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TW_+ AND RW_+ ARE DECIDABLE[1]

INTRODUCTION

Sequent calculi (consecution calculi, Gentzen systems) have been a powerful tool of formal research since their introduction in [7]. It was recognized early on that "standard" sequents or consecutions utilizing sequences of formulae could be used to provide Gentzen systems for pure implicational fragments of relevant logics. (See [3] and [10], for example.) However, it was a full fifteen years later before the secret to producing Gentzen systems with appropriate Cut Theorems for full positive relevant logics was discovered.

In a consecution calculus for, say, classical logic, a sequence of formulae can be thought of as implicitly representing the conjunction of those formulae when it occurs in the antecedent of a consecution and the disjunction of those formulae when it occurs in the consequent. In the relevant Gentzen systems previously mentioned sequences of formulae were being used alternately to represent *intensional* conjunction (fusion) and disjunction (fission). From this point of view, what Professors Dunn [5, 6] and Minc [16] discovered was the following: since the positive relevant logics contain both (truth-functional) conjunction and fusion, two types of sequences would be required to gentzenise them – extensional sequences to stand in for conjunction, and intensional sequences to stand in for fusion. (These systems are singular in the consequent.) Further, such sequences must be allowed to be nested within one another, i.e., the elements of these sequences must be allowed to be sequences of sequences of . . . . The particular character of each type of sequence is determined to be extensional or intensional according to the structural rules which govern it.

Another point that was discovered was that a constant truth (t as in [6] or its functional analogue as in [16]) would be needed for a correct Cut Theorem. The reason for this is that the straightforward Cut Rule is not admissible when consecutions are allowed to have empty antecedents.

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For the rule would then indirectly license the inference of the infamous modal fallacy $\vdash p \rightarrow . q \rightarrow q$ from the provable $\vdash (q \rightarrow q) \& p \rightarrow . q \rightarrow q$, for example. And of course such fallacies are anathema to relevant logics.

So such Gentzen systems are of a higher order of complexity than their progenitors. Consequently, they are more difficult to *use*. In this paper we modify techniques of [6] to gentzenise the positive, contractionless relevant logics TW_+^0 and RW_+^0, and for the first time put such complex systems to appropriate use, namely that of providing decision procedures. (For more discussion of these topics and for detailed proofs of the theorems to follow, see [8].)

It is worth noting that TW_+ and RW_+ will be (almost) the strongest decidable relevant systems. [19], a truly brilliant work, not only lays the decision problem for R to rest, but shows that relevant systems as weak as T_+ are also undecidable. Indeed, even the "deducibility problem" for TW_+ is undecidable.

TW_+^{ot} AND RW_+^{ot}

TW_+^{0t} and RW_+^{0t} are the positive fragments of the relevant logics T (Ticket Entailment) and R (Relevant Implication) with fusion and t, but without W, i.e., the Contraction Axiom $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$. (We adopt the notational conventions of [1].) They can be conveniently formulated from the following group of axioms and rules.

A1. $A \rightarrow A$

A2. $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$

A3. $B \rightarrow C \rightarrow . A \rightarrow B \rightarrow . A \rightarrow C$

A4. $A \rightarrow . A \rightarrow B \rightarrow B$

A5. $A \circ B \rightarrow C \rightarrow . A \rightarrow . B \rightarrow C$

A6. $A \& B \rightarrow A, \quad A \& B \rightarrow B$

A7. $(A \rightarrow B) \& (A \rightarrow C) \rightarrow . A \rightarrow B \& C$

A8. $A \rightarrow A \vee B, \quad B \rightarrow A \vee B$

A9. $(A \rightarrow C) \& (B \rightarrow C) \rightarrow . A \vee B \rightarrow C$

A10. $A \& (B \vee C) \rightarrow (A \& B) \vee C$

R1. $\dfrac{\vdash A \to B \quad \vdash A}{\vdash B}$ R3. $\dfrac{\vdash A \to . B \to C}{\vdash A \circ B \to C}$

R2. $\dfrac{\vdash A \quad \vdash B}{\vdash A \,\&\, B}$ R4. $A \Leftrightarrow t \to A$

TW^0_+ = A1–A3, A6–A11, R1, R2, and R3.
For TW^{0t}_+ add R4.

$RW^0_+ = TW^0_+$ + A4. For RW^{0t}_+ add R4.

Before moving to the corresponding Gentzen systems, a few words are in order about contractionless relevant logics. Such systems have been of interest from very early on in the study of relevant logics. We do not know the original date of Belnap's conjecture that *PW* ($TW_\to$) is minimal in the sense that if $A \to B$ and $B \to A$ are both theorems, then A and B are the same formula. However, progress toward its solution was reported by Powers as early as 1968. (Those results were eventually published in [17].) This problem remained one of the most interesting and recalcitrant problems in relevant logics until the conjecture was proved in [12]. (The solution is more accessible in [13].)

Interest in full contractionless systems was first stimulated by [15], where it is shown (with due acknowledgement to [4]) that a non-trivial naive set theory cannot be based on R, T nor E. The problem is that the contraction axiom in the presence of other rather minimal logical principles will collapse any theory containing the full Abstraction Principle.

