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INFERENCE AND NECESSITY[1]

I. INTRODUCTION

In recent years we have been concerned with the development of a natural generalization of the usual modal semantics. We have been impelled to undertake this enterprise by philosophical misgivings concerning the privileged status accorded certain modal principles by the received semantical framework. This concern led to a particular kind of generalization which has been set forth and defended elsewhere.[2] It has now become apparent that the large body of logical theory which this generalization places at our disposal satisfies a philosophical need which transcends the original motivation of the work.

That there is a connexion between modal concepts and inferential ones is indisputable. It is not at all unusual, for example, to describe a valid inference as one in which, if the premises are true, it is *impossible* for the conclusion to be false. (The earliest roots for this way of characterizing deductive validity lie in the logical works of Aristotle). In addition, it has often been felt that the concept of necessity has a much deeper connexion with the theory of inference than it's merely descriptive role. C. I. Lewis, for example, seemed to feel that a theory of 'implication', (and derivatively a theory of necessity) would be dictated by "the facts of deduction". Although such a scheme has obvious appeal, especially as regards the philosophical motivation of modal logic, it has, historically, encountered grave difficulties.

In what follows we derive the model theory of modal logic from a theory of inference. The main features of the derivation are first, that it is highly general and secondly, that the original theory of inference is recoverable from the derived modal logic.

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II. INFERENCE

II.1

'If our world were classical, then so might our logic be also.' Anyone who favors (for whatever reason) some non-classical logic is sooner or later tempted to deploy this slogan in his own defence. In many cases however the strategy does not succeed. For if we do not actually reason in accord with the classical canons the fault may lie not in the world but in our reasoning. Our inability to add or to reason well will only feebly motivate development of non-classical arithmetic or logic. If invention is to be mothered, then by all means let it be mothered, but through necessity – not through failure to take suitable precautions.

There are, however, moments when we abandon classical methods not because we are stupid, but because they fail us. If the Sorites paradox has a moral then this is it. It is senseless to reason as if all predicates were exact, for they are not, and classical procedures followed here lead us unto contradiction.

It has often been recognized that, from the viewpoint of applied logic, the classical treatment of inconsistency is deficient. This is not because we sometimes contradict ourselves but rather because often the best data available to us is contradictory. All that the classical logician can advise us to do is to start again from better data. Sometimes this is wise counsel, but more often it is of a piece with what the doctor said when told by a patient: "It hurts when I do this".

We take the position that consistency, classically conceived, is an ideal state of affairs often approximated rather than achieved. In particular, we concern ourselves with the question: what principles of inference are suitable for reasoning from inconsistent premises? Our approach is to introduce a more general notion of consistency, one which admits of levels or degrees. It is this generalized concept of consistency which leads to the notion of entailment that we wish to present.

II.2. Consistency and coherence

The classical conception of consistency may be given in terms of (classical) *provability* and the notion of an *absurd* sentence. The latter is also defined using provability:

A is absurd iff $A \vdash \alpha$ for every sentence α

Obviously all absurd sentences are provably equivalent and we may let "$\perp$" stand for an arbitrary absurdity. Next we say that a set, Γ, of sentences is *consistent* iff:

$$\Gamma \nvdash \perp.$$

Within the classical framework, if for some α, $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma$ then $\Gamma \vdash \perp$. Classical logic does not distinguish inferentially, between having among one's premises $\perp$, and having simultaneously α and $\neg\alpha$. To employ a colourful bit of recent jargon, both premise sets explode. On the classical view, reasoning from a set of premises ceases upon its detonation.

