Measures on Spaces of Surfaces

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1. Introduction

We define a geometrically natural probability measure on a function space of k dimensional surfaces in $\boldsymbol{R}^n$. This measure generalizes n dimensional circular Brownian motion to an arbitrary smoothness class and its domain to a compact manifold of arbitrary dimension and topological type. Then, with respect to this measure, we prove several geometric results including the following:

(7.1) *Almost every compact, connected, k dimensional $C^{2,\alpha}$ submanifold of $\boldsymbol{R}^n$ bounds a unique $k+1$ dimensional surface of least area.*

(8.9) *If $n > 2k$, almost every C^n map of a fixed compact C^∞ Riemannian manifold of dimension k into $\boldsymbol{R}^n$ is an embedding.*

Our methods combine geometric measure theory, probability theory, partial differential equations, and pseudo-differential operators.

1.1. The measure. There is a unique (up to scaling) measure on the space of continuous real-valued functions on $[0, \infty)$ which vanish at 0 such that

(1) behavior on disjoint intervals is equivalent and independent,

(2) increasing and decreasing are equally likely.

That measure is the celebrated *Brownian motion* [22, pp. 97, 420]. We will be interested in the version of Brownian motion defined on the space

$$C(\boldsymbol{S}^1, \boldsymbol{R}^n)$$

of continuous functions from the circle into $\boldsymbol{R}^n$. Like Brownian motion on the line, this measure can be defined by considering a set of random Fourier series

$$X_0 + \sum_{m=1}^{\infty} X_m m^{-1} \cos mt + Y_m m^{-1} \sin mt \qquad (X_m, Y_m \in \boldsymbol{R}^n) \tag{3}$$

under the measure obtained by giving the coefficients X_m, Y_m independent Gaussian distributions of mean zero and variance one. It follows from work of G. HUNT [25] that

(4) *almost surely the series* (3) *converges uniformly to an α-Hölder continuous function for* $0 < \alpha < \frac{1}{2}$.

The resulting probability measure induced on $C(\boldsymbol{S}^1, \boldsymbol{R}^n)$, called n dimensional Brownian motion on the circle, gives a measure on the space of continuous closed curves in $\boldsymbol{R}^n$. To generalize this measure to k dimensional C^q surfaces, we replace $\boldsymbol{S}^1$ by an arbitrary compact, connected C^∞ Riemannian manifold $\mathscr{M}$ of dimension k, and we replace sines and cosines by eigenfunctions ϕ_m of the Laplacian on $\mathscr{M}$. We consider formal series

$$\sum_{m=1}^{\infty} B_m \phi_m / \beta(m), \qquad B_m \in \boldsymbol{R}^n, \tag{5}$$

where the B_m have independent Gaussian distributions of mean zero and variance one and the weight function

$$\beta\colon \boldsymbol{Z}^+ \to \boldsymbol{R}^+$$

is chosen large enough to insure that almost every series (5) converges in the C^q norm (as can be accomplished trivially). The induced measure μ on $C^q(\mathscr{M}, \boldsymbol{R}^n)$ has many nice geometric properties (cf. Theorem 6.2). It is invariant under rotations of $\boldsymbol{R}^n$, and open sets in the C^q norm are measurable and have positive measure. In particular the subspace $\mathscr{E}$ of embeddings has positive measure, and thus

(6) *there is a measure* $\mu \llcorner \mathscr{E}$ *on the space of compact, connected C^q submanifolds of $\boldsymbol{R}^n$ of dimension k.*

The measure μ on $\mathscr{C}$ could not be translation invariant. Indeed, V. SUDAKOV ([31], presented in [23]) has proved that there is no σ-finite measure on an infinite dimensional, locally convex, Hausdorff linear space under which even the class of measure zero sets is translation invariant. However, for our measure μ, this class is invariant under translations by sufficiently smooth functions (Theorem 6.2(2)).

The construction of the measure μ works as well on the space $C^{q,\alpha}(\mathcal{M}, \boldsymbol{R}^n)$ of functions with α-Hölder continuous q^{th} derivatives. The choice of the weight function β determines the smoothness class of the measure's support quite finely (cf. (6.6)). In particular, we prove that (Theorem 6.3)

(7) *if* $r > (q + \alpha)/k + 1/2$, $c > 0$, *and* $\beta(m) \geqq c m^r$, *then almost surely the series* (5) *converges in* $C^{q,\alpha}(\mathcal{M}, \boldsymbol{R}^n)$,

a generalization of HUNT's result (4) to arbitrary dimension and smoothness class. This theorem leads naturally to a canonical choice for the weight function β, given in (6.4), which fits particularly well into the abstract theory of Gaussian measures as generalized by L. Schwartz (cf. 6.7). Indeed, one outcome of the present paper is to provide interesting applications of such measures.

Remark. A measure provides a discriminating indicator of small sets. Not every first category set, for example, has measure zero. Even on the real line there are open dense sets of arbitrarily small Lebesgue measure (let $q_1, q_2, \ldots$ be an enumeration of the rationals and put $E = \bigcup (q_m - \varepsilon/2^m, q_m + \varepsilon/2^m)$). It has been our experience with Theorem 7.1 that to prove a set has measure zero involves a more *uniform* version of a much simpler first category argument.

1.2. For almost every boundary there is a unique mass minimizing surface. Every compact manifold in $\boldsymbol{R}^n$ bounds a mass minimizing flat chain modulo two, that is roughly, a surface (not necessarily orientable) such that no other has less area. Large families of counterexamples show that such a surface need not be unique (cf. the introduction to [17]). Nevertheless we prove that, with respect to the above measure (1.1 (6)), almost every compact, connected $C^{2,\alpha}$ manifold does bound a *unique* mass minimizing flat chain modulo two (Theorem 7.1). This theorem generalizes our earlier result for two dimensional surfaces in $\boldsymbol{R}^3$ [17, Theorem 7.1]. The largest change in the proof involves a lemma (7.2) on uniqueness for the Cauchy problem for a second order elliptic system of partial differential inequalities with merely C^1 coefficients. This lemma, when applied to the minimal surface system, implies that two unoriented mass minimizing surfaces with the same manifold as boundary, which are tangent C^2 manifolds with boundary in some open subset of the boundary, are identical.

1.3. Transversality and immersions almost everywhere. We shall prove some generic geometric properties of the space $\mathscr{C} = C^q(\mathcal{M}, \boldsymbol{R}^n)$. The most basic arguments of geometric measure theory about dimension and product integration yield our first type of result:

(8.2) *If* $n \geqq 2k$ $(k = \dim \mathcal{M})$ *and* $q \geqq 2$, *then almost every* $B \in \mathscr{C}$ *is an immersion.*

Combining our methods with Sard's theorem proves, in the terminology of 8.1, the next result:

(8.7) *If* $q > 2k$, *then almost every immersion intersects itself transversally.*

The result (8.9) quoted at the outset of the paper follows as an immediate corollary of these two.

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2. Function spaces

We define the two spaces of functions fundamental to our work, the Hölder spaces $C^{q,\alpha}$ and the space L^p_u of Bessel potentials, and state the Sobolev Embedding Theorem 2.4 which relates them. SEELEY's article [10] provides a good introduction to the topics of Chapters 2 and 3.

2.1. Hölder spaces. A *multiindex* α is a k-tuple $(\alpha_1, \ldots, \alpha_k)$ of nonnegative integers. Put $|\alpha| = \Sigma\alpha_i$, and define the differential operator on $\boldsymbol{R}^k$,

$$D^\alpha = \left(-i\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(-i\frac{\partial}{\partial x_k}\right)^{\alpha_k}.$$

Let $\mathscr{M}$ be a C^∞ Riemannian manifold and denote the distance from y to z by $|y - z|$. Let V be $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$, and let $q \in \boldsymbol{Z}^+$, $0 \leqq \gamma \leqq 1$. We denote by

$$C^{q,\gamma}(\mathscr{M}, V),$$

the vectorspace of functions $f: \mathscr{M} \to V$ with γ-Hölder continuous derivatives of order q; i.e., functions f for which there is a constant C such that, for all multiindices α with $|\alpha| = q$ and for all $y, z \in \mathscr{M}$, we have

$$|D^\alpha f(y) - D^\alpha f(z)| \leqq C\,|y - z|^\gamma.$$

The least such C is called the Hölder constant. We sometimes omit reference to γ if $\gamma = 0$ or to V if $n = 1$. A subscript C indicates compact support. $C^{q,\gamma}$ becomes a Banach space under the norm

$$\|f\|_{C^{q,\gamma}} = \sup\{\text{Hölder constant of } |D^\alpha f(y)\|: |\alpha| \leqq q, \quad y \in \mathscr{M}\}.$$

Put $C^\infty = \bigcap_{q \in Z^+} C^q$. Then C^∞ is dense in C^q, but not in $C^{q,\gamma}$ for $\gamma > 0$, although the closure of C^∞ in $C^{q,\gamma}$ contains $C^{r,\beta}$ for $r + \beta > q + \gamma$. $C^{q,\gamma}$ $(0 \leqq \gamma \leqq 1)$ is not locally compact, but closed balls in $C^{q,\gamma}$ are compact under the $C^{r,\delta}$ norm if $q + \gamma > r + \delta$.

