On Inequalities of Korn's Type

I. Boundary-Value Problems for Elliptic Systems of Partial Differential Equations

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Preface

If one derives the fundamental properties of solutions of the boundary-value problems for elliptic systems of partial differential equations using a variational approach, as was done, for example, in the book [1] of J. NEČAS, one realizes the important role of the *coerciveness* inequality (compare also [2]). Part I of this work presents some consequences of the coerciveness inequality, and yields the existence, uniqueness and continuous dependence of the solution upon the given data.

In Part II, the general theory of Part I is applied to the linear theory of three-dimensional elasticity, where inequalities resulting from coerciveness have been called KORN's inequalities and have been studied by KORN [3], FRIEDRICHS [4], EYDUS [5], PAYNE & WEINBERGER [6], GOBERT [7] and others. We present here new proofs for the previous results and extend them to some mixed boundary-value problems with more general boundary conditions, in which displacements and tractions are prescribed in the normal and tangential directions to the boundary. The cases of elastic supports are also considered.

1. Notation. Preliminary Definitions

Let a Lipschitz* region Ω in N-dimensional Euclidian space E_N and positive integers m and κ_s, $s=1, 2, \ldots m$, be given. $L_2(\Omega)$ will denote the space of real functions which are square-integrable on Ω (in the Lebesgue-sense). $W_2^{(k)}(\Omega)$ denotes the subspace of $L_2(\Omega)$ consisting of functions whose derivatives up to order k, in the sense of distributions, are in $L_2(\Omega)$.

Let us introduce the scalar product on $W_2^{(k)}(\Omega)$ by means of

$$(v, u)_k = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha v\, D^\alpha u\, dX, \tag{1.1}$$

* We call a region $\Omega\subset E_N$ Lipschitz if it is bounded and its boundary Γ has the following properties: a) to each point $X\in\Gamma$ an open hypersphere S_X about X exists, such that the intersection $S_X\cap\Gamma$ may be described by means of a Lipschitz function, and b) $S_X\cap\Gamma$ divides S_X into exterior and interior parts with respect to Ω.

where
$$D^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}.$$

Let W denote the Cartesian product $\prod_{s=1}^{m} W_2^{(\kappa_s)}(\Omega)$, and let two bilinear forms $A(v, u)$, $a(v, u)$ be given on $W \times W$:

$$A(v, u) = \int_\Omega \sum_{r,s=1}^{m} \sum_{\substack{|i| \leqq \kappa_r \\ |j| \leqq \kappa_s}} a_{ij}^{rs} D^i v_r D^j u_s \, dX \tag{1.2}$$

where a_{ij}^{rs} are bounded and measurable on Ω, and

$$a(v, u) = \int_\Gamma \sum_{r,s=1}^{m} \sum_{i=0}^{\kappa_r - 1} \sum_{|\alpha| \leqq \kappa_s - 1} b_{i\alpha}^{rs} \frac{\partial^i v_r}{\partial n^i} D^\alpha u_s \, d\Gamma \tag{1.3}$$

where $b_{i\alpha}^{rs}$ are bounded and measurable on the boundary Γ. Furthermore, let $\mathscr{D}(\Omega)$ be the space of real functions with compact support in Ω which are infinitely differentiable, let $\mathring{W}_2^{(k)}(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ in $W_2^{(k)}(\Omega)$, and let $\mathring{W} = \prod_{s=1}^{m} \mathring{W}_2^{(\kappa_s)}(\Omega)$.

Let V be a closed subspace of W, such that $\mathring{W} \subset V \subset W$. Define

$$f(v) = \int_\Omega \sum_{s=1}^{m} f_s v_s \, dX$$

where $f_s \in L_2(\Omega)$. Let g be a linear continuous functional on V such that $g(v) = 0$ for $v \in \mathring{W}$. Moreover, let $u_0 \in W$.

We say that $u \in W$ is a *weak solution of the boundary-value problem* if

$$u - u_0 \in V \tag{1.4}$$

and if for each $v \in V$ the relation

$$A(v, u) + a(v, u) = f(v) + g(v) \tag{1.5}$$

holds.

Let the operators $N_l v$, mapping W into $L_2(\Omega)$, $l = 1, \dots h$, be given in the form

$$N_l v = \sum_{r=1}^{m} \sum_{|\alpha| \leqq \kappa_r} n_{lr\alpha} D^\alpha v_r$$

where $n_{lr\alpha}$ are bounded and measurable on Ω. Suppose that these operators form a *coercive system* on* W, i.e.,

$$v \in W \Rightarrow \sum_{l=1}^{h} \|N_l v\|_{L_2}^2 + \sum_{r=1}^{m} \|v_r\|_{L_2}^2 \geqq c \|v\|_W^2, \qquad c > 0. \tag{1.6}$$

We assume furthermore that

$$v \in W \Rightarrow A(v, v) \geqq c \sum_{l=1}^{h} \|N_l v\|_{L_2}^2, \tag{1.7}$$

$$v \in W \Rightarrow a(v, v) \geqq 0. \tag{1.8}$$

* See the theorems of Section 3 for necessary and sufficient conditions for coerciveness.