The ability to be used in investigating non-trivial but inconsistent theories, i.e., being weakly paraconsistent, has always been a motivating feature of relevant logics. And naive set theory has always been near the top of the list of interesting such theories – within and outside the relevant program. So contractionless relevant logics have found favor amongst those who want a logic suitable for such purposes.

Another point on which the contractionless systems commend themselves is that of being more Catholic than the Pope on a central feature of relevant logics. [1] begins with the claim that "the heart of logic [lies] in the notion of 'if . . . then –'; . . . ." We take the point to be that the main task of a logic is to separate out valid inferences from invalid ones. Contractionless systems can be seen as taking this point further. Distinguishing valid from invalid inferences is not simply the major task of logic, it is *the* task.

To be sure, logic may say something "about" truth functions, since they are needed in the vocabulary to express certain truths about implication. But whether or not Excluded Middle, for instance, is to be accepted is not a matter to be determined by logic. Accordingly, *TW*, *RW* and *EW* do not record any purely truth-functional formulae as logical truths. More precisely, no formulae in which an $\rightarrow$ does not occur is a theorem of these logics.

It is not that such logics *deny* any of the putative truths about truth-functions – or for that matter about quantifiers, or alethic modal operators, or what have you. Rather, it is that such matters are to be decided on *non-logical* grounds, and recorded in theories appropriate to *those* subjects.

And with respect to taking valid inference to be its proper subject, *TW* outshines its cousins for it is shown in [17] that every theorem of *TW* is equivalent to a conjunction of theorems each of which is a disjunction one of whose disjuncts is a valid implication.

As Professor Slaney [18] puts it, this fact "establishes a good sense in which *TW* is fundamentally implicational". Which is as a logic ought to be, we might add.

Finally, one should note that these systems are prime; that is, a disjunction is a theorem just in case one of its disjuncts is. Naturally, this is a point on which they recommend themselves to whose who cherish constructivism.

LTW_+^{ot} AND LRW_+^{ot}

The Gentzen systems for TW_+^{ot} and RW_+^{ot} are actually a modified blend of the work of Dunn [6] and of Meyer [14] and Belnap [2]. The reason for this is partially preference and partially convenience. We have come to prefer the structural connectives of [14] and [2], since they lay bare the direct relationship between structural elements and formulae. So we will have an intensional binary structural connective, rather than intensional sequences. However, sequences are far easier to deal with when it comes to show decidability. So we do retain the extensional sequences of [6].

So letting 'X', 'Y', 'Z', and 'W' with or without subscripts and/or superscripts be structural variables, a *structure* is defined recursively:

(1) A is a structure, for any formula A;

(2) if X and Y are structures, so is $(X; Y)$; and

(3) if $X_1, \ldots, X_n$ $(n \geqslant 2)$ are structures, then so is $E(X_1, \ldots, X_n)$.

Note that there are no null structures nor any structures of the form $E(X)$. With respect to the latter, we say that our structures are *denuded.* (This is a desirable simplification of [6] for the decidability argument.) Unless otherwise indicated, semicolons are taken to be associated to the left. Parentheses are used to disambiguate notation as necessary.

Structures of the form of (3) are called extensional structures, extensional sequences, or *e*-sequences. Those of the form of (2) are intensional structures or *i*-structures. And we say that a structure X *occurs in* a structure Y just in case

(1) X is Y; or

(2) Y is $(W; Z)$ and X occurs in W or in Z; or

(3) Y is $E(W_1, \ldots, W_n)$ and X occurs in some W_i.

Of course, if X occurs in Y, then X is a substructure of Y, and the appropriate occurrence(s) of X is/are a constituent(s) of Y. (The notion of a particular occurrence of a structure is taken as primitive. However, the distinction between a structure and a particular occurrence thereof is often ignored when it is not likely to cause confusion.) And for $1 \leqslant i \leqslant n$, the "displayed" occurrence of X_i in $E(X_1, \ldots, X_n)$ is an *immediate constituent* thereof. And for an intensional structure, say $X; Y$, we refer to X as the *left constituent* and to Y as the *right constituent.*

A *sequent* or *consecution* is an entity of the form $X \vdash A$. 'Σ', with or without scripting, is used as a variable ranging over consecutions. X is the *antecedent* and A is the *consequent* of $X \vdash A$. And Y *occurs in* a consecution just in case it occurs in its antecedent or consequent. The use of 'constituent' is similarly extended. And we say that structures and consecutions are *built up from* or *built up out of* the wffs that occur in them.

The following structural analogue to the notion of the length or complexity of a formula will be very useful. So define the *structural complexity* (sc) of a structure as follows.

(1) $\mathrm{sc}(A) = 1$, for any formula A;

(2) $\mathrm{sc}(X; Y) = \mathrm{sc}(X) + \mathrm{sc}(Y) + 1$, for any structures X and Y; and

(3) $\mathrm{sc}(E(X_1, \ldots, X_n)) = \mathrm{sc}(X_1) + \ldots + \mathrm{sc}(X_n) + 1$, for any structures $X_1, \ldots, X_n$.