2.2. Tempered distributions. Let $\mathscr{S}(\boldsymbol{R}^k)$ denote the set of $f \in C^\infty(\boldsymbol{R}^k, \boldsymbol{C})$ such that for all multiindices α, $m \in \boldsymbol{Z}^+$, the function $|x|^m\,|D^\alpha f(x)|$ is bounded. The neighborhoods of zero in $\mathscr{S}$ are generated by the sets

$$\{f \in \mathscr{S}: |x|^m\,|D^\alpha f(x)| < \varepsilon\}.$$

The Fourier transform $f \mapsto \hat{f}$ given by

$$\hat{f}(\xi) = (2\pi)^{-k/2} \int f(x)\, e^{-ix\cdot\xi}\, dx$$

maps $\mathscr{S}$ onto $\mathscr{S}$. We denote by $\mathscr{S}'$ the space of bounded linear functionals: $\mathscr{S} \to \boldsymbol{C}$, called *tempered distributions.* If g is a measurable function and

$$\int (1 + |x|)^{-m}\, |g(x)|\, dx < \infty$$

for some $m \in \boldsymbol{Z}$, then we have a corresponding linear functional $\tilde{g}$ in $\mathscr{S}'$ given by

$$\tilde{g}(f) = \int fg;$$

we identify g and $\tilde{g}$. The notions of differentiation, Fourier transform, etc., along with their usual properties, extend to the space $\mathscr{S}'$. For example, if $g \in \mathscr{S}'$, one defines

$$\hat{g}(f) = g(\hat{f}) \quad \text{for} \quad f \in \mathscr{S}.$$

The Fourier transform maps $\mathscr{S}'$ onto $\mathscr{S}'$.

2.3. Spaces of Bessel potentials. We follow CALDERON'S early treatment of the spaces L^p_u of Bessel potentials [2], corresponding to the development of Sobolev spaces in ADAMS [1] or STEIN [11, Chapter V].

For $u \in \boldsymbol{R}$, given $f \in \mathscr{S}'$, one defines its *Bessel potential of order u* $J^u f \in \mathscr{S}'$ by

$$(J^u f)^\wedge = (1 + 4\pi^2\, |\xi|^2)^{-u/2} \hat{f}.$$

For $1 < p < \infty$ we define the space of *Bessel potentials*

$$L^p_u(\boldsymbol{R}^k)$$

as the image of $L^p(\boldsymbol{R}^k)$ under J^u, with associated norm

$$\|f\|_{L^p_u} = \|J^{-u} f\|_{L^p}.$$

(1) The spaces L^p_u are complete and isometric with L^p. If $u < v$ then $L^p_u \supset L^p_v$, and for $f \in L^p_v$ we have $\|f\|_{L^p_v} \geqq \|f\|_{L^p_u}$. If $u \in \boldsymbol{Z}^+$, then $f \in L^p_u$ if and only if f has distribution derivatives of orders $\leqq u$ in L^p, and there is a constant $C_{p,u}$ such that

$$C^{-1}_{p,u} \|f\|_{L^p_u} \leqq \sum_{|\alpha| \leqq u} \|D^\alpha f\|_{L^p} \leqq C_{p,u} \|f\|_{L^p_u}. \tag{2}$$

For $u \in \boldsymbol{R}$, the usual duality between L^p and L^q (where $1/q + 1/p = 1$) generalizes to L^p_u and L^q_{-u}, with

$$\|fg\|_{L^1} \leqq \|f\|_{L^p_u} \|g\|_{L^q_{-u}}. \tag{3}$$

2.4. A Sobolev Embedding Theorem. The following results hold [cf. 2, Theorems 6, 5 (e)].

(1) *Suppose* $u \geqq v$ *and*

$$\frac{1}{q}=\frac{1}{p}-\frac{u-v}{k}>0.$$

Then

$$L_u^p \subset L_v^q$$

and the inclusion map is continuous.

(2) *Suppose* q *is a nonnegative integer and*

$$q+1>u-k/p>q.$$

Then

$$L_u^p \subset C^{q,u-q-k/p}$$

and the inclusion map is continuous.

3. Pseudo-differential operators

We introduce a space of pseudo-differential operators, and in particular treat negative and fractional powers of a modified Laplacian on a compact manifold, including various estimates on its eigenfunctions and eigenvalues.

The differential operator on $\boldsymbol{R}^k$

$$P=p(x)\left(-i\frac{\partial}{\partial x^1}\right)^{\alpha_1}\cdots\left(-i\frac{\partial}{\partial x^k}\right)^{\alpha_k}, \quad p\in C^\infty(\boldsymbol{R}^k)$$

satisfies, for any $f\in C_C^\infty(\boldsymbol{R}^k)$ with Fourier transform $\hat{f}$,

$$Pf(x)=(2\pi)^{-k}\int_{\mathbf{R}^k} e^{ix\cdot\xi}\,p(x)\,\xi_1^{\alpha_1}\cdots\xi_k^{\alpha_k}\hat{f}(\xi)\,d\xi.$$

Conversely, given any *symbol* $a(x,\xi)\in C^\infty(\boldsymbol{R}^k\times\boldsymbol{R}^k)$ (e.g. $p(x)\,\xi_1^{\alpha_1}\cdots\xi_k^{\alpha_k}$), we can define an operator $\mathrm{Op}(a)$ such that, for any $f\in C_C(\boldsymbol{R}^k)$,

$$\mathrm{Op}(a)f(x)=(2\pi)^{-k}\int_{\mathbf{R}^k} e^{ix\cdot\xi}\,a(x,\xi)\hat{f}(\xi)\,d\xi.$$

We define pseudo-differential operators as convergent sums of operators arising from symbols which are nearly homogeneous in ξ and often of fractional and negative degree. Thus the class of differential operators is enlarged to include fractional integration. Some authors study larger classes of operators arising from more general symbols. We have chosen the simplest definition consistent with our needs.

3.1. Definition. A *pseudo-differential operator* (or *Calderon-Zygmund operator*) in $\boldsymbol{R}^k$ of order $m\in\boldsymbol{R}$ is an operator

$$A\colon C_C^\infty(\boldsymbol{R}^k)\to C^\infty(\boldsymbol{R}^k)$$

for which there exists a sequence

$$a_m, a_{m-1}, \ldots \in C^\infty(\boldsymbol{R}^k\times\boldsymbol{R}^k)$$

satisfying

$$a_j(x, t\xi) = t^j a_j(x, \xi), \quad t \geqq 1, \quad |\xi| \geqq 1,$$

and such that, for each $l \in \boldsymbol{R}$, $J \in \boldsymbol{Z}$, we have

$$\text{(1)} \qquad \|Af - \sum_{j=0}^{J} \mathrm{Op}(a_{m-j}) f\|_{L^2_{l-m+J+1}} \leqq C \|f\|_{L^2_l},$$

for some constant C and for all $f \in C_C^\infty(\boldsymbol{R}^k)$. We require that these a_j have supports in x within a single compact set. The following lemma gives an example of an L^q type estimate sometimes included in the definition of a pseudo-differential operator.

3.2. Lemma. *Let A be a pseudo-differential operator of order $m \in \boldsymbol{R}$. Then given $q \geqq 2$ and a compact set K, there is a constant C such that for all $f \in C^\infty(\boldsymbol{R}^k)$ with support in K,*

$$\|Af\|_{L^q_{-m}} \leqq C \|f\|_{L^q}.$$

Proof. Let $\Theta \in C^\infty(\boldsymbol{R}^k)$ be such that

$$\Theta(\xi) = 0 \quad \text{if} \quad |\xi| \leqq 1,$$
$$\Theta(\xi) = 1 \quad \text{if} \quad |\xi| \geqq 2.$$

Choose an integer J such that

$$J + 1 \geqq \frac{k}{2}\left(1 - \frac{2}{q}\right).$$

We will view A as a sum

$$\text{(1)} \qquad A = \sum_{j=0}^{J} \mathrm{Op}(\Theta a_{m-j}) + \left(A - \sum_{j=0}^{J} \mathrm{Op}(a_{m-j})\right) + \mathrm{Op}(b)$$

where $b = \sum_{j=0}^{J} (1 - \Theta)\, a_{m-j} \in C^\infty(\boldsymbol{R}^k \times \boldsymbol{R}^k)$ has compact support in both variables. By a result of Seeley [8, Theorem 1], there are constants C_j such that for all $f \in C_C^\infty(\boldsymbol{R}^k)$,

$$\text{(2)} \qquad \|\mathrm{Op}(\Theta a_{m-j}) f\|_{L^q_{-m}} \leqq C_j \|f\|_{L^q_{-j}} \leqq C_j \|f\|_{L_q}.$$

Also, for all $f \in C^\infty(\boldsymbol{R}^k)$ with support in K, we obtain with the help of Theorem 2.4

$$\begin{aligned}
\text{(3)} \qquad & \left\|Af - \sum_{j=0}^{J} \mathrm{Op}\,(a_{m-j}) f\right\|_{L^q_{-m}} \\
& \leqq C' \left\|Af - \sum_{j=0}^{J} \mathrm{Op}\,(a_{m-j}) f\right\|_{L^2_{-m+\frac{k}{2}\left(1-\frac{2}{q}\right)}} \\
& \leqq C'' \|f\|_{L^2_{-(J+1)+\frac{k}{2}\left(1-\frac{2}{q}\right)}} \quad \text{(by 3.1(1))} \\
& \leqq C'' \|f\|_{L^2} \quad \text{(by choice of } J) \\
& \leqq C''' \|f\|_{L^q}.
\end{aligned}$$

