Nonlinear Buckled States of Rectangular Plates

GEORGE H. KNIGHTLY & D. SATHER

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Abstract

A constructive method is developed to establish the existence of buckled states of a thin, flat elastic plate that is rectangular in shape, simply supported along its edges, and subjected to a constant compressive thrust applied normal to its two short edges. Under the assumption that the stress function and the deformation of the plate are described by the nonlinear von Kármán equations, the approach used yields information regarding not only the number of buckled states near an eigenvalue of the linearized problem, but also the continuous dependence of such states on the load parameter and the possible selection of that buckled state "preferred" by the plate. In particular, the methods used provide a rigorous approach to studying the existence of buckled states near the first eigenvalue of the linearized problem (that is, near the "buckling load") even when the first eigenvalue is not simple.

1. Introduction

In this paper we study the nonlinear deflections of a thin, flat elastic plate that is rectangular in shape, simply supported along its edges and subjected to a constant compressive thrust applied normal to its two short edges. Our approach is a constructive one that, in some cases, yields information regarding not only the number of buckled states near an eigenvalue of the linearized problem but also the *continuous* dependence of such states on the load parameter and the possible selection of that buckled state "preferred" by the plate. In particular, we show how to treat cases wherein the first eigenvalue of the linearized problem (that is, the so-called "buckling load") is not simple but has multiplicity $n=2$; since the first eigenvalue cannot have multiplicity greater than two, the methods of the present paper together with some well known results for the simple eigenvalue case provide a rigorous approach to studying the multiplicity of buckled states near the buckling load of a rectangular plate.

The stress function and the deformation of the plate are assumed to be described by a coupled pair of nonlinear partial differential equations (the von Kármán equations) together with boundary conditions suitable for the simply supported plate. By using some techniques related to, but somewhat simpler than, those in BERGER & FIFE [5] it is possible to reformulate such a problem in an appropriate *real* Hilbert space $\mathcal{W}$ so that the basic problem of determining the buckled states of the plate is equivalent to one of finding nontrivial solutions of a single operator equation of the form

$$w - \lambda L w + C(w) = 0, \quad w \in \mathcal{W} \tag{1.1}$$

where $L: \mathcal{W} \to \mathcal{W}$ is a linear, self-adjoint, positive, compact operator, λ measures the edge load, and $C: \mathcal{W} \to \mathcal{W}$ is a continuous, homogenous polynomial operator of degree three which is the gradient of the functional $\tau(w) = \frac{1}{4}(C(w), w)$.

Let $\lambda_0 > 0$ be an eigenvalue of the linear problem $w - \lambda L w = 0$ and let the null space $\mathcal{N} \subset \mathcal{W}$ of $I - \lambda_0 L$ be n-dimensional. When $n = 1$ a standard application (e.g., see [9, page 56]) of methods used here shows that there is exactly one pair of nontrivial buckled states associated with λ_0. Accordingly, we consider in this paper only the case $n \geqq 2$. Then, since $\mathcal{N}$ is finite dimensional and L is self-adjoint, the method of LYAPUNOV-SCHMIDT (e.g., see [7; 9; 10; 11]) reduces the problem of finding solutions of equation (1.1) in $\mathcal{W}$ to that of finding sufficiently small solutions $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in $\mathbb{R}^n$ (Euclidean n-space) of a system

$$-\eta \xi_i + \left(C\left(V + \sum_{j=1}^{n} \xi_j v_j \right), v_i \right) = 0 \quad (i = 1, 2, \ldots, n). \tag{1.2}$$

Here $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for $\mathcal{N}$,

$$\eta = \frac{\lambda}{\lambda_0} - 1 \tag{1.3}$$

is a real parameter, and $V = V(\xi, \eta)$ is an element of $\mathcal{N}^{\perp}$ (the orthogonal complement of $\mathcal{N}$ in $\mathcal{W}$) which is analytic in ξ and η for $|\xi| < \rho_0$ and $|\eta| < \eta_0$ (e.g., see [11, p. 19]) and satisfies

$$\|V(\xi, \eta)\| \leqq K |\xi|^3, \quad |\xi| < \rho_0, \quad |\eta| < \eta_0 \tag{1.4}$$

where K is a constant depending only on ρ_0 and η_0. Thus the existence of buckled states in the problem is demonstrated by solving a system of n analytic equations involving the load parameter λ.

It is, of course, well known that finding nontrivial solutions $\xi = \xi(\eta)$ in $\mathbb{R}^n$ of the branching equations (1.2) may be a difficult problem; here, as in an earlier paper [7], the most definitive results are obtained when λ_0 has multiplicity two. For example, in the case of the plate of length $\sqrt{2}$ units and width one unit, the first eigenvalue (corresponding to the buckling load), $\lambda_0 = 9\pi^2/2$, has a two-dimensional null space $\mathcal{N}$. In §3 we shall show that on the unit circle S in $\mathcal{N}$ the functional $(C(w), w)$ assumes extreme values at four distinct pairs of points $\pm u_i$ $(i = 1, 2, 3, 4)$. If these extreme values θ_i are defined by

$$\theta_i = (C(u_i), u_i) \quad (i = 1, 2, 3, 4) \tag{1.5}$$

then $\min_{w \in S} (C(w), w) = \theta_1 < \theta_2 < \theta_3 = \theta_4 = \max_{w \in S} (C(w), w)$ and there are eight corresponding buckled states of the form

$$\pm w_i = \pm(\eta^{1/2} \theta_i^{-1/2} u_i + U_i), \quad 0 \leqq \eta \leqq \eta_1 \quad (i = 1, 2, 3, 4), \tag{1.6}$$

where U_i is an analytic function of $\eta^{1/2}$ and $\lim_{\eta\to 0^+} \eta^{-1/2} U_i = 0$. Moreover, for small positive η the potential energies $E(w_i)$ satisfy $E(w_1) < E(w_2) < E(w_j)$ $(j=3, 4)$ so that, under the assumption that a "Principle of Least Energy" holds, the buckled states $\pm w_1$ would be "preferred" near $\lambda_0 = 9\pi^2/2$ to the remaining six buckled states. Similar results hold, for example, for the eigenvalue $\lambda = 25\pi^2/4$ in the case of a rectangular plate of length two units and width one unit.

Our results supplement earlier work by BAUER & REISS [3] on rectangular plates and also the more general existence theorem of BERGER & FIFE [5]. In [3] buckled states are computed corresponding to the first few eigenvalues and, in particular, states related to those for $i=1, 2$ in (1.6) are found. In [5] for quite general plates it is shown that (at least) one pair of buckled states exists near each eigenvalue λ_0; these solutions are shown to be continuous in λ only at λ_0. Thus our proof that the states (1.6) actually bifurcate from the zero solution at $\lambda_0 = 9\pi^2/2$ and are continuous for λ in some interval with endpoint λ_0 may also be of interest.

Let us emphasize here that this continuous dependence on the load parameter λ is a stronger regularity result than that usually obtained by topological methods alone, and seems to require some new techniques. In order to establish such a continuous dependence result in this paper, we employ an implicit-function theorem due to MACMILLAN [8] and BLISS [6], which is based upon the WEIERSTRASS preparation theorem for systems of analytic functions.

This implicit-function theorem argument has wider applicability (e.g., see [10]) than the simply-supported rectangular plate problem considered here and hence yields continuous dependence results in other branching situations also. For example, in [7] the authors treated the problem of a clamped plate with the aid of finite-dimensional (topological) degree theory. The argument given there in the proof of Theorem 2 is incomplete and requires an additional step to establish the continuity, in the load parameter, of the solution, but the methods employed in the present paper can be used to give a full proof of that result.

