*Potential Analysis* 1: 93–113, 1992.



# Subsolutions for Abstract Evolution Equations

LOUISE BARTHÉLEMY and PHILIPPE BÉNILAN
*Equipe de Mathématiques de Besançon, U.A. C.N.R.S. 741, Université de Franche Compté, 25030 Besançon Cedex, France.*

(Received: 14 May 1991; accepted: 1 October 1991)

**Abstract.** We study different notions of subsolutions for an abstract evolution equation $\mathrm{d}u/\mathrm{d}t + Au \ni f$ where $A$ is an $m$-accretive nonlinear operation in an ordered Banach space $X$ with order-preserving resolvents. A first notion is related to the operator $\mathrm{d}/\mathrm{d}t + A$ in the ordered Banach space $L^1(0, T; X)$; a second one uses the evolution equation $\mathrm{d}u/\mathrm{d}t + A_{\rightarrow} u \ni f$ where $A_{\rightarrow}: x \rightarrow \{y; z \leqslant y \text{ for some } z \in Ax\}$; other notions are also considered.

**AMS subject classifications (1991).** 47H06, 47H15, 47H19, 47H20.

**Key words.** Nonlinear operator, accretivity, subsolution, evolution equation.

The nonlinear semigroup theory gives a general notion of solution, called 'mild solution', for abstract evolution equations of the form

$$\mathrm{d}u/\mathrm{d}t + Au \ni f \tag{1}$$

where $A$ is an operator in a Banach space $X$ and $f \in L^1(0, T; X)$; the exact definition will be recalled in Section 1. This theory can be applied as well to degenerate parabolic equations in divergence form

$$u_t = \operatorname{div} a(u, \operatorname{grad} u) \tag{2}$$

where $a(r, \xi)$ is monotone in the vector $\xi$, as to fully nonlinear parabolic equation of the form

$$u_t = H(Du, D^2u) \tag{3}$$

where $H(\xi, S)$ is monotone in the symmetric matrix $S$. In both cases, solutions formally satisfy a 'parabolic comparison principle'. If $u$ is a 'subsolution' and $v$ a 'supersolution' of the equation on a cylinder $Q = ]0, T[x\Omega$ and $u \leqslant v$ on the parabolic boundary $\partial_p Q = (\{0\}x\Omega) \cup ([0, T]x\partial\Omega)$, then $u \leqslant v$ on $Q$. Such a result is classical for sufficiently regular solutions, but its extension to generalized solutions is often a tricky problem. The aim of this paper is to make precise an abstract framework for this principle.

In a previous paper [2], we have introduced an abstract notion of subsolution for the 'stationary problem'

$$Ax \ni y \tag{4}$$

for a class of operators $A$ in an ordered Banach space $X$, which we called 'pregenerators'. An operator $A: X \to \mathscr{P}(X)$ is a pregenerator if for $\lambda > 0$ small enough, the operator $J_\lambda = (I + \lambda A)^{-1}$ is everywhere defined on $X$ single-valued and order-preserving; if $x, y \in X$, we say that $x$ is a subsolution for (4) if

$$x \leqslant J_\lambda(x + \lambda y) \text{ for all sufficiently small } \lambda > 0.$$

It is clear that if $x \in D(A)$ and there exists $z \in Ax$ such that $z \leqslant y$, then $x$ is a subsolution for (4); but the notion of subsolution is much wider. In order to emphasize this, let us recall the situation in the linear case (see [2], Section 2.B.): if

$-A$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))$ of positive linear bounded operators on $X$, (5)

then $A$ is a pregenerator; and if $x, y \in X$, then $x$ is a subsolution for (4) if and only if

$$\langle A'w, x\rangle \leqslant \langle w, y\rangle \quad \text{for any } w \in D(A') \text{ with } w \geqslant 0$$

where $A'$ is the adjoint of $A$.

In this paper we will consider the evolution equation (1) with a pregenerator $A$ and we will assume $A + \omega I$ to be accretive for some $\omega \in \mathbb{R}$ (i.e. for $\lambda > 0$ with $\lambda\omega < 1$, $J_\lambda$ is a Lipschitz continuous mapping in $X$ with Lipschitz constant $(1 - \lambda\omega)^{-1}$); then the Crandall–Liggett theorem guarantees for $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; X)$, the existence of a unique mild solution $u$ of (1) with $u(0) = u_0$. For $u_0 \in \overline{D(A)}$ we denote by $\mathscr{A}_{u_0}$ the operator in the Banach space $\mathscr{X} = L^1(0, T; X)$ defined by

$$(u, f) \in \mathscr{A}_{u_0} \text{ iff } u \text{ is a mild solution of (1) with } u(0) = u_0.$$

We will see (Theorem 1) that $\mathscr{A}_{u_0}$ is a pregenerator in $\mathscr{X}$: to this pregenerator corresponds a notion of subsolution which is a first notion of a subsolution for (1); it actually is a notion of subsolution for the abstract Cauchy problem:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au \ni f, \quad u(0) = u_0. \tag{6}$$

We will see that the comparison principle holds for this notion of subsolution. In the linear case, if (5) holds and $u, f \in \mathscr{X}$, we will show (see Proposition 7) that $u$ is a subsolution for (6) if and only if

$$\int \langle A'w, u(t) - \mathbf{u}(t)\rangle \zeta'(t)\,\mathrm{d}t \geqslant 0$$

for $w \in D(A')$ with $w \geqslant 0$ and $\zeta \in \mathscr{C}^1([0, T], X)$ with $\zeta \geqslant 0$ and $\zeta(T) = 0$, where $\mathbf{u}$ is the exact solution of (6) given by

$$\mathbf{u}(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s.$$

There are alternative ways to define a notion of subsolution for (1). A first way consists in considering the mild solutions for the evolution problem

$$\frac{\mathrm{d}u}{dt} + \underline{\mathrm{A}}u \ni f \tag{7}$$

where $\underline{\mathrm{A}}$ is the operator associated to subsolutions of (4), namely

$$y \in \underline{\mathrm{A}}x \quad \text{iff } x, y \in X \text{ and } x \text{ is a subsolution for (4).}$$

This operator $\underline{\mathrm{A}}$ contains the operator $A_{\rightarrow}$ defined by

$$A_{\rightarrow}: x \in X \rightarrow \{y \in X;\ z \leqslant y \text{ for some } z \in Ax\} = \cup_{z \in Ax}[z, \rightarrow[\, \in \mathscr{P}(X).$$

If $f \in L^1(0, T; X)$, we will see (Theorem 2) that a mild solution $u$ of (7) is a subsolution for (6) for any $u_0 \in \overline{D(A)}$ with $u(0) \leqslant u_0$. This result is interesting for applications, but the converse statement is more surprising. If $u \in \mathscr{C}([0, T], X)$ taking its values in $\overline{D(A)}$, is a subsolution for (6) with $u_0 = u(0)$, then $u$ is a mild solution of (7); $u$ is even more than that: it is an exact mild solution of $\mathrm{d}u/\mathrm{d}t + A_{\rightarrow}u \ni f$ (we will make precise the meaning of this assertion in Section 1).

A second way of defining subsolutions for (1) is to extend the notion of integral solutions as introduced in [3]. We will not make more precise this extension in this introduction (see Section 2), but as we will see (Theorem 3) it will give a characterization for a subsolution of (6) in the same way that the integral inequalities characterize the mild solutions of (6).

The content of this paper is the following: in Section 1 we introduced the definitions and state the main results; in Section 2 we extend the notion of integral solution; in Section 3, we give the proofs of the statements of Section 1; finally, in Section 4, we consider the linear case.

We will not consider examples in this paper which is long enough. A characterization of subsolutions for first order quasilinear equations in terms of Kruskov inequalities has been given by the first author in [1]. The notion of viscosity subsolutions for Hamilton Jacobi equation, or more generally for equations of type (3), as developed in [7], [8], [10], etc., appears also as a concrete example of the abstract framework introduced here. Other examples will be presented in forthcoming papers.