2. Fundamental Properties of the Weak Solution of the Boundary-Value Problem

Let us put

$$\mathscr{P}=\left\{v\in V,\ \sum_{l=1}^{h}\|N_l v\|_{L_2}^2+a(v,v)=0\right\}$$

and

$$((v,u))=A(v,u)+a(v,u),$$

and denote by $V/\mathscr{P}$ the factor-space of classes $\tilde{v}=\{v+p,\ v\in V,\ p\in\mathscr{P}\}$ with the norm

$$\|\tilde{v}\|_{V/\mathscr{P}}=\inf_{p\in\mathscr{P}}\|v+p\|_V.$$

Theorem 2.1. *Let the relations*

$$\begin{aligned}A(\tilde{v},\tilde{u})&=A(v,u),\\ a(\tilde{v},\tilde{u})&=a(v,u),\end{aligned}\qquad v\in\tilde{v},\quad u\in\tilde{u},\quad \tilde{v},\tilde{u}\in V/\mathscr{P},$$

define two bilinear forms on $V/\mathscr{P}\times V/\mathscr{P}$. *Let* (1.6), (1.7) *and* (1.8) *hold. Then*

$$(2.1)\qquad ((\tilde{v},\tilde{v}))\geqq c_1\left(\sum_{l=1}^{h}\|N_l\tilde{v}\|_{L_2}^2+a(\tilde{v},\tilde{v})\right)\geqq c_2\|\tilde{v}\|_{V/\mathscr{P}}^2$$

holds for each $\tilde{v}\in V/\mathscr{P}$.

Proof. Suppose the contrary holds. Then there exists a sequence of elements $\tilde{v}\in V/\mathscr{P}$ such that

$$(2.2)\qquad \|\tilde{v}_n\|_{V/\mathscr{P}}=1$$

and

$$(2.3)\qquad \sum_{l=1}^{h}\|N_l\tilde{v}_n\|_{L_2}^2+a(\tilde{v}_n,\tilde{v}_n)<\frac{1}{n}.$$

Let $V=K\oplus\mathscr{P}$ be the orthogonal decomposition of V by means of the scalar product (see (1.1))

$$\sum_{s=1}^{m}(v_s,u_s)_{\kappa_s}.$$

Denote by v_K the orthogonal projections of v on K. We have

$$\|\tilde{v}_n\|_{V/\mathscr{P}}=\|v_{nK}\|_V.$$

As the immersion of $W_2^{(1)}(\Omega)$ into $L_2(\Omega)$ and $L_2(\Gamma)$, respectively, is compact and V is weakly compact, we can choose a subsequence of v_{nK} (denoted again by v_{nK}) such that $v_{nK}\rightharpoonup v_K$ (weak convergence), $v_{nK}\to v_K$ in $\prod_{s=1}^{m}W_2^{(\kappa_s-1)}(\Omega)$ and $a(v_{nK},v_{nK})\to a(v_K,v_K)$. Then (2.3) implies $v_K\in\mathscr{P}$ and consequently $v_K=\mathbf{0}$. By virtue of (2.3) and (1.6), $v_{nK}\to\mathbf{0}$ in V, which contradicts (2.2).

Theorem 2.2. *Let the form* $((v,u))$ *define the bilinear form* $((\tilde{v},\tilde{u}))$ *for* $\tilde{v}\in W/\mathscr{P}$, $\tilde{u}\in W/\mathscr{P}$, $v\in\tilde{v}$, $u\in\tilde{u}$. *Let* (1.6)–(1.8) *hold. Then a necessary and sufficient condition for the existence of a weak solution of the boundary-value problem is*

$$(2.4)\qquad p\in\mathscr{P}\ \Rightarrow\ f(p)+g(p)=0.$$

In this case the solution is determined except for an element $p \in \mathscr{P}$. Moreover

$$\|\tilde{u}\|_{W/\mathscr{P}} \leqq c(\|u_0\|_W + \|f\|_{[L_2]^m} + \|g\|_{V'}). \tag{2.5}$$

Let $(v, u)_W$ and $(v, u)_V$ be scalar products in W and V, supplying norms equivalent to

$$\left[\sum_{s=1}^{m} (v_s, v_s)_{K_s}\right]^{\frac{1}{2}}$$

in W and V, respectively. Let Q and R denote the orthogonal complements of $\mathscr{P}$ in W and V, respectively, by means of these scalar products, and let u_Q, u_R be the corresponding orthogonal projections of u.