2. Formulation of the Problem SSP

The mathematical model adopted here for the plate problem is a dimensionless one determined in [3] by BAUER & REISS. Let $\Omega = \{(x, y): 0 < x < l, 0 < y < 1\}$ correspond to the middle plane of the undeflected plate and let $\partial\Omega$ denote its boundary. We suppose that the deformations of the plate are described by the following dimensionless version of the von Kármán equations

(vKa) $$\Delta^2 f = -\tfrac{1}{2}[w, w]$$

(vKb) $$\Delta^2 w = [w, f] - \lambda w_{xx},$$

where Δ denotes the Laplacian with respect to x and y and

(2.1) $$[u, v] = u_{xx} v_{yy} + u_{yy} v_{xx} - 2 u_{xy} v_{xy}.$$

Here $w = w(x, y)$ is a measure of the deflection of the middle plane of the plate out of the x, y-plane, $f = f(x, y)$ is an "excess" stress function corresponding to the effects of deformation on stress, and the parameter λ is a measure of the load. The determination of w and f in Ω by means of the two coupled nonlinear partial differential equations (vK), together with the "simply supported" boundary conditions

$$\text{(bc)} \qquad w=f=\Delta w=\Delta f=0 \quad \text{on } \partial\Omega$$

shall constitute the classical problem SSP.

Definition. A classical solution of the problem SSP is a pair of functions w, f that belong to $C^4(\Omega)\cap C^2(\bar{\Omega})$ and satisfy (vK) and (bc) pointwise.

In order to define a generalized solution of problem SSP we first introduce a real Hilbert space $\mathscr{W}$ defined as the closure, in the norm $\| \;\|_{2,2}$ of the Sobolev space $W_{2,2}(\Omega)$, of the set of smooth functions defined on $\bar{\Omega}$ and vanishing on $\partial\Omega$. Then it follows that $\mathscr{W}$ consists of functions in $W_{2,2}(\Omega)$ which, by the Sobolev embedding theorem, are continuous on $\bar{\Omega}$ and vanish on $\partial\Omega$. A more convenient norm and inner product for $\mathscr{W}$ may be obtained in the following way. Since

$$\int_\Omega (\Delta u)^2 = \int_\Omega (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) \qquad u\in\mathscr{W}$$

and the bilinear form $(u, v) = \int_\Omega \Delta u \Delta v$ is coercive over $\mathscr{W}$ (e.g., see [5, p. 232]), there exists a constant K_1 such that

$$\text{(2.2)} \qquad K_1 \|u\|_{2,2}^2 \leqq \int_\Omega (\Delta u)^2 \leqq \|u\|_{2,2}^2$$

for all $u\in\mathscr{W}$. Thus, the bilinear form (u, v) may be used as an inner product on $\mathscr{W}$; throughout the paper we denote the corresponding norm by $\| \;\|$. We point out that, although our notation and setting are somewhat similar to those in [5] and [7], the space $\mathscr{W}$ here is not identical to spaces in either of those papers (a similar situation exists for the operators to be defined in $\mathscr{W}$).

Let φ, Ψ be smooth functions in $\mathscr{W}$. Then upon multiplying equation (vKa) by φ and equation (vKb) by Ψ, and integrating by parts over Ω, one obtains

$$\text{(2.3a)} \qquad (f, \varphi) = -\tfrac{1}{2} b(w, w; \varphi)$$

$$\text{(2.3b)} \qquad (w, \Psi) = b(f, w; \Psi) + \lambda c(w; \Psi),$$

where

$$\text{(2.4)} \qquad b(u, v; \varphi) = \int_\Omega [(u_{xy} v_y - u_{yy} v_x)\varphi_x + (u_{xy} v_x - u_{xx} v_y)\varphi_y]$$

$$\text{(2.5)} \qquad c(u; \varphi) = \int_\Omega u_x \varphi_x.$$

The equations (2.3) suggest the following definition.

Definition. A generalized solution of the problem SSP is a pair of functions w, f in $\mathscr{W}$ satisfying (2.3a) and (2.3b) for all φ, Ψ in $\mathscr{W}$.

In Appendix B we sketch a proof that every classical solution of problem SSP is a generalized solution and, conversely, every generalized solution is a classical solution in Ω and on $\partial\Omega$ except at the corners; the arguments used are similar to those given in [5, p. 230 and Appendix A]. However, Appendix B may be of interest because it avoids the theory of non-Dirichlet boundary value problems by exploiting the special conditions of our particular problem. More specifically, because of the boundary conditions (bc) being considered here, one can reduce the indicated regularity problem to one for a coupled system of four equations in the four functions f, Δf, w, Δw and involving only Dirichlet boundary conditions, so that the L_p regularity theory for the Dirichlet problem may be applied.

Proceeding as in [5] and making use of (2.2) one can also show for any fixed $u, v \in \mathscr{W}$ that $b(u, v; \varphi)$ and $c(u; \varphi)$ are, in φ, bounded linear functionals on $\mathscr{W}$. From the RIESZ representation theorem it follows that the coupled system of equations (2.3) may be recast as two uncoupled operator equations in $\mathscr{W}$, namely

$$f = -\tfrac{1}{2} B(w, w) \tag{2.6}$$

$$w - \lambda L w + C(w) = 0. \tag{2.7}$$

Here $B\colon \mathscr{W} \times \mathscr{W} \to \mathscr{W}$ is a bounded bilinear operator and $L\colon \mathscr{W} \to \mathscr{W}$ is a bounded linear operator such that for all φ in $\mathscr{W}$

$$\big(B(u, v), \varphi\big) = b(u, v; \varphi) \qquad u, v \in \mathscr{W}, \tag{2.8}$$

$$(Lu, \varphi) = c(u; \varphi) \qquad u \in \mathscr{W}. \tag{2.9}$$

Then $C\colon \mathscr{W} \to \mathscr{W}$ is defined by

$$C(w) = \tfrac{1}{2} B\big(w, B(w, w)\big). \tag{2.10}$$

Thus, to determine a generalized solution of problem SSP, it is sufficient to determine a solution w in $\mathscr{W}$ of the single operator equation (2.7). However, application of the LYAPUNOV-SCHMIDT technique shows that finding solutions of equation (2.7) in $\mathscr{W}$ is equivalent to finding sufficiently small solutions of the system (1.2) in $\mathbb{R}^n$. Consequently, the question of existence of buckled states of problem SSP is reduced to a finite-dimensional problem.

3. The Branching Results for Problem SSP

The linearized eigenvalue problem associated with the generalized problem SSP is to determine nontrivial solutions of the equation

$$w - \lambda L w = 0 \qquad w \in \mathscr{W}. \tag{3.1}$$

This generalized eigenvalue problem can be solved completely for the rectangular plate. In fact it is well known that the classical problem

$$\begin{aligned} \Delta^2 w + \lambda w_{xx} &= 0 \quad \text{in } \Omega \\ w = \Delta w &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.2}$$

has the set of eigenfunctions $u_{mn} = C_{mn} \sin \dfrac{m\pi x}{l} \sin n\pi y$, $C_{mn}^{-1} = \pi^2 (m^2 + l^2 n^2)/2 l^{3/2}$, and associated eigenvalues $\lambda_{mn} = \pi^2 (m^2 + l^2 n^2)^2 / l^2 m^2$ $(m, n = 1, 2, \ldots)$. One sees readily that the compact operator L has the same eigenvalues and eigenfunctions and that, for any $l>0$, the first eigenvalue (which is equal to λ_{m1} for one of the two integers m "nearest" to l) cannot have multiplicity greater than two. In particular, let us note that in the case where $l=\sqrt{2}$, $\lambda_{11}=\lambda_{21}=9\pi^2/2$ is an eigenvalue of multiplicity two so that the null space $\mathcal{N}(I-\lambda_{11}L)$ is two-dimensional.