## 1. Definitions and Main Results

Let $X$ be an ordered Banach space with norm $\|\cdot\|$ and closed convex positive cone $X_+$; we assume that the norm is nondecreasing on the positive cone:

$$0 \leqslant x \leqslant y \Rightarrow \|x\| \leqslant \|y\|. \tag{8}$$

We will use the sublinear nondecreasing functional on $X$,

$$N_+(x) = \operatorname{dist}(-x, X_+) = \inf\{\|x + z\|; z \in X_+\} \tag{9}$$

and its directional derivative on $XxX$

$$N'_+(x, y) = \lim_{\lambda \to 0_+} \lambda^{-1}(N_+(x + \lambda y) - N_+(x)) = \inf_{\lambda > 0} \lambda^{-1}(N_+(x + \lambda y) - N_+(x)). \tag{10}$$

Let $A$ be an operator in $X$, that is a map $A: X \to \mathscr{P}(X)$ which is identified with its graph $\{(x, y); y \in Ax\}$. Recall the usual terminology: an operator $A$ is *accretive* if

$$\|x_1 - x_2\| \leqslant \|x_1 - x_2 + \lambda(y_1 - y_2)\| \quad \text{for } (x_1, y_1), (x_2, y_2) \in A \text{ and } \lambda > 0$$

and it is *m-accretive* if it is accretive and for any $\lambda > 0$ and $y \in X$ there exists a (unique) solution of

$$x + \lambda Ax \ni y. \tag{11}$$

Following [2], we say that an operator $A$ is a *pregenerator* if there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, for any $y \in X$ there exists a unique solution $x = J_\lambda y$ of (11) and the map $y \in X \to J_\lambda y \in X$ is order-preserving. It is clear that if $A$ is a pregenerator and $A + \omega I$ is accretive for some $\omega \in \mathbb{R}$, then $A + \omega I$ is $m$-accretive.

Let $A$ be a pregenerator and $x, y \in X$. It is clear that

$$y \in Ax \Leftrightarrow x = J_\lambda(x + \lambda y) \quad \text{for } 0 < \lambda < \lambda_0;$$

we say that $x$ is a *subsolution* (*resp. supersolution*) *for* $Ax \ni y$ iff

$$x \leqslant J_\lambda(x + \lambda y) \text{ (resp. } x \geqslant J_\lambda(x + \lambda y)) \quad \text{for } 0 < \lambda < \lambda_0.$$

One can prove this definition is independent of $\lambda_0$ (see [2], Proposition 1.1).

Let us recall the following characterization (see [2], Proposition 2.5):

LEMMA 1. *Let $A$ be a pregenerator and $x, y \in X$. Assume that $A + \omega I$ is accretive for some $\omega \in \mathbb{R}$. Then $x$ is a subsolution (resp. supersolution) for $Ax \ni y$ if and only if*

$$N'_+(x - \mathbf{x}, y - \mathbf{y}) + \omega N_+(x - \mathbf{x}) \geqslant 0 \, (\textit{resp.}\, N'_+(\mathbf{x} - x, \mathbf{y} - y) + \omega N_+(\mathbf{x} - x) \geqslant 0)$$

*for any* $\mathbf{y} \in A\mathbf{x}$.

Let $A$ be an operator in $X$ and $f \in L^1(0, T; X)$; a *strong solution* of (*1*) is a function $u \in W^{1,1}(0, T; X)$ satisfying

$$u'(t) + Au(t) \ni f(t) \quad a.e.\ t \in (0, T). \tag{12}$$

Following the terminology of [4] (see also [5]), a *mild solution* of (1) is a continuous function $u\colon [0, T] \to X$ satisfying the following property:

for all $\varepsilon > 0$, there exist a subdivision $t_0 = 0 < t_1 < \cdots < t_n \leqslant T < t_{n+1}$ with $t_i - t_{i-1} < \varepsilon$ and $x_0, x_1, \ldots, x_n$, $f_1, \ldots, f_n \in X$ such that $\sum \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|\, dt < \varepsilon$, $\dfrac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni f_i$ and $\max\limits_{t_{i-1} \leqslant t \leqslant t_i} \|u(t) - x_i\| < \varepsilon$ for $i = 1, \ldots, n$.

A mild solution takes its values in $\overline{D(A)}$. A strong solution is a mild solution, but the converse is false in general.

Here we introduce a stronger notion: an *exact mild solution* of (1) is a continuous function $u\colon [0, T] \to X$ satisfying the following property:

for some $\varepsilon_0 > 0$ and some continuous function $\eta\colon [0, \varepsilon_0[ \to [0, \infty[$ with $\eta(0) = 0$; for any $\varepsilon \in ]0, \varepsilon_0[$, $t_0 = 0 < t_1 < \cdots < t_n \leqslant T < t_{n+1}$ with $t_i - t_{i-1} < \varepsilon$ and $f_1, \ldots, f_n \in X$ with $\sum_i \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|\, dt < \varepsilon$, there exist $x_0 = u(0), x_1, \ldots, x_n \in X$ such that $\dfrac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni f_i$ and $\max\limits_{t_{i-1} \leqslant t \leqslant t_i} \|u(t) - x_i\| < \eta(\varepsilon)$ for $i = 1, \ldots, n$.

Since step functions are dense in $L^1(0, T; X)$, an exact mild solution is a mild solution, but the converse is false in general.

Let us recall the *Crandall–Liggett theorem* as follows:

LEMMA 2. *If $A + \omega I$ is m-accretive for some $\omega \in \mathbb{R}$, then for any $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; X)$ there exists a unique mild solution $u$ of (1) with $u(0) = u_0$. Moreover, any mild solution of (1) is an exact mild solution of (1).*

The first main result of this paper is the following.

THEOREM 1. *Let $A$ be a pregenerator in $X$ and assume that $A + \omega I$ is accretive for some $\omega \in \mathbb{R}$. Let $u_0 \in \overline{D(A)}$ and define the operator $\mathscr{A}_{u_0}$ in the Banach space $\mathscr{X} = L^1(0, T; X)$ by*

$$(u, f) \in \mathscr{A}_{u_0} \text{ iff } u \text{ is a mild solution of } (1) \text{ with } u(0) = u_0.$$

*Then*

(a) $\mathscr{A}_{u_0}$ *is a pregenerator in the ordered Banach space* $\mathscr{X}$.

(b) *If* $u, f \in \mathscr{X}$, *the following properties are equivalent:*

(i) *u is a subsolution (resp. supersolution) for* $\mathscr{A}_{u_0} u \ni f$,

(ii) $\int_0^T N'_+(u(t) - \mathbf{u}(t), f(t) - \mathbf{f}(t))e^{-\omega t}\,\mathrm{d}t \geqslant 0$

$\left(\textit{resp.}\ \int_0^T N'_+(\mathbf{u}(t) - u(t), \mathbf{f}(t) - f(t))e^{-\omega t}\,\mathrm{d}t \geqslant 0\right)$ *for any* $(\mathbf{u}, \mathbf{f}) \in \mathscr{A}_{u_0}$.

(c) *Let* $u_0, v_0 \in \overline{D(A)}$, $u, f, v, g \in \mathscr{X}$. *If u is a subsolution for* $\mathscr{A}_{u_0} u \ni f$, *v is a supersolution for* $\mathscr{A}_{v_0} v \ni g$, $u_0 \leqslant v_0$ *and* $f \leqslant g$, *then* $u \leqslant v$.

REMARK 1. The integrals in (b.ii) are well defined: indeed, the function $(x, y) \in XxX \to N'_+(x, y)$ is u.s.c. and satisfies $|N'_+(x, y)| \leqslant \|y\|$.

We now state the second main result of this paper.

THEOREM 2. *Under the assumptions and notations of Theorem 1, we consider the operators* $\underline{A}$ *and* $A_{\rightarrow}$ *in X defined by*

$$(x, y) \in \underline{A} \text{ iff } (x, y) \in X \text{ and } x \text{ is subsolution of } Ax \ni y$$

*and*

$$A_{\rightarrow}: x \in X \to \{y \in X;\ z \leqslant y \text{ for some } z \in Ax\} = \cup_{z \in Ax}[z, \to[ \in \mathscr{P}(X), \text{ respectively.}$$

*Let* $u_0 \in \overline{D(A)}$, $u \in \mathscr{C}([0, T], X)$ *and* $f \in L^1(0, T; X)$. *Then*

(a) *If* $u(0) \leqslant u_0$ *and u is a mild solution of* $du/dt + Au \ni f$, *then u is a subsolution for* $\mathscr{A}_{u_0} u \ni f$.

(b) If $u(t) \in \overline{D(A)}$ *for* $t \in [0, T]$, *then the following assertions are equivalent:*

(i) *u is a subsolution for* $\mathscr{A}_{u_0} u \ni f$,

(ii) $u(0) \leqslant u_0$ *and u is a mild solution of* $du/dt + \underline{A}u \ni f$,

(iii) $u(0) \leqslant u_0$ *and u is an exact mild solution of* $du/dt + A_{\rightarrow} u \ni f$.

REMARK 2. We do not know if (i) $\Rightarrow$ (ii) is true in general. Of course one may state the corresponding result for supersolutions.

Let $A + \omega I$ be $m$-accretive for some $\omega \in \mathbb{R}$ and $f \in L^1(0, T; X)$; let us recall (see [3]) that $u$ is a mild solution of (1) if and only if $u \in \mathscr{C}([0, T], X)$ satisfies the inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u - x\| \leqslant [u - x, f - y] + \omega\|u - x\| \text{ in } \mathscr{D}'(]0, T[) \text{ for any } (x, y) \in A \quad (13)$$

where $[\,,]$ is the directional derivative of the norm; a function $u \in \mathscr{C}([0, T], X)$

satisfying the inequalities (13) is called an integral solution of (1).