Then we may choose u_Q and $u' = u_0 + (u - u_0)_R$, respectively, as the representatives of $\tilde{u}$, and

$$\|u_Q\|_W + \|u'\|_W \leqq c(\|u_0\|_W + \|f\|_{[L_2]^m} + \|g\|_{V'}). \tag{2.6}$$

Proof. The necessity of (2.4) follows from (1.5). If (2.4) is satisfied, let us seek a $\tilde{w} \in V/\mathscr{P}$ such that

$$((\tilde{v}, \tilde{w})) = F(\tilde{v}) \tag{2.7}$$

holds for $\tilde{v} \in V/\mathscr{P}$, where

$$F(\tilde{v}) = f(v) + g(v) - ((v, u_0)).$$

The condition (2.1) enables us to use the Lax-Milgram theorem. Hence there exists a unique element $\tilde{w}$ which satisfies (2.7). Evidently, any element u of the class $\tilde{u} = \tilde{u}_0 + \tilde{w}$ is a solution of the problem. The remaining part of the theorem is obvious.

Remark 1. Let us take $(v, u)_V = (v, u)_W$ for $v, u \in V$. Then applying this scalar product to the orthogonal decomposition of W, we obtain

$$W = \mathscr{P} \oplus Q = \mathscr{P} \oplus R \oplus D$$

and consequently,

$$u' = u_0 + u_R - u_{0R} = u_{0D} + u_{0\mathscr{P}} + u - u_D - u_{\mathscr{P}} = u_Q + u_{0\mathscr{P}} + (u_0 - u)_D = u_Q + u_{0\mathscr{P}}.$$

Remark 2. Put

$$\mathscr{P}_A = \left\{ v \in V, \ \sum_{l=1}^{h} \|N_l v\|_{L_2}^2 = 0 \right\}.$$

Let

$$A(v, u) = A(u, v),$$

$$0 \leqq A(v, v) \leqq c \sum_{l=1}^{h} \|N_l v\|_{L_2}^2$$

hold for $v \in W$. Then a necessary condition for the existence of a solution is

$$p \in \mathscr{P}_A \ \Rightarrow \ a(p, u) = f(p) + g(p). \tag{2.8}$$

In fact, the assertion follows from (1.5) when we note that the Schwarz inequality for $A(v, u)$ implies $A(p, u)=0$.

Theorem 2.3. *Let* (1.6)–(1.8) *hold and let* $p_i \in V'$, $i=1, 2, \dots r$ *be a system of linear continuous functionals on* V *such that*

$$v \in V, \ \sum_{l=1}^{h} \|N_l v\|_{L_2}^2 + a(v, v) + \sum_{i=1}^{r} (p_i(v))^2 = 0 \ \Rightarrow \ v = 0. \tag{2.9}$$

Then

$$\sum_{l=1}^{h} \|N_l v\|_{L_2}^2 + a(v, v) + \sum_{i=1}^{r} (p_i(v))^2 \geqq c \|v\|_W^2 \tag{2.10}$$

holds for each $v \in V$.

Proof. The method is analogous to that of Theorem 2.1. Suppose the contrary holds. Then elements $v_n \in V$ exist such that $\|v_n\|_V = 1$, $v_n \rightharpoonup v$ in V, $v_n \to v$ in $\prod_{s=1}^{m} W_2^{(\kappa_s - 1)}(\Omega)$, $a(v_n, v_n) \to a(v, v)$, and

$$\sum_{l=1}^{h} \|N_l v_n\|_{L_2}^2 + a(v_n, v_n) + \sum_{i=1}^{r} (p_i(v_n))^2 \leqq \frac{1}{n}. \tag{2.11}$$

Hence by transition to the limit $n \to \infty$, $v = 0$ follows. (2.11) and (1.6) yield $v_n \to 0$ in V, and we have a contradiction.

Remark 3. Let the hypotheses of Theorem 2.3 hold. Put

$$V_p = \left\{ v \in V, \sum_{i=1}^{r} (p_i(v))^2 = 0 \right\}.$$

Then

$$v \in V_p \ \Rightarrow \ ((v, v)) \geqq c_1 \left(\sum_{l=1}^{h} \|N_l v\|_{L_2}^2 + a(v, v) \right) \geqq c_2 \|v\|_W^2.$$

Remark 4. Let the hypotheses of Theorem 2.3 hold. Furthermore, let the system p_i be linearly independent on $\mathscr{P}$, i.e., let $\sum_{i=1}^{r} \alpha_i p_i(p) = 0$ for each $p \in \mathscr{P}$ imply $\sum_{i=1}^{r} \alpha_i^2 = 0$. Let us choose in Theorem 2.2

$$(v, u)_V = \sum_{l=1}^{h} \int_{\Omega} N_l v \, N_l u \, dX + \tfrac{1}{2}\big(a(v, u) + a(u, v)\big) + \sum_{i=1}^{r} p_i(v)\, p_i(u).$$