The following lemma shows that the analogs of Lemmas 1 and 2 in [7, pp. 69–70] hold for problem SSP.

Lemma 1. (a) *If* $\alpha, \beta, \gamma \in \mathcal{W}$, *then the form* $(B(\alpha, \beta), \gamma)$ *is symmetric in* α, β, γ. *In particular, if* $w \in \mathcal{W}$ *then*

$$(C(w), w)=\tfrac{1}{2}\|B(w, w)\|^2. \tag{3.3}$$

(b) *If* τ *is the real-valued functional defined by*

$$\tau(w)=\tfrac{1}{4}(C(w), w) \qquad w \in \mathcal{W} \tag{3.4}$$

then

1) *C is the (strong) gradient of τ (see* [4, p. 696])
2) *C is a continuous homogeneous polynomial operator of degree three (see* [11, p. 17]).
3) *$\tau(u)=0$ for $u \in \mathcal{N}(I-\lambda_{mn}L)$ if and only if $u=0$ (see* [5, p. 233]).

(c) *For each $w \in \mathcal{W}$ the nonlinear operator C has a differential D_w, which satisfies*

$$D_w(h)=B(w, B(w, h))+\tfrac{1}{2}B(h, B(w, w))$$

for all $h \in \mathcal{W}$ and is Lipschitz continuous in w (see [7, p. 70]).

Proof. As indicated, parts (b) and (c) are proved in the same way as related results in [4; 5; 7]. We sketch a proof of the important symmetry property in part (a); then the identity (3.3) and property (b1) are immediate consequences. Let α, β, γ be smooth functions in $\mathcal{W}$. Then upon using (2.4), (2.8) and integrating by parts one sees first of all that

$$(B(\alpha, \beta), \gamma)=\int_\Omega [\alpha, \beta]\gamma=\int_\Omega [\beta, \alpha]\gamma=(B(\beta, \alpha), \gamma).$$

The remaining relationship required, namely $(B(\alpha, \beta), \gamma)=(B(\alpha, \gamma), \beta)$, follows from (2.4) and (2.8) merely by rearranging the indicated terms. The symmetry property in part (a) is then obtained for α, β, γ in $\mathcal{W}$ by a standard limiting argument.

Let λ_{mn} be an eigenvalue of the linear problem (3.1), let $\eta=(\lambda/\lambda_{mn})-1$, let S denote the unit sphere in the null space $\mathcal{N}(I-\lambda_{mn}L)$ and let Q be the orthogonal projection of $\mathcal{W}$ onto $\mathcal{N}(I-\lambda_{mn}L)$. The following theorem is analogous to results in [7; 9] and is a consequence of the ordinary implicit-function theorem.

Theorem 1. *Suppose that $\mathcal{N}(I-\lambda_{mn}L)$ is k-dimensional. Let the functional $(C(u), u)$ restricted to S have a relative extremum at $u^* \in S$ and suppose that $\theta \equiv (C(u^*), u^*)$ is not an eigenvalue of QD_{u^*}. Then there exists a positive constant δ such that for $\lambda_{mn}<\lambda<\lambda_{mn}+\delta$ the equation* (2.7) *has a nontrivial solution of the*

form
$$w(\lambda)=(\eta/\theta)^{1/2}u^*+U^*,$$
where U^* *is an analytic function of* $\eta^{1/2}$ *and* $\lim_{\eta\to 0^+}\eta^{-1/2}U^*=0$.

Other sufficient conditions for the existence of nontrivial solutions can be formulated as in [7].

This theorem may be deduced using arguments given in [7] and [9] (see, in particular, Theorem 4 and Remark 5 in [9] and Theorem 1 in [7]) and we provide here only the central ideas. Instead of equation (2.7) it is sufficient to consider the associated system (1.2) in $\mathbb{R}^k$. If one sets $\xi_i=\eta^{1/2}\beta_i$ $(i=1, \ldots, k)$ in (1.2) and divides by $\eta^{3/2}$, then the resulting system of equations may be written as

$$f^i(\beta,\eta)\equiv -\beta_i+\left(C\left(\eta^{-1/2}V+\sum_{j=1}^{k}\beta_j v_j\right), v_i\right)=0 \qquad (i=1, \ldots, k). \tag{3.5}$$

Because of the extremum property of u^* it is possible to find a special orthonormal basis $v_1, \ldots, v_k$ for $\mathcal{N}(I-\lambda_{mn}L)$ with $v_1=u^*$ and such that

$$\big(C(v_1), v_j\big)=0 \qquad (j=2,3,\ldots,k)$$
and
$$\big(D_{v_1}(v_i), v_j\big)=0 \qquad \text{if } i\neq j.$$

With this special basis and with $\eta=0$ the functions $f^i(\beta, 0)$ become, for $i=1, \ldots, k$,

$$\begin{aligned} f^i(\beta,0)&=-\beta_i+\left(C\left(\sum_{j=1}^{k}\beta_j v_j\right), v_i\right)\\ &=-\beta_i+\big(C(u^*),u^*\big)\beta_1^3\delta_{i1}+\{\text{terms vanishing if } (\beta_2,\ldots,\beta_n)=(0,\ldots,0)\}. \end{aligned}$$

It follows that $(\beta_1, \beta_2, \ldots, \beta_k)=(\theta^{-1/2}, 0, \ldots, 0)$ satisfies

$$f^i(\beta,0)=0 \qquad (i=1,2,\ldots,k) \tag{3.6}$$

and that the Jacobian $J\equiv\partial(f^1, \ldots, f^k)/\partial(\beta_1, \ldots, \beta_k)$ at this point is

$$J=\prod_{i=2}^{k}\left[-1+\big(D_{u^*}(v_i), v_i\big)/\theta\right]. \tag{3.7}$$

Since θ is not an eigenvalue of QD_{u^*}, one sees easily that $J\neq 0$ and that the existence of a solution of (3.5) of the form $\beta^*(\eta)=\theta^{-1/2}+b(\eta^{1/2})$, where b is an analytic function of $\eta^{1/2}$ and $\lim_{\eta\to 0^+}b(\eta^{1/2})=0$, follows from the implicit function theorem. Clearly such a solution β^* generates also a solution $\xi^*(\eta)=\eta^{1/2}\beta^*(\eta)$ of (1.2), which in turn generates the desired solution w^* of (2.7).

The number of relative extrema of $(C(u), u)$ on S satisfying the eigenvalue condition of Theorem 1 may, of course, be determined by direct calculation for specific eigenvalues λ_{mn}. For example, if $l=\sqrt{2}$ then the first eigenvalue $\lambda_{11}=\lambda_{21}=9\pi^2/2$ has multiplicity two and we have the following corollary to Theorem 1 (see also Remark 3 below):

Corollary 1. *There exists a positive constant* δ *such that for* $\lambda_{11}<\lambda<\lambda_{11}+\delta$ *problem* SSP *possesses eight buckled states which are of the form* (1.6) *and depend analytically on* $(\lambda-\lambda_{11})^{1/2}$.