We now state an extension of this result for the problem of subsolutions for (1).

THEOREM 3. *Under assumptions and notations of Theorems 1 and 2, let* $u, f \in L^1(0, T; X)$ *and consider the following assertions:*

(i) $u \in \mathscr{C}([0, T], X)$ *is a mild solution of* $\mathrm{d}u/\mathrm{d}t + \underline{A}u \ni f$.

(ii) *u satisfies the inequalities*

$$\frac{\mathrm{d}}{\mathrm{d}t} N_+(u - \mathbf{u}) \leqslant N'_+(u - \mathbf{u}, f - \mathbf{f}) + \omega N_+(u - \mathbf{u}) \quad \text{in } \mathscr{D}'(]0, T[) \tag{14}$$

*for any* $(\mathbf{u}, \mathbf{f}) \in \mathscr{X} x \mathscr{X}$ *satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t} N_+(x - \mathbf{u}) \leqslant N'_+(x - \mathbf{u}, y - \mathbf{f}) + \omega N_+(x - \mathbf{u}) \quad \text{in } \mathscr{D}'(]0, T[) \quad \text{for any } (x, y) \in A. \tag{15}$$

(iii) *u satisfies the inequalities (14) for any* $(\mathbf{u}, \mathbf{f}) \in \mathscr{C}([0, T], X) \times L^1(0, T; X)$ *with* $\mathbf{u}$ *mild solution of* $\mathrm{d}\mathbf{u}/\mathrm{d}t + A\mathbf{u} \ni \mathbf{f}$.

(iv) *u satisfies the inequalities*

$$\frac{\mathrm{d}}{\mathrm{d}t} N_+(u - x) \leqslant N'_+(u - x, f - y) + \omega N_+(u - x) \quad \text{in } \mathscr{D}'(]0, T[) \text{ for any } (x, y) \in A. \tag{16}$$

(v) *u is a subsolution for* $\mathscr{A}_{u_0} u \ni f$ *for any* $u_0 \in \overline{D(A)}$ *satisfying*

$$\lim_{t \to 0_+} \text{ess } N_+(u(t) - u_0) = 0. \tag{17}$$

*Then*

(a) *The following properties hold*

$$\{\text{(i) or (ii)}\} \Rightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Rightarrow \text{(v)}.$$

(b) *If* $u(t) \in \overline{D(A)}$ *a.e.* $t \in (0, T)$ *and there exists* $u_0 \in \overline{D(A)}$ *such that (17) holds, the assertions* (ii), (iii), (iv) *and* (v) *are equivalent.*

(c) *If* $u \in \mathscr{C}([0, T], X)$ *and* $u(t) \in \overline{D(A)}$ *for* $t \in [0, T]$, *the assertions* (i), (ii), (iii), (iv) *and* (v) *are equivalent.*

REMARK 3. The inequalities (14), (15), (16) are well defined; indeed if $u, f \in L^1(0, T; X)$ then $N_+(u)$ and $N'_+(u, f)$ are integrable functions on $(0, T)$. Recall that for $\phi, \psi \in L^1(0, T)$, the inequality

$$\frac{d\phi}{dt} \leqslant \psi \text{ in } \mathscr{D}'(]0, T[)$$

is equivalent to

$$\phi \in BV_{\mathrm{loc}}(]0, T[) \text{ and } \phi(t+) \leqslant \phi(s+) + \int_a^t \psi(\tau)\,d\tau \quad \text{for any } 0 < s < t < T.$$

In Theorem 3(a), the property $\{(\mathrm{iii}) \Rightarrow (\mathrm{iv})\}$ is immediate and the property $\{(\mathrm{iii}) \Rightarrow (\mathrm{iv})\}$ is an easy corollary of Theorem 1(b); one can also see that, according to (a) and (b), the part (c) of Theorem 3 is a restatement of the property $\{(\mathrm{i}) \Leftrightarrow (\mathrm{ii})\}$ in Theorem 2(b).

## 2. Extension of the Notion of Integral Solution

In this section we extend the notion of integral solution as introduced in [3]. In this section $X$ is a Banach space, $A$ is an operator in $X$ and $N: X \to \mathbb{R}$ is a Lipschitz continuous convex functional; for $u, v \in X$, we denote by $N'(u, v)$ the directional derivative

$$N'(u,v) = \lim_{\lambda \to 0_+} \lambda^{-1}(N(u + \lambda v) - N(u)).$$

We state the extension of the fundamental 'uniqueness theorem' of [3] as follows:

THEOREM 4. *Let* $u, f \in L^1(0, T; X)$ *satisfy*

$$\frac{d}{dt} N(u - x) \leqslant N'(u - x, f - y) + \omega N(u - x) \quad \text{in } \mathscr{D}'(]0, T[) \quad \text{for any } (x, y) \in A. \tag{18}$$

*Then*

$$\frac{d}{dt} N(u - \mathbf{u}) \leqslant N'(u - \mathbf{u}, f - \mathbf{f}) + \omega N(u - \mathbf{u}) \quad \text{in } \mathscr{D}'(]0, T[) \tag{19}$$

*for any* $(\mathbf{u}, \mathbf{f}) \in \mathscr{C}([0, T], X) \times L^1(0, T; X)$ *with* $\mathbf{u}$ *mild solution of* $d\mathbf{u}/dt + A\mathbf{u} \ni \mathbf{f}$.

In the statement of Theorem 4, we do not assume any accretivity of the operator $A$; let us state a corollary assuming some accretivity:

COROLLARY 5. *For* $i = 1, 2$, *let* $A_i$ *be an operator in* $X$, $f_i \in L^1(0, T; X)$ *and* $u_i$ *be a mild solution of* $du_i/dt + A_i u_i \ni f_i$. *Assume that*

$$N'(x_1 - x_2, y_1 - y_2) + \omega N(x_1 - x_2) \geqslant 0 \quad \text{for any } (x_1, y_1) \in A_1, (x_2, y_2) \in A_2. \tag{20}$$

*Then*

$$\frac{d}{dt}N(u_1 - u_2) \leqslant N'(u_1 - u_2, f_1 - f_2) + \omega N(u_1 - u_2) \quad \textit{in } \mathscr{D}'(]0, T[). \tag{21}$$

*Proof of Corollary 5.* Applying Theorem 4 with $A = A_2$, $(u, f) = (x_1, y_1) \in A_1$ we deduce, using (20), that

$$\frac{d}{dt}N(x_1 - u_2) \leqslant N'(x_1 - u_2, y_1 - f_2) + \omega N(x_1 - u_2) \quad \text{in } \mathscr{D}'(]0, T[)$$

for any $(x_1, y_1) \in A_1$.

Then apply Theorem 4 with $A = A_1$, $(u, f) = (u_2, f_2)$ and $N(x)$ replaced by $N(-x)$ to obtain (21). ■

*Proof of Theorem 4.* Let $(\mathbf{u}, \mathbf{f}) \in \mathscr{C}([0, T], X) \times L^1(0, T; X)$ with $\mathbf{u}$ a mild solution of $d\mathbf{u}/dt + A\mathbf{u} \ni \mathbf{f}$; we prove that (19) holds. We follow closely the proof of Theorem 1.1 in [3]; the difference is that we take a general Lipschitz continuous convex functional $N$ here (instead of the norm) and $u$ is only assumed to be integrable (instead of continuous). Notice that $N$ being Lipschitz continuous, the functions $N(u - \mathbf{u})$ and $N'(u - \mathbf{u}, f - \mathbf{f})$ are integrable functions on $(0, T)$, such that (19) is well defined.

