Cauchy's Theorem in Classical Physics

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1. Introduction

Basic in all of classical physics are balance laws of the form

$$\int_{\partial B} f(x, n(x))\, dA_x + \int_B b(x)\, dV_x = 0, \tag{1.1}$$

where $n(x)$ is the outward unit normal to the boundary ∂B of B. In mechanics f represents the *surface force* per unit area on ∂B; in thermodynamics f gives the *heat flow* per unit area into B across its boundary.

In 1823 CAUCHY[1] established what is probably the most important theorem in continuum mechanics; he proved that if $f(x, n)$, defined for each x in an open region R and every unit vector n, is *continuous* in x, if $b(x)$ is uniformly bounded on R, and if (1.1) is satisfied for every sufficiently nice region B in R, then $f(x, n)$ must necessarily be *linear* in n; that is,

$$f(x, n) = T(x)\, n \tag{1.2}$$

with $T(x)$ a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^N$, the codomain of f.

When f is the surface force, the field T is called the *stress tensor*; its value $T(x)$ at x is a linear transformation from $\mathbb{R}^3$ into itself. For this case CAUCHY also proved that if f is consistent with the moment balance law

$$\int_{\partial B} (y-x) \wedge f(y, n(y))\, dA_y + \int_B (y-x) \wedge b(y)\, dV_y = 0 \tag{1.3}$$

for every B, then $T(x)$ must necessarily be *symmetric*.

When f represents the heat flow, $T(x)$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}$ and (hence) can be represented as the inner product with a vector $q(x)$:

$$f(x, n) = q(x) \cdot n.$$

The field q is the *heat flux vector*, a concept which plays a major role in the theory of heat transfer.

While CAUCHY's monumental contribution is uncontested, one cannot help but notice the one major drawback[2] to Cauchy's Theorem: the assumed continuity of the function $x \mapsto f(x, n)$. Indeed, as recent studies in the foundations of me-

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[1] CAUCHY (1823), (1827).

[2] As a matter of fact, NOLL (1973), p. 79 remarks: "It is unfortunate that nobody has been able, so far, to [establish the linearity of f] ... without the *ad hoc* continuity assumption ..."

chanics have shown, the *basic* concept is not f but rather the *total force*

$$F(S)=\int_S f(x, n(x))\, dA_x \tag{1.4}$$

on any oriented surface S; the density f is a *derived* quantity computed by taking the derivative dF/dA of F with respect to Lebesgue area measure A. It therefore seems pertinent to ask the following two questions:

(I) Is there a physically reasonable hypothesis that one can place on F that yields, as a *consequence*, the continuity of f?

(II)[1] Starting with a general balance law, can one establish, without further assumptions, the linearity of f, at least almost everywhere?

To answer these questions succinctly, we introduce the notion of a *Cauchy flux* F, which is a vector-valued, additive,[2] area-bounded function whose domain is the collection of all (plane,[3] oriented) surface elements in R. The density $f(x, n)$ is then defined to be the limit as $r\to 0$ (where it exists) of the *average density*

$$\frac{F(D_r(x, n))}{A(D_r(x, n))}$$

over oriented discs $D_r(x, n)$ centered at x with radius r and positive unit normal n. It then follows that the density exists almost everywhere on each cross section of R, is uniformly bounded, and delivers F through the relation (1.4).

In terms of F the balance law (1.1) has the form

$$F(\partial B)+\int_B b\, dV=0, \tag{1.5}$$

and, when b is uniformly bounded on R,

$$|F(\partial B)|\leqq K\, V(B) \tag{1.6}$$

for every body B. We use the term *weakly balanced* for a Cauchy flux that satisfies (1.6); for our purposes (1.6) is simpler to work with than (1.5).

Similarly, (1.3) implies that for B a cube containing x,

$$\lim_{V(B)\to 0} \frac{\int_{\partial B} (y-x)\wedge f(y, n(y))\, dA_y}{V(B)}=0. \tag{1.7}$$

In our theory a Cauchy flux that satisfies (1.7) at every x in R is said to be *moment balanced*.

Using these ideas, we are able to answer the questions (I) and (II). In answer to (I) we show that a necessary and sufficient condition for f to be a *continuous* function of position is that F have *uniform average density* in the following sense: for each fixed n the one-parameter family

$$\left\{\frac{F(D_r(x, n))}{A(D_r(x, n))}\right\}_{r>0}$$

[1] A partial answer to this question was given by GURTIN, MIZEL & WILLIAMS (1968) (*cf.* Lemma 2), whose results form an essential part of our analysis.

[2] Precise definitions of the terms we use can be found in the text.

[3] Because of NOLL's Theorem (1957), Theorem 4, it suffices to define F only on *plane* surfaces.

of functions of x is a uniform Cauchy family[1] as $r \to 0$ for x in any compact subset of R. As an answer to (II) we show that a weakly balanced Cauchy flux has a density which is *linear almost everywhere*; a weakly balanced Cauchy flux that is moment balanced has a density which is *linear and symmetric almost everywhere.*

2. Notation

We write $\mathbb{R}$ for the reals, $a \cdot b$ and $|a|$ for the inner product and norm on $\mathbb{R}^N$, and

$$\text{Unit}(\mathbb{R}^N) = \{n \in \mathbb{R}^N : |n| = 1\}$$

for the set of all unit vectors in $\mathbb{R}^N$. Further, $\text{Lin}(\mathbb{R}^N, \mathbb{R}^Q)$ is the space (with standard topology) of linear transformations of $\mathbb{R}^N$ into $\mathbb{R}^Q$,

$$\text{Lin}(\mathbb{R}^N) = \text{Lin}(\mathbb{R}^N, \mathbb{R}^N),$$

and $L \in \text{Lin}(\mathbb{R}^N)$ is symmetric if L equals its transpose L^T. Given $a \in \mathbb{R}^Q$ and $b \in \mathbb{R}^N$,

$$a \otimes b \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^Q)$$

is defined by

$$(a \otimes b)\, u = (b \cdot u)\, a$$

for all $u \in \mathbb{R}^N$, and for the case in which $N = Q$ we write

$$a \wedge b = a \otimes b - b \otimes a.$$

An **oriented plane set (o.p.s.)** S is a pair (P, n) consisting of a plane Borel set $P \subset \mathbb{R}^3$ and a unit normal vector n to P. We call P the **underlying set** of S and $n_S = n$ the **positive unit normal** to S. As is natural, we write $-S$ and $S + x\,(x \in \mathbb{R}^3)$ for the o.p.s.'s $-S = (P, -n_S)$ and $S + x = (P + x, n_S)$, where

$$P + x = \{y \in \mathbb{R}^3 : y - x \in P\}.$$

Further, $-S + x$ denotes $(-S) + x$. We say that $S_1 = (P_1, n_1)$ and $S_2 = (P_2, n_2)$ are **compatible** if P_1 and P_2 lie in the same plane and $n_1 = n_2$. When such is the case we define $S_1 \cup S_2$ and $S_1 \cap S_2$ to be the o.p.s.'s $(P_1 \cup P_2, n_1)$ and $(P_1 \cap P_2, n_1)$, respectively; similar definitions apply to other set theoretic operations. As is customary, in operations involving the underlying set P of an o.p.s. S, we will usually write S in place of P; thus $x \in S$ means $x \in P$, $\int_S$ means $\int_P$, *etc.*

We say that a sequence $\{S_k\}$ of o.p.s.'s tends to an o.p.s. S **regularly** if each S_k is compatible with S and if the area of the symmetric difference $(S - S_k) \cup (S_k - S)$ tends to zero as $k \to \infty$.