One way to prove the corollary is to exhibit four pairs of points on S at each of which the functional $(C(w), w)$ restricted to S has an extremum that satisfies the eigenvalue condition of Theorem 1. Such a program is feasible because each w in S has the form $w=pu_{11}+qu_{21}$ with $p^2+q^2=1$ so that, by using the bilinearity of the operator B, $(C(w), w)$ can be written as

$$(3.8)\qquad (C(w), w)=ap^4+2bp^2q^2+cq^4, \quad (p^2+q^2=1).$$

Here

$$(3.9)\qquad \begin{aligned} a&=\tfrac{1}{2}\|B(u_{11}, u_{11})\|^2\\ b&=\|B(u_{11}, u_{21})\|^2+\tfrac{1}{2}(B(u_{11}, u_{11}), B(u_{21}, u_{21}))\\ c&=\tfrac{1}{2}\|B(u_{21}, u_{21})\|^2 \end{aligned}$$

and we have also used the relationships

$$(3.10)\qquad (B(u_{11}, u_{11}, B(u_{11}, u_{21}))=(B(u_{21}, u_{21}), B(u_{11}, u_{21}))=0.$$

Setting $q^2=1-p^2$ in (3.8) and completing the square, we obtain

$$(C(w), w)=-F(p^2-G)^2+K, \quad (-1\leqq p\leqq 1),$$

where $F=(2b-a-c)$, $G=(b-c)/F$ and $K=(b^2-ac)/F$. If $b>a>c$, it follows that, on S, $(C(w), w)$ has extrema at precisely eight points: a minimum θ_1 at $\pm u_1\equiv\pm u_{21}$, a minimum $\theta_2>\theta_1$ at $\pm u_2\equiv\pm u_{11}$ and a single maximum value θ_3 assumed at four points

$$\pm u_3\equiv\pm[G^{1/2}u_{11}+(1-G)^{1/2}u_{21}] \quad\text{and}\quad \pm u_4\equiv\pm[G^{1/2}u_{11}-(1-G)^{1/2}u_{21}].$$

Moreover, again using the bilinearity of B, one can in this case reduce the system (3.6) to

$$(3.11\text{a})\qquad f^1(\beta, 0)=\beta_1[-1+a\beta_1^2+b\beta_2^2]$$

$$(3.11\text{b})\qquad f^2(\beta, 0)=\beta_2[-1+b\beta_1^2+c\beta_2^2]$$

so that the eight points $\pm u_i$ correspond to the eight intersections of the curves: $\beta_1=0$ and $-1+b\beta_1^2+c\beta_2^2=0$, $\beta_2=0$ and $-1+a\beta_1^2+b\beta_2^2=0$, $-1+a\beta_1^2+b\beta_2^2=0$ and $-1+b\beta_1^2+c\beta_2^2=0$. Since the inequalities $b>a>c$ imply that the normals to these curves are independent at the points of intersection, each of the points $\pm u_i$ must satisfy the eigenvalue condition of Theorem 1. Thus, it remains to establish the following lemma.

Lemma 2. *On S, $(C(w), w)$ has the form* (3.8) *with $b>a>c$.*

The proof of this lemma is given in Appendix A. However, let us observe here that the principal difficulty in establishing both (3.10) and $b>a>c$ lies in the fact that B is available only in the weak form (2.8) so that inner products of the form $(B(f, g), B(u, v))$ are not easily estimated. On the other hand, equation (2.8) (together with (2.4)) does enable us to calculate certain Fourier coefficients. with respect to the eigenfunctions $\{u_{mn}\}$ of (3.1), and this suffices for our purposes

Remark 1. It is natural to ask whether the eight buckled states of Corollary 1 are the only buckled states branching from the unbuckled state at λ_{11} and depending continuously on the load parameter λ. Our methods show that this is indeed the case. In fact, Theorem 1 of [9] implies that every nontrivial solution of (2.7) continuous and vanishing at $\lambda=\lambda_{11}$ comes from a continuous solution $\beta(\eta)$ of (3.5) with $\beta(0)\neq 0$. These values $\beta(0)$ must therefore satisfy the system (3.11), which has only the eight nontrivial solutions corresponding to the points $\pm u_i (i=1,2,3,4)$ at which $(C(w),w)$ has extreme values.

The situation encountered in Theorem 1 and Corollary 1 represents the nondegenerate case in which the existence of solutions continuous in the eigenparameter can be inferred from the nonvanishing of certain Jacobians of the system $f^i(\beta,0)=0$ $(i=1,2,\ldots,k)$. The degenerate situation where such Jacobians do vanish is a more difficult problem; however, in the case of a two-dimensional null space, we shall present next an approach that yields continuous solutions even in the degenerate situation.

Let us then consider in more detail the case where the null space $\mathcal{N}(I-\lambda_{mn}L)$ is two-dimensional. Let $\{v_1, v_2\}$ be an orthonormal basis for $\mathcal{N}(I-\lambda_{mn}L)$. Then the system (1.2) can be written as

$$-\eta\xi_i+(C(\xi_1 v_1+\xi_2 v_2), v_i)+r^i(\xi,\eta)=0 \quad (i=1,2), \tag{3.12}$$

where the r^i are analytic for $|\xi|<\rho_1$ and $|\eta|<\eta_1$. The r^i are also higher order terms in the sense that if $|\eta|<\eta_0$ then $r^i(\xi,\eta)/|\xi|^3\to 0$ uniformly in η as $|\xi|\to 0$ (see (1.4) and part (c) of Lemma 1).

For $\eta\geqq 0$ it is again useful to make the substitution

$$\xi_i=\eta^{1/2}\beta_i \quad (i=1,2) \tag{3.13}$$

and to consider instead of (3.12) the system

$$f^i(\beta,\eta)\equiv -\beta_i+(C(\beta_1 v_1+\beta_2 v_2), v_i)+s^i(\beta,\eta^{1/2})=0 \quad (i=1,2), \tag{3.14}$$

where $s^i(\beta,\eta^{1/2})=\eta^{-3/2}r^i(\eta^{1/2}\beta,\eta)$, $s^i(\beta,\eta^{1/2})\to 0$ uniformly for β in compact subsets of $|\beta|<\rho_0/(\eta_0^{1/2})$, and the $s^i(\beta,\sigma)$ are analytic for $|\beta|<R=\rho_1/(\eta_1^{1/2})$ and $|\sigma|<\sigma_0$. By restricting η_1 to be sufficiently small we may assume also that $R>(C(v_1),v_1)^{-1/2}$.

The following lemma (see also [10, Sec. 4A]) will be useful in determining continuous solutions of the system (3.14). The lemma is due to MACMILLAN [8] and BLISS [6] and is based upon the Weierstrass preparation theorem for systems of real analytic functions. It is convenient to state the lemma in connection with the problem of the existence of solutions $x=x(\sigma)$ of systems of the form

$$\Phi^i(x,\sigma)=0 \quad x\in\mathbb{R}^n,\ \sigma\in\mathbb{R}^1\ (i=1,2,\ldots,n), \tag{3.15}$$

where the Φ^i are real and analytic in a ball $|x|^2+\sigma^2<\rho^2$ in $\mathbb{R}^{n+1}$.

Lemma 3. *Suppose that* $\Phi^i(x,0)=\varphi^i_{k_i}(x)+\rho^i(x)$ $(i=1,2,\ldots,n)$, *where the* $\varphi^i_{k_i}$ *are homogeneous polynomials of degree* k_i *and the* ρ^i *satisfy* $|\rho^i(x)|/(|x|^{k_i})\to 0$ *as* $|x|\to 0$. *Suppose that* $\prod_{i=1}^n k_i$ *is odd and the resultant of the homogeneous polynomials*

$\varphi^i_{k_i}$ does not vanish. Then there exist positive constants x_0 and σ_0 such that for $|x|<x_0$ and $|\sigma|<\sigma_0$ the system (3.15) *has at least one continuous real solution $x=x(\sigma)$ with $x(0)=0$.*

The next result implies that, even in the degenerate case, there are at least as many nontrivial solutions of (2.7) as there are distinct elements v in S corresponding to relative extrema of $(C(u), u)$.

