$$

where $\mathscr{SO}_{\{e,h,\hat{h},V,\hat{V},A\}}$ is the invariance group defined by (3.5). There are two possibilities for $\mathscr{H}$:

1. e, h, and $\hat{h}$ are all parallel and V and $\hat{V}$ are both degenerate. In this case the invariance group is $\mathscr{SO}_e$, and the admissible space is the space of tensors of the form $c\mathbf{1}$, where $\mathbf{1}$ is the identity tensor on $h^\perp$.

2. e, h, and $\hat{h}$ are not all parallel or V and $\hat{V}$ are not both degenerate. In this case the invariance group consists in either the identity map only or the identity map and the rotation of π about e, h, and $\hat{h}$, and the admissible space is the space of all symmetric tensors on $h^\perp$. We consider the following two possibilities:

2a. e, h, and $\hat{h}$ are not all parallel, say e and h are not parallel. As explained in the preceding section, a basis for $h^\perp$ is

$$\{e - (e \cdot h)h,\ e \times h\}. \tag{4.21}$$

The product basis of (4.21) then forms a basis for $\mathscr{H}_{\{e,h,\hat{h},V,\hat{V},A\}}$.

2b. V and $\hat{V}$ are not both degenerate; say V is non-degenerate. In this case a basis for $h^\perp$ is

$$\{v_1, v_2\}. \tag{4.22}$$

Again the product basis of (4.22) then forms a basis for $\mathscr{H}_{\{e,h,\hat{h},V,\hat{V},A\}}$.

Since (4.21) and (4.22) obey the transformation rule

$$\left.\begin{array}{l}\{e, h, \hat{h}\} \mapsto Q\{e, h, \hat{h}\}\\ \{V, \hat{V}\} \mapsto Q\{V, \hat{V}\}Q^T\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\{e-(e \cdot h)h,\ e \times h\} \mapsto Q\{e-(e \cdot h)h,\ e \times h\},\\ \{v_1, v_2\} \mapsto Q\{v_1, v_2\},\end{array}\right. \tag{4.23}$$

when we express $\hat{H}$ in component form relative to the product bases of (4.21) or (4.22), the components are scalar-valued hemitropic functions. Consequently, we have the following

Representation Theorem 4.2. *A symmetric, tensor-valued, response function* H *of the form* (4.10) *obeys the identities* (4.4) *and* (4.5) *if and only if its components relative to the product bases of* (4.21) *or* (4.22) *are scalar-valued hemitropic functions.*

Clearly, the preceding theorem can be modified for tensor-valued response functions of the form (4.3) also.

5. Reversibility in Shells

As explained before, the mathematical model for shells used in this paper is the double-membrane model developed in [1]. In that model, generally, a shell is not reversible in the sense that the upper membrane is distinguished from the lower membrane by an orientation condition, and their local states are characterized separately by the deformation gradients F and $\hat{F}$. We cannot interchange the roles of F and $\hat{F}$ without violating the orientation condition.

Now suppose that we wish to establish a mathematical model for a shell by a double-membrane in which the role of the upper membrane and the role of the lower membrane are mathematically equivalent. Then the shell may be called *reversible*. To make this concept precise we introduce a new symmetry condition, called *reversibility*. We say that an isotropic point X in a shell is *reversible* if for any rotation Q'' of the form

$$Q''n = -n \tag{5.1}$$

the local states $(F, \hat{F})$ and $(\hat{F}Q'', FQ'')$ are materially equivalent, *i.e.*,

$$\Phi(\hat{F}Q'', FQ'') = \Phi(F, \hat{F}) \tag{5.2}$$

for any local state $(F, \hat{F})$. Notice that the order of F and $\hat{F}$ on the right-hand side is opposite to that on the left-hand side in (5.2).