Finally, for all $f \in C^\infty(\mathbf{R}^k)$ with support in K,

$$\|\operatorname{Op}(b)f\|_{L^q_{-m}} \leqq \sum_{\substack{|\alpha| \leqq -m+1 \\ \alpha = 0}} \|D_x^\alpha (2\pi)^{-k} \int e^{ix\cdot\xi} b(x,\xi) \hat{f}(\xi)\, d\xi\|_{L^q} \quad \text{(by 2.3(2))}$$

$$\leqq K_1 \sup \left| \int D_x^\alpha (e^{ix\cdot\xi}\, b(x,\xi))\, \hat{f}(\xi)\, d\xi \right|,$$

where the supremum is over $|\alpha| \leqq -m+1$ or $\alpha = 0$, $x \in \mathbf{R}^k$. Hence by Hölder's inequality

$$\text{(4)} \qquad \|\operatorname{Op}(b)f\|_{L^q_{-m}} \leqq K_1 \,[\sup \|D_x^\alpha (e^{ix\cdot\xi}\, b(x,\xi))\|_{L^2}]\, \|\hat{f}\|_{L^2} \leqq K_2 \|f\|_{L^q}.$$

Combining the above results proves the lemma.

3.3. Pseudo-differential operators on manifolds. Let $\mathcal{M}$ be a compact C^∞ manifold of dimension k. By a *local coordinate neighborhood* we mean an open subset of $\mathcal{M}$ identified with an open subset of $\mathbf{R}^k$ by means of a C^∞ bijection with a C^∞ inverse. We say that $A\colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ ia *pseudo-differential operator* of order m on $\mathcal{M}$ if for every local coordinate neighborhood U and for every pair of C^∞ functions ψ, ω supported in U

$$f \mapsto \omega A(\psi f), \qquad f \in C_C^\infty(\mathbf{R}^k),$$

is a pseudo-differential operator of order m on $\mathbf{R}^k$.

3.4. The Laplacian. Let $\mathcal{M}$ be a compact, connected C^∞ Riemannian manifold of dimension k. We denote by Δ the Laplace-Beltrami operator on $\mathcal{M}$, which in normal coordinates at a point is simply $\sum \frac{\partial^2}{\partial x_i^2}$. P is the operator of projection onto the space of constants; that is

$$Pf = \int_{\mathcal{M}} f \Big/ \int_{\mathcal{M}} 1.$$

$P - \Delta$ is a normal, second order, elliptic differential operator with positive eigenvalues $\lambda_1 = 1$, $0 < \lambda_2 \leqq \lambda_3 \leqq \cdots$ corresponding to C^∞ eigenfunctions ϕ_1 (constant), $\phi_2, \phi_3, \ldots,$ which form an orthonormal basis for $L^2(\mathcal{M})$. By a result of HÖRMANDER [4, Theorem 1.1], the function

$$e(x, y, \lambda) = \sum_{\lambda_i \leqq \lambda} \phi_i(x)\, \phi_i(y)$$

satisfies (for some constant C_1)

$$\text{(1)} \qquad e(x, x, \lambda) = C_1 \lambda^{k/2} + O(\lambda^{(k-1)/2}),$$

uniformly in x. Integrating over $\mathcal{M}$ yields

$$\sum_{\lambda_i \leqq \lambda} 1 = C_2 \lambda^{k/2} + O(\lambda^{(k-1)/2}),$$

from which one obtains positive constants c_3, C_3 such that

$$\text{(2)} \qquad c_3 m^{2/k} \leqq \lambda_m \leqq C_3 m^{2/k}.$$

Now by (1) and (2), there is a constant C_4 such that

$$\sum_{m=1}^{M} \|\phi_m\|_{C^0}^2 \leqq C_4 M,$$

so that for $\delta > 0$,

$$\sum_{m=1}^{\infty} \|\phi_m\|_{C^0}^2 \, m^{-1-\delta} < \infty. \tag{3}$$

Using a technique of SEELEY [7] we can define $(P - \Delta)^s$ for negative s as a pseudo-differential operator of order $2s$, via the formula

$$A^s = \frac{i}{2\pi} \int_\Gamma \lambda^s (A - \lambda)^{-1} \, d\lambda, \tag{4}$$

where λ is a complex variable and Γ is the curve beginning at $-\infty$, passing along the real axis to a small circle about the origin, then clockwise about the circle, and back to $-\infty$ along the real axis. Note that

$$(P - \Delta)^s \, \phi_m = \lambda_m^s \phi_m. \tag{5}$$

4. Approximation

We prove that equicontinuous families of functions on the manifold $\mathscr{M}$ can be uniformly approximated by linear combinations of a finite set of eigenfunctions of the Laplacian (Theorem 4.4). In applications this result reduces measure theoretic problems to the manageable finite dimensional case (cf. [17, 1.3]). The proof requires some estimates on norms of the eigenfunctions (Lemma 4.1), which later play a vital role in the construction of measures on function spaces (Chapter 6).

In what follows $\mathscr{M}$ is a compact, connected C^∞ Riemannian manifold of dimension k. As in (3.4), $P - \Delta$ has eigenvalues $\lambda_1 = 1$, $0 < \lambda_2 \leqq \lambda_3 \leqq \ldots$ and corresponding eigenfunctions $\phi_1, \phi_2, \ldots$. Let $\{U_i, 1 \leqq i \leqq L\}$ be a covering of $\mathscr{M}$ by local coordinate neighborhoods identified with disjoint open subsets of $\boldsymbol{R}^k$. Let $\{\chi_i \colon \mathscr{M} \to [0, 1]\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. We identify a function f on $\mathscr{M}$ with the function $\Sigma \chi_i f$ on $\boldsymbol{R}^k$. Notice that, given q, α, there is a constant C such that for all f

$$C^{-1} \|f\|_{C^{q,\alpha}(\boldsymbol{R}^k)} \leqq \|f\|_{C^{q,\alpha}(\mathscr{M})} \leqq C \|f\|_{C^{q,\alpha}(\boldsymbol{R}^k)}. \tag{1}$$

4.1. Lemma. *Given $u \in \boldsymbol{R}$, $p \in (1, \infty)$, q a non-negative integer, $0 \leqq \alpha \leqq 1$, and $\varepsilon > 0$, there are constants C_1, C_2 such that for all $\delta > 0$,*

$$\|\phi_m\|_{L_u^p(\boldsymbol{R}^k)} \leqq C_1 m^{u/k} \|\phi_m\|_{C^0}, \tag{1}$$

and hence $\Sigma \|\phi_m\|_{L_u^p}^2 m^{-1-\delta-2u/k} < \infty$. Moreover

$$\|\phi_m\|_{C^{q,\alpha}(\mathscr{M})} \leqq C_2 m^{(q+\alpha)/k} \|\phi_m\|_{C^0}. \tag{2}$$

Proof. By taking a smaller ε if necessary we can assume that $0 < \alpha < 1$. Now for some constant C_1', we have by (3.4(4)), (3.3), and (3.2),

$$\|\lambda_m^{-u/2}\phi_m\|_{L_u^p} = \|(P-\Delta)^{-u/2}\phi_m\|_{L_u^p} \qquad \text{(by 3.4(5))}$$

$$= \left\|\sum_{i,j=1}^{L} \chi_j(P-\Delta)^{-u/2}\chi_i\phi_m\right\|_{L_u^p} \leqq C_1' \sum_{i=1}^{L} \|\chi_i\phi_m\|_{L^p}$$

$$\leqq C_1'' \sum_{i=1}^{L} \|\chi_i\phi_m\|_{C^0(\mathbf{R}^k)} \quad \text{(for some constant } C_1'')$$

$$\leqq C_1''L\,\|\phi_m\|_{C^0(\mathcal{M})}.$$

Therefore by (3.4(2))

$$\|\phi_m\|_{L_u^p} \leqq LC_1''\lambda^{u/2}\,\|\phi_m\|_{C^0(\mathcal{M})} \leqq C_1 m^{u/k}\,\|\phi_m\|_{C^0},$$

for some constant C_1. The second inequality follows from (3.4(3)), proving (1). Furthermore, if $p > k/\varepsilon$, the Sobolev Embedding Theorem 2.4 shows that there is a constant C_2' such that

$$\|\phi_m\|_{C^{q,\alpha}(\mathbf{R}^k)} \leqq C_2' m^{(q+\alpha)/k}\,\|\phi_m\|_{C^0}.$$

Hence by 4(1), (2) holds for some constant C_2.

4.2. Lemma. *For any* $B \in L^2(\mathcal{M})$, *put*

$$a_m = \int_{\mathcal{M}} B(t)\,\phi_m(t)\,dt,$$

so that in the L^2 *norm*

(1) $$\Sigma a_m\phi_m \to B.$$

Given $\varepsilon > 0$ *and nonnegative integers* $u, r < u - k$, *there is a positive integer* N *such that*

$$\left\|B - \sum_{m=1}^{N} \phi_m\right\|_{C^r} \leqq \varepsilon\,\|B\|_{C^u}.$$

for all $B \in C^u$.