An o.p.s. S is **polygonal** if its underlying set is a plane closed polygonal region. More generally, S is **regular** if S is closed (in the plane) and if there exists a sequence $\{S_k\}$ of polygonal o.p.s.'s $S_k \subset S$ and a constant $K_0 > 0$ such that $\{S_k\}$ tends to S regularly and

$$p(S_k) < K_0 \tag{2.1}$$

[1] It is a tribute to the genius of CAUCHY that we use a notion of his in analysis to prove a result of his in continuum mechanics.

for all k, where $p(S_k)$ is the perimeter of S_k. (Note that since $S_k \subset S$, the symmetric difference of S and S_k is $S - S_k$.) An example of a regular o.p.s. is the **oriented plane disc** $D_r(x, n)$ with center at x, radius $r > 0$, and positive unit normal n.

We consider throughout a bounded open set R in $\mathbb{R}^3$.

By a **surface element** S we mean an o.p.s. contained in R. A surface element C is then a **subelement** of S if C is compatible with S and $C \subset S$. Finally, a **cross section** π is a surface element whose underlying set is the (non-empty) intersection of R with a plane.

A **body** B is a compact polyhedron in R. A surface element $S \subset \partial B$ is **oriented by** ∂B if n_S coincides with the outward unit normal to ∂B on S. The **faces** $S_1, S_2, \ldots, S_Q$ of B will always be considered as polygonal surface elements oriented by ∂B.

We will consistently use the notation

$$R_r = \{x \in R \colon R \text{ contains the closed ball with center at } x \text{ and radius } r\}. \tag{2.2}$$

Clearly, R_r is open, $R_r \subset R_l$ if $r > l$, and $\bigcup_{r>0} R_r = R$.

We write V for the Lebesgue volume measure in $\mathbb{R}^3$ and A for the Lebesgue surface measure on planes in $\mathbb{R}^3$. As is customary, we write, *e.g.*, "$[V]$ almost everywhere" to signify "almost everywhere with respect to the measure V"; similar abbreviations have obvious meaning.

Let G be an $\mathbb{R}^N$-valued set function whose domain consists of Borel subsets of R. Then G is **volume-bounded** if there is a $K > 0$ such that

$$|G(D)| \leqq K V(D)$$

for every D in the domain of G. Similarly, if G is an $\mathbb{R}^N$-valued function whose domain is a set of surface elements, then G is **area-bounded** if there is a $K > 0$ such that

$$|G(S)| \leqq K A(S)$$

for every surface element S in the domain of G.

Let A be a set, and let f_r $(0 < r < r_0)$ be a one-parameter family of $\mathbb{R}^N$-valued functions each of whose domain contains A. Then f_r $(0 < r < r_0)$ is a **uniform Cauchy family** on A as $r \to 0$ if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f_r(x) - f_l(x)| < \varepsilon$ for all $r, l < \delta$ and $x \in A$.

3. Properties of the Density

We assume throughout that an integer $N > 0$ is given.

A **Cauchy flux** is a function F that assigns to each surface element S a vector $F(S)$ in $\mathbb{R}^N$ and has the following properties:

(C_1) *F is area bounded.*

(C_2) F is *additive* on compatible surface elements; *i.e.*,

$$F(S_1 \cup S_2) = F(S_1) + F(S_2)$$

whenever S_1 and S_2 are disjoint, compatible surface elements.

Trivially, since a Cauchy flux is area-bounded, it must necessarily be *countably additive* on compatible surface elements.

Let F be a Cauchy flux. In applications $F(S)$ represents the net flux across S; thus given a body B with faces $S_1, S_2, \ldots, S_Q$, the vector $F(\partial B)$, defined by

$$F(\partial B) = \sum_{i=1}^{Q} F(S_i), \tag{3.1}$$

gives the net flux into B. We say that F is **weakly balanced** if the map $B \mapsto F(\partial B)$ is *volume bounded.*

By definition, when F is a Cauchy flux there exists a constant $K>0$ such that

$$|F(S)| \leqq KA(S) \tag{3.2}$$

for every surface element S. For a weakly balanced Cauchy flux we have, in addition,

$$|F(\partial B)| \leqq KV(B) \tag{3.3}$$

for every body B. Henceforth K will always denote this constant. A trivial consequence of (3.2) and (C_2) is

Lemma 1. *Let F be a Cauchy flux. Further, let S be a surface element, and let $\{S_k\}$ be a sequence of surface elements that tends to S regularly. Then*

$$F(S_k) \to F(S) \quad \text{as } k \to \infty. \tag{3.4}$$

Crucial to the results of this section, and of interest in itself, is

Theorem 1. *Let F be a weakly balanced Cauchy flux. Then given any regular surface element S*

$$F(S) = -F(-S) \qquad (3.5)^1$$

and the mapping $x \mapsto F(S+x)$ is continuous.

Proof. Before beginning the proof note that, given any regular surface element S, the domain of the map $x \mapsto F(S+x)$ is the open set $\mathscr{D}(S)$ in $\mathbb{R}^3$ consisting of all x such that $S+x \subset R$.

Choose a *polygonal* surface element $S=(P,n)$, and let B_1, B_2, and B be the polygonal prisms

$$B_1 = \bigcup_{\alpha \in [0,\varepsilon]} (P+\alpha n), \qquad B_2 = \bigcup_{\alpha \in [0,\varepsilon]} (P-\alpha n), \qquad B = B_1 \cup B_2$$

(Fig. 1). Clearly, for ε sufficiently small B_1, B_2, and B are bodies. Moreover, $-S$

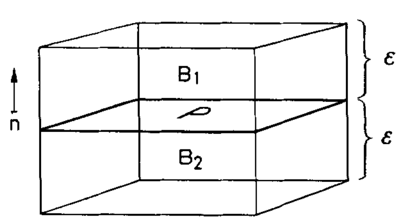

Fig. 1

[1] *Cf.* NOLL (1957), Theorem 3; GURTIN & WILLIAMS (1967), Theorem 4. Actually, (3.5) holds for every surface element, regular or not.

is a face of B_1, S is a face of B_2, and $P = B_1 \cap B_2$. Thus

$$F(\partial B_1) + F(\partial B_2) - F(\partial B) = F(S) + F(-S),$$

and, since

$$V(B) + V(B_1) + V(B_2) = 4\varepsilon A(P),$$

if we apply (3.3) to B_1, B_2, and B, we conclude that

$$|F(S) + F(-S)| \leqq 4\varepsilon A(P) K.$$

Letting $\varepsilon \to 0$, we see that (3.5) must hold for *polygonal* surfaces.

Let S again be a polygonal surface element and choose $x \in \mathscr{D}(S)$. Then $\mathscr{D}(S)$ contains an open ball Ω centered at x. Choose $y \in \Omega$ with $(y-x) \cdot n_S = 0$. Then $S+x$ and $S+y$ are compatible, and a simple computation based on (3.2) and (C_2) yields the inequality

$$|F(S+x) - F(S+y)| \leqq K p(S) |x-y|, \tag{3.6}$$

where $p(S)$ is the perimeter of S.