Let $t_0 = 0 < t_1 < \cdots < t_n \leqslant T < t_{n+1}$, $x_0, x_1, \ldots, x_n$, $f_1, \ldots, f_n \in X$ with

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni f_i$$

and define the step functions $v$, $g$ on $]0, t_n]$ by $v = x_i$, $g = f_i$ on $]t_{i-1}, t_i]$. *By* (18), *for* $i = 1, \ldots, n$ the function $N(u - x_i)$ is of bounded variation on $[0, T]$ and for $0 \leqslant a < b < T$, one has

$$N(u - x_i)(b+) \leqslant N(u - x_i)(a+) + \int_a^b \left\{ N'\left( u(\sigma) - x_i, f(\sigma) - f_i + \frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right) + \omega N(u(\sigma) - x_i) \right\} d\sigma.$$

On the other hand

$$N'\left( u(\sigma) - x_i, f(\sigma) - f_i + \frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right) \leqslant N'(u(\sigma) - x_i, f(\sigma) - f_i) + \frac{N(u(\sigma) - x_{i-1}) - N(u(\sigma) - x_i)}{t_i - t_{i-1}}$$

which one can see, by using $w \in \partial N(u(\sigma) - x_i)$ such that

$$N'\left( u(\sigma) - x_i, f(\sigma) - f_i + \frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right) = \left\langle w, f(\sigma) - f_i + \frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right\rangle.$$

Then, by definition of $v$, $g$, one has

$$\begin{aligned}\int_{t_{i-1}}^{t_i} N(u - v(\tau))(b+)\,\mathrm{d}\tau \leqslant &\int_{t_{i-1}}^{t_i} N(u - v(\tau))(a+)\,\mathrm{d}\tau \\ &+ \int_{t_{i-1}}^{t_i} \mathrm{d}\tau \int_a^b \{N'(u(\sigma) - v(\tau), f(\sigma) - g(\tau)) + \omega N(u(\sigma) - v(\tau))\}\,\mathrm{d}\sigma \\ &+ \int_a^b N(u(\sigma) - x_{i-1})\,\mathrm{d}\sigma - \int_a^b N(u(\sigma) - x_i)\,\mathrm{d}\sigma.\end{aligned}$$

Adding these inequalities, one gets for any $0 \leqslant i < j \leqslant n$

$$\begin{aligned}\int_{t_i}^{t_j} N(u - v(\tau))(b+)\,\mathrm{d}\tau \leqslant &\int_{t_i}^{t_j} N(u - v(\tau))(a+)\,\mathrm{d}\tau \\ &+ \int_{t_i}^{t_j} \mathrm{d}\tau \int_a^b \{N'(u(\sigma) - v(\tau), f(\sigma) - g(\tau)) + \omega N(u(\sigma) - v(\tau))\}\,\mathrm{d}\sigma \\ &+ \int_a^b N(u(\sigma) - x_i)\,\mathrm{d}\sigma - \int_a^b N(u(\sigma) - x_j)\,\mathrm{d}\sigma.\end{aligned}$$

By definition of a mild solution, one may approximate $(\mathbf{u}, \mathbf{f})$ by functions $(v, g)$ of the type above; passing to the limit in these inequalities and using the upper semicontinuity of the derivative $N'$, one has for any $0 \leqslant s \leqslant t \leqslant T$

$$\begin{aligned}\int_s^t N(u - \mathbf{u}(\tau))(b+)\,\mathrm{d}\tau \leqslant &\int_s^t N(u - \mathbf{u}(\tau))(a+)\,\mathrm{d}\tau \\ &+ \int_s^t \mathrm{d}\tau \int_a^b \{N'(u(\sigma) - \mathbf{u}(\tau), f(\sigma) - \mathbf{f}(\tau)) + \omega N(u(\sigma) - \mathbf{u}(\tau))\}\,\mathrm{d}\sigma \\ &+ \int_a^b N(u(\sigma) - \mathbf{u}(s))\,\mathrm{d}\sigma - \int_a^b N(u(\sigma) - \mathbf{u}(t))\,\mathrm{d}\sigma.\end{aligned}$$

This being true for any $0 \leqslant a \leqslant b < T$, $0 \leqslant s \leqslant t < T$, one deduces that

$$\left(\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}\right) N(u(\sigma) - \mathbf{u}(\tau)) \leqslant N'(u(\sigma) - \mathbf{u}(\tau), f(\sigma) - \mathbf{f}(\tau)) + \omega N(u(\sigma) - \mathbf{u}(\tau)) \quad \text{in } \mathscr{D}'(]0, T[\times]0, T[).$$

Using the following Lemma with $F(x, \mathbf{x}) = N(x - \mathbf{x})$, $G(x, \mathbf{x}, y, \mathbf{y}) = N'(x - \mathbf{x}, y - \mathbf{y})$, will conclude the proof of (19). ■

LEMMA 3. *Let* $F: X \times X \to \mathbb{R}$ *be continuous and let* $G: X \times X \times X \times X \to \mathbb{R}$ *be u.s.c. with*

$$|F(x, \mathbf{x})| + |G(x, \mathbf{x}, y, \mathbf{y})| \leqslant C(1 + \|x\| + \|\mathbf{x}\| + \|y\| + \|\mathbf{y}\|). \tag{22}$$

*Let* $u, \mathbf{u}, f, \mathbf{f} \in L^1(0, T; X)$ *satisfy*

$$\left(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\tau}\right)F(u(\sigma),\mathbf{u}(\tau))\leqslant G(u(\sigma),\mathbf{u}(\tau),f(\sigma),f(\tau))\quad \text{in } \mathscr{D}'(]0,T[x]0,T[). \tag{23}$$

*Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}F(u(t),\mathbf{u}(t))\leqslant G(u(t),\mathbf{u}(t),f(t),\mathbf{f}(t))\quad \text{in } \mathscr{D}'(]0,T[). \tag{24}$$

*Proof of Lemma 3.* Let $\zeta\in\mathscr{D}(]0,T[)$, $\zeta\geqslant 0$ and $\rho\in\mathscr{D}(]-1,1[)$, $\rho\geqslant 0$, $\int\rho=1$. Consider the functions $\zeta_n(\sigma,\tau)=\zeta((\sigma+\tau)/2)\rho(n(\sigma-\tau)/2)$. For $n$ large enough, $\zeta_n\in\mathscr{D}(]0,T[x]0,T[)$ and $(\partial/\partial\sigma+\partial/\partial\tau)\zeta_n(\sigma,\tau)/2)=\zeta'(\sigma+\tau)/2)\rho(n(\sigma-\tau)/2)$. Applying (23),

$$\iint\left\{G(u(\sigma),\mathbf{u}(\tau),f(\sigma),\mathbf{f}(\tau))\zeta_n(\sigma,\tau)+F(u(\sigma),\mathbf{u}(\tau))\left(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\tau}\right)\zeta_n(\sigma,\tau)\right\}\mathrm{d}\sigma\,\mathrm{d}\tau\geqslant 0,$$

and then changing variables $\sigma=t+(s/n)$, $\tau=t-(s/n)$, one has

$$\iint\{G_n(t,s)\zeta(t)+F_n(t,s)\zeta'(t)\}\rho(s)\,\mathrm{d}s\,\mathrm{d}t\geqslant 0 \tag{25}$$

with $G_n(t,s)=G(u(t+(s/n)),\ \mathbf{u}(t-(s/n)),\ f(t+(s/n)),\ \mathbf{f}(t-(s/n)))$ and $F_n(t,s)=F(u(t+(s/n)),\mathbf{u}(t-(s/n)))$.

But if $v\in L^1(0,T;X)$ and $0<a<b<T$, for $n$ large enough, the functions $v_n(t,s)=v(t+(s/n))$ (resp. $v(t-(s/n))$) are in $L^1(]a,b[x]-1,1[;X)$ and converge in $L^1(]a,b[x]-1,1[;X)$ to the function $\mathbf{v}(t,s)=v(t)$. Thanks to the continuity of $F$, the upper semicontinuity of $G$ and (22), at the limit in (25) one obtains

$$\iint\{G(u(t),\mathbf{u}(t),f(t),\mathbf{f}(t))\zeta(t)+F(u(t),\mathbf{u}(t))\zeta'(t)\}\rho(s)\,\mathrm{d}s\,\mathrm{d}t\geqslant 0$$

and then

$$\int\{G(u(t),\mathbf{u}(t),f(t),\mathbf{f}(t))\zeta(t)+F(u(t),\mathbf{u}(t))\zeta'(t)\}\,\mathrm{d}t\geqslant 0.$$

This proves (24). ∎

REMARK 4. The Lemma may be extended easily to more than one variable: if $\Omega$ is an open set in $\mathbb{R}^N$, $F$ takes values in $\mathbb{R}^N$, $u$, $\mathbf{u}$, $f$, $\mathbf{f}\in L^1_{\mathrm{loc}}(\Omega;X)$ satisfy

$$(\operatorname{div}_\sigma+\operatorname{div}_\tau)F(u(\sigma),\mathbf{u}(\tau))\leqslant G(u(\sigma),\mathbf{u}(\tau),f(\sigma),\mathbf{f}(\tau))\quad\text{in } \mathscr{D}'(\Omega x\Omega)$$

then $\operatorname{div}F(u,\mathbf{u})\leqslant G(u,\mathbf{u},f,\mathbf{f})$ in $\mathscr{D}'(\Omega)$.

This result is closely related to the Kruskov's uniqueness Theorem for entropy solutions of first order quasilinear equations ([9], see also [6], [3], [1]).

## 3. Proofs of Theorems 1, 2 and 3

In this section we use the notations of Section 1. One of the main arguments in the proofs is application of Theorem 4 with the functional $N(x) = N_+(x)$ (or $N_+(-x)$) which is convex and Lipschitz continuous.