Theorem 2. *Suppose that the null space $\mathcal{N}=\mathcal{N}(I-\lambda_{mn}L)$ is two-dimensional and suppose that*

$$(C(u), u) \not\equiv \text{constant for } u \text{ in } S \tag{3.16}$$

(that is, $(C(u), u) \not\equiv \text{const.}\, \|u\|^4$ for u in $\mathcal{N}$). Let the functional $(C(u), u)$ restricted to S have a relative extremum at u^* and set $\theta=(C(u^*), u^*)$. Then there exists a positive constant δ such that for $\lambda_{mn}<\lambda<\lambda_{mn}+\delta$ the equation (2.7) *has a nontrivial continuous solution of the form*

$$w^*(\lambda)=(\eta/\theta)^{1/2} u^* + U^* \tag{3.17}$$

where U^* depends continuously on η and $\lim_{\eta\to 0^+} \eta^{-1/2} U^*=0$. In particular, corresponding to the absolute maximum and minimum of $(C(u), u)$ on S, the equation (2.7) *has at least four nontrivial continuous solutions, two of which are analytic in $(\lambda-\lambda_{mn})^{1/2}$.*

Proof. Let $v_1=u^*$ and choose v_2 so that $\{v_1, v_2\}$ is an orthonormal basis for $\mathcal{N}$. If $b\equiv\theta$ is not an eigenvalue of QD_{v_1} then Theorem 1 applies and one sees easily that, in fact, v_1 generates a nontrivial continuous solution of the form (3.17). Therefore, let us suppose, without loss of generality, that the degenerate case $(D_{v_1}(v_2), v_2)=(C(v_1), v_1)\equiv b$ holds. By using the bilinearity of the operator B, the system (3.14) can be written as (see also [7, pp. 70–72])

$$f^1(\beta,\eta)=-\beta_1+b\beta_1^3+3e\beta_1^2\beta_2+b\beta_1\beta_2^2+d\beta_2^3+s^1(\beta,\eta^{1/2})=0 \tag{3.18a}$$

$$f^2(\beta,\eta)=-\beta_2+e\beta_1^3+b\beta_1^2\beta_2+3d\beta_1\beta_2^2+c\beta_2^3+s^2(\beta,\eta^{1/2})=0, \tag{3.18b}$$

where $c=(C(v_2), v_2)$, $d=(C(v_2), v_1)$ and $e=(C(v_1), v_2)$. But, since b is an extremum of $(C(u), u)$ on S, one necessarily has $c=d=0$ (see [7, (2.21a) and Lemma 3]). Thus it suffices to show that the system

$$f^1(\beta,\eta)=-\beta_1+b\beta_1^3+b\beta_1\beta_2^2+s^1(\beta,\eta^{1/2})=0 \tag{3.19a}$$

$$f^2(\beta,\eta)=-\beta_2+b\beta_1^2\beta_2+c\beta_2^3+s^2(\beta,\eta^{1/2})=0 \tag{3.19b}$$

has a continuous solution that generates a solution of (2.7) of the form (3.17).

Let us emphasize here that although the points $(\beta_1, \beta_2, \eta)=(\pm a, 0, 0)$ $(a\equiv b^{1/2})$ are nontrivial solutions of the system (3.19), $\partial(f^1, f^2)/\partial(\beta_1, \beta_2)$ vanishes at these points so that the ordinary implicit function theorem used to establish Theorem 1 does not apply.

In order to apply the more general implicit function theorem contained in Lemma 3, we first transform $\{f^1, f^2\}$ into a vector field $\Psi=\{\Psi^1, \Psi^2\}$ under the change of variables $\beta_1 = x_1 + a$ and $\beta_2 = x_2$. Simple calculations then yield

$$\begin{aligned}\Psi^1(x,\eta) &\equiv f^1(x_1+a, x_2, \eta)\\ &= 2x_1 + a^{-1}(3x_1^2 + x_2^2) + a^{-2}(x_1^3 + x_1 x_2^2) + t^1(x, \eta^{1/2})\end{aligned} \tag{3.20a}$$

$$\begin{aligned}\Psi^2(x,\eta) &\equiv f^2(x_1+a, x_2, \eta)\\ &= 2a^{-1} x_1 x_2 + a^{-2} x_1^2 x_2 + c x_2^3 + t^2(x, \eta^{1/2}),\end{aligned} \tag{3.20b}$$

where the $t^i(x,\sigma) = s^i(x_1+a, x_2, \sigma)$ are analytic in a ball $|x|^2 + \sigma^2 < \rho^2$ in $\mathbb{R}^3$, and satisfy $t^i(x,0)=0$. Since Lemma 3 does not apply to the vector field Ψ, we introduce another vector field $\Phi = \{\Phi^1, \Phi^2\}$ with $\Phi^1 \equiv \Psi^1$ and

$$\Phi^2(x,\eta) \equiv \Psi^2(x,\eta) - a^{-1} x_2 \Psi^1(x,\eta) = \varphi_3^2(x) + q^2(x, \eta^{1/2}), \tag{3.21}$$

where $\varphi_3^2(x) = (c-b)x_2^3 - 2a^{-2} x_1^2 x_2$ and $q^2(x,\sigma) = t^2(x,\sigma) - a^{-3}(x_1^3 x_2 + x_1 x_2^3) - a^{-1} x_2 t^1(x,\sigma)$. Clearly, if we set $\varphi_1^1(x) = 2x_1$ then the vector field Φ satisfies the hypotheses of Lemma 3 if and only if $b \neq c$. But if $b = c$ then (see [7, p. 71])

$$\big(C(\tau_1 v_1 + \tau_2 v_2), \tau_1 v_1 + \tau_2 v_2\big) = b(\tau_1^2 + \tau_2^2)^2 \tag{3.22}$$

so that $(C(u), u)$ is constant on S, contrary to assumption (3.16). Therefore, since $\Phi = 0$ if and only if $\Psi = 0$, Lemma 3 implies that the system (3.20) has at least one continuous real solution $x = x^*(\eta)$ $(0 \leqq \eta < \eta_1)$ with $x^*(0) = 0$. The solution $x^*(\eta)$ generates a nontrivial continuous solution $\beta^*(\eta) = (b^{-1/2} + x_1^*(\eta), x_2^*(\eta))$ of the system (3.19), which in turn generates the desired solution of the form (3.17).

Finally, proceeding as in the proof of Theorem 2 in [7], one can show that if $(C(u), u) \neq$ constant on S then either the absolute maximum or the absolute minimum of $(C(u), u)$ on S must correspond to the non-degenerate situation in which Theorem 1 applies. This completes the proof of the theorem.