Upper case Greek letters (except 'Σ') are used to range over (possibly empty) sequences of symbols drawn from: formula variables and parameters, 'E', left and right parentheses, the comma and the semicolon. For example $\Gamma_1 X \Gamma_2 \vdash A$ represents a consecution. Further, a particular occurrence of X is taken to have been *displayed* therein.

Now LTW_+^{0t} and LRW_+^{0t} can be formulated from the following set of axioms and rules. (Two sided rules are indicated by ⇔.) Following the tradition of [7], there are two types of rules, Logical and Structural. But since there are two types of structures, there are more structural rules than usual – a group for extensional structures, designated with an 'e', and a group for intensional ones, designated with an 'i'. The names of the structural rules derive from the names of the associated combinatorial rules.

AXIOMS

$A \vdash A$, for any formula A.

RULES

Structural Rules

$Ke\vdash$ $$\frac{\Gamma_1 X \Gamma_2 \vdash C}{\Gamma_1 E(X, Y)\Gamma_2 \vdash C}$$

$Ce\vdash$ $$\frac{\Gamma_1 E(X_1, \ldots, Y, W, \ldots, X_n)\Gamma_2 \vdash C}{\Gamma_1 E(X_1, \ldots, W, Y, \ldots, X_n)\Gamma_2 \vdash C}, \quad n \geqslant 0$$

$We\vdash$ $$\frac{\Gamma_1 E(X, X)\Gamma_2 \vdash C}{\Gamma_1 X \Gamma_2 \vdash C}$$

$ee\vdash$ $$\Gamma_1 E(X_1, \ldots, E(Y_1, \ldots, Y_m), \ldots, X_n)\Gamma_2 \vdash C$$
$$\Leftrightarrow \Gamma_1 E(X_1, \ldots, Y_1, \ldots, Y_m, \ldots, X_n)\Gamma_2 \vdash C,$$

$CIi\vdash$ $$\frac{\Gamma_1(X; Y)\Gamma_2 \vdash C}{\Gamma_1(Y; X)\Gamma_2 \vdash C}$$

$Bi\vdash$ $$\frac{\Gamma_1(X; (Y; Z))\Gamma_2 \vdash C}{\Gamma_1(X; Y; Z)\Gamma_2 \vdash C}$$ $B'i\vdash$ $$\frac{\Gamma_1(X; (Y; Z))\Gamma_2 \vdash C}{\Gamma_1(Y; X; Z)\Gamma_2 \vdash C}$$

Logical Rules

& $\vdash$ $\quad \dfrac{\Gamma_1 A \Gamma_2 \vdash C}{\Gamma_1 A \,\&\, B \Gamma_2 \vdash C} \qquad \dfrac{\Gamma_1 B \Gamma_2 \vdash C}{\Gamma_1 A \,\&\, B \Gamma_2 \vdash C}$

$\vdash$ & $\quad \dfrac{X \vdash A \qquad X \vdash B}{X \vdash A \,\&\, B}$

$\vee \vdash$ $\quad \dfrac{\Gamma_1 A \Gamma_2 \vdash C \quad \Gamma_1 B \Gamma_2 \vdash C}{\Gamma_1 A \vee B \Gamma_2 \vdash C}$

$\vdash \vee$ $\quad \dfrac{X \vdash A}{X \vdash A \vee B} \qquad \dfrac{X \vdash B}{X \vdash A \vee B}$

$\rightarrow \vdash$ $\quad \dfrac{Y \vdash A \qquad \Gamma_1 B \Gamma_2 \vdash C}{\Gamma_1 (A \rightarrow B; Y) \Gamma_2 \vdash C}$

$\vdash \rightarrow$ $\quad \dfrac{X; A \vdash B}{X \vdash A \rightarrow B}$

$\circ \vdash$ $\quad \dfrac{\Gamma_1 (A; B) \Gamma_2 \vdash C}{\Gamma_1 A \circ B \Gamma_2 \vdash C}$

$\vdash \circ$ $\quad \dfrac{X \vdash A \qquad Y \vdash B}{X; Y \vdash A \circ B}$

$t \vdash$ $\quad \dfrac{\Gamma_1 X \Gamma_2 \vdash C}{\Gamma_1 (t; X) \Gamma_2 \vdash C}$

$t - \vdash$ $\quad \dfrac{\Gamma_1 (t; X) \Gamma_2 \vdash C}{\Gamma_1 X \Gamma_2 \vdash C}$

The axioms, all of the logical rules and $Ke \vdash$, $We \vdash$, $Ce \vdash$ and $ee \vdash$ are common to the L-systems. To get LTW_+^{0t} add $Bi \vdash$ and $B'i \vdash$. For LRW_+^{0t} replace $B'i$ by $CIi \vdash$.

Next we should establish some terminology. $Ke \vdash$ is an extensional rule of *weakening*, and we speak in the obvious way of a structure having been *weakened in*. $Ce \vdash$ and $CIi \vdash$ are permutation rules, and $We \vdash$ is a rule of extensional *contraction*. Again we speak of a permuted structure and of a contracted structure.