Since Q'' obeys (5.1), it must be a rotation of π about an axis q_1'' in $n^{\perp}$, *i.e.*, Q'' may be characterized by

$$Q''\{q_1'', q_2'', n\} = \{q_1'', -q_2'', -n\}, \tag{5.3}$$

where $\{q_1'', q_2''\}$ is a positive basis in $n^{\perp}$. When we change the reference configuration by (5.3), the transformation rule for the vectors $\{e, h, \hat{h}\}$ is

$$(F, \hat{F}) \mapsto (\hat{F}Q'', FQ'') \Rightarrow e \mapsto -e, \ h \mapsto -\hat{h}, \ \hat{h} \mapsto -h, \tag{5.4}$$

which shows that the operation on the left-hand side of (5.4), indeed, corresponds to a reversal of the sense of the double-membrane. Hence, we call that operation a *reversal operation* on $(F, \hat{F})$.

Given any pair $(F, \hat{F})$, we consider a particular reversal operation given by the rotation $\bar{Q}$ such that

$$\bar{Q}\{u_1, u_2, n\} = \{\hat{u}_1, -\hat{u}_2, -n\}. \tag{5.5}$$

For this particular $\bar{Q}$ the axis $\bar{q}_1$ is in the direction of $u_1+\hat{u}_1$, while $\bar{q}_2$ is in the direction of $u_2+\hat{u}_2$. It can be shown that $\bar{Q}$ gives rise to the following transformation rule:

$$(F,\hat{F})\mapsto(\hat{F}\bar{Q},F\bar{Q})\Rightarrow\begin{cases}(G,R)\mapsto\hat{P}(\hat{G},\hat{R}),(\hat{G},\hat{R})\mapsto P(G,R),\\(U,V,\hat{U},\hat{V})\mapsto(\hat{U},\hat{V},U,V),\\(\alpha_1,\alpha_2,\hat{\alpha}_1,\hat{\alpha}_2)\mapsto(\hat{\alpha}_1,\hat{\alpha}_2,\alpha_1,\alpha_2),\\(v_1,v_2,\hat{v}_1,\hat{v}_2)\mapsto(\hat{v}_1,-\hat{v}_2,v_1,-\hat{v}_2),\\(u_1,u_2,\hat{u}_1,\hat{u}_2)\mapsto(\hat{u}_1,-\hat{u}_2,u_1,-u_2),\\(e,h,\hat{h})\mapsto(-e,-\hat{h},-h),A\mapsto PA^T\hat{P},\end{cases}\tag{5.6}$$

where P and $\hat{P}$ are rotations characterized by

$$P\{v_1,v_2,h\}=\{v_1,-v_2,-h\}\tag{5.7}$$

and

$$\hat{P}\{\hat{v}_1,\hat{v}_2,\hat{h}\}=\{\hat{v}_1,-\hat{v}_2,-\hat{h}\}.\tag{5.8}$$

Notice that P and $\hat{P}$ are symmetric and commute with V and $\hat{V}$, respectively. In view of the transformation rule (5.6) we have the following

Representation Theorem 5.1. *An isotropic point X in a shell is reversible if and only if its reduced response function $\hat{\Phi}$ relative to an undistorted reference satisfies the condition*

$$\hat{\Phi}(e,h,\hat{h},V,\hat{V},A)=\hat{\Phi}(-e,-\hat{h},-h,\hat{V},V,PA^T\hat{P}).\tag{5.9}$$

Necessity can be read off from (5.2), (5.6), and the particular choice

$$Q''=\bar{Q}.\tag{5.10}$$

Conversely, suppose that (5.9) holds. Then from (5.6) we have

$$\Phi(\hat{F}\bar{Q},F\bar{Q})=\Phi(F,\hat{F}).\tag{5.11}$$

But since X is isotropic, we have also

$$\Phi(FQ',\hat{F}Q')=\Phi(F,\hat{F})\tag{5.12}$$

for all $Q'\in\mathscr{SO}_n$, *cf.* (1.21). Combining (5.11) and (5.12), we then obtain

$$\Phi(\hat{F}Q'\bar{Q}Q'^T,FQ'\bar{Q}Q'^T)=\Phi(\hat{F}Q'\bar{Q},FQ'\bar{Q})=\Phi(FQ',\hat{F}Q')=\Phi(F,\hat{F})\tag{5.13}$$

for all $Q'\in\mathscr{SO}_n$. Now in general if K is a rotation of angle κ about the axis k, then $Q'KQ'^T$ is a rotation of the same angle κ but about the axis $Q'k$. Hence if Φ satisfies (5.11) in a particular reversal operation $\bar{Q}$, then (5.13) implies that it satisfies (5.2) in all reversal operations Q''. Thus the proof of the representation theorem is complete.