We first remark the elementary lemma:

LEMMA 4. *Let $u, f \in X$ satisfy*

$$\frac{d}{dt} N_+(u) \leqslant N'_+(u, f) + \omega N_+(u) \quad \textit{in } \mathscr{D}'(]0, T[) \tag{26}$$

*and*

$$\limsup_{t \to 0} \operatorname{ess} N_+(u(t)) = 0. \tag{27}$$

*Then*

(a) *If $f(t) \leqslant 0$ a.e. $t$, then $u(t) \leqslant 0$ a.e. $t$*

(b) $\int N'_+(u(t), f(t)) e^{-\omega t} \, dt \geqslant 0.$

Throughout this Section, $A$ is a pregenerator with $A + \omega I$ accretive and $\underline{A}$, $A_{\rightarrow}$ are defined as in Theorem 2. We state in a Proposition the corollaries of Theorem 4 corresponding to this particular situation:

PROPOSITION 6. *Let $u$, $\mathbf{u}$, $f$, $\mathbf{f} \in \mathscr{X}$.*

(a) *If $u$ is a mild solution of* $du/dt + \underline{A}u \ni f$, *then (16) holds.*

(b) *If $\mathbf{u}$ is a mild solution of* $d\mathbf{u}/dt + \bar{A}\mathbf{u} \ni f$, *then*

$$\frac{d}{dt} N_+(x - \mathbf{u}) \leqslant N'_+(x - \mathbf{u}, y - \mathbf{f}) + \omega N_+(x - \mathbf{u}) \quad \textit{in } \mathscr{D}'(]0, T[) \textit{ for any } (x, y) \in \underline{A} \tag{28}$$

*and in particular (15) holds.*

(c) *If $u$ is a mild solution of* $du/dt + \underline{A}u \ni f$ *and (28) holds, then* (14) *holds.*

(d) *If (16) holds and $\mathbf{u}$ is a mild solution of* $d\mathbf{u}/dt + A\mathbf{u} \ni \mathbf{f}$, *then (14) holds.*

*Proof.* Recall Lemma 1:

$$N'_+(x - \mathbf{x}, y - \mathbf{y}) + \omega N_+(x - \mathbf{x}) \geqslant 0 \quad \text{for any } (x, y) \in \underline{A}, (\mathbf{x}, \mathbf{y}) \in A.$$

To obtain the different parts (a), (b), (c), (d) we apply Theorem 4 taking for the data $\{N(x), A, (u, f), (\mathbf{u}, \mathbf{f})\}$ in the Theorem

for (a): $\{N_+(-x), \underline{A}, (x, y), (u, f)\}$ where $(x, y) \in A$

for (b): $\{N_+(x), A, (x, y), (\mathbf{u}, \mathbf{f})\}$ where $(x, y) \in \underline{A}$

for (c): $\{N_+(-x), \underline{A}, (\mathbf{u}, \mathbf{f}), (u, f)\}$

for (d): $\{N_+(x), A, (u, f), (\mathbf{u}, \mathbf{f})\}$. ■

*Proof of Theorem 1.* We first prove that $\mathscr{A}_{u_0}$ is a pregenerator in $\mathscr{X}$. Let $\lambda > 0$ and $f \in \mathscr{X}$. The inclusion $u + \lambda\mathscr{A}_{u_0}u \ni f$ means that

$$u \text{ is a mild solution of } \frac{\mathrm{d}u}{\mathrm{d}t} + Au \ni \frac{f - u}{\lambda} \quad \text{with } u(0) = u_0. \tag{29}$$

Using the definition of a mild solution, it is easy to see (and well-known) that it is equivalent to say that $u$ is a mild solution of

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \left(A + \frac{I}{\lambda}\right)u \ni \frac{f}{\lambda},\ u(0) = u_0.$$

But $(A + I/\lambda)$ is a pregenerator and $(A + I/\lambda) + (\omega - \lambda^{-1})I = A + \omega I$ is accretive; applying Lemma 2 to $(A + I/\lambda)$, there exists a unique solution $u$ of (29); applying Corollary 5 with $N(x) = N_+(x)$ and $A_1 = A_2 = A + I/\lambda$, by Lemma 1 and Lemma 4(a), we see that the map $f \to u$ is order-preserving. This proves part (a).

Now let $\mathscr{N}$ be the functional on $\mathscr{X}$ defined by

$$\mathscr{N}(u) = \int N_+(u(t))\, e^{-\omega t}\, \mathrm{d}t.$$

It is clear that $\mathscr{N}$ is convex nondecreasing, $\mathscr{X}_+ = \{u \in \mathscr{X};\ \mathscr{N}(-u) = 0\}$ and we have $\mathscr{N}'(u, f) = \int N_+'(u(t), f(t))\, e^{-\omega t}\, \mathrm{d}t$ for $(u, f) \in \mathscr{X}$. Using Corollary 5 with $N(x) = N_+(x)$ and $A_1 = A_2 = A$, because of Lemma 1 and Lemma 4(b), one has

$$\mathscr{N}'(u - \mathbf{u}, f - \mathbf{f}) \geqslant 0 \text{ for any } (u, f),\ (\mathbf{u}, \mathbf{f}) \in \mathscr{A}_{u_0}.$$

Then part (b) is a particular case of Proposition 2.5(ii) in [2].

To prove part (c), use the definition of a subsolution (resp. supersolution): we have $u \leqslant u_\lambda$ (resp. $v \geqslant v_\lambda$) where $u_\lambda = (I + \lambda\mathscr{A}_{u_0})^{-1}(u + \lambda f)$ (resp. $v_\lambda$) is the mild solution of

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} + Au_\lambda \in f_\lambda = f + \frac{u - u_\lambda}{\lambda},\ u_\lambda(0) = u_0$$

$$\left(\text{resp. } \frac{\mathrm{d}v_\lambda}{\mathrm{d}t} + Av_\lambda \ni g_\lambda = g + \frac{v - v_\lambda}{\lambda},\ v_\lambda(0) = v_0\right).$$

In particular $f_\lambda \leqslant f$ and $g \leqslant g_\lambda$; thus $f_\lambda \leqslant g_\lambda$ and $u_0 \leqslant v_0$ so that, as above, $u_\lambda \leqslant v_\lambda$. This gives $u \leqslant v$ and finishes the proof of the theorem. ■

*Proof of Theorem 2.* Part (a) follows directly from Proposition 6(c) and (b); actually if $u$ is a mild solution of $du/dt + \underline{A}u \ni f$ then the assertion (iii) of Theorem 3 is satisfied; if moreover $u(0) \leqslant u_0$, using Lemma 4(b) and Theorem 1(b), we see that $u$ is a subsolution for $\mathscr{A}_{u_0}u \ni f$. For part (b), (iii) ⇒ (ii) is clear since $A_\rightarrow \subset \underline{A}$, and we already have proved (ii) ⇒ (i); it remains to prove (i) ⇒ (iii).

Assume that $u$ is a subsolution for $\mathscr{A}_{u_0}u \ni f$ and $u(t) \in \overline{D(A)}$ for $t \in [0, T]$. For $\lambda > 0$, set $u_\lambda = (I + \lambda\mathscr{A}_{u_0})^{-1}(u + \lambda f)$; by definition, $u \leqslant u_\lambda$ where $u_\lambda$ is the mild solution of $du_\lambda/dt + Au_\lambda \ni f + (u - u_\lambda)/\lambda$ with $u_\lambda(0) = u_0$. In particular we have $u(0) \leqslant u_0$. We have to prove that $u$ is an exact mild solution of $du/dt + A_\rightarrow u \ni f$.

We first prove that

$$u_\lambda(t) \to u(t) \text{ in } X \text{ for } t \in ]0, T] \text{ as } \lambda \to 0. \tag{30}$$

Fix $t \in ]0, T]$; since $u_\lambda$ is a mild solution of $du_\lambda/dt + Au_\lambda \ni f + (u - u_\lambda)/\lambda$, we have the integral inequalities

$$\frac{d}{dt}\|u_\lambda - x\| \leqslant \left[u_\lambda - x, f + \frac{u - u_\lambda}{\lambda} - y\right] + \omega\|u_\lambda - x\|$$
$$= \left[u_\lambda - x, f - y + \frac{u - x}{\lambda}\right] + (\omega - \lambda^{-1})\|u_\lambda - x\| \text{ in } \mathscr{D}'(]0, T[)$$

for any $(x, y) \in A$.