A derivation is a finite tree branching upward with the normal sorts of properties, and a proof of A is a derivation of $t \vdash A$. We take the notion of a consecution occurrence being *immediately above* (*below*) another consecution (occurrence) as primitive. Being *above* (*below*) is the transitive closure of immediately above (below). So where Der is a derivation and x is a particular occurrence of some consecution therein, the *subderivation determined by* x is the derivation that one would get by deleting from Der all consecutions except x and those above it. A *branch* of a derivation is a sequence $x_1, \ldots, x_n$ of consecution occurrences such that x_1 has no consecution occurrence above it and x_n has none below it, and for all $1 \leqslant i < n$, x_i is immediately above x_{i+1}.

The *weight of a derivation*, say Der, is the length of a longest branch, and the *weight of a consecution occurrence* x in Der is the weight of the subderivation determined by x. The conclusion (bottom node) of a derivation that has weight n is said to be *derivable with weight n.*

Finally, the *height* of a consecution occurrence, say x, in a derivation Der is the length of the branch segment consisting of x and all consecution occurrences below it.

A Cut Theorem in the style of [6] can be established for these systems using the argument there with only minor variations. The systems can then be shown to be equivalent to their axiomatic counterparts, more or less as in [6]. The major difference is that $t - \vdash$ is required to show that *modus ponens* is admissible in these systems, and to show that importation (R3) is admissible in LTW_+^{0t}.

The equivalence theorem can be stated as follows:

THEOREM 1. $t \vdash A$ is derivable in LTW_+^{0t} (LRW_+^{0t}) iff A is a theorem of TW_+^{0t} (RW_+^{0t}).

The above Gentzen systems are convenient for proving the desired equivalence, but present several problems for the decidability argument. The first problem is $t - \vdash$. Not only does this rule block the straightforward subformula property for the systems, but also fails to be degree preserving – a property essential to the decidability argument below.

Reformulation 1

The easiest solution to this problem is to now do away with t and all of its works. We keep the definition of structures as before. There will be *no* null or empty structure. We simply allow consecutions to be entities either of the form $X \vdash A$ or of the form $\vdash A$. To do otherwise is to introduce the ridiculous question of whether or not there are structures of the form $E(X_1, \ldots, X_n)$, for instance, where each X_i is empty. Of course, the adopted policy is not without its own headache. Technically, whenever we want to say something about consecutions in general we must often speak double, once about those of the form $X \vdash A$ and once about those of the form $\vdash A$.

Of course, when one has a headache, the sensible thing to do is to take asprin. Our asprin will be to use double-speak rather than speak double. We now allow structural variables to be existentialist variables, that is, they range over structures and the dreaded Nothingness. Otherwise, notation remains the same.

We must still occasionally restrict structural variables to range only over structures. But with a bit of good will on the part of the reader and a few conventions, this is not so cumbersome. In the first place, we insist that structural variables never range over Nothingness when used to represent an immediate constituent of an e-sequence. And likewise for structural variables that occur in the statement of structural rules.

The simplest method for getting rid of t is to first leave it in and make a few modifications (including being empty on the left) to the original formulations, and then show that we no longer need t. So let the L'-systems come from the L-systems by

I. Adding $\vdash t$ as an axiom;

II. Leaving the structural rules as they are (but note the conventions on Nothingness);

III. For $L'TW_+^{\circ t}$, insisting that (1) the antecedent of the left premise of $\to\vdash$ is never empty, and (2) the antecedent of the right premise of $\vdash\circ$ is empty only if the antecedent of the left premise is; and

IV. For $L'RW_+^{\circ t}$, replacing $t\vdash$ by the more general

$t\#\vdash$ $$\frac{\Gamma_1 X \Gamma_2 \vdash C}{\Gamma_1(X; Y)\Gamma_2 \vdash C} \qquad \frac{\Gamma_1 X \Gamma_2 \vdash C}{\Gamma_1(Y; X)\Gamma_2 \vdash C}$$

where Y is a t-structure, and a t-structure of course is one built up from t.

It is easy to show that $L'TW_+^{\circ t}$ ($L'RW_+^{\circ t}$) is contained on appropriate translation in $TW_+^{\circ t}$ ($RW_+^{\circ t}$); and the L'-systems are supersystems of the

L-systems. (We now say that a formula A is provable iff $\vdash A$ is derivable.) So using Theorem 1 and $t - \vdash$, $t \vdash$ and $t \# \vdash$, we have:

THEOREM 2. A is provable in $L'TW_+^{0t}$ ($L'RW_+^{0t}$) iff A is a theorem of TW_+^{0t} (RW_+^{0t}).

But note that this theorem does not allow us to do away with t, since there is yet no guarantee that there is a t-free derivation of each provable t-free formula. To rectify this situation, one first shows:

FACT 1. Let Der be an L'-derivation of a consecution Σ satisfying the following conditions:

(1) t is not a subformula of the consequent of Σ;

(2) t is not a *proper* subformula of any formula occurring in the antecedent of Σ;

(3) Σ is not of the form $\Gamma_1 E(Y_1, \ldots, X, \ldots, Y_n)\Gamma_2 \vdash C$, where X is a t-structure and some Y_i is not a t-structure.

Then every consecution in Der satisfies (1), (2), and (3).
Proof. By induction on height.