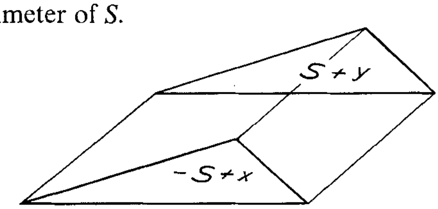

Fig. 2

Next, choose $y \in \Omega$ with $(y-x) \cdot n_S > 0$, and let B denote the prism, constructed in the obvious manner (Fig. 2), with end faces $S+y$ and $-S+x$ and lateral faces $S_1, S_2, \ldots, S_Q$. Then B is a body, and

$$F(\partial B) = F(S+y) + F(-S+x) + \sum_{i=1}^{Q} F(S_i).$$

In view of (3.2), the last term in this relation is bounded in magnitude by

$$K p(S) |x-y|.$$

Further, $V(B) \leqq A(S)|x-y|$. Thus we conclude, with the aid of (3.5) applied to the polygonal surface element $-S+x = -(S+x)$, that

$$|F(S+y) - F(S+x)| \leqq K\{A(S) + p(S)\} |x-y|. \tag{3.7}$$

Clearly, (3.7) also holds when $(y-x) \cdot n_S < 0$. We therefore conclude from (3.6) that (3.7) is satisfied for all $y \in \Omega$. Thus (3.5) and (3.7) are satisfied whenever S is a polygonal surface element.

Now let S be a regular surface element. By definition there exists a sequence $\{S_k\}$ of *polygonal* subelements of S that tends regularly to S and satisfies (2.1). If we apply (3.5) to S_k and let $k \to \infty$, we conclude, with the aid of (3.4), that S satisfies (3.5). To establish the continuity of the map $x \mapsto F(S+x)$, choose $x \in \mathscr{D}(S)$, let Ω be an open ball in $\mathscr{D}(S)$ centered at x, and let $y \in \Omega$. Since $S_k \subset S$, $\mathscr{D}(S) \subset \mathscr{D}(S_k)$ and $\Omega \subset \mathscr{D}(S_k)$. Thus, since

$$K\{A(S_k) + p(S_k)\} \leqq C = K\{A(S) + K_0\},$$

where K_0 is the constant appearing in (2.1), (3.7) applied to S_k yields

$$|F(S_k+y)-F(S_k+x)| \leqq C\,|x-y|. \tag{3.8}$$

Next, since $\{S_k\}$ tends regularly to S, $\{S_k+z\}$ tends regularly to $S+z$ for every $z \in \Omega$, and we conclude from Lemma 1 that $F(S_k+z) \to F(S+z)$. Thus (3.8) implies

$$|F(S+y)-F(S+x)| \leqq C\,|x-y|,$$

and the map $x \mapsto F(x+S)$ is continuous. □

The function f_r: $R_r \times \text{Unit}(\mathbb{R}^3) \to \mathbb{R}^N$ defined by

$$f_r(x,n) = \frac{F(D_r(x,n))}{A(D_r(x,n))} \tag{3.9}$$

represents the **average density** over discs of radius r. Here, of course, $D_r(x,n)$ is the oriented disc with center at x, radius r, and positive unit normal n, while R_r is defined in (2.2). Note that, by (3.2) and (3.9),

$$|f_r(x,n)| \leqq K \tag{3.10}$$

whenever F is a Cauchy flux, while (3.5) yields

$$f_r(x,n) = -f_r(x,-n) \tag{3.11}$$

if, in addition, F is weakly balanced. Some less trivial properties of f_r are given by

Theorem 2 (Properties of the average density). *Let F be a weakly balanced Cauchy flux. Then the map $(r,x,n) \mapsto f_r(x,n)$ is separately*[1] *continuous in each of its three arguments. Moreover, for all $x \in R_r$ and any pair of unit vectors m and n*

$$|f_r(x,m)-f_r(x,n)| \leqq K\,\theta(2+r), \tag{3.12}$$

where θ is the angle between m and n.

Proof. The continuity of the map $x \mapsto f_r(x,n)$ follows from (3.9), Theorem 1, and the fact that $D_r(x,n)$ is a regular surface element. Next, write D_r for $D_r(x,n)$ and note that, by Lemma 1, $F(D_{r+\delta_k}) \to F(D_r)$ for any sequence $\{\delta_k\}$ with $\delta_k \to 0$. Thus $r \mapsto F(D_r)$ is continuous. Clearly, so also is $r \mapsto A(D_r)$. Thus the map $r \mapsto f_r(x,n)$ is continuous. To complete the proof we have only to establish (3.12), since the continuity of $n \mapsto f_r(x,n)$ follows from this inequality.

Let $m, n\,(m \neq n)$ be unit vectors with angle θ between them *acute*, choose $r>0$ with $R_r \neq \emptyset$, and let $x \in R_r$. Let $D(m)$ and $D(n)$ denote the oriented discs with center at x, radius r, and positive unit normals m and n, respectively. Then $D(m)$ and $D(n)$ intersect in a diameter d; for $p=m,n$ this diameter cuts $D(p)$ into two

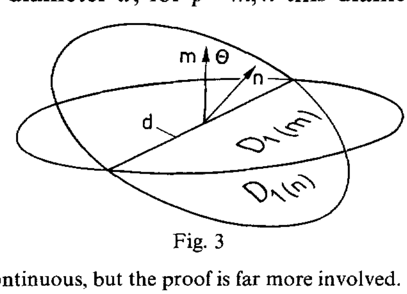

Fig. 3

[1] Actually, f is jointly continuous, but the proof is far more involved.

regular subelements $D_1(p)$ and $D_2(p)$. Here the subscripts 1 and 2 are chosen so that $D_1(m)$ and $D_1(n)$ form a wedge with interior angle θ (Fig. 3). Divide the circumference of each disc into $2k$ arcs of equal length in such a way that d is a diagonal for each of the two inscribed polygons defined by this division. For $p=m,n$ let $D_{1k}(p)$ denote the polygonal subelement of $D_1(p)$ whose underlying set is the intersection of $D_1(p)$ with the closed polygonal region inscribed in $D(p)$. By Lemma 1,

$$F(D_{1k}(p)) \to F(D_1(p)) \tag{3.13}$$

as $k \to \infty$. Next, fix k and consider the wedge-shaped body B_k whose faces (without loss in generality) are $D_{1k}(m)$, $-D_{1k}(n)$, and k other regular surface elements $S_1, S_2, \ldots, S_k$ (Fig. 4). Clearly,

$$V(B_k) \leqq \tfrac{1}{2}\pi r^3 \theta, \qquad \sum_{i=1}^{k} A(S_i) \leqq \pi r^2 \theta;$$