Remark 2. Let us take the potential energy of a buckled state of problem SSP to be (e.g., see [3, p. 608])

$$E = \int_\Omega [(\Delta w)^2 - \lambda w_x^2 + (\Delta f)^2].$$

Then, in terms of a generalized solution of problem SSP, the energy may be written as

$$E(w) = (w,w) - \lambda(Lw, w) + \tfrac{1}{2}(C(w), w) \tag{3.23}$$

so that the potential energy of an unbuckled state is $E(0) = 0$, whereas the potential energy of a buckled state w_0 satisfies $E(w_0) = -\frac{1}{2}(C(w_0), w_0) < 0$. Let the functional $(C(u), u)$ restricted to S have relative extrema θ at u_1 and Θ at z_1, suppose that $\theta < \Theta$ and let u and z be the corresponding buckled states as determined in Theorem 2. Since $\lim_{\eta \to 0^+} \eta^{-1/2} u = \theta^{-1/2} u_1$ and $\lim_{\eta \to 0^+} \eta^{-1/2} z = \Theta^{-1/2} z_1$ and since

$$\big(C(\theta^{-1/2} u_1), \theta^{-1/2} u_1\big) = \frac{1}{\theta} > \frac{1}{\Theta} = \big(C(\Theta^{-1/2} z_1), \Theta^{-1/2} z_1\big)$$

then for λ sufficiently close to $\lambda_{mn}(\lambda>\lambda_{mn})$ we have $E(u)<E(z)$. Thus, under the assumption that a "Principle of Least Energy" holds (that is, that the plate selects a buckled state with minimum potential energy), the methods of the present paper predict that, among the buckled states generated by the relative extrema of $(C(w), w)$ on S, the "preferred" buckled states of problem SSP are those states that correspond to the absolute minimum of $(C(w), w)$ on S.

Remark 3. As an immediate consequence of Remark 2, we see that the buckled states $w_1, \ldots, w_4$ obtained in the Corollary to Theorem 1 satisfy

$$E(w_1)<E(w_2)<E(w_i) \quad (i=3,4).$$

(Since $E(w_3)$ and $E(w_4)$ are equal in the limit as $\eta\to 0^+$, their order is not determined by Remark 2). Thus, if a "Principle of Least Energy" holds, the buckled states $\pm w_1$ would be preferred near $\lambda_0=9\pi^2/2$ to the other six buckled states $\pm w_j$ $(j=2,3,4)$ (of course, the possibility remains that under this "Principle" the plate actually selects some state distinct from all the $\pm w_j$).

Appendix A

Proof of Lemma 2. Let us note first of all that the eigenfunctions $\{u_{mn}\}$ of (3.1) form a complete orthonormal set in $\mathscr{W}$. In fact, if $(v, u_{mn})=0$ $(m, n=1,2,\ldots)$ then $\int_\Omega u_{mn}\Delta v=0$ $(m,n=1,2,\ldots)$ so that, by the completeness of the $\{u_{mn}\}$ in $L_2(\Omega)$, $\Delta v=0$ in $\mathscr{L}^2(\Omega)$ which implies that $v=0$ in $\mathscr{W}$.

An elementary but lengthy calculation using (2.4) and (2.8) now reveals that $(B(u_{mn}, u_{pq}), u_{jk})=0$ unless $(m+p+j)$ is odd and $(n+q+k)$ is odd, in which case

$$(B(u_{mn}, u_{pq}), u_{jk}) = \frac{2^{31/4}\, m p j n q k[(m^2+p^2+j^2)(n^2+q^2+k^2)-}{\pi^4(m^2+2n^2)(p^2+2q^2)(j^2+2k^2)[m^4+p^4+j^4-2(m^2p^2+m^2j^2+p^2j^2)]\cdot} \frac{-2(m^2n^2+p^2q^2+j^2k^2)]}{\cdot[n^4+q^4+k^4-2(n^2q^2+n^2k^2+q^2k^2)]}.$$

Equation (3.10) now follows easily from Parseval's equation. In addition, if we set $\kappa=(9\pi^4)/(2^{31/4})$, $\alpha(j,k)\equiv\kappa(B(u_{11},u_{11}),u_{jk})$, $\beta(j,k)\equiv\kappa(B(u_{11},u_{21}),u_{jk})$ and $\gamma(j,k)\equiv\kappa(B(u_{21},u_{21}),u_{jk})$, then we find from (3.9) that

$$2\kappa^2 a=\sum_{j,k\text{ odd}}\alpha^2(j,k)$$

$$2\kappa^2 b=2\sum_{\substack{j\text{ even}\\ k\text{ odd}}}\beta^2(j,k)+\sum_{j,k\text{ odd}}\alpha(j,k)\gamma(j,k)$$

$$2\kappa^2 c=\sum_{j,k\text{ odd}}\gamma^2(j,k),$$

where the summation indices are positive and odd or even as indicated.

We consider separately the inequalities $a>c$ and $b>a$. The first of these follows if we can prove that

$$\text{(A.1)} \qquad \sum_{j,k\text{ odd}}\delta(j,k)>0,$$

where

$$\delta(\xi,\eta)\equiv\alpha^2(\xi,\eta)-\gamma^2(\xi,\eta)=\frac{12[\xi^4(2-\eta^2)+15\xi^2\eta^2-20\xi^2-32\eta^2]}{\eta^2(\eta^2-4)(\xi^2+2\eta^2)^2(\xi^2-4)^2(\xi^2-16)^2}.$$

The first quadrant of the (ξ,η)-plane is divided into six parts by the curves $\eta=2$, $(\xi^4-15\xi^2+32)\eta^2=2\xi^2(\xi^2-10)$ across each of which $\delta(\xi,\eta)$ changes sign; $\delta(\xi,\eta)$ is positive on three of these regions and negative on the remaining three. It is convenient to separate the set A of points (j,k) in the first quadrant satisfying $\delta(j,k)<0$ in (A.1) into four subsets $A_1,\ldots,A_4$ defined by

$$A_1=\{(\xi,\eta)\colon \xi,\eta \text{ odd integers}, \xi\geqq 5, \eta\geqq 5\}$$

$$A_2=\{(\xi,3)\colon \xi \text{ an odd integer}, \xi\geqq 5\}$$

$$A_3=\{(\xi,1)\colon \xi \text{ an odd integer}, \xi\geqq 3\}$$

$$A_4=\{(1,\eta)\colon \eta \text{ an odd integer}, \eta\geqq 3\}.$$

Then

$$\sum_{j,k\text{ odd}}\delta(j,k)\geqq\delta(1,1)-\sum_A|\delta(j,k)|.$$

One easily checks that $\delta(1,1)\geqq 78\times10^{-4}$ so that (A.1) is proved if $\sum_A|\delta(j,k)|<78\times10^{-4}$. This is verified by estimating separately the $\sum_{A_i}|\delta(j,k)|$ $(i=1,2,3,4)$.

We shall indicate the estimation procedure for $i=1$. Now

$$\delta(j,k)=\frac{1+\dfrac{1}{j^2}\left(\dfrac{32}{j^2}+\dfrac{20}{k^2}\right)-\left(\dfrac{2}{k^2}+\dfrac{15}{j^2}\right)}{\left(1-\dfrac{4}{k^2}\right)\left(1-\dfrac{4}{j^2}\right)^2\left(1-\dfrac{16}{j^2}\right)^2}\,\frac{-12}{j^4k^2(j^2+2k^2)^2}.$$

For any j,k there holds $j^2+2k^2>2\sqrt{2}\,jk$, so that, for $(j,k)\in A_1$, we have

$$|\delta(j,k)|\leqq 12\left[1+\frac{52}{625}\right]\left[8\left(\frac{21}{25}\right)^3\left(\frac{9}{25}\right)^2 j^6k^4\right]^{-1}<20j^{-6}k^{-4}.$$

Use of the integral test yields

$$\sum_{A_1}|\delta(j,k)|\leqq 20\int_4^\infty\int_4^\infty \xi^{-6}\eta^{-4}\,d\xi\,d\eta<5\times10^{-4}. \tag{A.2}$$

Similarly, one finds that $\sum_{A_2}|\delta(j,k)|<21\times10^{-4}$, $\sum_{A_3}|\delta(j,k)|<23\times10^{-4}$ and $\sum_{A_4}|\delta(j,k)|<8\times10^{-5}$. Hence $\sum_A|\delta(j,k)|<5\times10^{-3}$, which proves that $a>c$.