But consider how imperfectly this approach reflects the way that we actually reason. Our normal data set is the set of sentences that we believe to be true, and few of us imagine that among our beliefs are lurking no inconsistent pairs. Of course we recognize that we do not bring into play all of our beliefs simultaneously. But activate enough beliefs for any but the most trivial ratiocination, and the likelihood is not negligible that among them or their assumptions is a contradictory pair. Nor are such pairs always hidden from us. Often enough in philosophy, evidence which we think is good evidence yields contradictory consequences. Our response is sometimes to reject one thesis but on other occasions we have *no* basis on which to decide what must be discarded. In these cases we decide to live with the embarrassment against the discovery of an *experimentum crucis* or the development of a more perspicuous way of speaking. In the meantime the data remains and if our decisions are to be rational we must reason with what we have. We do naturally distinguish between believing a contradiction and having contradictory beliefs, avoiding the one with ease and getting no ulcers over the other.

To make the distinction more formally, we employ the idea of (level of) *coherence*; a concept which embraces the classical notion of consistency as a special case. More precisely a *coherence function* c is a function having as its domain the set of all finite sets of sentences and as a codomain the set $Nat \cup \{\omega\}$, where Nat is the set of natural numbers (including 0). The pointwise definition of c is:

for $\perp \notin \Gamma$

$c(\Gamma) = m$ iff m is the least integer such that there are sets $a_1, \ldots, a_m, a_i \nvdash \perp (1 \leqslant i \leqslant m)$

and $\bigcup_{i=1}^{m} a_i = \Gamma$.

If $\perp \in \Gamma$ then we adopt the convention: $c(\Gamma) = (\omega)$.

The idea of an hierarchy of levels of coherence (or incoherence) leads to a new concept of derivability. We define a relation between finite sets of sentences and sentences written Γ [⊢ α (read Γ forces α[3]):

for $c(\Gamma) = n(\omega)\Gamma$ [⊢ α iff for every n-fold (ω-fold) decomposition of $\Gamma, a_1, \ldots, a_n$, there is some i such that $a_i \vdash \alpha$ $(1 \leqslant i \leqslant n(\omega))$.

As an attempt to define a kind of entailment in terms of coherence this may not be the only one which springs to mind. We hope to show however, that the relation indicated by '[⊢' is the most natural one among its competitors. As a path to a better understanding of '[⊢' we compare it to the classical notion of provability '⊢'. The comparison will be facilitated if we specify the two relations by means of rules of inference. These fall into two classes, *structural* rules and *operator* rules. For the classical propositional calculus the following completely determine '⊢'.

Structural rules

[*Ref*] $\alpha \in \Gamma \Rightarrow \Gamma \vdash \alpha$

[*Mon*] $\Gamma \vdash \alpha \Rightarrow \Gamma \cup \Delta \vdash \alpha$

[*Trans*] $\Gamma \cup \{\alpha\} \vdash \beta \;\&\; \Gamma \vdash \alpha \Rightarrow \Gamma \vdash \beta$

Operator rules

($\wedge I$) $\dfrac{\Gamma \vdash \alpha, \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}$ ($\wedge E$) $\dfrac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha, \Gamma \vdash \beta}$

($\vee I$) $\dfrac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta}$ ($\vee E$) $\dfrac{\Gamma \vdash \alpha \vee \beta, \Gamma \cup \{\alpha\} \vdash \gamma, \Gamma \cup \{\beta\} \vdash \gamma}{\Gamma \vdash \gamma}$

($\rightarrow I$) $\dfrac{\Gamma \cup \{\alpha\} \vdash \beta}{\Gamma \vdash \alpha \rightarrow \beta}$ ($\rightarrow E$) $\dfrac{\Gamma \vdash \alpha \rightarrow \beta, \Gamma \vdash \alpha}{\Gamma \vdash \beta}$

($\neg I$) $\dfrac{\Gamma \cup \{\alpha\} \vdash \perp}{\Gamma \vdash \neg\alpha}$ ($\neg E$) $\dfrac{\Gamma \cup \{\neg\alpha\} \vdash \perp}{\Gamma \vdash \alpha}$

II.3. Forcing

'[⊢' is determined by a set of rules very like these. The difference is that we must stipulate in [*Mon*] that $c(\Gamma \cup \Delta) = c(\Gamma)$ and in addition we add one rule to the structural set and delete certain of the operator rules. This is a source of some satisfaction since it is well known that any relation satisfying the classical structural rules determines a *consequence* relation in Tarski's sense.[4] Thus we can say that there are grounds for calling '[⊢' an entailment relation of *some* sort.