Of course, (1), (2), and (3) above are conditions which must be met by consecutions occurring in a subderivation of a proof of a t-free formula. Now, this fact can be used to show by induction on n:

LEMMA 1. *Vanishing-t Lemma.* Let X be a t-structure and let Σ be a consecution satisfying conditions (1), (2), and (3) of Lemma 5.1. If $\Sigma = \Gamma_1(X; Y)\Gamma_2 \vdash C$ is derivable in either L'-system with weight n, or if $\Sigma = \Gamma_1(Y; X)\Gamma_2 \vdash C$ is derivable in $L'RW_+^{0t}$ with weight n, then $\Sigma' = \Gamma_1 Y \Gamma_2 \vdash C$ is derivable with weight $\leqslant n$ – where Y is possibly empty if Γ_1 and Γ_2 are.

The Vanishing-t Lemma makes short work of a proof (by induction on weight) of:

LEMMA 2. A t-free sequent is derivable in $L'TW_+^{0t}$ ($L'RW_+^{0t}$) iff there is a t-free derivation of it.

So by Theorem 2, Lemma 2 and known conservative extension results:

THEOREM 3. If t is not a subformula of A, then A is a theorem of TW_+^0 (RW_+^0) iff there is a t-free proof of A in $L'TW_+^{0t}$ ($L'RW_+^{0t}$).

Then drop t from the language and let LTW_+^0 and LRW_+^0 come from the corresponding L'-systems by dropping the axioms $t \vdash t$ and $\vdash t$ and the t-rules. Obviously,

THEOREM 4. A is provable in LTW_+^0 (LRW_+^0) iff A is a theorem of TW_+^0 (RW_+^0).

Reformulation 2

With t out of the way, we now turn to the next problem for decidability, namely the nesting of e-sequences. Even if e-sequences were limited to reduced form, as they soon shall be, there are still an infinite number of distinct e-sequences that can be built up from a single formula, e.g.,

$$E(p,p), E(p,E(p,p)), E(p,E(p,E(p,p))), \ldots .$$

To circumvent this difficulty, let $L'TW_+^0$ ($L'RW_+^0$) come from LTW_+^0 (LRW_+^0) by adding the further extensional weakening and contraction rules:

$$K'e \vdash \qquad \frac{\Gamma_1 E(X_1, \ldots, X_n)\Gamma_2 \vdash C}{\Gamma_1 E(X_1, \ldots, X_n, Y)\Gamma_2 \vdash C}$$

$$W'e \vdash \qquad \frac{\Gamma_1 E(X_1, \ldots, X_n, Y, Y)\Gamma_2 \vdash C}{\Gamma_1 E(X_1, \ldots, X_n, Y)\Gamma_2 \vdash C}, \quad n \geqslant 1.$$

We call $Ke \vdash$ and $K'e \vdash$ (for instance) the *companion* of one another. Naturally, $K'e \vdash$ is the *prime companion* of $Ke \vdash$.

It is easy to see that the prime rules are admissible in the L-systems (use the companion rule and $ee \vdash$); and of course, the L-systems are subsystems of the corresponding L'-systems. So by Theorem 4:

THEOREM 5. *L' Equivalence Theorem.* $X \vdash A$ is derivable in $L'TW_+^0$ ($L'RW_+^0$) just in case it is derivable in LTW_+^0 (LRW_+^0). Hence, A is provable in $L'TW_+^0$ ($L'RW_+^0$) iff it is provable in TW_+^0 (RW_+^0).

Now let us say that a structure X is *denested* just in case it has no substructure of the form $E(Y_1, \ldots, E(W_1, \ldots, W_m), \ldots, Y_n)$. Then for any structure X, define the *denestation* of X (dn(X)) as follows:

(1) $\text{dn}(A) = A$, for any formula A;

(2) $\text{dn}(X; Y) = \text{dn}(X); \text{dn}(Y)$;

(3) $\text{dn}(E(X_1, \ldots, X_n)) = E(\text{dn}(X_1), \ldots, \text{dn}(X_n))$, where no X_i is an e-sequence;

(4) $\text{dn}(E(X_1, \ldots, E(Y_1, \ldots, Y_m), \ldots, X_n)) = \text{dn}(E(X_1, \ldots, Y_1, \ldots, Y_m, \ldots, X_n))$.

And let us say that a consecution is denested just in case its antecedent is (or is empty); and for any consecution $X \vdash A$, define $\text{dn}(X \vdash A)$ as $\text{dn}(X) \vdash \text{dn}(A)$, i.e., $\text{dn}(X) \vdash A$. (Of course, $\text{dn}(\vdash A) = \vdash \text{dn}(A) = \vdash A$.)

The reader will no doubt have noticed that for any consecution Σ, $\text{dn}(\Sigma)$ either is Σ or follows from it by one or more applications of *ee* $\vdash$, in which case Σ as follows from $\text{dn}(\Sigma)$ by a sequence of applications of *ee* $\vdash$. So the following important fact is immediate.

FACT 2. *Denestation Fact.* A consecution is derivable iff its denestation is.

This fact shows that every consecution has an equivalent extensional canonical form. But the decidability argument will require that *derivations* have an extensional canonical form. So let us say that a derivation is *denested* just in case each consecution that occurs in it is denested. And let us say, that an occurrence of a substructure X of a structure Y is a *nested e-sequence* (in Y) just in case it is an occurrence of an e-sequence as an immediate constituent of an e-sequence (in Y). Some further facts and lemmas can now be gathered toward proving what is required.