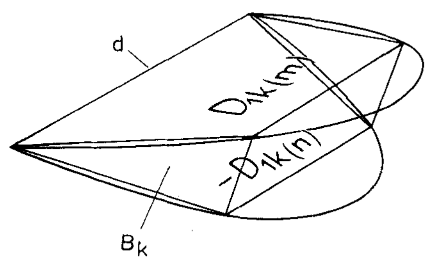

Fig. 4

thus, since

$$F(\partial B_k) = F(D_{1k}(m)) + F(-D_{1k}(n)) + \sum_{i=1}^{k} F(S_i),$$

we conclude from (3.2), (3.3), and (3.5) that

$$|F(D_{1k}(m)) - F(D_{1k}(n))| \leqq \tfrac{1}{2}\pi r^2 \theta K(2+r).$$

Therefore, by (3.13),

$$|F(D_1(m)) - F(D_1(n))| \leqq \tfrac{1}{2}\pi r^2 \theta K(2+r).$$

Obviously, $D_2(m)$ and $D_2(n)$ obey an identical inequality; hence

$$\begin{aligned} |F(D(m)) - F(D(n))| &= |F(D_1(m)) + F(D_2(m)) - F(D_1(n)) - F(D_2(n))| \\ &\leqq \sum_{\alpha=1}^{2} |F(D_\alpha(m)) - F(D_\alpha(n))| \\ &\leqq \pi r^2 \theta K(2+r). \end{aligned}$$

If we divide this inequality by $A(D(m)) = A(D(n)) = \pi r^2$ and use (3.9), we are led to (3.12). Thus (3.12) holds when θ is acute. For θ not acute we simply choose a unit vector p in the plane spanned by m and n with p a bisector of θ, and apply (3.12) to m, p and n, p separately. □

Let us agree to call (x, n) a **density pair** if the limit

$$f(x,n) = \lim_{r \to 0} f_r(x,n) \tag{3.14}$$

exists. The function f so defined *on the set of all density pairs* is called the **density** of F. When (x, n) is a density pair for every unit vector n, then x is called a **point of density**, and when every $x \in R$ is a point of density, then F has **density everywhere.** If this is the case, and if $x \mapsto f(x, n)$ is continuous on R for each fixed n, we say that f is a **continuous function of position.**

By (3.10), for a Cauchy flux

$$|f(x, n)| \leqq K, \tag{3.15}$$

while (3.11) implies, for a weakly balanced Cauchy flux, that if (x, n) is a density pair, then so also is $(x, -n)$ and

$$f(x, n) = -f(x, -n). \tag{3.16}$$

In general the density will neither be continuous nor defined everywhere. The next theorem shows, however, that for a weakly balanced Cauchy flux the density is defined at enough points and is sufficiently nice to make analysis meaningful.

Theorem 3 (Properties of the density). *Let f be the density of a Cauchy flux F. Then:*

(i) *for each cross section π the density $f(x, n_\pi)$ exists at $[A]$ almost every $x \in \pi$, and $f(\cdot, n_\pi) \in L^1(\pi)$;*

(ii) *for each surface element S*

$$F(S) = \int_S f(x, n_S)\, dA_x. \tag{3.17}$$

If, in addition, F is weakly balanced, then:

(iii) *for each unit vector n, $f(x, n)$ exists at $[V]$ almost every $x \in R$, $f(\cdot, n)$ is a Borel function, and $f(\cdot, n) \in L^1(R)$;*

(iv) *$[V]$ almost every $x \in R$ is a point of density;*

(v) *for x a point of density $n \mapsto f(x, n)$ is continuous on Unit $(\mathbb{R}^3)$.*

Proof. Let π be a cross section, and let $\mathscr{B}(\pi)$ denote the Borel subsets of (the underlying set of) π. For $P \in \mathscr{B}(\pi)$ let P_π be the *subelement* of π whose underlying set is P, and define F_π on $\mathscr{B}(\pi)$ by

$$F_\pi(P) = F(P_\pi).$$

Then, since F is countably additive on compatible surface elements, F_π is a measure on $\mathscr{B}(\pi)$. Further, if $D_r(x)$ denotes a (non-oriented) disc in π with center at x and radius r,

$$f_r(x, n_\pi) = \frac{F_\pi(D_r(x))}{A(D_r(x))}.$$

Therefore $f(x, n_\pi)$ is the symmetric derivative[1] of F_π at x (with respect to the measure A). This derivative clearly exists at every point x at which the (ordinary measure-theoretic) derivative[2] DF_π exists, and (i) and (ii) follow[3] from the absolute continuity of F_π with respect to A.

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[1] *Cf.* RUDIN (1974), p. 165.
[2] *Cf.* RUDIN (1974), Definition 8.3.
[3] *Cf.* RUDIN (1974), Theorem 8.6 and its Corollary.

To establish (iii) fix $n \in \mathrm{Unit}(\mathbb{R}^3)$ and let $R(n)$ be the set of all $x \in R$ for which the limit (3.14) exists; thus $R(n)$ is the domain of $f(\cdot, n)$. In view of the continuity of $r \mapsto f_r(x, n)$ (Theorem 2), $R(n)$ consists exactly of those points x for which

$$\lim_{q \to 0} f_q(x, n) \quad \text{exists } (q \text{ rational}). \tag{3.18}$$

Choose $r > 0$ with $R_r \neq \emptyset$. Since $R_r \subset R_q$ for $q < r$, we conclude from the continuity of $x \mapsto f_\lambda(x, n)$ on R_λ (Theorem 2) that $f_q(\cdot, n)$ is continuous on R_r. Thus the set of all points $x \in R_r$ for which (3.14) holds is a Borel measurable subset of R_r, and the restriction of $f(\cdot, n)$ to $R_r \cap R(n)$ is a Borel function. Thus, since $R = \bigcup_{k=1}^{\infty} R_{1/k}$, $R(n)$ is itself a Borel set and $f(\cdot, n)$ a Borel function. Further, it follows from (i) and Fubini's Theorem that

$$V(R - R(n)) = 0, \tag{3.19}$$

and we conclude from (3.15) that $f(\cdot, n) \in L^1(R)$. This completes the proof of (iii).

Suppose that (x, m) and (x, n) are density pairs with angle θ between m and n. Then by (3.12) and (3.14),

$$|f(x, m) - f(x, n)| \leqq 2K\theta,$$

which yields (v).

To prove (iv) let $\mathcal{M}$ be a countably dense subset of $\mathrm{Unit}(\mathbb{R}^3)$, and let

$$R(\mathcal{M}) = \bigcap_{n \in \mathcal{M}} R(n), \tag{3.20}$$

so that, by (3.19),

$$V(R - R(\mathcal{M})) = 0.$$

To complete the proof, it suffices to show that every $x \in R(\mathcal{M})$ is a point of density. Thus choose $x \in R(\mathcal{M})$, and note that, by (3.20), (x, m) is a density pair for every $m \in \mathcal{M}$. Let n be an arbitrary unit vector. Then there exists a sequence $\{m_i\}$ with $m_i \in \mathcal{M}$ such that $m_i \to n$. Let θ_i denote the angle between m_i and n. Then for r, l sufficiently small, say $r, l < r_0$, $x \in R_r \cap R_l$, and we conclude from (3.12) that

$$\begin{aligned} |f_r(x, n) - f_l(x, n)| &\leqq |f_r(x, n) - f_r(x, m_i)| + |f_r(x, m_i) - f_l(x, m_i)| + |f_l(x, m_i) - f_l(x, n)| \\ &\leqq K\theta_i(4 + r + l) + |f_r(x, m_i) - f_l(x, m_i)|. \end{aligned}$$

Choose $\varepsilon > 0$. Then (for $r, l < r_0$) we can choose i large enough to have the first term strictly less than $\varepsilon/2$, and, since (x, m_i) is a density pair for each i, there exists a $\delta \in (0, r_0)$ such that the last term is less than $\varepsilon/2$ for $r, l < \delta$. Thus

$$|f_r(x, n) - f_l(x, n)| < \varepsilon,$$

and the limit (3.14) exists. Therefore (x, n) is a density pair, and, since n is arbitrary, x is a point of density. This completes the proof of Theorem 3. □

As we shall see in the next section, Cauchy's Theorem is based on the assumption that f be a continuous function of position. We now show that this assumption is equivalent to a condition of uniformity concerning the average density.