We shall delay discussion of the operator rules since our new structural rule requires some additional technical vocabulary.

For a any finite set we say that $C \subseteq 2^a$ (C, a set of subsets of a) is an *m-cluster* iff: $m \in Nat$ and

$$\forall f \in m^a,\ \exists x \in C,\ \exists y \leqslant m\colon\ x \subseteq f^{-1}[y].$$

In words: for any way of dividing a into m subsets there is a member x of C (i.e., a subset of a which belongs to C) such that x is included in at least one of the m subsets into which a has been divided. We say, derivatively, that a set of sets of sentences $\{\{\alpha_i, \alpha_j, \ldots, \alpha_m\}, \ldots, \{\alpha_n, \ldots, \alpha_p\}\}$ drawn from $\alpha_i, \ldots, \alpha_q$ is an m-cluster iff the set $\{\{i, j, \ldots, m\}, \ldots, \{n, \ldots, p\}\} \subseteq 2^q$ is an m-cluster.

By means of this terminology we now frame a structural rule for '[⊢' which depends upon the coherence level of a set Γ.

[C] If $C = \{c_1, c_2, \ldots, c_n\}$ is an m-cluster constructed out of $\alpha_1, \ldots, \alpha_k \in \Gamma$ and $c(\Gamma) = m$, then

$$\frac{c_1 \vdash \beta, c_2 \vdash \beta, \ldots, c_n \vdash \beta}{\Gamma\ [\!\vdash \beta}$$

By inspecting our definition of '[⊢' we see that we lose the classical operator rule (∧*I*); from Γ [⊢ α and Γ [⊢ β it no longer follows that Γ [⊢ $\alpha \wedge \beta$. We have this only in the special case that $c(\Gamma) = 1$ (given rule [C]).

It is worthwhile emphasizing that our characterization of '[⊢' is a true generalization of the classical '⊢' and that each of the rules which fails in general does hold for the "classical" case: $c(\Gamma) = 1$.

11.4. Theories

Making a distinction between absurd premises and contradictory pairs of premises leads to a new and more general concept of a *theory*. The narrower concept is defined in terms of the classical '$\vdash$'.

$$\Delta \text{ is a theory iff for every sentence } \alpha, \Delta \vdash \alpha \Rightarrow \alpha \in \Delta.$$

One of the things which makes the notion of a theory mathematically attractive is that it may be independently characterized by two closure conditions which do not mention $\Delta \vdash \alpha$. These conditions are:

(a) $\alpha \in \Delta \;\&\; \vdash \alpha \to \beta \Rightarrow \beta \in \Delta,$

(b) $\alpha \in \Delta \;\&\; \beta \in \Delta \Rightarrow \alpha \wedge \beta \in \Delta.$

In mathematical logic theories are employed as generalizations of logics. Instead of restricting our consideration to the consequences of some axiom set detailing the behavior of logical constants, (a logic), we may wish to allow the inclusion of so-called non-logical principles as well. In this case we study the whole class of theories which include our base logic.

Within the approach we have developed we may define:

$$\Delta \text{ is an } m\text{-theory iff } c(\Delta) = m \text{ and } \Delta \;[\!\vdash \alpha \Rightarrow \alpha \in \Delta.$$

The classical concept of a theory is what we would call a 1-theory. The closure condition characterization of 1-theories turns out to be a special case of a more general condition, viz., Δ is an m-theory iff

(a) $\alpha \in \Delta \;\&\; \vdash \alpha \to \beta \Rightarrow \beta \in \Delta$

(b) $\{c_1, \ldots, c_k\} \subseteq 2^{\Delta}$ is an m-cluster

$$\Rightarrow \bigvee_1^k \{\wedge c_1, \ldots, \wedge c_k\} \in \Delta$$

when "$\wedge c_i$" denotes "$\alpha_{i_1} \wedge \ldots \wedge \alpha_{i_j}$" for $c_i = \{\alpha_{i_1}, \ldots, \alpha_{i_j}\}$.