FACT 3. If $Z = \Gamma_1 X \Gamma_2$ is such that the displayed occurrence of X is not a nested e-sequence, then $\text{dn}(\Gamma_1 X \Gamma_2) = \text{dn}(\Gamma_1\, \text{dn}(X) \Gamma_2) = \Delta_1\, \text{dn}(X) \Delta_2$ for some Δ_1 and Δ_2, with the "displayed occurrence of $\text{dn}(X)$" in $\Delta_1\, \text{dn}(X) \Delta_2$ corresponding, in the obvious sense, to the displayed occurrence of X in Z.

Proof. By a straightforward induction on complexity of Z which is left to the reader.

FACT 4. $\text{dn}(\Gamma_1 E(X_1, \ldots, E(Y_1, \ldots, Y_m), \ldots, X_n)\Gamma_2) = \text{dn}(\Gamma_1 E(X_1, \ldots, Y_1, \ldots, Y_m, \ldots, X_n)\Gamma_2)$.

Proof. Again by a straightforward induction on complexity.

Loosely speaking, what lies behind these two facts (and the proof of the upcoming lemma) is this: for any $\Gamma_1, \Gamma_2, X_1, \ldots, X_n$, there are some Δ_1, $\Delta_2, Y_1, \ldots, Y_m$ such that dn $(\Gamma_1 X_1, \ldots, X_n \Gamma_2) = \Delta_1 Y_1, \ldots, Y_m \Delta_2$. And further, Δ_1 and Δ_2 are functions of Γ_1 and Γ_2 only. That is, if dn $(\Gamma_1 X_1, \ldots, X_n \Gamma_2) = \Gamma_1 Y_1, \ldots, Y_m \Delta_2$, then dn $(\Gamma_1 Z_1, \ldots, Z_k \Gamma_2) = \Delta_1 W_1, \ldots, W_j \Delta_2$, for some $W_1, \ldots, W_j$.

LEMMA 3. *dn-Substitution Lemma.* Let Z be a structure containing an occurrence y of some structure Y such that y is not a nested e-sequence. By Fact 5, let $\mathrm{dn}(Z) = \Delta_1\, \mathrm{dn}(Y) \Delta_2$. Let y' be the displayed occurrence of $\mathrm{dn}(Y)$. Then for all structures X such that the substituted occurrence of X in $Z[X/y]$ is not a nested e-sequence, $\mathrm{dn}(Z[X/y]) = (\mathrm{dn}(Z))[\mathrm{dn}(X)/y']$.

Proof. By a long and tedious induction on the complexity of Z utilizing the previous facts.

This lemma can then be used to show by cases:

LEMMA 4. *Denestation Lemma.* If Σ follows from $\Sigma_1(\Sigma_2)$ by an application of a rule Ru, then either $\mathrm{dn}(\Sigma) = \Sigma_1$ or it follows from $\mathrm{dn}(\Sigma_1)$ $(\mathrm{dn}(\Sigma_2))$ by a sequence of applications of Ru and/or its companion (and possibly $Ce \vdash$), such that the conclusion of each such inference is denested.

Finally we have

THEOREM 6. *Denestation Theorem.* For any consecution Σ, Σ is derivable in $L'TW_+^0$ $(L'RW_+^{0t})$ iff $\mathrm{dn}(\Sigma)$ has a denested derivation.

Proof. Right to left is obvious by the Denestation Fact. Left to right proceeds by induction on the weight of derivation of Σ. The base step is simple and the cases for the inductive step are straightforward using the Denestation Fact and Denestation Lemma.

The essence of Gentzen's original argument for decidability in [7] lies in getting control over the length of sequences and hence the number of sequences that can occur in consecutions which could occur in a proof search tree for a given formula. We must get analogous control over the complexity of, and hence the number of, such structures. The Vanishing-t and Denestation Theorems were necessary, but unprecedented steps. What remains is more closely analogous to Gentzen's procedure.

However, we still have two distinct kinds of structures, where Gentzen had only one type of sequence. Moreover, since these types of structures can be nested within one another, we must in a sense get simultaneous control over the number of distinct e- and i-structures that can enter into a proof search tree. Happily, the needed "simultaneity" is not literal.

Reduction

The Denestation Theorem allows extensional sequences to be reduced more or less as in [7]. So let us say that a structure is ***reduced*** just in case no structure occurs more than twice as an immediate constituent of any given extensional substructure of it. Then a structure is ***e-reduced*** iff it is denested and reduced. Of course, a consecution is reduced (e-reduced) just in case its antecedent and consequent are (or its antecedent is empty); and a derivation is reduced (e-reduced) iff each consecution occurring therein is.