The condition (5.9) suggests that we define a local state characterized by $\{e,h,\hat{h},V,\hat{V},A\}$ to be *symmetric* with respect to reversal if there exists a rotation Q such that

$$\{Qe,Qh,Q\hat{h},QVQ^T,Q\hat{V}Q^T,QAQ^T\}=\{-e,-\hat{h},-h,\hat{V},V,PA^T\hat{P}\}.\tag{5.14}$$

Such a local state can be characterized in the following way: First, since $\boldsymbol{Q}$ reverses the orientation of $\boldsymbol{e}$ as required by (5.14), it must be a rotation of angle π about some axis orthogonal to $\boldsymbol{e}$. For definiteness, let us denote that axis by $\boldsymbol{q}_1$, and we put

$$\boldsymbol{q}_2=\boldsymbol{e}\times\boldsymbol{q}_1. \tag{5.15}$$

Then $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{e}\}$ form a positive orthogonal basis in $\mathscr{R}^3$, relative to which the component matrix of $\boldsymbol{Q}$ is the diagonal matrix diag $(1, -1, -1)$, *i.e.*, $\boldsymbol{Q}$ can be characterized by

$$\boldsymbol{Q}\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{e}\}=\{\boldsymbol{q}_1, -\boldsymbol{q}_2, -\boldsymbol{e}\}. \tag{5.16}$$

Now suppose that the components of $\boldsymbol{h}$ relative to the basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{e}\}$ is $\{h^1, h^2, h^3\}$, *viz*,

$$\boldsymbol{h}=h^1\boldsymbol{q}_1+h^2\boldsymbol{q}_2+h^3\boldsymbol{e}. \tag{5.17}$$

Then (5.14) requires that the components of $\hat{\boldsymbol{h}}$ relative to the same basis be $\{-h^1, h^2, h^3\}$, *viz*,

$$\hat{\boldsymbol{h}}=-h^1\boldsymbol{q}_1+h^2\boldsymbol{q}_2+h^3\boldsymbol{e}=-\boldsymbol{Q}\boldsymbol{h}. \tag{5.18}$$

Of course, the component forms (5.17) and (5.18) are consistent with the condition (2.23) if and only if

$$\boldsymbol{h}\cdot\boldsymbol{e}=\hat{\boldsymbol{h}}\cdot\boldsymbol{e}=h^3>0. \tag{5.19}$$

Next let $\{\alpha_1, \alpha_2\}$ be the principal values and $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be a principal basis for $\boldsymbol{V}$ in $\boldsymbol{h}^\perp$. As usual, we choose $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ in such a way that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{h}\}$ is positive in $\mathscr{R}^3$. In component form $\boldsymbol{V}$ is then given by

$$\boldsymbol{V}=\alpha_1\boldsymbol{v}_1\otimes\boldsymbol{v}_1+\alpha_2\boldsymbol{v}_2\otimes\boldsymbol{v}_2+\boldsymbol{h}\otimes\boldsymbol{h}. \tag{5.20}$$

From (5.14), $\hat{\boldsymbol{V}}$ is related to $\boldsymbol{V}$ by

$$\hat{\boldsymbol{V}}=\boldsymbol{Q}\boldsymbol{V}\boldsymbol{Q}^T, \tag{5.21}$$

which means that the principal values of $\hat{\boldsymbol{V}}$ are also $\{\alpha_1, \alpha_2\}$, and that a principal basis for $\hat{\boldsymbol{V}}$ in $\hat{\boldsymbol{h}}^\perp$ is $\{\boldsymbol{Q}\boldsymbol{v}_1, \boldsymbol{Q}\boldsymbol{v}_2\}$. Since

$$\boldsymbol{Q}\boldsymbol{h}=-\hat{\boldsymbol{h}}, \tag{5.22}$$

we may choose

$$\hat{\boldsymbol{v}}_1=\boldsymbol{Q}\boldsymbol{v}_1, \qquad \hat{\boldsymbol{v}}_2=-\boldsymbol{Q}\boldsymbol{v}_2; \tag{5.23}$$