Proof. Choose $\delta > 0$ such that $r < u - k - \delta k$.

$$|a_m| \leqq \int_{\mathcal{M}} |B(t)\,\phi_m(t)|\,dt$$

$$\leqq C_3 \int_{\mathbf{R}^k} |B(t)\,\phi_m(t)|\,dt \quad \text{(for some constant } C_3)$$

$$\leqq C_3\,\|B\|_{L_u^2}\,\|\phi_m\|_{L_{-u}^2} \qquad \text{(by 2.3(3))}$$

$$\leqq C_4\,\|B\|_{C^u(\mathcal{M})}\,\|\phi_m\|_{C^0}\,m^{-u/k}$$

for some constant C_4, where we have used (2.3(2)), (4.1(1)), and (4(1)). Consequently, by (4.1(2)), there is a constant C_5 such that

$$\begin{aligned} \|a_m\phi_m\|_{C^r(\mathcal{M})} &\leqq C_5 \|B\|_{C^u(\mathcal{M})} \|\phi_m\|_{C^0}^2 m^{-u/k} m^{r/k} \\ &\leqq C_5 \|B\|_{C^u(\mathcal{M})} \|\phi_m\|_{C^0}^2 m^{-1-\delta}. \end{aligned}$$

Hence by (3.4(3)), (1) holds in the C^r norm. If we choose N so large that $C_5 \sum_{N+1}^{\infty} \|\phi_m\|_{C^0}^2 m^{-1-\delta} \leqq \varepsilon$, then

$$\left\|B - \sum_1^N a_m\phi_m\right\|_{C^r} \leqq \left\|\sum_{N+1}^{\infty} a_m\phi_m\right\|_{C^r} \leqq \varepsilon \|B\|_{C^u}.$$

4.3. Lemma. *Let H be a set of equicontinuous functions: $\mathcal{M} \to \mathbf{R}^n$ and suppose $u \in \mathbf{Z}^+$. Given $\eta > 0$, we can find a constant C with the property that, for any $h \in H$, there exists an element $\tilde{h} \in C^\infty(\mathcal{M}, \mathbf{R}^n)$ such that*

$$\|h - \tilde{h}\|_{C^0} < \frac{1}{2}\eta \quad \textit{and} \quad \|\tilde{h}\|_{C^u} \leqq C \|h\|_{C^0}.$$

We omit the proof, a standard regularization argument.

4.4. Approximation Theorem. *Let H be a set of equicontinuous functions: $\mathcal{M} \to \mathbf{R}^n$. Then given $\eta > 0$ we can find a positive integer N with the property that, for any $h \in H$, there exist constants $a_1, a_2, \ldots, a_N \in \mathbf{R}^n$ such that*

$$\left\|h - \sum_{m=1}^N a_m\phi_m\right\|_{C^0} \leqq \eta \max\{\|h\|_{C^0}, 1\}.$$

This result follows easily from (4.2) and (4.3).

5. Geometric preliminaries

After presenting the most basic concepts of geometric measure theory, namely measures and rectifiable sets, we introduce the concept of mass minimizing surfaces. In general we follow the notation of FEDERER's treatise [14] and our previous paper [17].

5.1. Measures [24; Sections 7, 10, 11], [14; 2.1.2, 2.2.1, 2.2.3]. A *Borel measure* on a topological space X is a countably additive function

$$\phi: \{\text{Borel subsets}\} \to [0, \infty].$$

Associated to ϕ there is a countably subadditive outer measure defined on all subsets of X, called *Borel regular measure*. Examples of Borel regular measures on $\mathbf{R}^n$ are *Lebesgue measure* $\mathcal{L}^n$ and, for any nonnegative real number m, m dimensional *Hausdorff measure* $\mathcal{H}^m$ [14; 2.10.2]. The *Hausdorff dimension* of a nonempty subset E of $\boldsymbol{R}^n$ is given by

$$\inf\{m \geqq 0 \colon \mathscr{H}^m(E) = 0\}.$$

A measure ϕ is called a *probability* measure if $\phi(X) = 1$.

5.2. Rectifiable sets [14; 3.2.14]. Let $E \subset \boldsymbol{R}^n$ and $m \in \boldsymbol{Z}^+$. Then E is $\mathscr{H}^m$ *rectifiable* if $\mathscr{H}^m(E) < \infty$ and if $\mathscr{H}^m$ almost all of E can be covered by countably many images of bounded subsets of $\boldsymbol{R}^m$ under $C^{0,1}$ maps.

5.3. Lemma. *Let W be an $\mathscr{H}^m$ rectifiable measurable subset of $\boldsymbol{R}^n \cong \boldsymbol{R}^{n-\nu} \times \boldsymbol{R}^\nu$, with $m \geqq \nu$. If $\mathscr{H}^m(W) = 0$, then for $\mathscr{L}^\nu$ almost all $z \in \boldsymbol{R}^\nu$ we have*

$$\mathscr{H}^{m-\nu}\{t \in \boldsymbol{R}^{n-\nu} \colon (t, z) \in W\} = 0.$$

Proof. This is a special case of [14; 3.2.22(3)] with $\mu = \nu$, $Z = \boldsymbol{R}^\nu$, f a projection, and $g = 1$.

5.4. Jacobians [14; 3.2.1]. Suppose $\mathscr{M}$ is a k dimensional C^∞ Riemannian manifold and $f \in C^1(\mathscr{M}, \boldsymbol{R}^n)$. For $a \in \mathscr{M}$ we introduce the mappings

$$Df(a) \colon T_a\mathscr{M} \to \boldsymbol{R}^n,$$

$$\Lambda_k Df(a) \colon \Lambda_k T_a\mathscr{M} \to \Lambda_k \boldsymbol{R}^n,$$

where T_a and Λ_k have the meanings of [14]. Put

$$J_k f(a) = \|\Lambda_k Df(a)\|.$$

Then the area formula [14; 3.2.22] holds:

$$\text{(1)} \qquad \int_{\mathscr{M}} J_k f(x)\, d\mathscr{H}^k(x) = \int_{\boldsymbol{R}^n} \operatorname{card}(f^{-1}(y))\, d\mathscr{H}^k(y).$$

5.5. Mass minimizing surfaces [14; 4.1.24, 4.2.26]. In $\boldsymbol{R}^n$ we introduce the space $\boldsymbol{I}_{k+1}(\boldsymbol{R}^n)$ of oriented $k+1$ dimensional surfaces, called *integral currents*, and the corresponding space $\boldsymbol{I}^2_{k+1}(\boldsymbol{R}^n)$ of unoriented surfaces, called *flat chains modulo two*. There is a continuous boundary operator

$$\partial \colon \boldsymbol{I}_{k+1} \to \boldsymbol{I}_k,$$

$$\partial \colon \boldsymbol{I}^2_{k+1} \to \boldsymbol{I}^2_k$$

and a lower semicontinuous mass function

$$\boldsymbol{M} \colon \boldsymbol{I}_{k+1} \to [0, \infty),$$

$$\boldsymbol{M} \colon \boldsymbol{I}^2_{k+1} \to [0, \infty)$$

which gives the area or volume of the surfaces counting multiplicities [14; 4.1.7]. Associated with any surface $S \in \boldsymbol{I}_{k+1}$ (or $\boldsymbol{I}^2_{k+1}$) there is the set $\operatorname{spt} S \subset \boldsymbol{R}^n$. The surface $S \in \boldsymbol{I}_{k+1}(\boldsymbol{I}^2_{k+1})$ is *mass minimizing* if, whenever $S' \in \boldsymbol{I}_{k+1}(\boldsymbol{I}^2_{k+1})$ and $\partial S' = \partial S$, we have

$$\boldsymbol{M}(S') \geqq \boldsymbol{M}(S).$$

If $B \in I_k(I_k^2)$ and $\partial B = 0$, then there is a mass minimizing surface $S \in I_{k+1}(I_{k+1}^2)$ with $\partial S = B$. If a portion of spt S is a C^2 manifold viewed as the graph of a C^2 function

$$u: U \to R^{n-k},$$

where U is a domain in R^k, then u satisfies the *minimal surface system* [19, Theorem 2.2]

$$\sum_{i,j=1}^{k} G^{ij} u_{ij} = 0, \tag{1}$$

where subscripts on u denote partial differentiation, $g_{ij} = \delta_{ij} + u_i \cdot u_j$, and G^{ij} is the corresponding matrix of cofactors.

5.6. Interior regularity [15], [13], [6, Theorem 6.7.6]. If S is a mass minimizing surface in R^n, then spt S — spt ∂S is an analytic manifold except for a set E where

(1) if $S \in I_{k+1}^2$, then Hausdorff dimension $E \leqq k - 1$,
(2) if $S \in I_{k+1}$, then $\mathscr{H}^{k+1}(E) = 0$, and
(3) if $S \in I_{n-1}$, then Hausdorff dimension $E \leqq n - 8$.

5.7. Boundary Regularity Theorem. *Suppose that*

(1) $S \in I_{k+1}$ *(or* I_{k+1}^2*) is mass minimizing,*
(2) ∂S *corresponds to a* $C^{q,\alpha}$ *manifold* $\mathscr{M}$ *with* $q \geqq 2$, $0 < \alpha < 1$, *and*
(3) spt S *is a* $C^{1,\alpha}$ *manifold with boundary at a point* $b \in \mathscr{M}$.