To see that this is indeed a generalization we need only notice that when $m = 1$ any finite subset of Δ is an m-cluster so that (b) amounts to closure under finite conjunctions.

III. NECESSITY

III.1

Here we make precise the connexion between forcing and relational semantics. As "$[\vdash$" is a generalization of "$\vdash$" so our semantics is a generalization of Kripke's. It is the concept of an *m*-theory which provides the key to the relation between inference and modality.

The language of modal propositional logic $PC(\Box)$ results from the language of classical propositional logic by the addition of a new unary sentence operator '$\Box$'. Thus it is stipulated that $\Box\alpha$ is well-formed whenever α is.

Sentences of the language are evaluated by objects called models which may be described as follows:

A frame is a pair (U, R) where U is a non-empty set and $R \subseteq U^2$ is a binary relation. ***Truth*** is defined for the atomic sentences of the language (the set of which is called At) relative to members of U (often called informally, possible worlds). More specifically we get a model $\mathfrak{M}$ on a frame whenever we give a function V: $At \to 2^U$ which associates with every atomic sentence a set of worlds (informally: the set of worlds in which the sentence is true).

V is extended (uniquely) to a function $\|\cdot\|^{\mathfrak{M}}$ which evaluates all sentences of the language by means of the *truth-conditions.* These are the obvious counterparts of the usual truth-table definitions for the classical operators, e.g.,

$$\|\alpha \wedge \beta\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}} \cap \|\beta\|^{\mathfrak{M}}.$$

For the modal operator we have $\models^{\mathfrak{M}}_{u} \Box\alpha \Rightarrow \forall\, v \in U$: $uRv \Rightarrow \models^{\mathfrak{M}}_{v}\alpha$ where "$\models^{\mathfrak{M}}_{u}\alpha$" abbreviates "$u \in \|\alpha\|^{\mathfrak{M}}$".

In words: α is necessarily true in u iff α is true in all the alternatives of u. We ensure that particular axioms and rules will hold, by placing restrictions on the frame relation R. Some principles, however, hold even when no restrictions are imposed.

[RR] $\vdash \alpha \to \beta \Rightarrow \vdash \Box\alpha \to \beta$,

[K] $\Box p \wedge \Box q \to \Box(p \wedge q)$,

are among them.

Now if we represent by $\Box(u)^{\mathfrak{M}}$ the set of necessary truths with respect to the world u in the model $\mathfrak{M}$, i.e., $\left\{\alpha \mid \overset{\mathfrak{M}}{\underset{u}{\models}} \Box\alpha\right\}$, then on the usual semantics $\Box(u)^{\mathfrak{M}}$ is a 1-theory.

Many philosophers have objected to the representation of belief and obligation, for example, by means of modal operators on the grounds that [K] is not a principle which governs such concepts. We may restate this objection in the form: belief sets and sets of sentences which ought to be true, if they form theories, do not form 1-theories.

Suppose that such sets form m-theories for some m. Then we shall want to uncover the truth-condition of the associated modal operator.

III.2. *Premodel theory for modal logic*

If we are to derive truth-conditions for $\Box$-formulae we must give away the truth-conditions we have, and trust to our derivation to render them unto us n-fold. Accordingly, we do so.