Next let us say that a structure is *super reduced* just in case it contains no e-sequence with two distinct immediate constituents that are occurrences of the same structure. Again, the definition is extended to consecutions in the obvious way. (Obviously a super reduced structure or consecution is reduced.) Then define the *super reduct* of any *denested* structure as follows:

(1) $\mathrm{sr}(A) = A$, for any formula A;

(2) $\mathrm{sr}(Y;Z) = \mathrm{sr}(Y);\mathrm{sr}(Z)$, for any structures Y and Z; and

(3) for any structures $Y_1, \ldots, Y_n$, $\mathrm{sr}(E(Y_1, \ldots, Y_n)) = \mathrm{sr}(Y_1)$, if for all $1 \leqslant i \leqslant n$, $Y_1 = Y_i$; otherwise, $\mathrm{sr}(E(Y_1, \ldots, Y_n)) = E(W_1, \ldots, W_m)$, where $E(W_1, \ldots, W_m)$ is as follows: For each Y_i, let k_i be the number of occurrences of $\mathrm{sr}(Y_i)$ as an immediate constituent of $E(\mathrm{sr}(Y_1), \ldots, \mathrm{sr}(Y_n))$. Then $E(W_1, \ldots, W_m)$ is the result of deleting the first $k_i - 1$ occurrences of $\mathrm{sr}(Y_i)$ for $E(\mathrm{sr}(Y_1), \ldots, \mathrm{sr}(Y_m))$.

What this tedious definition comes down to is the following: the super reduct of a structure is formed by deleting *all* repetitions of immediate constituents of e-sequences and working one's way out – making sure that structures remain denuded throughout the process. Naturally, for any formula A and denested structure X, $\mathrm{sr}(X \vdash A) = \mathrm{sr}(X) \vdash \mathrm{sr}(A) = \mathrm{sr}(X) \vdash A$; and $\mathrm{sr}(\vdash A) = \vdash A$.

Given $Ce\vdash$ and the extensional contraction and weakening rules, it is clear that

FACT 5. *The Super Reduction Fact.* A denested consecution is derivable iff its super reduct is.

But the fact gives no reduction control over entire derivations. What we want to show is that sr (Σ) has an e-reduced derivation if it has one at all. The following lemma will clear the way.

LEMMA 5. *Reduction Lemma.* Let Inf $\Sigma_1(\Sigma_2)/\Sigma_3$ be an instance of some rule *Ru*, such that $\Sigma_1(\Sigma_2)$ and Σ_3 are denested. Then either sr $(\Sigma_3) =$ sr (Σ_1) or sr $(\Sigma_3) =$ sr (Σ_2) or sr (Σ_3) follows from sr (Σ_1) (and/or sr (Σ_2)) by a sequence of applications of $Ce \vdash$, $Ke \vdash$, $K'e \vdash$, $We \vdash$, $W'e \vdash$, and/or at most one application of *Ru* (if *Ru* be distinct from the aforementioned rules), the conclusion of each of which is e-reduced. Further, if no antecedent of a premise nor of the conclusion of *Inf* is an e-sequence, then neither is the antecedent of the conclusion of any inference in the above-mentioned sequence.

Proof. By induction on the complexity of Σ.

THEOREM 7. *Reduction Theorem.* A denested consecution is derivable iff its super reduct has an e-reduced derivation.

Proof. Right to left is straightforward by the Super Reduction Fact. Left to right proceeds by induction on weight of derivation. The base step is simple, and the inductive step is straightforward by the Super Reduction Fact and the Reduction Lemma.

Degree and Decidability

The Reduction Theorem will provide a finite upper bound on the number of (denested) e-sequences built up from a finite number of formulae that need to be examined in a proof search provided that we have control over the number of intensional structures that can be built up from the said formulae and need to be considered. To take care of this problem we introduce a notion of degree. So define the ***degree*** (deg) of a formula as follows:

(1) $\deg(A) = 1$, if A is an atom;

(2) $\deg(B \,\&\, C) = \deg(B \vee C) = \deg(B) + \deg(C)$, for any formulae B and C; and

(3) $\deg(B \rightarrow C) = \deg(B \circ C) = \deg(B) + \deg(C) + 1$, for any formulae B and C.

Noting that the degree of a formula is supposed to indicate its *intensional* complexity, the definition is obviously felicitous. And given that the structural connective is "standing in" for fusion, it is clear that the degree of a structure should be defined as follows:

(1) $\deg(A)$ is of course the degree of the formula A as defined above;

(2) $\deg(X; Y) = \deg(X) + \deg(Y) + 1$, for any structures X and Y; and

(3) $\deg(E(X_1, \ldots, X_n)) = \max\{\deg(X_1), \ldots, \deg(X_n)\}$, for any structures $X_1, \ldots, X_n$.

And $\deg(X \vdash A) = \deg(X) + \deg(A)$, and $\deg(\vdash A) = \deg(A)$.

Now let us say that a rule is degree preserving just in case for any instance of the rule, the degree of the conclusion is greater than or equal to that of any premise. Then it is clear that

LEMMA 6. *Degree Lemma.* The rules of $L'RW_+^0$ and $L'TW_+^0$ are degree preserving.

Now reduction and degree will work in tandem to give us the needed control on the total complexity of structures that could occur in an e-reduced derivation of a given consecution. The virtual coup de grace is delivered by

LEMMA 7. *Counting Lemma.* For any formula A and any $n \geqslant 0$, there are at most finitely many e-reduced structures of degree $\leqslant n$ built up from subformulae of A.