Then spt S *is a* $C^{q,\alpha}$ *manifold with boundary at* b.

Proof. Locally we may view the mass minimizing surface as the graph of a function u satisfying the minimal surface system (5.5(1)) at interior points. In particular each component of u solves an associated linear Dirichlet problem which arises by viewing the G^{ij} as given $C^{0,\alpha}$ functions. We infer from the estimates of Schauder [5, Chapter 3, Theorems 1.1, 1.2] that u is of class $C^{2,\alpha}$. In turn G^{ij} is of class $C^{1,\alpha}$, and hence u is $C^{3,\alpha}$, etc.

Remark. By a result of ALLARD [12; 5.2, 4], conditions (1) and (2) imply (3) for certain $b \in \mathscr{M}$, including all b in a neighborhood of the point of $\mathscr{M}$ farthest from the origin (cf. [17, 6.2]).

6. Measures on spaces of surfaces

We give a very simple construction of a class of probability measures on the space $\mathscr{C}$ of k dimensional surfaces in R^n, which generalize n dimensional Brownian motion to an arbitrary smoothness class and its domain to a compact manifold of arbitrary dimension. The useful properties of these measures include the positivity of every open set and a product decomposition. More delicate estimates (Theorem 6.3) select a canonical measure (6.4), which fits particularly well into the theory of abstract Gaussian measures as generalized by L. SCHWARTZ (6.7). Finally we present in (6.8) another useful notion of "sets of measure zero" due to J. P. R. CHRISTENSEN.

6.1. Construction of the measure. Let n, k be positive integers. Let $\mathscr{M}$ be a compact, connected, k dimensional C^∞ Riemannian manifold, let Δ denote the Laplacian on $\mathscr{M}$, and let P denote projection onto the space of constants. Let ϕ_1 (constant), $\phi_2, \phi_3, \ldots$ be the eigenfunctions of $P-\Delta$, normalized with $\|\phi_m\|_{L^2}=1$, with corresponding eigenvalues $\lambda_1 = 1$, $0<\lambda_2 \leq \lambda_3 \leqq \ldots$. Finally, let $0 \leqq \alpha \leqq 1$, and let q be a nonnegative integer (which in geometric applications will be at least 2). We consider as the space of surfaces $\mathscr{C}$ the closure of C^∞ in $C^{q,\alpha}(\mathscr{M}, \boldsymbol{R}^n)$. $\mathscr{C}$ is a separable Banach space since by (4.2) finite rational linear combinations of the ϕ_m are dense. If $\alpha = 0$ then $\mathscr{C} = C^q$; but if $\alpha > 0$,

$$C^{q,\alpha} \supsetneqq \mathscr{C} \supset \bigcup_{r+\gamma>q+\alpha} C^{r,\gamma}.$$

Let $\mathscr{L}^n$ denote Lebesgue measure on $\boldsymbol{R}^n$. Define the Gaussian probability measure $\mathscr{G}^n$ on Borel sets E by

$$\mathscr{G}^n(E) = (2\pi)^{-n/2} \int_E e^{-|x|^2/2}\, d\mathscr{L}^n x.$$

Note that

1) $$E(x^2) \equiv \int x^2\, d\mathscr{G}^n x = n.$$

We may introduce the product Borel measure $\Pi\mathscr{G}^n$ on $\prod_{m=1}^{\infty} \boldsymbol{R}^n$ (cf. [24, Chapter 7]). Choose a weight function

$$\beta\colon \boldsymbol{Z}^+ \to \boldsymbol{R}^+$$

such that, for $\Pi\mathscr{G}^n$ almost all $(B_1, B_2, \ldots) \in \Pi\boldsymbol{R}^n$,

2) $$\Sigma B_m \phi_m/\beta(m)$$

converges in $\mathscr{C}$ (or more generally such that the image of the map Υ below has measure one). This can be accomplished easily by taking $\beta(m) = \|\phi_m\|_{C^{q,\alpha}}\, m^2$; cf. (6.3) and (6.6). Define the continuous linear injection

$$\Upsilon\colon \mathscr{C} \to \Pi\boldsymbol{R}^n,$$

$$\Upsilon\colon B \to (B_1, B_2, \ldots),$$

where $B_m = \beta(m) \int_{\mathscr{M}} B(t)\, \phi_m(t)\, dt$. By (1),

$$\Pi\mathscr{G}^n(\text{image } \Upsilon) = 1.$$

Since Υ gives an isomorphism between the Borel σ-algebras of $\mathscr{C}$ and $\Upsilon(\mathscr{C})$ [14; 2.2.10], the measure $\Pi\mathscr{G}^n$ induces a Borel probability measure μ on $\mathscr{C}$. The measure has the following nice properties:

6.2. Theorem. *The following results hold*:

(1) *μ is invariant under rotations of* $\boldsymbol{R}^n$.

(2) *Let* $f \in \mathscr{C}$ *and suppose* Υf *is square summable (such functions are dense by* 4.4). *Then the translation of* μ *by* f *is absolutely continuous with respect to* μ.

(3) *Open sets have positive* μ *measure.*

Proof. (1) holds for μ because it holds for $\mathscr{G}^n$. One verifies (2) by applying KAKUTANI's criterion for the absolute continuity of infinite product measures [27]. To prove (3), consider the countable set F of finite rational linear combinations of the eigenfunctions ϕ_m of the Laplacian Δ. If (3) fails, there is an $\varepsilon > 0$ and a function $\phi \in F$ such that

$$(4) \qquad \mu\{f \in \mathscr{C} : \|f - \phi\|_{C^{q,\alpha}} < \varepsilon\} = 0.$$

By (2) it follows that (4) holds for all $\phi \in F$, and we have a countable cover of $\mathscr{C}$ by sets of measure zero. This contradiction of the condition $\mu(\mathscr{C}) = 1$ proves (3).

The product decomposition. As in [17, 4.4], we can view $\mathscr{C}$ as a product of measure spaces

$$(\mathscr{C}, \mu) = (\mathscr{C}_N, \mu_N) \times (\mathscr{C}^N, \mu^N),$$

where

$$\mathscr{C}_N = \left\{ \sum_{m=1}^{N} a_m \phi_m : a_m \in \mathbf{R}^n \right\},$$

$$\mathscr{C}^N = \{B \in \mathscr{C} : B_1 = \ldots = B_N = 0\}.$$

We make the key observation that, by Fubini's Theorem,

(5) *a Borel subset* $E \subset \mathscr{C}$ *has measure zero if, for some finite set of natural numbers* $\{m_1, m_2, \ldots, m_N\}$ *and for all* $B \in \mathscr{C}$, *we have*

$$\mathscr{L}^{nN}\left\{(a_1, \ldots, a_N) \in \mathbf{R}^{nN} : B + \sum_{i=1}^{N} a_i \phi_{m_i} \in E\right\} = 0.$$

This result will prove to be a powerful tool in the applications of Chapters 7 and 8.

6.3. Theorem. *Let* q *be a nonnegative integer, let* $0 \leqq \alpha \leqq 1$, $c > 0$, $r > (q + \alpha)/k + 1/2$, *and suppose* $\beta(m) \geqq cm^r$. *Then for* $\Pi\mathscr{G}^n$ *almost all* $(B_1, B_2, \ldots) \in \Pi\mathbf{R}^n$,

$$(1) \qquad \Sigma B_m \phi_m / \beta(m)$$

converges in $\mathscr{C}$.

Proof. Choose $1 < p < \infty$, $u \in \mathbf{R}$ such that

$$q + \alpha + k/p < u < k(r - 1/2).$$

Then by the Sobolev Embedding Theorem (2.4) and (4(1)), it suffices to establish convergence in $L^p_u(\mathbf{R}^k)$. By a theorem of DELPORTE [21, Cor. 6.4B], which applies to L^p spaces and hence by (2.3(1)) to L_u^p spaces, this is the case if we can show that

$$\sum_{m=1}^{\infty} \boldsymbol{E}\| B_m\phi_m/\beta(m)\|_{L_u^p}^2 < \infty,$$

or equivalently by (6.1(1)), that

$$\Sigma(\|\phi_m\|_{L_u^p}/\beta(m))^2 < \infty.$$

But this follows from (4.1(1)) because $\beta(m) \geqq cm^r$ and $u < k(r - 1/2)$.

Remark. If $\mathscr{M}$ is real analytic, so are the eigenfunctions ϕ_m, and $\beta(m) = e^{-m}$ produces a measure on the space of analytic maps: $\mathscr{M} \to \boldsymbol{R}^n$.

6.4. The canonical measure. A canonical choice for the measure μ is given by

$$\beta(m) = \lambda_m^s$$

where $s > (2q + 2\alpha + k)/4$. It is clear from (6.3) and (3.4(2)) that B satisfies the condition (6.1(2)). In this case the map Υ is given by

$$B_m = \int f(P - \Delta)^s \phi_m,$$

which is well defined for any smooth orthonormal basis $\{\phi_m\}$ for $L^2(\mathscr{M})$. *The induced measure is independent of the choice of basis*, as may be verified by a direct probability argument or by more abstract methods (cf. e.g. (6.7) below).