Note that given any compact set $A \subset R$ the domain of $f_r(\cdot, n)$ contains A for all sufficiently small r. We say that F has **uniform average density** if given any unit vector n and any compact set A in R the one-parameter family $f_r(\cdot, n)(r > 0$ sufficiently small) is uniformly Cauchy convergent on A as $r \to 0$. This is, of course, equivalent to the requirement that F have density everywhere and that the limit (3.14) be uniform on any compact subset of R.

Theorem 4. *Let F be a weakly balanced Cauchy flux. Then the following two statements are equivalent:*

(i) *F has uniform average density.*

(ii) *F has density f everywhere, and f is a continuous function of position.*

Proof. That (i) implies (ii) is a direct consequence of the above definition and the continuity of $x \mapsto f_r(x, n)$ (Theorem 2). To prove the converse assertion assume that (ii) holds. By (3.17) and (3.9)

$$|f(x,n)-f_r(x,n)| \leqq \frac{\int_{D_r(x,n)} |f(x,n)-f(y,n)|\, dA_y}{A(D_r(x,n))} \leqq \sup_{y\in D_r(x,n)} |f(x,n)-f(y,n)|,$$

and (i) follows from the uniform continuity of $f(\cdot, n)$ on each set of the form R_l, $l>0$. □

4. Linearity of the Density

Let F be a Cauchy flux with density f. We say that f is **linear** at x if x is a point of density and if the map $n \mapsto f(x,n)$ is the restriction to Unit$(\mathbb{R}^3)$ of a linear transformation $T(x)$ from $\mathbb{R}^3$ into $\mathbb{R}^N$:

$$f(x,n) = T(x)\, n \tag{4.1}$$

for every unit vector n. If, in addition, $N=3$ and $T(x)$ is symmetric, then f is **linear and symmetric** at x.

Not every Cauchy flux has linear density. Indeed[1], consider the Cauchy flux F with values in $\mathbb{R}^3$ defined by

$$F(S) = A(S)(u \cdot n_S)\, n_S$$

where $u \neq 0$ belongs to $\mathbb{R}^3$. F has density $f(x,n)=(u\cdot n)\,n$ everywhere, and f is obviously linear *nowhere* on R. If we consider a cube B in R of width ε with one pair of parallel faces perpendicular to u, then

$$\frac{F(\partial B)}{V(B)} = \frac{2}{\varepsilon}\, u,$$

so that F is *not* weakly balanced. The next two theorems show that being weakly balanced is a sufficient condition for linearity everywhere when f is continuous and for linearity almost everywhere in general.

Given a Cauchy flux F with values in $\mathbb{R}^3$ and a point $x \in \mathbb{R}^3$, consider the function MF_x, with values in Lin$(\mathbb{R}^3)$, defined by

$$MF_x(S) = \int_S (y-x) \wedge f(y, n_S)\, dA_y \tag{4.2}$$

for every surface element S. (The integral exists by (i) of Theorem 3.) If we identify Lin$(\mathbb{R}^3)$, in the usual manner, with $\mathbb{R}^9$, then we may conclude from (3.15) and the

[1] This example arose in private communications with R. LANGNER.

boundedness of R that MF_x is a Cauchy flux with values in $\mathbb{R}^9$. We call MF_x the **moment of** F **about** x. We say that F is **moment balanced** if, given any $x \in R$ and any sequence $\{B_k\}$ of cubes with $x \in B_k$ and $V(B_k) \to 0$,

$$\lim_{k \to \infty} \frac{MF_x(\partial B_k)}{V(B_k)} = 0.$$

Theorem 5 (Cauchy's Theorem). *Let F be a weakly balanced Cauchy flux and assume that:*

(i) *F has density everywhere;*

(ii) *the density f is a continuous function of position.*

Then f is linear at each point of R. If, in addition, F has values in $\mathbb{R}^3$ and is moment balanced, then f is linear and symmetric at each point of R.

Proof. Let $\{e_i\}$ be an orthonormal basis for $\mathbb{R}^3$. Choose $x \in R$ and let n be a unit vector with

$$n \cdot e_i > 0 \tag{4.3}$$

for $i = 1, 2, 3$. Following the classical proof of Cauchy's Theorem we choose $h > 0$ and consider the tetrahedron B_h, shown in Fig. 5, with faces S_h, S_{1h}, S_{2h}, and S_{3h}, with n and $-e_i$ the positive unit normals to S_h and S_{ih}, respectively, and with x the vertex opposite to S_h. Then B_h is a body for h sufficiently small and hence, by (3.3),

$$|F(\partial B_h)| = \left| F(S_h) + \sum_{i=1}^{3} F(S_{ih}) \right| \leqq K V(B_h).$$

If we divide this relation by $A(S_h)$, let $h \to 0$, and use (3.16), (3.17), and the fact that f is a continuous function of position, we find that

$$f(x, n) = \sum_{i=1}^{3} (n \cdot e_i) f(x, e_i). \tag{4.4}$$

If we have $e_i \cdot n < 0$ for one or more values of i, by considering the new basis with each such e_i replaced by $-e_i$, we conclude, with the aid of (3.16), that (4.4) remains valid. Thus (4.4) holds as long as n does not lie in a coordinate plane. But by (v)

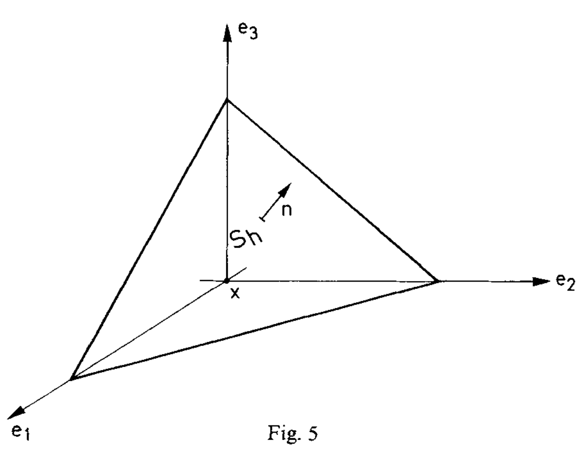

Fig. 5

of Theorem 3 the map $n \mapsto f(x, n)$ is continuous on Unit $(\mathbb{R}^3)$; thus (4.4) holds for all n. If we define $T(x)\colon \mathbb{R}^3 \to \mathbb{R}^N$ by

$$T(x)v = \sum_{i=1}^{3} (v \cdot e_i) f(x, e_i),$$

then $T(x)$ is linear and (4.1) holds. Thus f is linear at each point of R.