In order to prove that $b>a$ we obtain a lower bound for b and an upper bound for a by methods similar to those above. If A now denotes the set of odd, positive integer pairs (j,k) for which $\alpha(j,k)\,\gamma(j,k)<0$ then by plotting the curves, in the first quadrant of the (ξ,η)-plane, across which either $\alpha(\xi,\eta)$ or $\gamma(\xi,\eta)$ change sign, one finds that A consists of the points $(3,1)$ and $(3,k)$, $k=5,7,9,\ldots$. Furthermore

$$\sum_A\alpha(j,k)\gamma(j,k)=\alpha(3,1)\gamma(3,1)+\sum_{\substack{k\geqq5\\ k\text{ odd}}}\alpha(3,k)\gamma(3,k)$$

$$>-5.5\times10^{-4}-9\times10^{-3}\sum_{k\geqq5}k^{-6}>-6\times10^{-4}$$

so that

$$2\kappa^2 b > 2\beta^2(2,1) + \alpha(1,1)\gamma(1,1) - 6\times 10^{-4}$$
$$> 2(44\times 10^{-4}) + 74\times 10^{-4} - 6\times 10^{-4} > 15\times 10^{-3}$$

To show that $2\kappa^2 a < 15\times 10^{-3}$ it suffices to prove the following estimates:

(A.3) $$\alpha^2(1,1) < 124\times 10^{-4}$$

(A.4) $$\sum_{k\geqq 3,\,\text{odd}} \alpha^2(1,k) < 3\times 10^{-4}$$

(A.5) $$\sum_{j\geqq 3,\text{odd}} \alpha^2(j,1) < 10\times 10^{-4}$$

(A.6) $$\sum_{\substack{j\geqq 3,\,\text{odd}\\ k\geqq 3,\,\text{odd}}} \alpha^2(j,k) < 3\times 10^{-4}.$$

The estimate (A.3) follows by a direct calculation, while (A.4), (A.5) and (A.6) may be obtained in essentially the same way as (A.2). For example, using the estimate

$$\alpha^2(j,k) = \frac{\left(1-\frac{2}{j^2}-\frac{2}{k^2}\right)^2}{j^2k^2(j^2+2k^2)^2\left(1-\frac{4}{j^2}\right)^2\left(1-\frac{4}{k^2}\right)^2} \leqq \frac{81}{200}\,\frac{1}{j^4k^4}$$

valid for $j\geqq 3$, $k\geqq 3$, we have

$$\sum_{\substack{j\geqq 3,\,\text{odd}\\ k\geqq 3,\,\text{odd}}} \alpha^2(j,k) \leqq (5\times 10^{-1}) \sum_{\substack{j\geqq 3,\,\text{odd}\\ k\geqq 3,\,\text{odd}}} \frac{1}{j^4k^4} < (5\times 10^{-1})\left(\frac{1}{81}+\int_4^\infty \frac{d\xi}{\xi^4}\right)^2 < 3\times 10^{-4},$$

proving (A.6). This completes the proof of Lemma 2.

Appendix B

If we denote by $\partial'\Omega$ the boundary of Ω with the four corners deleted, then the regularity result we wish to prove may be stated as follows.

Theorem B.1. *Every classical solution of problem* SSP *is a generalized solution. Every generalized solution is a classical solution in Ω and on $\partial'\Omega$.*

For smooth φ, ψ in $\mathscr{W}$ equations (2.3a, b) follow from (vKa, b) by integration by parts. Then equations (2.3a, b) are obtained for general φ, ψ in $\mathscr{W}$ by a standard limiting argument, using the fact that $b(u, v; \varphi)$ and $c(u; \varphi)$ are continuous linear functionals of $\varphi\in\mathscr{W}$ for fixed $u, v\in\mathscr{W}$. This establishes the first statement in the lemma.

In the proof of the second part of the lemma, much of the detail is the same as in the proof of regularity in [5, p. 231 and Appendix A]. We refer the reader to that work as well as to the regularity theorems of Agmon in [1] and [2, §9].

For any open set G, $W_{m,p}(G)$ denotes the Sobolev space of functions which, together with their derivatives of order less than or equal to m, lie in $L_p(G)$. The norm in $W_{m,p}(G)$ is denoted by $\|u\|^G_{m,p}$. By Σ_R we mean any open half disc of radius R in Ω with boundary $\partial\Sigma_R = \partial_1\Sigma_R + \partial_2\Sigma_R$ where $\partial_1\Sigma_R$ is the straight part of $\partial \Sigma_R$ and $\partial_1 \Sigma_R \subset \partial \Omega$. We shall need the class of functions $\mathscr{B}(\Sigma_R)=\{\varphi \in C^\infty(\bar{\Sigma}_R)$: $\varphi=0$ on $\partial_1 \Sigma_R$, $\varphi \equiv 0$ in a neighborhood of $\partial_2 \Sigma_R$ and in $\Omega-\Sigma_R\}$ and the Hilbert space $\overset{\circ}{W}_{1,2}(\Sigma_R)$ which is the completion of $C_0^\infty(\Sigma_R)$ under the norm $\|\cdot\|_{1,2}^{\Sigma_R}$. Finally, in this appendix the inner product $(\psi, \varphi)_0$ is that of $L_2(\Omega)$.

Let f, w be a generalized solution of problem SSP. Since $f, w \in \mathscr{W}$ we recall that they are continuous on $\bar{\Omega}$ and vanish on $\partial \Omega$.

Now set $u=\Delta f$, $v=\Delta w$. Introducing the weak forms of these equations we obtain the following system of four second order (weak) Dirichlet problems in place of (2.3a, b):

(B.1) $$(f, \Delta \varphi)_0=(u, \varphi)_0$$

(B.2) $$(w, \Delta \varphi)_0=(v, \varphi)_0$$

(B.3) $$(u, \Delta \varphi)_0=-\tfrac{1}{2} b(w, w; \varphi)$$

(B.4) $$(v, \Delta \varphi)_0=b(f, w; \varphi)+\lambda c(w; \varphi)$$

which are valid for all φ in $\mathscr{W}$. The derivation of (B.1) and (B.2) also uses a limiting process, however, we omit the details.

Let Σ_R be a fixed half disc as described above. For $R'<R$, $\Sigma_{R'}$ henceforth denotes a half disc of the same type having the same center as Σ_R; in particular then $\Sigma_{R'} \subset \Sigma_R$ and $\partial_1 \Sigma_{R'} \subset \partial_1 \Sigma_R$. Each point of the rectangle Ω is interior to some such $\Sigma_{R'}$, except for the center when Ω is a square. The interior regularity theory needed for such a point is easier than the boundary regularity sketched here and will be omitted.

We may complete the proof of Theorem B.1 by proving the following sequence of facts. For all $R'<R$,

(i) $u, v \in W_{1,\frac{4}{3}}(\Sigma_{R'}) \subset L_4(\Sigma_{R'})$
(ii) $f, w \in W_{2,4}(\Sigma_{R'})$
(iii) $u, v \in W_{2,2}(\Sigma_{R'}) \subset C^0(\bar{\Sigma}_{R'})$
(iv) $u=v=0$ on $\partial_1 \Sigma_{R'}$
(v) $f, w \in W_{4,2}(\Sigma_{R'})$
(vi) $u, v \in W_{4,2}(\Sigma_{R'})$
(vii) $f, w \in W_{6,2}(\Sigma_{R'}) \subset C^4(\bar{\Sigma}_{R'})$.