6.5. Brownian motion. We note that the canonical measure (6.4) generalizes circular Brownian motion to an arbitrary smoothness class and its domain to a compact manifold of arbitrary dimension. Indeed, one obtains the trigonometric expansion for Brownian motion on the circle (cf. [17, 4.7])

$$\pi^{1/2}B(t) = X_0 + \sum_{m=1}^{\infty} X_m m^{-1} \cos mt + Y_m m^{-1} \sin mt \tag{1}$$

by taking $\mathscr{M} = S^1$, with eigenfunctions of $P - \Delta$ $\{1, \cos mt, \sin mt\}$, eigenvalues $\{1, m^2, m^2\}$, and $q = 0$, $s = 1/2$, $\alpha < 1/2$. (The usual Brownian motion on the real line involves an inconsequential modification.) Theorem 6.3 gives the sharp result that the series (1) converges almost surely in the $C^{0,1/2-\varepsilon}$ norm $(0 \leqq \varepsilon < 1/2)$, stronger than the usual statement that the series converges uniformly to a $C^{0,1/2-\varepsilon}$ function, almost surely [25].

We remark that, taking as the domain of our process a compact manifold rather than a Euclidean space, imposes an additional demand that the process fit together globally: e.g., circular Brownian motion must return to its origin after 2π. For such a construction local methods do not suffice.

6.6. The weight function β. As indicated by Theorem 6.3, the choice of the weight function β is intimately related to the smoothness class on which the resulting measure is concentrated, often quite finely. We have already mentioned that Brownian motion is concentrated on functions which are $(\frac{1}{2} - \varepsilon)$-Hölder continuous for all $\varepsilon > 0$. In fact, it is known that (putting $\log\log = \log_2$, etc.)

$$|B(t) - B(0)| \leqq \sqrt{2}\, t^{\frac{1}{2}} \sqrt{\log_2 \frac{1}{t} + \frac{3}{2} \log_3 \frac{1}{t} + \log_4 \frac{1}{t} + \dots + (1 + \delta) \log_n \frac{1}{t}}$$

for small t, almost surely if $\delta > 0$ and almost never if $\delta = 0$! (cf. [26, 1.8]). Of course, for geometric problems, the wide dispersal of the measure in a geometric sense (6.2(3)) is of first importance. Nevertheless, it is satisfying to know that, as we would expect, the geometric measure theoretic results of Chapters 7 and 8 are independent of the precise smoothness class, in that they hold for arbitrary admissible choice of the weight function β.

6.7. Abstract Gaussian measures. The canonical measure (6.4) fits particularly well into the theory of abstract Gaussian measures (cf. [28]) as generalized by L. SCHWARZ. Given $\mathscr{C}$ as before, choose $s > (2q + 2\alpha + k)/4$ as in (6.4). Now choose $0 < \delta < 1/2$, $1 < p < \infty$ such that

$$v \equiv 2s - k/2 - \delta > q + \alpha + k/p.$$

Consider the composite continuous linear map

$$\begin{aligned} u\colon L^2(\mathscr{M}, \boldsymbol{R}^n) &\xrightarrow{(P-\Delta)^{-k/4-\delta}} L^\infty \hookrightarrow L^p \\ &\xrightarrow{(P-\Delta)^{-v/2}} L^p_v \hookrightarrow \mathscr{C}. \end{aligned}$$

By (3.4(4)) and (3.2), $(P - \Delta)^{-v/2}$ maps L^p into L^p_v, and $(P - \Delta)^{-k/4-\delta}$ maps L^2 into $L^2_{k/2+2\delta} \subset C^{0,2\delta} \subset L^\infty$. The inclusion $L^p_v \hookrightarrow \mathscr{C}$ follows from the Sobolev Embedding Theorem 2.4. Now according to the terminology and theory of SCHWARTZ [30], the map u is p-summing because the inclusion $L^\infty \hookrightarrow L^p$ is p-summing and the composition of continuous linear maps is p-summing if one of them is. Therefore [30, Theorem 2.3, Proposition 2.7, Proposition 2.8] Gaussian cylindrical probability Γ on L^2 induces a Radon measure μ' of order p on $\mathscr{C}$.

We claim that μ' equals the canonical measure μ of (6.4). To prove this we consider the composite map

$$\begin{aligned} \Upsilon \circ u\colon L^2 &\to \Pi \boldsymbol{R}^n \\ : \phi_m &\to (0, 0, \dots, 1_m, \dots). \end{aligned}$$

One notes that, via $\Upsilon \circ u$, Γ induces the measure $\Pi \mathscr{G}^n$ on $\Pi \boldsymbol{R}^n$ $\left(\text{it suffices to verify equality on Borel sets of the form } E \times \prod_{M+1}^{\infty} \boldsymbol{R}^n, \text{ where } E \subset \prod_{1}^{M} \boldsymbol{R}^n\right)$. Thus $\mu' = \mu$.

The abstract approach of SCHWARTZ shows immediately that the canonical measure μ of (6.4) is independent of the choice of a smooth basis for L^2. It also implies that μ has order p for all $p < \infty$, namely

$$\int_{\mathscr{C}} \|f\|^p_{C^{q,\alpha}}\, d\mu < \infty.$$

6.8. Christensen zero sets. J. P. R. CHRISTENSEN [20] has proposed for any abelian "Polish" group G a good general notion of a "set of measure zero", although there is no associated measure unless G is locally compact.

Definitions. A topological group G is called *Polish* if G is a complete, separable metric space. A Borel subset A of an abelian Polish group G is a *Christensen zero set* if there is a Borel probability measure ν on G such that every translate of A has ν measure zero. A property which holds outside a Christensen zero set is said to hold *almost everywhere.*

Theorem [20]. *The following results hold*:

(1) *The class of Christensen zero sets is translation invariant.*
(2) *If G is locally compact, then the Christensen zero sets are precisely the sets of Haar measure zero.*
(3) *A countable union of Christensen zero sets is a Christensen zero set.*
(4) (*A partial Fubini Theorem*). *Suppose $G = H \times T$ is the product of two abelian Polish groups, T is compact, and A is a Borel subset of G. Then A is a Christensen zero set if and only if the Haar measure of*

$$\{(h, t) \in A : h = h_0\}$$

is zero for almost all $h_0 \in H$.

Remark. Although we know of no general relationship between Christensen zero sets and our sets of measure zero (6.1), properties (2)–(4) suffice for the proofs of this paper. *All the results of this paper hold as well when "almost all" is understood in the sense of Christensen zero sets.*

7. Mass minimizing surfaces

As the first application of our measure on the space of surfaces we prove that almost every k dimensional, compact, connected manifold in $\boldsymbol{R}^n$ bounds a *unique* unoriented mass minimizing surface ("Theorem" 7.1), notwithstanding the examples of families of manifolds which bound two or more mass minimizing surfaces (cf. [17, Intro.]). This theorem generalizes our earlier result for two dimensional surfaces in $\boldsymbol{R}^3$ [17, "Theorem" 7.1]. The largest change in the proof involves a lemma (7.2) on uniqueness for the Cauchy problem for a second order elliptic system of partial differential inequalities with merely C^1 coefficients, to be applied to the minimal surface system.

Let n, k be positive integers with $k \leqq n - 2$. Let $\mathscr{M}$ be a compact, connected, k dimensional C^∞ Riemannian manifold, $q \in \boldsymbol{Z}$, $0 \leqq \alpha \leqq 1$, $q + \alpha > 2$. As our space of boundaries we consider the set $\mathscr{E}$ of embeddings

$$\mathscr{E} = \{B \in C^{q,\alpha}(\mathscr{M}, \boldsymbol{R}^n) : J_k B > 0 \quad \text{and} \quad B(s) = B(t) \Rightarrow s = t\}.$$

Since $\mathscr{E}$ is open, (6.1) gives a nonzero measure μ on $\mathscr{E}$. Associated to every $B \in \mathscr{E}$ is a flat chain modulo two and, if $\mathscr{M}$ is oriented, an integral current, both of which we shall also denote by B. The following theorem extends our previous results for mass minimizing hypersurfaces in $\boldsymbol{R}^3$ [17, "Theorem" 7.1] to arbitrary dimension and codimension.

7.1. Theorem. *Almost every* $B \in \mathscr{E}$ *bounds a unique mass minimizing flat chain modulo two of dimension* $k+1$. *If* $\mathscr{M}$ *is oriented and* $k = n-2$, *then almost every* $B \in \mathscr{E}$ *bounds a unique mass minimizing integral current.*

Remark. The restriction to hypersurfaces for integral currents stems from the current lack of regularity results in higher codimensions (cf. (5.6)). To remove the restriction, it would be enough to know that the set of interior singular points of a $k+1$ dimensional mass minimizing integral current (where its support fails to be a C^∞ manifold) has $\mathscr{H}^k$ measure zero.

Proof sketch. In general, the proof proceeds as in [17, 1.2, Chapt. 7]. Some more judicious approximations center on the new Approximation Theorem 4.4 (cf. [17, 4.6]), but the major change occurs in the use of partial differential equation theory to show that (cf. [17, "Theorem" 5.1])

(1) *two mass minimizing surfaces with the same manifold as boundary, which are tangent* C^2 *manifolds with boundary in some open subset of the boundary, coincide locally.*

The old technique, employing the Legendre transformation, does not generalize to higher dimensions or codimensions, while the more general alternate method, following Aronszajn, requires stronger smoothness assumptions.