Assume now that F has values in $\mathbb{R}^3$. By (4.2), given any body B,

$$M F_x(\partial B) = I^T - I, \tag{4.5}$$

where

$$I = \int_{\partial B} f(y, n(y)) \otimes (y - x)\, dA_y$$

with $n(y)$ the outward unit normal to ∂B at y. In view of (4.1),

$$f(y, n(y)) \otimes (y - x) = [T(y) n(y)] \otimes (y - x) = T(y)[n(y) \otimes (y - x)];$$

thus

$$I = T(x) \int_{\partial B} n(y) \otimes (y - x)\, dA_y + \int_{\partial B} [T(y) - T(x)]\, [n(y) \otimes (y - x)]\, dA_y. \tag{4.6}$$

Let I_1 and I_2, respectively, denote the first and second terms on the right-hand side of (4.6). A trivial application of the divergence theorem yields

$$I_1 = V(B)\, T(x). \tag{4.7}$$

Assume now that B is a *cube* containing x. Then

$$|I_2| \leqq \sup_{y \in B} |T(y) - T(x)| \int_{\partial B} |y - x|\, dA_y \leqq 6\sqrt{3}\, V(B) \sup_{y \in B} |T(y) - T(x)|. \tag{4.8}$$

Since f is a continuous function of position, T is continuous on R. Thus we may conclude from (4.5)–(4.8) that

$$\frac{M F_x(\partial B)}{V(B)} \to T(x)^T - T(x) \tag{4.9}$$

as $V(B) \to 0$ (with B a cube containing x), and hence, if F is moment balanced, T is symmetric on R. □

Remark. It is clear from (4.9) that if F is a weakly balanced Cauchy flux, with values in $\mathbb{R}^3$, consistent with hypotheses (i) and (ii) of Cauchy's Theorem, and if f is *symmetric* at each point of R, then F is moment balanced.

In view of Theorem 4, Cauchy's Theorem has the following interesting

Corollary. *Let F be a weakly balanced Cauchy flux with uniform average density. Then the density f is linear at each point of R. If, in addition, F has values in $\mathbb{R}^3$ and is moment balanced, then f is linear and symmetric at each point of R.*

Both Cauchy's Theorem and the last result require hypotheses over and above the assumption that F be a weakly balanced Cauchy flux. One can ask if it is possible to establish linearity, at least almost everywhere, without such additional hypotheses. The next theorem, which is our main result, shows that the answer to this question is yes.

Theorem 6 (Linearity almost everywhere). *Let F be a weakly balanced Cauchy flux with density f. Then f is linear at $[V]$ almost every point of R. If, in addition, F has values in $\mathbb{R}^3$ and is moment balanced, then f is linear and symmetric at $[V]$ almost every point of R.*

Our proof is based on

Lemma 2.[1] *Let F be a weakly balanced Cauchy flux with density f. Then there exists a field $T: R \to \mathrm{Lin}(\mathbb{R}^3, \mathbb{R}^N)$ and for each unit vector n a subset $R_*(n)$ of R such that*

$$V(R - R_*(n)) = 0 \tag{4.10}$$

and

$$f(x, n) = T(x)n \tag{4.11}$$

for every $x \in R_*(n)$. If, in addition, $N=3$ and F is moment balanced, then T can be chosen to be symmetric $[V]$ almost everywhere on R.

Proof of the Lemma. It is not difficult to exhibit, for each $\delta > 0$, a class C^∞ function $\rho^\delta: \mathbb{R}^3 \to \mathbb{R}$ such that $\rho^\delta \geqq 0$, $\rho^\delta = 0$ outside the closed ball of radius δ centered at the origin, and

$$\int_{\mathbb{R}^3} \rho^\delta \, dV = 1.$$

Recall that the domain $R(n)$ of $f(\cdot, n)$ is a Borel set and that $f(\cdot, n)$ is a Borel function (*cf.* the discussion in the paragraph containing (3.19)). Thus if we extend $f(\cdot, n)$ from $R(n)$ to all of $\mathbb{R}^3$ by requiring that $f(\cdot, n) = 0$ on $\mathbb{R}^3 - R(n)$, then the extended function is also a Borel function. We also write $f(\cdot, n)$ for the extended function, so that $f: \mathbb{R}^3 \times \mathrm{Unit}(\mathbb{R}^3) \to \mathbb{R}^N$ and the integral

$$f^\delta(x, n) = \int_{\mathbb{R}^3} \rho^\delta(x-y) f(y, n) \, dV_y$$

defines a function $f^\delta: \mathbb{R}^3 \times \mathrm{Unit}(\mathbb{R}^3) \to \mathbb{R}^N$. Observe that $f^\delta(\cdot, n)$ is of class C^∞ and approaches $f(\cdot, n)$ in $L^1(R)$ as $\delta \to 0$, and that, by (3.15),

$$|f^\delta(x, n)| \leqq K \tag{4.12}$$

for every $x \in \mathbb{R}^3$ and each unit vector n.

Now choose $\varepsilon > 0$ with $R_\varepsilon \neq \emptyset$ and confine δ to the interval $(0, \varepsilon)$. For each surface element $S \subset R_\varepsilon$ let

$$F^\delta(S) = \int_S f^\delta(x, n_S) \, dA_x,$$

so that, by (4.12), F^δ is a Cauchy flux on R_ε. Moreover, f^δ is a continuous function of position and is, therefore, the density of F^δ everywhere. We now show that F^δ is weakly balanced; once this is done we will be in a position to apply Cauchy's Theorem to the density f^δ. With this in mind, we establish the following identity,

[1] The portion of this lemma concerning the linearity of f is due to GURTIN, MIZEL & WILLIAMS (1968), and we follow their proof almost verbatim.

in which Φ: $\mathbb{R}^3 \to \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^Q)$ is a continuous map and S is an arbitrary surface element:

$$\int_S \Phi(x) f^\delta(x, n_S)\, dA_x = \int_{\mathbb{R}^3} \rho^\delta(z) \int_{S-z} \Phi(x+z) f(x, n_S)\, dA_x\, dV_z. \tag{4.13}$$

To prove (4.13) let I denote the left-hand side and write n for n_S; then

$$I = \int_S \int_{\mathbb{R}^3} \rho^\delta(x-y)\Phi(x) f(y,n)\, dV_y\, dA_x = \int_S \int_{\mathbb{R}^3} \rho^\delta(z)\, \Phi(x) f(x-z, n)\, dV_z\, dA_x.$$

Clearly, $(x, z) \mapsto f(x-z, n)$ is a Borel function, since it is the composition of a continuous function $(x, z) \mapsto x-z$ followed by a Borel function $x \mapsto f(x, n)$. Therefore, by Fubini's Theorem,

$$I = \int_{\mathbb{R}^3} \rho^\delta(z) \int_S \Phi(x) f(x-z, n)\, dA_x\, dV_z.$$

But

$$\int_S \Phi(x) f(x-z, n)\, dA_x = \int_{S-z} \Phi(x+z) f(x, n)\, dA_x,$$

and (4.13) follows. Let B be a body. If S is a face of B, then $n_S = n_{S-z}$, and (4.13) yields

$$\int_{\partial B} \Phi(x) f^\delta(x, n_{\partial B}(x))\, dA_x = \int_{\mathbb{R}^3} \rho^\delta(z) \int_{\partial B - z} \Phi(x+z) f(x, n_{\partial B - z}(x))\, dA_x\, dV_z, \tag{4.14}$$

where $n_{\partial B}$ and $n_{\partial B - z}$ are the outward unit normal fields to ∂B and $\partial B - z$, respectively.