then $\{\hat{\boldsymbol{v}}_1, \hat{\boldsymbol{v}}_2, \hat{\boldsymbol{h}}\}$ is a positive principal basis for $\hat{\boldsymbol{V}}$ in $\mathscr{R}^3$. That is, $\hat{\boldsymbol{V}}$ has the component form

$$\hat{\boldsymbol{V}}=\alpha_1\hat{\boldsymbol{v}}_1\otimes\hat{\boldsymbol{v}}_1+\alpha_2\hat{\boldsymbol{v}}_2\otimes\hat{\boldsymbol{v}}_2+\hat{\boldsymbol{h}}\otimes\hat{\boldsymbol{h}}. \tag{5.24}$$

From (5.22) and (5.23), $\boldsymbol{Q}$ can be characterized also by

$$\boldsymbol{Q}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{h}\}=\{\hat{\boldsymbol{v}}_1, -\hat{\boldsymbol{v}}_2, -\hat{\boldsymbol{h}}\}. \tag{5.25}$$

Having treated the first five variables in (5.14), we consider next the last variable in the same condition:

$$\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T=\boldsymbol{P}\boldsymbol{A}^T\hat{\boldsymbol{P}}. \tag{5.26}$$

We recall first that A is a rotation satisfying the condition (2.24). Hence the axis of rotation $\boldsymbol{a}_1$ of A must be on the bisecting plane of $\boldsymbol{h}$ and $\hat{\boldsymbol{h}}$. From (5.17) and (5.18) that bisecting plane is obviously the plane $\boldsymbol{q}_1^{\perp}$. In view of (5.16), we see that $\boldsymbol{Q}A\boldsymbol{Q}^T$ is a rotation of the same angle as that of A but the axis of rotation of $\boldsymbol{Q}A\boldsymbol{Q}^T$ is $-\boldsymbol{a}_1$. This result means simply

$$\boldsymbol{Q}A\boldsymbol{Q}^T = A^T. \tag{5.27}$$

Comparing (5.27) with (5.26), and noting the fact that both $\boldsymbol{P}$ and $\hat{\boldsymbol{P}}$ are symmetric, *cf.* (5.7) and (5.8), we obtain

$$\hat{\boldsymbol{P}}A = A\boldsymbol{P}. \tag{5.28}$$

If we now apply both sides of (5.28) to the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{h}\}$, the result is

$$\hat{\boldsymbol{P}}\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, \hat{\boldsymbol{h}}\} = A\{\boldsymbol{v}_1, -\boldsymbol{v}_2, -\boldsymbol{h}\} = \{A\boldsymbol{v}_1, -A\boldsymbol{v}_2, -\hat{\boldsymbol{h}}\}. \tag{5.29}$$

This equation and the equation (5.8) imply that $A\boldsymbol{v}_1$ is parallel to $\hat{\boldsymbol{v}}_1$ and $A\boldsymbol{v}_2$ is parallel to $\hat{\boldsymbol{v}}_2$. Thus there are two possibilities:

$$A\boldsymbol{v}_1 = \pm\hat{\boldsymbol{v}}_1, \qquad A\boldsymbol{v}_2 = \pm\hat{\boldsymbol{v}}_2, \tag{5.30}$$

where the signs are either both $+$ or both $-$. Since the principal axes of $\boldsymbol{V}$ and $\hat{\boldsymbol{V}}$ are not oriented, by a judicious choice of the directions of $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ relative to those of $\{\hat{\boldsymbol{v}}_1, \hat{\boldsymbol{v}}_2\}$ we can select a definite sign in (5.30), say $+$. This particular choice does not affect the component forms (5.20) and (5.24) but it may reverse the signs of $\hat{\boldsymbol{v}}_1$ and $\hat{\boldsymbol{v}}_2$ in (5.23) and (5.25).