The lemma below, a modification of work of Hörmander, when applied to the minimal surface system (5.51), implies (1). Then, for flat chains modulo two, global equality follows by regularity (cf. the remark above).

7.2. Lemma (cf. [3, Thm. 8.9.1]). *Let* P *be a second order elliptic differential operator with real* C^1 *coefficients in a neighborhood* Ω *of a point* x^0. *Let* $\psi \in C^2(\Omega)$ *have a nonvanishing gradient. Then there is a neighborhood* Ω_1 *of* x^0 *with the following property: if* $u^{(1)}, u^{(2)}, \ldots, u^{(l)} \in C^2(\Omega)$ *vanish on* $\{x \in \Omega\colon \psi(x) > \psi(x^0)\}$ *and satisfy*

$$|Pu^{(j)}| \leqq K_1 \sum_{\substack{|\alpha| \leqq 1 \\ 1 \leqq i \leqq l}} |D^\alpha u^{(i)}| \quad (1 \leqq j \leqq l) \tag{1}$$

for some fixed constant K_1, *then* $u^{(j)} \mid \Omega_1 = 0$ $(1 \leqq j \leqq l)$.

Proof. First we note that (1) implies

$$|Pu^{(j)}|^2 \leqq l K_1^2 \sum_{\substack{|\alpha| \leqq 1 \\ 1 \leqq i \leqq l}} |D^\alpha u^{(i)}|^2 \quad (1 \leqq j \leqq l). \tag{2}$$

Moreover, any C^2 hypersurface is strongly pseudoconvex for a real elliptic differential operator of order two. Hence Hörmander's theorem [3, 8.9.1] and proof apply. As in that proof, we obtain a neighborhood Ω' of x^0, a function $\phi \in C^\infty(\Omega')$, and an open set ω, with the property

(3) $$\{x \in \Omega' - \{x^0\}: \phi(x) \geqq \phi(x^0)\} \subset \omega \subset \{x \in \Omega': \psi(x) > \psi(x^0)\},$$

and also such that for any $v \in C_C^2(\Omega')$ and any τ large the Carleman estimate

(4) $$\tau \sum_{|\alpha| \leqq 1} \int_{\Omega'} |D^\alpha v|^2 e^{2\tau\phi} \leqq K_2 \int_{\Omega'} |Pv|^2 e^{2\tau\phi}$$

holds.

Choose $\chi \in C_C^\infty(\Omega')$ such that χ equals 1 in a neighborhood Ω'' of x^0, and put $v^{(j)} = \chi u^{(j)} \mid \Omega'$ $(j = 1, 2, \ldots, l)$. Then $v^{(j)} \in C^2(\Omega')$ and $v^{(j)} = u^{(j)}$ in $\omega \cup \Omega''$, this set containing $\{x \in \Omega': \phi(\chi) \geqq \phi(x^0)\}$. Hence there is an $\varepsilon > 0$ such that if $\Omega_1 = \{x \in \Omega': \phi(x) > \phi(x^0) - \varepsilon\}$ then $\Omega_1 \subset \omega \cup \Omega''$ and $u^{(j)} \mid \Omega_1 = v^{(j)} \mid \Omega_1$. Summing (4) over all the $v^{(j)}$'s yields

$$\begin{aligned}
\tau \sum_{\substack{|\alpha| \leqq 1 \\ 1 \leqq j \leqq l}} \int_{\Omega_1} |D^\alpha v^{(j)}|^2 e^{2\tau\phi}
&\leqq K_2 \sum_{1 \leqq j \leqq l} \int_{\Omega_1} |Pv^{(j)}|^2 e^{2\tau\phi} + K_2 \sum_{1 \leqq j \leqq l} \int_{\Omega' - \Omega_1} |Pv^{(j)}|^2 e^{2\tau\phi} \\
&\leqq l^2 K_1^2 K_2 \sum_{\substack{|\alpha| \leqq 1 \\ 1 \leqq j \leqq l}} |D^\alpha v^{(j)}|^2 e^{2\tau\phi} + K_2 \sum_{1 \leqq j \leqq l} \int_{\Omega' - \Omega_1} |Pv^{(j)}|^2 e^{2\tau\phi},
\end{aligned}$$

by (2), since $v^{(j)} \mid \Omega_1 = u^{(j)} \mid \Omega_1$. Thus if $\tau > 2l^2 K_1^2 K_2$,

$$(\tau - l^2 K_1^2 K_2) \sum_{\substack{|\alpha| \leqq 1 \\ 1 \leqq j \leqq l}} \int_{\Omega_1} |D^\alpha v^{(j)}|^2 e^{2\tau(\phi(x^0) - \varepsilon)} \leqq K_2 \sum_{1 \leqq j \leqq l} \int_{\Omega' - \Omega_1} |Pv^{(j)}|^2 e^{2\tau(\phi(x^0) - \varepsilon)},$$

and hence

$$\sum_{\substack{|\alpha| \leqq 1 \\ 1 \leqq j \leqq l}} \int_{\Omega_1} |D^\alpha v^{(j)}|^2 \leqq 2K_2 \tau^{-1} \sum_{1 \leqq j \leqq l} \int_{\Omega' - \Omega_1} |Pv^{(j)}|^2.$$

Letting $\tau \to \infty$, we conclude that $u \mid \Omega_1 = v \mid \Omega_1 = 0$.

8. Transversality and immersions almost everywhere

As a second application of our measure on the function space $\mathscr{C} = C^q(\mathscr{M}, \mathbf{R}^n)$ we prove some generic results about embeddings, immersions, and transversality. For example we shall obtain the following conclusions.

(8.2) *If* $n \geqq 2k$ *(where* $k = \dim \mathscr{M}$*) and* $q \geqq 2$, *then almost every* $B \in \mathscr{C}$ *is an immersion.*

(8.9) *If* $n > 2k$ *and* $q > 2k$, *then almost every* $B \in \mathscr{C}$ *is an embedding.*

As usual, corresponding transversality results are based on Sard's Theorem.

8.1. Definitions. Let n, k, q be positive integers, with $q \geqq 2$. Let $\mathscr{M}$ be a compact, connected C^∞ Riemannian manifold of dimension k. Put

$$\mathscr{C} = C^q(\mathscr{M}, \mathbf{R}^n),$$

with measure μ as defined in (6.1). Let $\mathscr{I}$ be the subspace of immersions

$$\mathscr{I} = \{B \in \mathscr{C}: J_k B > 0\}$$

and let $\mathscr{E}$ be the subspace of embeddings

$$\mathscr{E} = \{B \in \mathscr{I}: B(s) = B(t) \Rightarrow s = t\}.$$

Now suppose $\mathscr{N}$ is a manifold and $F \in C^1(\mathscr{N}, \mathbf{R}^m)$. An element $a \in \mathbf{R}^m$ is a *regular value* of F if $DF(t)$ is surjective whenever $F(t) = a$. Two submanifolds $\mathscr{N}_1$, $\mathscr{N}_2$ of $\mathbf{R}^n$ *intersect transversally* ($\mathscr{N}_1 \pitchfork \mathscr{N}_2$) if $T_x\mathscr{N}_1 + T_x\mathscr{N}_2 = \mathbf{R}^n$ whenever $x \in \mathscr{N}_1 \cap \mathscr{N}_2$. A submanifold $\mathscr{N}$ of $\mathbf{R}^n$ *intersects* $B \in \mathscr{C}$ *transversally* ($\mathscr{N} \pitchfork B$) if $\operatorname{im} DB(t) + T_{B(t)}\mathscr{N} = \mathbf{R}^n$ whenever $B(t) \in \mathscr{N}$. An immersion $B \in \mathscr{I}$ *intersects itself transversally* ($B \pitchfork B$) if $\operatorname{im} DB(s) + \operatorname{im} DB(t) = \mathbf{R}^n$ whenever $B(s) = B(t)$ for $s \neq t$.

8.2. Theorem. *If* $n \geqq 2k$, *then almost every* $B \in \mathscr{C}$ *is an immersion.*

The proof requires the following lemma.

8.3. Lemma. *Let* $t \in \mathscr{M}$. *There is a sequence of integers* $1 \leqq I_1 < I_2 < \dots$ $\dots < I_k \leqq N$ *such that* $\{\phi_{I_i}\}$ *gives local coordinates for* $\mathscr{M}$ *at* t. *The integer* N *is independent of* t.

Proof. Assume for contradiction that the lemma is false. Then for all N there is a point $t_N \in \mathscr{M}$ and a vector $\xi_N \in T_{t_N}\mathscr{M}$ such that

$$\langle D\phi_i(t_N), \xi_N \rangle = 0, \quad 1 \leqq i \leqq N.$$

Then by compactness there is a point $t_\infty \in \mathscr{M}$ and a vector $\xi_\infty \in T_\infty\mathscr{M}$ such that

$$\langle D\phi_i(t_\infty), \xi_\infty \rangle = 0, \quad i = 1, 2, 3, \dots.$$

It follows by Lemma 4.2 that $\langle Df(t_\infty), \xi_\infty \rangle = 0$ for every sufficiently smooth function f, which is impossible.