We now show that F^δ is weakly balanced. Let B be a body in R_ε. Then, since $\delta \in (0, \varepsilon)$,

$$B - z \subset R \qquad \text{whenever } \rho^\delta(z) \neq 0. \tag{4.15}$$

Thus, if we apply (4.14) with $\Phi(x)$, for each x, the identity map on $\mathbb{R}^N$, and use the fact that F is weakly balanced, we find that

$$\left| \int_{\partial B} f^\delta(x, n_{\partial B}(x))\, dA_x \right| \leqq K V(B).$$

Thus F^δ is weakly balanced, and we conclude from Cauchy's Theorem that there exists a field T^δ: $R_\varepsilon \to \mathrm{Lin}(\mathbb{R}^3, \mathbb{R}^N)$ such that

$$f^\delta(x, n) = T^\delta(x) n \tag{4.16}$$

for all $(x, n) \in R_\varepsilon \times \mathrm{Unit}(\mathbb{R}^3)$.

Let $\{e_i\}$ be an orthonormal basis for $\mathbb{R}^3$. Then (4.16) implies

$$T^\delta(x) = \sum_{i=1}^{3} f^\delta(x, e_i) \otimes e_i,$$

and, since $f^\delta(\cdot, n) \to f(\cdot, n)$ in $L^1(R)$ (and hence $L^1(R_\varepsilon)$), T^δ tends to the field

$$T(x) = \sum_{i=1}^{3} f(x, e_i) \otimes e_i$$

is $L^1(R_\varepsilon)$. Therefore, letting $\delta \to 0$ in (4.16) we see that (for each fixed n) (4.11) is satisfied for $[V]$ almost every $x \in R_\varepsilon$. Since the only requirement on ε is that it be sufficiently small, and since $R = \bigcup_{k=1}^{\infty} R_{1/k}$, (4.11) holds for all x in a set $R_*(n)$ which differs from R by a set of zero volume.

Our next step will be to establish the symmetry of T almost everywhere. Thus let F have values in $\mathbb{R}^3$. Then, using (4.2), (4.12), and the same estimate as was used to derive (4.8), we find that

$$|MF_x(\partial B_0)| \leqq 12\sqrt{3}\, K V(B_0) \tag{4.17}$$

for any $x \in R$ and any cube $B_0 \subset R$ containing x.

As before, choose $\varepsilon > 0$, confine δ to the interval $(0, \varepsilon)$, and fix $y \in R_\varepsilon$. Let Φ in (4.14) be the field defined, at each $x \in \mathbb{R}^3$, by

$$\Phi(x) w = (x - y) \wedge w$$

for every $w \in \mathbb{R}^3$. Then (4.14) yields

$$MF_y^\delta(\partial B) = \int_{\mathbb{R}^3} \rho^\delta(z) \int_{\partial B - z} [x - (y - z)] \wedge f(x, n_{\partial B - z}(x)) \, dA_x \, dV_z \tag{4.18}$$

for every body $B \subset R_\varepsilon$. The inner integral in (4.18) is simply $MF_{y-z}(\partial B - z)$ at those z for which $B - z \subset R$. Thus, since ρ^δ (for fixed δ) is bounded, and since $V(B - z) = V(B)$, we conclude from (4.15), (4.17), and Lebesgue's dominated convergence theorem that: if F is moment balanced, then F^δ is moment balanced (on R_ε). Assume this is the case. Then Cauchy's Theorem implies that T^δ is symmetric on R_ε. Thus, since $T^\delta \to T$ in $L^1(R_\varepsilon)$, T is symmetric [V] almost everywhere on R_ε, and hence, arguing as before, [V] almost everywhere on R. □

Proof of Theorem 6. It suffices to show that

$$(4.11) \textit{ holds for all } (x, n) \in G \times \text{Unit}(\mathbb{R}^3), \tag{4.19}$$

where G is a subset of R that differs from R by a set of zero volume. Let $\mathcal{M}$ be a countably dense subset of Unit$(\mathbb{R}^3)$, and let G be the set of all points in $\bigcap_{n \in \mathcal{M}} R_*(n)$ that are points of density (*cf.* the statement of Lemma 2). By (iv) of Theorem 3, (4.10), and the countability of $\mathcal{M}$, $V(R - G) = 0$. Thus to complete the proof we have only to establish (4.19). But this follows from (4.11), (v) of Theorem 3, and the fact that $\mathcal{M}$ is dense in Unit$(\mathbb{R}^3)$. □

5. Additional Results

The classical proofs of Cauchy's Theorem are based on the assumption that the density f of F obey a balance law of the form

$$\int_{\partial B} f(x, n(x)) \, dA_x + \int_B b(x) \, dV_x = 0 \tag{5.1}$$

for every body B, where $n(x)$ is the outward unit normal to ∂B at x, and where b is a bounded integrable function on R. When this is the case we shall say that F

obeys a classical balance law. Our next theorem shows that for a Cauchy flux the notion of being weakly balanced is not as weak as it might first appear.

Theorem 7.[1] *Let F be a Cauchy flux. Then a necessary and sufficient condition that F obey a classical balance law is that F be weakly balanced.*

Proof. Necessity is obvious, since b in (5.1) is required to be bounded.

Conversely, assume that F is weakly balanced. To complete the proof it clearly suffices to show that there exists a Borel measure μ on R such that

$$F(\partial B) = \mu(B) \tag{5.2}$$

for every body B and

$$|\mu(D)| \leqq K V(D) \tag{5.3}$$

for every Borel set $D \subset R$. Indeed, were (5.2) and (5.3) true, then, by the Radon-Nikodym Theorem, there would exist a bounded integrable function b on R such that

$$\mu(D) = -\int_D b(x)\, dV_x$$

and (5.2) would yield

$$F(\partial B) + \int_B b(x)\, dV_x = 0,$$

which is (5.1). We now establish (5.2) and (5.3). For convenience, we restrict our attention to the case in which F is scalar-valued ($N=1$); the more general case then follows as an immediate corollary.

Let $\mathscr{B}$ denote the collection of all sets formed by taking finite unions of bodies. Then the boundary of $B \in \mathscr{B}$ is the union of a finite number of closed polygonal faces $S_1, S_2, \ldots, S_Q$ and $F(\partial B)$ can still be defined by (3.1). Choose $B, C \in \mathscr{B}$ with $V(B \cap C) = 0$; then $B \cap C$ must be the union of a finite number of plane polygonal regions $P_1, P_2, \ldots, P_Q$ and a finite number of plane sets of zero area. Let $S_1, S_2, \ldots, S_Q$ be the surface elements oriented by ∂B that have $P_1, P_2, \ldots, P_Q$ as underlying sets. Then by (3.5)

$$F(\partial(B \cup C)) - F(\partial B) - F(\partial C) = -\sum_{i=1}^{Q} [F(S_i) + F(-S_i)] = 0.$$

Thus the map $F_*: \mathscr{B} \to \mathbb{R}$ defined by

$$F_*(B) = F(\partial B)$$

has the following properties:

$$\begin{aligned} &F_*(B \cup C) = F_*(B) + F_*(C) \quad \text{whenever } V(B \cap C) = 0, \\ &|F_*(B)| \leqq K V(B). \end{aligned} \tag{5.4}$$

The inequality $(5.4)_1$, of course, follows from (3.3), (5.4), and the fact that every $B \in \mathscr{B}$ is the finite union of bodies.

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[1] The non-trivial portion of this theorem (sufficiency), within a slightly different framework, is due to GURTIN & WILLIAMS (1967), Theorem 6.