Then

$$R = V_p.$$

Indeed, let $v \in R$. Then $(v, p)_V = 0$ for $p \in \mathscr{P}$. Schwarz's inequality implies that

$$\sum_{l=1}^{h} \int_{\Omega} N_l v \, N_l p \, dX + \tfrac{1}{2}\big(a(v, p) + a(p, v)\big) = 0; \tag{2.12}$$

consequently,

$$\sum_{i=1}^{r} p_i(v)\, p_i(p) = 0$$

for every $p \in \mathscr{P}$. Hence

$$\sum_{i=1}^{r} (p_i(v))^2 = 0$$

and therefore $v \in V_p$.

Conversely, let $v \in V_p$. Then (2.12) yields $(v, p)_V = 0$ for every $p \in \mathscr{P}$, and therefore $v \in R$.

3. Coerciveness of the System $N_l v$

Let us consider operators $N_l v$ having the form

$$N_l v = \sum_{s=1}^{m} \sum_{|\alpha| \leq \kappa_s} n_{l_s \alpha} D^\alpha v_s, \tag{3.1}$$

and let us define

$$N_{l_s} \xi = \sum_{|\alpha| = \kappa_s} n_{l_s \alpha}(X) \xi_1^{\alpha_1} \xi_2^{\alpha_2} \ldots \xi_N^{\alpha_N}. \tag{3.2}$$

NECAS ([1], [2]) proved and BESOV generalized ([8]) the following results.

Theorem 3.1. *Let $n_{l s \alpha}$ be constants for $|\alpha| = \kappa_s$. Then the system $N_l v$ is coercive on W if and only if*

$$\textit{the rank of the matrix } (N_{l_s} \xi) = m \tag{3.3}$$

for each $\xi \in C_N$, $\xi \neq 0$, where C_N denotes complex N-dimensional space.

Theorem 3.2. *Let $n_{l s \alpha} \in C(\bar{\Omega})$* for $|\alpha| = \kappa_s$. Then the system $N_l v$ is coercive on W if and only if* (3.3) *holds for $\xi \in E_N$ if $X \in \Omega$, and for $\xi \in C_N$ if $X \in \Gamma$.*

Example. Let $m = N$, $\kappa_1 = \kappa_2 = \cdots = \kappa_m = 1$. Let the system $N_l v$ be defined by the operators $\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, $i, j = 1, \ldots N$, $i \neq j$ and $\frac{\partial v_i}{\partial x_i}$, $i = 1, \ldots N$. If we order these operators as follows:

$$\frac{\partial v_1}{\partial x_1}, \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}, \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}, \ldots \frac{\partial v_1}{\partial x_N} + \frac{\partial v_N}{\partial x_1},$$

$$\frac{\partial v_2}{\partial x_2}, \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}, \ldots \frac{\partial v_2}{\partial x_N} + \frac{\partial v_N}{\partial x_2}, \frac{\partial v_3}{\partial x_3}, \ldots \frac{\partial v_N}{\partial x_N},$$

then the transposed matrix $N_{l s} \xi$ will have the form

$$\begin{pmatrix}
\xi_1, & \xi_2, & \xi_3, & & \ldots \xi_N, & 0, & & & \ldots 0, & & & & \ldots & & \ldots 0 \\
0, & \xi_1, & 0, & & \ldots 0, & \xi_2, & \xi_3, & & \ldots \xi_N, & 0, & 0, & & \ldots & & 0 \\
0, & 0, & \xi_1, & 0, & 0, & 0, & \xi_2, & 0 & \ldots 0, & \xi_3, & \xi_4, & & \ldots \xi_N, & 0, & \ldots 0 \\
0, & 0, & 0, & \xi_1 & 0, & 0, & 0, & \xi_2 & \ldots 0, & 0, & \xi_3, & 0, \ldots 0, & \xi_4, \ldots & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & & & \ddots & \vdots & \vdots & \vdots & \ddots & & & \\
0, & 0, & & & \ldots \xi_1, & 0, & 0, & & \xi_2, & 0, & 0, & & \ldots \xi_3, & & \ldots
\end{pmatrix}.$$

Obviously, the rank of $(N_{l s} \xi)$ is equal to N for each $\xi \in C_N$, $\xi \neq 0$. Hence the system $N_l v$ is coercive on W.

* $C(\bar{\Omega})$ denotes the space of functions, which are continuous on Ω and have continuous extensions to Γ.

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