Proof of Theorem 8.2. It suffices to show that every point of $\mathscr{M}$ has a neighborhood $\mathscr{N}$ on which almost every $B \in \mathscr{C}$ is an immersion. By Lemma 8.3, for every $r \in \mathscr{M}$ there is a neighborhood $\mathscr{N}$ of r (parametrized by a $C^{1,1}$ diffeomorphism s from an open set $U \subset \mathbf{R}^k$ onto $\mathscr{N}$) and a sequence $1 \leqq I_1 < I_2 < \dots$ $\dots < I_K$ such that $\{\phi_{I_i}\}$ gives coordinates on $\mathscr{N}$.

By (6.2(5)) it suffices to show that, given $f \in \mathscr{C}$ the expression

$$g = f + \sum_{i=1}^{k} B_{I_i}\phi_{I_i}$$

is an immersion on $\mathscr{N}$ for $\mathscr{L}^{nk}$ almost all $(B_{I_1}, \dots, B_{I_k}) \in (\mathbf{R}^n)^k$. Put

$$W = \{(x, B_{I_1}, \dots, B_{I_k}) \in U \times (\mathbf{R}^n)^k \mid J_k(g \circ s) = 0\}.$$

By Lemma 5.3, it now is enough to show that W is $\mathscr{H}^{nk}$ rectifiable, with $\mathscr{H}^{nk}(W) = 0$.

In turn, therefore, the proof will follow if we show that the set

$$W_j = \left\{(x, B_{I_1}, \ldots, B_{I_k}) \in U \times (\boldsymbol{R}^n)^k : \frac{\partial g}{\partial \phi_{I_j}}(s(x)) \right.$$
$$\left. \text{is a linear combination of the } \frac{\partial g}{\partial \phi_{I_l}}, \quad l \neq j \right\}$$

is $\mathscr{H}^{nk}$ rectifiable with $\mathscr{H}^{nk}(W_j) = 0$, because $W = \bigcup W_j$. To this end, assuming for convenience that $j = k$, we consider the map

$$F\colon U \times (\boldsymbol{R}^n)^{k-1} \times \boldsymbol{R}^{k-1} \to U \times (\boldsymbol{R}^n)^k$$

$$F\colon (x; a_1, \ldots, a_{k-1}; \alpha_1, \ldots, \alpha_{k-1}) \mapsto (x; B_{I_1}, \ldots, B_{I_k}),$$

where

$$B_{I_l} = a_l - \frac{\partial f}{\partial \phi_{I_l}}(s(x)), \quad 1 \leqq l \leqq k-1,$$

$$B_{I_k} = \sum_l \alpha_l a_l - \frac{\partial f}{\partial \phi_{I_k}}(s(x)).$$

Now F is a $C^{0,1}$ map from a subset of $\boldsymbol{R}^\mu$, where $\mu = k + n(k-1) + k - 1$. Obviously $\mu \leqq nk - 1$ since $n \geqq 2k$. Furthermore we observe that the image of F contains W_k. Therefore W_k is $\mathscr{H}^{nk}$ rectifiable and $\mathscr{H}^{nk}(W_k) = 0$ [14, 2.10.11].

8.4. Sard's Theorem [14, 3.4.3]. *Let $\mathscr{N}$ be a C^m manifold of dimension p. Let $f \in C^m(\mathscr{N}, \boldsymbol{R}^q)$. If $m > p - q \geqq 0$, then almost every point in $\boldsymbol{R}^q$ is a regular value of f.*

8.5. Transversality Theorem (cf. [16, p. 68]). *Let X, Y, Z be C^m manifolds with $\dim X = l$ and $Z \subset Y$. Let $F \in C^m(X \times \boldsymbol{R}^q, Y)$ and suppose $F \pitchfork Z$. If $m > l \geqq 0$, then for $\mathscr{L}^q$ almost every $s \in \boldsymbol{R}^q$ we have $F(\cdot, s]) \pitchfork Z$.*

Proof. Because $F \pitchfork Z$ it is evident that $W = F^{-1}(Z)$ is a C^m submanifold of X. Let $\pi\colon X \times \boldsymbol{R}^q \to \boldsymbol{R}^q$ denote the projection map. Since $\pi \in C^m(W, \boldsymbol{R}^q)$, Sard's Theorem shows that almost every $s \in \boldsymbol{R}^q$ is a regular value for π, and one checks that $F(\cdot, s) \pitchfork Z$ for every such s.

8.6. Theorem. *Let $\mathscr{N}$ be any submanifold of $\boldsymbol{R}^n$. If $q > k$, then almost every $B \in \mathscr{C}$ intersects $\mathscr{N}$ transversally.*

Proof. For fixed $C \in \mathscr{C}$, consider the C^{k+1} map

$$F\colon \mathscr{M} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$$

$$F\colon (t, a) \mapsto C(t) + a\phi_1(t).$$

Since ϕ_1 is constant, DF is always surjective, and hence $F \pitchfork \mathscr{N}$. By (8.5) we have $C + a\phi_1 \pitchfork \mathscr{N}$ for $\mathscr{L}^n$ almost all $a \in \boldsymbol{R}^n$. The result now follows by (6.2(5)).

8.7. Theorem. *If* $q > 2k$, *then almost every* $B \in \mathscr{I}$ *intersects itself transversally.*

8.8. Lemma. *Given* $\delta > 0$, *there is a positive integer* N *such that*

$$(\phi_1(s), \ldots, \phi_N(s)) \notin \boldsymbol{R}(\phi_1(t), \ldots, \phi_N(t))$$

whenever $|s - t| \geqq \delta$.

Proof. Assume for contradiction that the result does not hold. Then there are sequences s_j, t_j such that $|s_j - t_j| \geqq \delta$ and

$$(\phi_1(s_j), \ldots, \phi_j(s_j)) \in \boldsymbol{R}(\phi_1(t_j), \ldots, \phi_j(t_j)).$$

By compactness we can then find points s, t with $|s - t| \geqq \delta$, $r \in \boldsymbol{R}$, such that

$$\phi_j(s) = r\phi_j(t) \text{ for all } j \in \boldsymbol{Z}^+.$$

Consequently by the Approximation Theorem 4.4 we have $f(s) = rf(r)$ if $f \in C(\mathscr{M})$, which is impossible.

Proof of Theorem 8.7. Let $C \in \mathscr{I}$. It suffices to show that there is an $r > 0$ such that

$$\|B - C\| < r \Rightarrow B \pitchfork B$$

for almost all $B \in \mathscr{I}$. Choose a finite open covering U_i of $\mathscr{M}$ and an $r > 0$ so that whenever $\|B - C\| < r$ the following two properties hold:

(1) B embeds $\tilde{U}_i = \cup \{U_j : U_i \cap U_j \neq \phi\}$ in $\boldsymbol{R}^n$,
(2) If $U_i \cap U_j = \phi$, then $\operatorname{dist}(U_i, U_j) > 0$.

By (8.8) we may choose a positive integer N with the property that, if $U_i \cap U_j = \phi$, $s \in U_i$, and $t \in U_j$, then

$$(\phi_1(s), \ldots, \phi_N(s)) \notin \boldsymbol{R}(\phi_1(t), \ldots, \phi_N(t)). \tag{3}$$

Let A be in $\mathscr{C}$. For N-tuples $(a_1, \ldots, a_N)$ in $(\boldsymbol{R}^n)^N$ we put

$$B = A + \sum_{i=1}^{n} a_i \phi_i.$$

By (6.2(5)) the proof will be completed if we can show that, for almost every N-tuple $(a_1, \ldots, a_N)$ the condition

$$\|B - C\| < r$$

implies $B(U_i) \pitchfork B(U_j)$.

We define a C^q map

$$F \colon U_i \times U_j \times (\boldsymbol{R}^n)^N \to \boldsymbol{R}^n \times \boldsymbol{R}^n$$
$$F \colon (s, t, a) \to (B(s), B(t)).$$

We note that the relation $B(U_i) \pitchfork B(U_j)$ is equivalent to the condition that $F(\cdot, \cdot, B) \pitchfork \Delta$ (the diagonal of $\boldsymbol{R}^n \times \boldsymbol{R}^n$). Hence by (8.5) we must prove that $F \pitchfork \Delta$. We show in fact that DF is always surjective. Indeed

$$\langle DF(s, t, B), (0, 0, b_1, \ldots, b_N)\rangle = (\Sigma b_i \phi_i(s), \Sigma b_i \phi_i(t)),$$

for $(b_1, \ldots, b_N) \in (\boldsymbol{R}^n)^N$, and by (3) such combinations span $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

As an immediate consequence of Theorems 8.2 and 8.7 we obtain the following result.

8.9. Corollary. *If $n > 2k$ and $q > 2k$, then almost every $B \in \mathscr{C}$ is an embedding.*

Note added in proof. The results on generic uniqueness for minimizing integral currents of codimension one have been generalized from the area integrand to any C^4 positive elliptic integrand Ψ with constant coefficients (F. MORGAN, "Generic uniqueness for hypersurfaces minimizing the integral of an elliptic integrand with constant coefficients", *Indiana U. Math. J.* **30** (1981), 29–45). Furthermore, recent regularity results of ALMGREN seem to establish generic uniqueness for area minimizing integral currents of arbitrary codimension (cf. the remark following Theorem 7.1),

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