From (5.30) and (2.24), we see that A can be characterized by

$$A\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{h}\} = \{\hat{\boldsymbol{v}}_1, \hat{\boldsymbol{v}}_2, \hat{\boldsymbol{h}}\}. \tag{5.31}$$

Then from the definition (2.18) for A, the condition (5.31) means that

$$\boldsymbol{R}^T\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{h}\} = \hat{\boldsymbol{R}}^T\{\hat{\boldsymbol{v}}_1, \hat{\boldsymbol{v}}_2, \hat{\boldsymbol{h}}\} \tag{5.32}$$

or, equivalently,

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \{\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2\}. \tag{5.33}$$

The equation (5.33) and the fact that

$$\{\alpha_1, \alpha_2\} = \{\hat{\alpha}_1, \hat{\alpha}_2\} \tag{5.34}$$

mean also

$$\boldsymbol{U} = \hat{\boldsymbol{U}}. \tag{5.35}$$

Summarizing the analysis for a symmetric state, we have the following

Theorem 5.2. *Given any* $\{\boldsymbol{e}, \boldsymbol{h}, \boldsymbol{V}\}$ *satisfying the restrictions* (2.23) *and* (2.24) *and given any rotation* $\boldsymbol{Q}$ *such that*

$$\boldsymbol{Q}\boldsymbol{e} = -\boldsymbol{e}, \tag{5.36}$$

there exist uniquely $\{\hat{\boldsymbol{h}}, \hat{V}, A\}$ *satisfying* (2.23) *and* (2.24) *such that the local state corresponding to* $\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A\}$ *is symmetric. In component form* $\{\hat{\boldsymbol{h}}, \hat{V}, A\}$ *are given by* (5.18), (5.24), *and* (5.31).

If $\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A\}$ define a symmetric local state, and if Q is a rotation which satisfies the condition (5.14), then from (5.9) we have

$$\hat{\Phi}(Q\boldsymbol{e}, Q\boldsymbol{h}, Q\hat{\boldsymbol{h}}, QVQ^T, Q\hat{V}Q^T, QAQ^T) = \hat{\Phi}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A). \tag{5.37}$$

But since isotropy and material frame-indifference imply that $\hat{\Phi}$ be hemitropic, the condition (5.37) restricts further the value of Φ in a symmetric state. In particular, if $\hat{\Phi}$ is vector-valued, say $\hat{\Phi} = \hat{g}$, then (5.37) and (3.4) imply

$$Q\hat{g}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A) = \hat{g}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A), \tag{5.38}$$

which means that the vector $\hat{g}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A)$ at the symmetric state must be parallel to the axis $\boldsymbol{q}_1$ of the rotation Q. In the case $\hat{\Phi}$ is tensor-valued, the situation is somewhat more complex. When we switch the roles of F and $\hat{F}$, a tensor of the form (4.2) is transformed into a tensor of the form (4.3). So a tensor-valued response function associated with a reversible shell must consist in a pair of tensors which are of the forms (4.2) and (4.3), respectively. The condition (5.37) and (4.8) now imply

$$Q\hat{H}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A)Q^T = \hat{H}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A) \tag{5.39}$$

which means that Q transform the tensor of the form (4.2) to the corresponding tensor of the form (4.3), and *vice versa*.

As an example let us consider the symmetric state characterized by $\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A\}$ such that $\boldsymbol{e}$, $\boldsymbol{h}$, and $\hat{\boldsymbol{h}}$ are all parallel, V and $\hat{V}$ are identical, and A is the identity map I. For this symmetric state we may choose Q to be a rotation of angle π about any principal axis of V and $\hat{V}$. Then (5.38) implies immediately that

$$\hat{g}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A) = 0 \tag{5.40}$$

and (5.39) implies that $\hat{H}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A)$ is coaxial with V and $\hat{V}$, *i.e.*,

$$\hat{H}(\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A) = \hat{f}_0 I + \hat{f}_1 V, \tag{5.41}$$

where $\hat{f}_0$ and $\hat{f}_1$ are isotropic functions of $\{\boldsymbol{e}, \boldsymbol{h}, \hat{\boldsymbol{h}}, V, \hat{V}, A\}$.

A special case of the representation formula (5.41) is a representation formula for symmetric tensor-valued response functions of isotropic membranes. In that theory every local state is a symmetric state as defined here, since the only pertinent variable there is the tensor V; *cf.* [5].

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