To facilitate the remainder of the proof,[1] we introduce some notation. Given $x \in \mathbb{R}^3$ and $\delta > 0$ we call

$$Q(x,\delta) = \{y \in \mathbb{R}^3 : 0 \leqq (x-y)_j \leqq \delta,\ j=1,2,3\} \tag{5.5}$$

the *box* with corner at x and length δ. Here, of course, $(x-y)_j$ are the coordinates of $(x-y)$. For $n=1,2,\ldots$, let P_n be the set of all $x \in \mathbb{R}^3$ such that the coordinates of x are integral multiples of 2^{-n}, let Ω_n denote the collection of all boxes with corners at points of P_n and length 2^{-n}, and for any set $A \subset \mathbb{R}^3$ let $P_n(A)$ denote the set of all $x \in P_n$ for which $Q(x, 2^{-n}) \subset \mathring{A}$. Finally, let $\hat{Q}(x,\delta)$ denote the set (5.5) when the inequality is replaced by $0 \leqq (y-x)_j < \delta$. We call $\hat{Q}(x,\delta)$ a semi-box, and we denote by $\hat{\Omega}_n$ the collection of semi-boxes with corners at points of P_n and length 2^{-n}. For $B \in \mathscr{B}$ the total volume of all boxes $Q(x, 2^{-n})$ that intercept ∂B and have $x \in P_n$ can be bounded by 2^{-n} times a constant (that depends on B but is independent of n). Thus we conclude from the properties (5.4) that

$$\Big|F_*(B) - \sum_{x \in P_n(B)} F_*(Q(x, 2^{-n}))\Big| \leqq K_B 2^{-n}. \tag{5.6}$$

Now let $C_c(R)$ denote the space of all continuous real-valued functions on R with compact support, and equip $C_c(R)$ with the sup norm. For $n=1,2,\ldots$ and $g \in C_c(R)$ let

$$\Lambda_n g = \sum_{x \in P_n(R)} g(x)\, F_*(Q(x, 2^{-n})).$$

Then

$$|\Lambda_n g| \leqq \sup_{x \in R} |g(x)|\, K V(R),$$

and hence Λ_n is a continuous linear functional on $C_c(R)$. Further, $(5.4)_2$, the uniform continuity of $g \in C_c(R)$, and the additivity of F_* imply that

$$\Lambda g = \lim_{n \to \infty} \Lambda_n g$$

exists and defines a continuous linear functional on $C_c(R)$. Thus, by the Riesz Representation Theorem, there exists a (unique) real regular[2] Borel measure μ such that

$$\Lambda g = \int_R g\, d\mu.$$

Our next step will be to show that μ satisfies (5.2) and (5.3). Choose a box Δ in R and for each integer $k > 0$ let $g_k \in C_c(R)$ be a non-negative function, bounded by 1, with $g_k(x) = 1$ for all $x \in \Delta$, and with support in a box that is concentric with Δ and has volume $V(\Delta) + 1/k$. Then, as $k \to \infty$, g_k tends pointwise to the characteristic function of Δ and

$$\Lambda g_k = \int_R g_k\, d\mu \to \mu(\Delta).$$

Moreover, $|F_*(\Delta) - \Lambda_n g_k| \leqq 2K/k$ for all sufficiently large n; thus $|F_*(\Delta) - \Lambda g_k| \leqq 2K/k$ and $\Lambda g_k \to F_*(\Delta)$. Therefore

$$F_*(\Delta) = \mu(\Delta).$$

[1] At this point we could simply appeal to a theorem of GURTIN & WILLIAMS ((1967), p. 113); we prefer, however, to give a different proof based on a technique, now standard, for the construction of Lebesgue measure on $\mathbb{R}^M$ with the aid of the Riesz Representation Theorem (*cf.* RUDIN (1974), p. 53).

[2] *Cf.*, *e.g.*, RUDIN (1974), p. 139.

Since

$$\hat{Q}(x,\delta)=\bigcup_{n=2}^{\infty} Q\left(x, d-\frac{\delta}{n}\right),$$

any semi-box $\hat{Q}(x,\delta)$ with closure in R must satisfy

$$\mu(\hat{Q}(x,\delta))=\lim_{n\to\infty}\mu\left(Q\left(x,\delta-\frac{\delta}{n}\right)\right)=\lim_{n\to\infty}F_*\left(Q\left(x,\delta-\frac{\delta}{n}\right)\right) \\ =F_*(Q(x,\delta)); \tag{5.7}$$

therefore

$$|\mu(\hat{Q}(x,\delta))|\leqq KV(Q(x,\delta))$$

for all semi-boxes with closure in R, and hence for all semi-boxes in R. Observe now that every non-empty open set in R is a countable union of disjoint semi-boxes belonging to $\bigcup_{n=1}^{\infty}\hat{Q}_n$. Thus for every such open set D

$$|\mu(D)|\leqq KV(D), \tag{5.8}$$

and, since μ is regular, (5.8) must hold for every Borel subset of R, so that (5.3) is satisfied. Thus for every $B\in\mathscr{B}$, $\mu(\partial B)=0$, and we conclude from (5.6) and (5.7) that

$$\mu(B)=\mu(\mathring{B})=\lim_{n\to\infty}\sum_{x\in P_n(B)}\mu(\hat{Q}(x,2^{-n}))=\lim_{n\to\infty}\sum_{x\in P_n(B)}F_*(Q(x,2^{-n})) \\ =F_*(B),$$

which yields (5.2). □

Remark 1. In the above proof we established the *existence* of a measure μ consistent with (5.2) and (5.3). *Uniqueness* also holds: there is at most one Borel measure μ on R that satisfies $\mu(B)=F(\partial B)$ for every $B\in\mathscr{B}$. Indeed, if $\hat{\mu}$ is a second Borel measure with this property, then, since $|\mu(B)|$ and $|\hat{\mu}(B)|$ are bounded by $KV(B)$ on every $B\in\mathscr{B}$, μ and $\hat{\mu}$ are regular[1] and coincide on every semi-box in R and hence on every Borel set in R. This shows that for a weakly balanced Cauchy flux F there exists *exactly one*[2] bounded integrable field b on R such that (5.1) holds.

Remark 2. While the above analysis establishing the existence of a unique extension μ of F_* is carried out in $\mathbb{R}^3$, it is clear that the essential items are additivity and boundedness with respect to Lebesgue measure V on $\mathbb{R}^3$ (*cf.* (5.4)), and it is therefore equally clear that completely analogous results hold for $\mathbb{R}^M$.

To define $F(\partial B)$ on a body B it is necessary only to define F on *polygonal* surface elements. Therefore our notion of a Cauchy flux, in which F is defined on all surface elements, might, at first sight, appear artificial. We now show that our definition involves no loss in generality.

Let $\mathscr{P}$ denote the collection of all surface elements which are finite unions of polygonal surface elements. Then as a direct consequence of Remark 2 we have

[1] *Cf., e.g.*, RUDIN (1974), p. 50.

[2] Equating functions that are equal $[V]$ almost everywhere.

Theorem 8. *Let F: $\mathscr{P}\to\mathbb{R}^N$ be area-bounded and satisfy*

$$F(S_1 \cup S_2) = F(S_1) + F(S_2)$$

whenever S_1 and S_2 are compatible and have $A(S_1 \cap S_2) = 0$. Then F is the restriction to $\mathscr{P}$ of a unique Cauchy flux.

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