Proof. By induction on n. The base step is trivial. So choose an arbitrary $m > 0$ and assume:

INDUCTIVE HYPOTHESIS (IH). For any formula B and any $k < m$, there are at most finitely many e-reduced structures of degree $\leqslant k$ built up from subformulae of B. Now choose an arbitrary formula A. It will then suffice to show that there are at most finitely many e-reduced structures of degree $\leqslant m$ built out of subformulae of A.

But any such structure is either

(1) a subformula of A, of which there are only finitely many;

(2) an intensional structure, whose left and right constituents are of degree $< m$ (by the definition of degree) and of course are built out of subformulae of A. But by IH there are at most finitely many such structures to serve as left and right constituents. Whence there are but finitely many intensional structures of the required kind; or

(3) an e-sequence, each of whose immediate constituents is a *non-extensional* structure of degree $\leqslant m$ (by the definition of e-reduced and degree) and again built out of subformulae of A. By IH and (1) and (2) above, there are at most finitely many structures to serve as immediate constituents; and by the definition of e-reduced, none can occur more than twice as such. So there are at most finitely many e-sequences of the requisite sort.

And finitely many + finitely many + finitely many = finitely many. So we are finished.

Of course, the lemma holds equally well for e-reduced consecutions built up from subformulae of any of a finite number of formulae.

Decidability is now clearly in sight. All that remains to be shown are well-known and/or by now obvious facts. First, let us say that a derivation is *irredundant* just in case no consecution occurs more than once on a branch thereof. Recalling the Denestation and Reduction Theorems, it is clear that

THEOREM 8. *Irredundancy Theorem.* Any sequent Σ is derivable iff $\mathrm{sr}(\mathrm{dn}(\Sigma))$ has an irredundant, e-reduced derivation.

Next, let us specify as follows a proof search procedure which produces the LRW_+^0 (LTW_+^0) proof search tree of Σ for any consecution Σ:

(1) Enter $\mathrm{sr}(\mathrm{dn}(\Sigma))$ as the bottom node;

(2) above each Σ' occurring with height k (in the tree so far constructed) (a) enter nothing, if Σ' is an axiom, (b) otherwise enter (in some assumed order) all e-reduced Σ'' such

that Σ'' is a premise of some $L'RW_+^0$ ($L'TW_+^0$) rule of which Σ' is the conclusion and such that the tree remains irredundant.

Obviously

LEMMA 8. *Effectiveness Lemma.* The proof search procedure thus specified is effective.

Now let us say that a (possibly null) tree T' is a *subtree* of a tree T iff it is the result of deleting some (possibly no) consecution occurrences in T and all consecution occurrences above them. Then by the Irredundancy Theorem and the above specification

LEMMA 9. *Completeness Lemma.* The proof search procedure is complete, i.e., Σ is derivable iff some subtree of the proof search tree of Σ is a derivation of sr (dn (Σ)).

Now, by inspection of the rules

LEMMA 10. The proof search tree of any consecution has the finite fork property.

Of course $L'TW_+^0$ and $L'RW_+^0$ have the Subformula Property. But more important for our purposes

LEMMA 11. For any rule of these systems, every formula constituent of a premise thereof is a subformula of a formula constituent of the conclusion.

At last we have

LEMMA 12. The proof search tree of any consecution has the finite branch property.

Proof. Choose an arbitrary consecution, say Σ, and let $m =$ deg (sr (dn (Σ))). By the Counting Lemma there are at most finitely many e-reduced structures of degree $\leqslant m$ built up from the subformulae of the formulae occurring in (sr (dn (Σ))). Whence by the specification of the proof search procedure, Lemma 10 and the Degree Lemma, there can be but a finite number of different consecutions (occurring no more than once) on any given branch of the proof search tree of E – which completes the proof.

So by the Lemmas 10 and 12 and König's Lemma

LEMMA 13. *Finitude Lemma.* The proof search tree of any consecution is finite.

Finally, by the Effectiveness, Completeness and Finitude Lemmas we get out main result

THEOREM 9. $L'TW_+^0$ and $L'RW_+^0$ are decidable.

Whence by the L′ Equivalence Theorem

THEOREM 10. TW_+^0 and RW_+^0 are decidable.

Given known conservative extension results, it follows that TW_+, RW_+, $T_{\to\&}$, $R_{\to\&}$, PW and $RW_\to$ are likewise decidable. However, the reader will have noted the conspicuous absence of EW_+ so far. The straightforward way to gentzenise EW_+^{0t} is to add

$$Clt \vdash \qquad \frac{\Gamma_1(X;t)\Gamma_2 \vdash C}{\Gamma_1(t;X)\Gamma_2 \vdash C}$$

to the formulations for TW_+^{0t}. The resulting systems are equivalent to EW_+^{0t}. However, the proof of Vanishing-t breaks down. So, not being able to eliminate $t - \vdash$, the decidability argument breaks down since that rule is not degree preserving. So the decision question for EW_+ remains open at this point.[2]

NOTES

[1] These results were claimed in [11] on the basis of subscripted Gentzen systems and an argument for decidability for them. However [9] shows that those systems are not equivalent to TW_+ and RW_+, and that the decidability argument is in fact unsound.

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