Applying this with $x = x_\lambda = (I + \lambda A)^{-1}u(t)$, $y = (u(t) - x_\lambda)/\lambda$, and integrating the differential inequality between 0 and $t$, we obtain

$$\|u_\lambda(t) - x_\lambda\| \leqslant e^{(\omega - \lambda^{-1})t}\|u_0 - x_\lambda\| + \int_0^t e^{(\omega - \lambda^{-1})(t-s)}\left[u_\lambda(s) - x_\lambda, f(s) + \frac{u(s) - u(t)}{\lambda}\right] ds$$

and then

$$\|u_\lambda(t) - u(t)\| \leqslant \|u(t) - x_\lambda\| + e^{(\omega - \lambda^{-1})t}\|u_0 - x_\lambda\| + \int_0^t e^{(\omega - \lambda^{-1})(t-s)}\left\{\|f(s)\| + \frac{\|u(s) - u(t)\|}{\lambda}\right\} ds. \tag{31}$$

Since $u(t) \in \overline{D(A)}$, we have $x_\lambda \to u(t)$ in $X$ as $\lambda \to 0$ and then (30) follows from (31).

Now remark that $u_\lambda(t)$ decreases as $\lambda$ decreases (this follows from Proposition 1.1 in [2]); then, thanks to (8), $\|u_\lambda(t) - u(t)\|$ decreases as $\lambda$ decreases. It follows from Dini's theorem, that

$$u_\lambda(t) \to u(t) \text{ in } X \text{ uniformly for } t \in [\delta, T] \text{ as } \lambda \to 0, \text{ for } \delta > 0 \text{ and}$$
$$u_0 = u(0) \Rightarrow u_\lambda(t) \to u(t) \text{ in } X \text{ uniformly for } t \in [0, T] \text{ as } \lambda \to 0. \tag{32}$$

Using this, we prove that for $\delta > 0$, $u$ is an exact mild solution of $du/dt + A_{\rightarrow} u \ni f$ on $[\delta, T]$ and that if $u_0 = u(0)$, $u$ is an exact mild solution of $du/dt + A_{\rightarrow} u \ni f$ on $[0, T]$. This will complete the proof of the theorem; indeed, it will follow that $u$ is a mild solution of $du/dt + \underline{A} u \ni f$ on $[\delta, T]$ for any $\delta > 0$, and then, by basic properties of mild solutions (see [4]), since $u$ is continuous on $[0, T]$, it is a mild solution of $du/dt + \underline{A} u \ni f$ on $[0, T]$; then using part (a) of the Theorem, it is a subsolution of $\mathscr{A}_{u(0)} u \ni f$ and this will finish the proof.

Let $\delta = 0$ if $u_0 = u(0)$ and fix $\delta > 0$ if $u_0 \neq u(0)$. We prove that $u$ is an exact mild solution of $du/dt + A_{\rightarrow} u \ni f$ on $[\delta, T]$. Fix $\varepsilon > 0$ with $\varepsilon\omega < 1$ and, according to (32), consider $\lambda > 0$ such that

$$\|u_\lambda(t) - u(t)\| \leqslant \varepsilon \quad \text{for } t \in [\delta, T]. \tag{33}$$

Let $t_0 = \delta < t_1 < \cdots < t_n \leqslant T < t_{n+1}$ with $t_i - t_{i-1} < \varepsilon$, and $f_1, \ldots, f_n \in X$ with $\Sigma_i \int_{t_{i-1}}^{t_i} \|f(t) - f_i\| \, dt < \varepsilon$. For $i = 1, \ldots, n$, set $g_i = f_i + (u(t_i) - u_\lambda(t_i))/\lambda$ and, using the fact that $A + \omega I$ is $m$-accretive, define by induction $x_0, x_1, \ldots, x_n \in X$ satisfying

$$x_0 = u(\delta), \frac{x_i - x_{i-1}}{t_i - t_{i-1}} + A x_i \ni g_i.$$

Since $g_i \leqslant f_i$, we have

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + A_{\rightarrow} x_i \ni f_i.$$

Set $g = f + (u - u_\lambda)/\lambda$. Since $u$, $u_\lambda$ are continuous on $[\delta, T]$, we have

$$\sum \int_{t_{i-1}}^{t_i} \|g(t) - g_i\| \, dt < \sigma(\varepsilon),$$

where $\sigma$ is a continuous function (depending on $\lambda$, but not on $(t_i, f_i)$) with $\sigma(0) = 0$; we may assume $\sigma(\varepsilon) \geqslant \varepsilon$. Since $u_\lambda$ is an exact mild solution of $du_\lambda/dt + A u_\lambda \ni g$ (see Lemma 2), we have

$$\max_i \left( \max_{t_{i-1} \leqslant t \leqslant t_i} \|u_\lambda(t) - x_i\| \right) \leqslant e^{\omega^+(T-\delta)} \|u_\lambda(\delta) - x_0\| + \eta(\sigma(\varepsilon)),$$

where $\eta$ is some continuous function with $\eta(0) = 0$. Then, using (33), we have

$$\max_i \left( \max_{t_{i-1} \leqslant t \leqslant t_i} \|u(t) - x_i\| \right) \leqslant (1 + e^{\omega^+ T})\varepsilon + \eta(\sigma(\varepsilon))$$

and this concludes the proof. ∎

*Proof of Theorem 3.* Part (a) is already proved:
(i) ⇒ (iii) follows from Proposition 6(c) and (b) and was already proved above (proof of Theorem 2(a));
(ii) ⇒ (iii) follows immediately from Proposition 6(b)
(iii) ⇒ (iv) is immediate as noted in Remark 3
(iv) ⇒ (iii) is exactly the statement of Proposition 6(d)
(iii) ⇒ (v) follows from Lemma 4(b) and Theorem 1(b).
As noted in Remark 3, according to the parts (a) and (b), the part (c) is only a restatement of {(i) ⇔ (ii)} in Theorem 2(b). To complete the proof of Theorem 3, we only have to show that if $u$ is a subsolution for $\mathscr{A}_{u_0}u \ni f$ for some $u_0 \in \overline{D(A)}$ and $u(t) \in \overline{D(A)}$ a.e. $t$, then (ii) holds.

Set $u_\lambda = (I + \lambda\mathscr{A}_{u_0})^{-1}(u + \lambda f)$. By Theorem 4, since $u_\lambda$ is a mild solution of $\mathrm{d}u_\lambda/\mathrm{d}t + Au_\lambda \ni f + (u - u_\lambda)/\lambda$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}N_+(u_\lambda - \mathbf{u}) \leqslant N'_+\left(u_\lambda - \mathbf{u}, f + \frac{u - u_\lambda}{\lambda} - \mathbf{f}\right) + \omega N_+(u_\lambda - \mathbf{u}) \quad \text{in } \mathscr{D}'(]0, T[)$$

for any $(\mathbf{u}, \mathbf{f}) \in \mathscr{X}x\mathscr{X}$ such that (15) holds.

But $u \leqslant u_\lambda$, so that $N'_+(u_\lambda - \mathbf{u}, f + (u - u_\lambda)/\lambda - \mathbf{f}) \leqslant N'_+(u_\lambda - \mathbf{u}, f - \mathbf{f})$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}N_+(u_\lambda - \mathbf{u}) \leqslant N'_+(u_\lambda - \mathbf{u}, f - \mathbf{f}) + \omega N_+(u_\lambda - \mathbf{u}) \quad \text{in } \mathscr{D}'(]0, T[). \tag{34}$$

We obtain (14) at the limit in (34) as $\lambda \to 0$ by using the following Lemma, and this will conclude the proof of the Theorem. ■

LEMMA 5. *Let $(u, f) \in \mathscr{X}x\mathscr{X}$, $u_0 \in \overline{D(A)}$ and for $\lambda > 0$, $u_\lambda = (I + \lambda\mathscr{A}_{u_0})^{-1}(u + \lambda f)$. If $u(t) \in \overline{D(A)}$ a.e. $t$, then $u_\lambda \to u$ in $L^1(0, T; X)$.*

*Proof.* Since $u_\lambda$ is a mild solution of

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} + Au_\lambda \ni f + \frac{u - u_\lambda}{\lambda},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\lambda - x\| \leqslant \left[u_\lambda - x, f + \frac{u - u_\lambda}{\lambda} - y\right] + \omega\|u_\lambda - x\| = \left[u_\lambda - x, f - y + \frac{u - x}{\lambda}\right]$$
$$+ (\omega - \lambda^{-1})\|u_\lambda - x\| \quad \text{in } \mathscr{D}'(]0, T[) \quad \text{for any } (x, y) \in A.$$

Setting $x = x_\lambda(t) = (I + \lambda A)^{-1}u(t)$, $y = (u(t) - x_\lambda(t))/\lambda$, and integrating, one gets

$$\|u_\lambda(t) - u(t)\| \leqslant \|u(t) - x_\lambda(t)\| + e^{(\omega - \lambda^{-1})t}\|u_0 - x_\lambda(t)\| + \int_0^t e^{(\omega-\lambda^{-1})(t-s)}\|f(s)\|\,ds$$
$$+ \int_0^t e^{(\omega-\lambda^{-1})(t-s)}\|u(s) - u(t)\|\frac{ds}{\lambda}. \tag{35}$$

Since $u(t) \in \overline{D(A)}$ a.e. $t$, we have $x_\lambda(t) \to u(t)$ a.e. $t$ as $\lambda \to 0$ and by dominated convergence, $x_\lambda \to u$ in $L^1(0, T; X)$ as $\lambda \to 0$ (note that $\|x_\lambda - (I + \lambda A)^{-1}u_0\| \leqslant (I - \lambda\omega)^{-1}\|u - u_0\|$ and $(I + \lambda A)^{-1}u_0 \to u_0$ in $X$ as $\lambda \to 0$).