From (iv) and (vii) it follows easily that $f, w \in C^4(\Omega \cup \partial' \Omega)$ and $\Delta f=\Delta w=0$ on $\partial' \Omega$; we have already seen that $f=w=0$ on $\partial \Omega$. That f, w satisfy equations (vKa, b) pointwise in Ω now follows as in [3, p. 231]. Thus it remains to verify (i)–(vii) above.

It is convenient to state here an *ad hoc* version of Theorem 6.2 in [1].

Lemma B.2. *Let $g \in L_q(\Sigma_R)$ for some $q>1$ and suppose that*

$$|(g, \Delta \varphi)_0| \leqq C \|\varphi\|_{2-j, p'}^{\Sigma_R}$$

for all $\varphi \in \mathscr{B}(\Sigma_R)$ and for some $p'>1$, some constant C, and an integer $j \in \{1, 2\}$. Let $p=\left(1-\frac{1}{p'}\right)^{-1}$. Then $g \in W_{j,p}(\Sigma_{R'})$ for every $R'<R$.

We now restrict φ in (B.1–4) to $\mathscr{B}(\Sigma_R)\supset\mathscr{B}(\Sigma_{R'})$. Then properties (i), (ii), and (iii) may be obtained successively by using Lemma B.2. For example, terms on the right side of (B.4) may be estimated as follows, using Hölder inequalities and the embedding $W_{1,2}(\Sigma_R)\subset L_4(\Sigma_R)$:

$$\begin{aligned}|(f_{xy}, w_y\varphi_x)_0| &\leqq \|f_{xy}\|_{0,2}^{\Sigma_R}\|w_y\|_{0,4}^{\Sigma_R}\|\varphi_x\|_{0,4}^{\Sigma_R}\\ &\leqq \tilde{C}\|f\|_{2,2}^{\Sigma_R}\|w\|_{2,2}^{\Sigma_R}\|\varphi\|_{1,4}^{\Sigma_R}.\end{aligned}$$

In this way (B.4) yields $|(v, \Delta\varphi)_0|\leqq C(\Sigma_R, f, w, \lambda)\|\varphi\|_{1,4}^{\Sigma_R}$ for all $\varphi\in\mathscr{B}(\Sigma_R)$ so that Lemma B.2 with $j=1$ and $p=4/3$ implies (i) for v; the result for u is proved in the same way (the inclusions in (i), (iii) and (vii) are from the Sobolev theorem). Property (ii) follows in the same way starting from (B.1, 2), using property (i) in estimating the right hand sides of (B.1, 2), and then employing Lemma B.2 with $j=2$ and $p=4$. From (ii) it follows that $[w, w]$ and $[w,f]-\lambda w_{xx}$ are in $L_2(\Sigma_{R'})$ so that integration by parts in the right sides of (B.3, 4) gives, for $\varphi\in\mathscr{B}(\Sigma_{R'})$,

(B.5) $$(u, \Delta\varphi)_0=(-\tfrac{1}{2}[w, w], \varphi)_0$$

(B.6) $$(v, \Delta\varphi)_0=([w,f]-\lambda w_{xx}, \varphi)_0$$

Property (iii) may be proved along the lines of the proof of (i) starting from (B.5, 6) and using Lemma B.2 with $j=2$ and $p=2$.

Because of (iii) we may, for $\varphi\in C_0^\infty(\Sigma_{R'})$, transfer the Laplacians on the left in (B.5, 6) from φ to u and v. It follows that in $L_2(\Sigma_{R'})$

(B.7) $$\Delta u=-\tfrac{1}{2}[w, w] \quad\text{and}\quad \Delta v=[w,f]-\lambda w_{xx}.$$

On the other hand, for $\varphi\in\mathscr{B}(\Sigma_{R'})$,

$$(u, \Delta\varphi)_0=(\Delta u, \varphi)_0+\int_{\partial_1\Sigma_{R'}} u\frac{\partial\varphi}{\partial n}$$

whereas from (B.5) and (B.7) we see that $(u, \Delta\varphi)_0=(\Delta u, \varphi)_0$. Hence $\int_{\partial_1\Sigma_{R'}} u\frac{\partial\varphi}{\partial n}=0$ for all $\varphi\in\mathscr{B}(\Sigma_{R'})$ and (iv) follows.

Properties (v), (vi) and (vii) are proved using Theorem 9.7 of [2]. (Hypothesis 2° of that theorem can be verified in each case since $f=w=u=v=0$ on $\partial_1\Sigma_R$ and $\partial\Sigma_R$ is piecewise smooth.) For example, (B.1) may be written as $\int_{\Sigma_{R'}}\nabla f\cdot\nabla\varphi=-(u, \varphi)_0$ for $\varphi\in\mathscr{B}(\Sigma_R)$ so that, because of property (iii), the theorem applies with $m=1$ and $k=2$; we conclude that property (v) holds. A consequence of (v) and the Sobolev embedding theorem is that the terms on the right sides in (B.7) lie in $W_{2,2}(\Sigma_{R'})$. Then property (vi) follows, in the same way as (v), starting from (B.5) and (B.6). By use of (vi), the process that led to (v) now yields (vii), which completes the proof.

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References

1. AGMON, S., The L_p approach to the Dirichlet problem. Ann. Scuola Norm. Sup. Pisa **13**, 405–448 (1959).
2. AGMON, S., Lectures on Elliptic Boundary Value Problems. Princeton, N. J.: Van Nostrand, 1965.
3. BAUER, L., & E. REISS, Nonlinear buckling of rectangular plates. SIAM J. **13**, 603–626 (1965).
4. BERGER, M., On von Kármán's equations and the buckling of a thin elastic plate. I. The clamped plate. Comm. Pure Appl. Math. **20**, 687–719 (1967).
5. BERGER, M., & P. FIFE, On VON KÁRMÁN's equations and the buckling of a thin elastic plate. II. Plate with general edge conditions. Comm. Pure Appl. Math. **21**, 227–241 (1968).
6. BLISS, G., A generalization of WEIERSTRASS' preparation theorem for a power series in several variables. Trans. Amer. Math. Soc. **13**, 133–145 (1912).
7. KNIGHTLY, G., & D. SATHER, On nonuniqueness of solutions of the VON KÁRMÁN equations. Arch. Rational Mech. Anal. **36**, 65–78 (1970).
8. MACMILLAN, W., A reduction of a system of power series to an equivalent system of polynomials. Math. Ann. **72**, 157–179 (1912).
9. SATHER, D., Branching of solutions of an equation in Hilbert space. Arch. Rational Mech. Anal. **36**, 47–64 (1970).
10. SATHER, D., Branching of solutions of nonlinear equations, Proceedings of the Symposium on Nonlinear Eigenvalue Problems, Santa Fe, New Mexico, 1971, Rocky Mtn. J. Math., (2) **3**, 203–250 (1973).
11. VAINBERG, M. M., & V. A. TRENOGIN, The methods of Lyapunov and Schmidt in the theory of non-linear equations and their further development. Russian Math. Surveys (2) **17**, 1–60 (1962).

Department of Mathematics
University of Massachusetts
Amherst, Massachusetts

and

Department of Mathematics
University of Colorado
Boulder, Colorado

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