It is clear that $\int e^{(\omega-\lambda^{-1})t}\|u_0 - x_\lambda(t)\|\,dt \to 0$ and also that $\int dt \int_0^t e^{(\omega-\lambda^{-1})(t-s)}\|f(s)\|\,ds \to 0$. Using (35), the proof of the lemma will be completed if we show that

$$R_\lambda(u) = \int dt \int_0^t e^{(\omega-\lambda^{-1})(t-s)}\|u(s) - u(t)\|\frac{ds}{\lambda} \to 0 \quad \text{as } \lambda \to 0. \tag{36}$$

One easily sees that if $u \in \mathscr{C}([0, T]; X)$, then $R_\lambda(u) \to 0$ as $\lambda \to 0$; on the other hand, the functionals $R_\lambda$ are equicontinuous seminorms on $\mathscr{X}$; thus, by density, (36) holds for any $u \in \mathscr{X}$. ■

REMARK 5. Lemma 5 is true for any operator $A$ such that $A + \omega I$ is $m$-accretive. It actually means that the closure of $D(\mathscr{A}_{u_0})$ in $\mathscr{X}$ is exactly $\{u \in \mathscr{X};\, u(t) \in \overline{D(A)} \text{ a.e. } t\}$.

## 4. The Linear Case

We discuss the particular situation when (5) holds; in the next Proposition we use the notation and assumptions of Section 1.

PROPOSITION 7. *Assume that $-A$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))$ of bounded positive linear operators on $X$ and denote by $A'$ the adjoint of $A$ in the dual space $X'$. Let $u_0 \in X$ and $f \in L^1(0, T; X)$. Then*

(a) *$A$ is a pregenerator in $X$ and in the notation of Theorem 1, $\mathscr{A}_{u_0}$ is a single-valued pregenerator with dense domain in $\mathscr{X}$.*

(b) *If $u \in L^1(0, T; X)$, then the following assertions are equivalent:*

 (i) *$u$ is subsolution for $\mathscr{A}_{u_0}u = f$*

 (ii) *$u$ satisfies (17) and*

$$\frac{d}{dt}\langle w, u - \mathbf{u}\rangle + \langle A'w, u - \mathbf{u}\rangle \leqslant 0 \text{ in } \mathscr{D}'(]0, T[) \quad \text{for any } w \in D(A') \text{ with } w \geqslant 0 \tag{37}$$

*where $\mathbf{u}(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds$,*

(iii) *for any* $w \in D(A')$ *with* $w \geqslant 0$ *and* $\zeta \in \mathscr{C}^1([0, T])$ *with* $\zeta \geqslant 0$ *and* $\zeta(T) = 0$,

$$\int \langle A'w, u(t)\rangle \zeta(t)\,\mathrm{d}t \leqslant \int \langle w, u(t)\rangle \zeta'(t)\,\mathrm{d}t + \int \langle w, f(t)\rangle \zeta(t)\,\mathrm{d}t. \tag{38}$$

(c) *If* $u \in \mathscr{C}([0, T], X)$, *then the following assertions are equivalent:*

(i) *u is subsolution for* $\mathscr{A}_{u_0} u \ni f$

(ii) $u(0) \leqslant u_0$ *and for any* $\varepsilon > 0$, *there exists a subdivision* $t_0 = 0 < t_1 < \cdots < t_n \leqslant T < t_{n+1}$ *with* $t_i - t_{i-1} < \varepsilon$ *and* $x_0, x_1, \ldots, x_n$, $f_1, \ldots, f_n \in N$ *with* $\dfrac{\langle w, x_i - x_{i-1}\rangle}{t_i - t_{i-1}} + \langle A'w, x_i\rangle \leqslant \langle w, f_i\rangle$ *for any* $w \in D(A')$ *with* $w \geqslant 0$ *such that* $\max_i \left( \max_{t_{i-1} \leqslant t \leqslant t_i} \|u(t) - x_i\| \right) < \varepsilon$, $\sum \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|\,\mathrm{d}t < \varepsilon.$

(iii) $u(0) \leqslant u_0$ *and for some continuous function* $\eta$ *with* $\eta(0) = 0$, *for all sufficiently small* $\varepsilon > 0$, *for all* $t_0 = 0 < t_1 < \cdots < t_n \leqslant T < t_{n+1}$ *with* $t_i - t_{i-1} < \varepsilon$, *and all* $f_1, \ldots, f_n \in X$ *with* $\sum \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|\,\mathrm{d}t < \varepsilon$, *there exists* $x_0 = u(0), x_1, \ldots, x_n \in D(A)$ *such that*

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \leqslant f_i \text{ and } \max_i \left( \max_{t_{i-1} \leqslant t \leqslant t_i} \|u(t) - x_i\| \right) < \eta(\varepsilon).$$

*Proof.* It is clear by the representation of the resolvent by Laplace transform that $A$ is a pregenerator. Notice that, for some $\omega \in \mathbb{R}$, $|||u||| = \sup\{e^{-\omega t}\|S(t)u\|;\ t \geqslant 0\}$ is an equivalent norm on $X$ satisfying (8) and $A + \omega I$ is accretive for this new norm; also if $u \in L^1(0, T; X)$, the property (17) is the same for the two norms. In other words, without loss of generality we may assume that $A + \omega I$ is accretive and then use the results of Theorem 1. In particular $\mathscr{A}_{u_0}$ is a pregenerator in $\mathscr{X}$ (this may actually be shown directly). The density of $D(\mathscr{A}_{u_0})$ in $\mathscr{X}$ follows from Remark 5 (it may be seen directly since $D(\mathscr{A}_{u_0})$ contains $\mathbf{u}_0 + D(A) \otimes \mathscr{D}(]0, T[)$, with $\mathbf{u}_0(t) = S(t)u_0$, and $D(A) \otimes \mathscr{D}(]0, T[)$ is dense in $\mathscr{X}$). Singlevaluedness of $\mathscr{A}_{u_0}$ follows from linearity: if $f, g \in \mathscr{A}_{u_0} u$, then $f - g \in \mathscr{A}_0 0$; but $\mathscr{A}_0$ is an $m$-accretive linear graph in the Banach space $\mathscr{X}$, endowed with the equivalent norm $\int e^{-\omega t}\|u(t)\|\,\mathrm{d}t$; since $D(\mathscr{A}_0)$ is dense in $\mathscr{X}$, singlevaluedness of $\mathscr{A}_0$ follows by classical arguments. This completes the proof of (a).

Part (c) is an immediate application of Theorem 2. Indeed using Proposition 2.3 in [2],

$$(x, y) \in \underline{\mathrm{A}} \Leftrightarrow x, y \in X \text{ and } \langle A'w, x\rangle \leqslant \langle w, y\rangle \text{ for any } w \in D(A') \text{ with } w \geqslant 0,$$

so that (ii) in Proposition 7(c) is exactly (ii) in Theorem 2(b); on the other hand, according to the definitions of $A_{\rightarrow}$ and of an exact mild solution, (iii) in Proposition

7(c) is exactly (iii) in Theorem 2(b).

To prove (b), let us state the following result in a lemma:

LEMMA 6. *Under the assumptions of Proposition 7, $v \in \mathscr{X}_+$ if and only if $v \in \mathscr{X}$ and*

$$\int \langle w, v(t)\rangle \zeta(t)\,\mathrm{d}t \geqslant 0 \text{ for any } \zeta \in \mathscr{D}(]0, T[),\ \zeta \geqslant 0 \text{ and } w \in D(A'),\ w \geqslant 0. \tag{39}$$

*Proof of Proposition 7 continued.* We prove (b). Notice first that for $\lambda > 0$, $(I + \lambda\mathscr{A}_{u_0})^{-1}(u + \lambda f) = \mathbf{u} + (I + \lambda\mathscr{A}_0)^{-1}(u - \mathbf{u})$ and hence, by definition of a subsolution, $u$ is a subsolution for $\mathscr{A}_{u_0}u = f$ if and only if $v = u - \mathbf{u}$ is a subsolution for $\mathscr{A}_0 v = 0$. Notice also (see for instance [5]) that for $\zeta \in \mathscr{D}(]0, T[)$ and $w \in D(A')$

$$\int \langle A'w, \mathbf{u}(t)\rangle \zeta(t)\,\mathrm{d}t = \int \langle w, \mathbf{u}(t)\rangle \zeta'(t)\,\mathrm{d}t + \int \langle w, f(t)\rangle \zeta(t)\,\mathrm{d}t$$

and hence $u$ satisfies (38) if and only if $v = u - \mathbf{u}$ satisfies (38) with $f = 0$. In other words we may assume $u_0 = 0$ and $f = 0$ (so that $\mathbf{u} = 0$). For simplicity we denote $\mathscr{A} = \mathscr{A}_0$.

If $u$ is a subsolution for $\mathscr{A}u = 0$, then $u \leqslant 0$ (use Theorem 1(c)) and hence (17) holds and for any $\lambda > 0$, $u \leqslant u_\lambda = (I + \lambda\mathscr{A})^{-1}u$ and so

$$\int \langle w, (u - u_\lambda)(t)\rangle \zeta(t)\,\mathrm{d}t \leqslant 0 \text{ for any } \zeta \in \mathscr{D}(]0, T[),\ \zeta \geqslant 0 \text{ and } w \in D(A'),\ w \geqslant 0. \tag{40}$$

But if $\zeta \in \mathscr{D}(]0, T[)$ and $w \in D(A')$, the linear functional on $\mathscr{X}$

$$\mathbf{w}: v \longrightarrow \int \langle w, v(t)\rangle \zeta(t)\,\mathrm{d}t$$

is in $D(\mathscr{A}')$ and

$$\langle \mathscr{A}'\mathbf{w}, v\rangle = \int \{\langle A'w, v(t)\rangle \zeta(t) - \langle w, v(t)\rangle \zeta'(t)\}\,\mathrm{d}t.$$

Then, using $u - u_\lambda = \lambda A u_\lambda$, (40) may be written

$$\int \langle A'w, u_\lambda(t)\rangle \zeta(t)\,\mathrm{d}t \leqslant \int \langle w, u_\lambda(t)\rangle \zeta'(t)\,\mathrm{d}t$$

$$\text{for any } \zeta \in \mathscr{D}(]0, T[),\ \zeta \geqslant 0 \text{ and } w \in D(A'),\ w \geqslant 0.$$

Passing to the limit as $\lambda \to 0$, this gives (37). Thus we have shown that (i) implies (ii).

To prove (ii) ⇒ (iii), let $w \in D(A')$, $w \geqslant 0$ and $\zeta \in \mathscr{C}^1([0, T])$ with $\zeta \geqslant 0$ and $\zeta(T) = 0$; using (37) we have

$$\int_s^T \langle A'w, u(t)\rangle \zeta(t)\,\mathrm{d}t \leqslant \langle w, u(s)\rangle \zeta(s) + \int_s^T \langle w, u(t)\rangle \zeta'(t)\,\mathrm{d}t \quad \text{a.e. } s. \tag{41}$$

But $\langle w, u(s)\rangle \leqslant \langle w, u(s) + z\rangle \leqslant \|w\| \|u(s) + z\|$ for $z \in X_+$, and using (17), $\liminf \mathrm{ess}_{s\to 0} \langle w, u(s)\rangle \leqslant 0$; passing to the limit in (41), we obtain (38).

Let us now assume that (iii) holds. By convex combination and density, we have

$$\int \langle A'w(t), u(t)\rangle \, dt \leqslant \int \langle w'(t), u(t)\rangle \, dt \quad \text{for any } w \in \mathscr{Y}_+, \tag{42}$$

where

$$\mathscr{Y}_+ = \{w \in \mathscr{C}^1([0, T]; X');\ w(T) = 0,\ w(t) \in D(A') \text{ for } t \in [0, T],$$
$$A'w \in \mathscr{C}([0, T]; X') \text{ and } \langle w(t), x\rangle \geqslant 0 \quad \text{for } t \in [0, T] \text{ and } x \in X_+\}.$$

But $\mathscr{Y}_+$ may be identified with a convex cone in the dual space $\mathscr{X}'$ contained in $D(\mathscr{A}')$ and if $w \in \mathscr{Y}_+$, $\mathscr{A}'w$ is identified with the continuous function $-dw/dt + A'w$; in other words (42) may be written

$$\langle \mathscr{A}'w, u\rangle \leqslant 0 \quad \text{for any } w \in \mathscr{Y}_+.$$

For any $\lambda > 0$, $(I + \lambda\mathscr{A}')^{-1}(\mathscr{Y}_+) \subset \mathscr{Y}_+$ and applying the inequality above with $(I + \lambda\mathscr{A}')^{-1}w$, one gets

$$\langle w, u - (I + \lambda\mathscr{A}^{-1}u\rangle \leqslant 0 \quad \text{for any } w \in \mathscr{Y}_+.$$

In particular (40) holds, and then thanks to Lemma 6, $u$ is a subsolution for $\mathscr{A}u = 0$. ■

*Proof of Lemma 6.* The necessary condition is clear. Let us assume that (39) holds. We will prove that $v \geqslant 0$. It is clear that we have

$$\langle w, v(t)\rangle \geqslant 0 \text{ a.e. } t, \text{ for any } w \in D(A') \text{ with } w \geqslant 0.$$

Let $w \in X'$ and $w \geqslant 0$; for any $\lambda > 0$, $w_\lambda = (I + \lambda A')^{-1}w$ is in $D(A')$, $w_\lambda \geqslant 0$ and we have $\langle w_\lambda, x\rangle \to \langle w, x\rangle$ as $\lambda \to 0$. It follows that

$$\langle w, v(t)\rangle \geqslant 0 \text{ a.e. } t, \text{ for any } w \in X' \text{ with } w \geqslant 0. \tag{43}$$

Using standard arguments we now show that $v(t) \geqslant 0$ a.e. $t$. Since $v$ is strongly measurable, there exists a countable set $D$ in $X$ such that $v(t) \in \bar{D}$ a.e. $t$; by the Hahn–Banach Theorem, for all $x \in D$ there exists $w_x \in X'$ such that

$$\langle w_x, x\rangle = -N_+(-x),\ w_x \leqslant N_+. \tag{44}$$

In particular $w_x \geqslant 0$, and $D$ being countable, we have

$$\langle w_x, v(t)\rangle \geqslant 0 \quad \text{for any } x \in D, \text{ a.e. } t$$

and thus

$$\langle w, v(t)\rangle \geqslant 0 \quad \text{for any } w \in C', \text{ a.e. } t, \tag{45}$$

where $C'$ is the $\sigma(X', X)$-closed convex cone generated by $\{w_x;\ x \in D\}$. But $v(t) \in \bar{D}$ a.e. $t$ and thus, using (44), we have for a.e. $t$, the existence of $w \in C'$ such that $\langle w, v(t) \rangle = -N_+(-v(t))$. Then, using (45), one obtains $N_+(-v(t)) \leqslant 0$ a.e. $t$, which means $v(t) \geqslant 0$ a.e. $t$. ■

## Acknowledgment

We thank W. Arendt, M. G. Crandall and R. Gariepy for helpful discussions.

## References

1. Barthélemy, L.: Problème d'obstacle pour une équation quasi-linéaire du premier ordre, *Ann. Fac. Sc. Toulous.* **IX**(2), (1988), 137–159.
2. Barthélemy, L., and Bénilan, Ph.: Sous-potentiels d'un opérateur nonlinéaire, *Israel J. Math.* **61**(1), (1988), 85–111.
3. Bénilan, Ph.: Equations d'evolution dans un espace de Banach quelconque et applications, Thèse Orsay (1972).
4. Bénilan, Ph., Crandall, M. G., and Pazy, A.: *Evolution Equations Governed by Accretive Operators*, book in preparation.
5. Bénilan, Ph., Crandall, M. G., and Pazy, A.: 'Bonnes solutions' d'un problème d'évolution semi-linéaire, *C.R.Ac. Sc. Paris* **306**(1), (1988), 527–530.
6. Crandall, M. G.: The semigroup approach to first order quasilinear equations in several space variables, *Israel J. Math.* **12** (1972), 108–122.
7. Crandall, M. G.: Semidifferentials, quadratic forms and fully nonlinear elliptic equations of second order, to appear.
8. Crandall, M. G., and Lions, P. L.: Viscosity solutions of Hamilton Jacobi equation. *Trans. Am. Math. Soc.* **277** (1983), 1–42.
9. Kruskov, S. N.: First order quasilinear equation in several independant variables, *Math. U.R.S.S. Sb.* **10** (1970), 217–243.
10. Lions, P. L.: *Generalized Solutions of Hamilton Jacobi Equations*, Pitman (1984).