A premodel $\mathfrak{P}$ for $PC(\Box)$ is like a model except that no relation is specified and no truth condition for $\Box$-formulae beyond the stipulation that $\|\Box\alpha\|^{\mathfrak{P}} \in 2^U$. Clearly '$\Box$' is not behaving semantically as a logical constant. Nevertheless, even in these reduced circumstances, it plays a crucial role. It permits us to transform inconsistent sets of sentences into sets with a non-zero semantical representation. For example, in $PC(\Box)$ premodels, $\|\{\alpha, \neg\alpha\}\|^{\mathfrak{P}} = \phi$, but $\|\{\Box\alpha, \Box\neg\alpha\}\|^{\mathfrak{P}}$ need not be null. Thus the $\Box$-operator performs the task of guarding inconsistent formulae, lest their contradictoriness prove semantically untoward. We note that in modal logics which subscribe to the principle [K] of complete aggregation, the guard duty of the $\Box$-operator is entirely ceremonial.

Our aim now is to define a modal logic engendered by the forcing relation, by singling out from the class of $PC(\Box)$ premodels the models for the logic. We do this in two stages:

(1) We restrict the class of premodels to a special subclass called full premodels.

(2) We derive the underlying structure of the desired model and the truth conditions for $\Box\alpha$ from the restrictions imposed by $[\vdash$.

Full premodels

$\mathfrak{P}$ is a full premodel iff, $\mathfrak{P}$ is a premodel and $\forall u$ s.t. $\Box(u)^{\mathfrak{P}}$ is an n-theory, $a \subseteq \Box(u)^{\mathfrak{P}}$ s.t. $a \nvdash \bot, a \subseteq b \;\&\; b \nvdash \bot \Rightarrow \|b\|^{\mathfrak{P}} \neq \phi$. The point of this restriction will become clear in what follows:

Definition of the n-natural relation

Let $\mathfrak{P}$ be a full $PC(\Box)$ premodel. For each $n \in Nat$, we define pointwise a function r: $\{x \mid c(\Box(x)^{\mathfrak{P}}) = n\} \to U^n$. Let u be an element of U such that $c(\Box(u)^{\mathfrak{P}}) = n$. Further, let $\Delta(u) = \{\delta \mid \delta\colon \Box(u)^{\mathfrak{P}} \twoheadrightarrow n\}$ be the set of non-trivial n-fold decompositions of $\Box(u)$.

Then $r(u) = \{\langle x_1, \ldots, x_n\rangle \mid x_i \in \|\delta^{-1}[i]\|^{\mathfrak{P}} (1 \leqslant i \leqslant n)$ for some $\delta \in \Delta(u)\}$.

Finally if $\langle x_1, \ldots, x_n\rangle \in r(u)$ we write $uRx_1 \ldots x_n$ and call R the n-natural relation of u.

Remarks on the definition

(1) The restriction of $\Delta(u)$ to the set of onto functions does not incur any loss of generality since $c(\Box(u)^{\mathfrak{P}}) = n$.

(2) Where $c(\Box(u)^{\mathfrak{P}}) = 1$ the relation obtained is the 1-natural relation which is binary.

THEOREM. If $\mathfrak{P}$ is a full $PC(\Box)$ premodel and $\Box(u)^{\mathfrak{P}}$ is an n-theory and R the n-natural relation, then $\models^{\mathfrak{P}}_{u} \Box\alpha \Leftrightarrow \forall x_1, \ldots, x_m, uRx_1 \ldots x_m \Rightarrow \models^{\mathfrak{P}}_{x_1} \alpha$ or $\ldots$ or $\models^{\mathfrak{P}}_{x_n} \alpha$.

Proof

($\Rightarrow$) Suppose that $\models^{\mathfrak{P}}_{u} \Box\alpha$. Then $\alpha \in \Box(u)^{\mathfrak{P}}$. Therefore $\Box(u)^{\mathfrak{P}}\ [\vdash \alpha$ (by [*Ref*]).

Therefore, for all decompositions δ of $\Box(u)^{\mathfrak{P}}$, $\exists i$: $\delta^{-1}[i] \vdash \alpha$.
Therefore, $\forall x_1, \ldots, x_n, uRx_1 \ldots x_n \Rightarrow \exists j$: $x_j \in \|\delta^{-1}[j]\|^{\mathfrak{P}}$ and $\delta^{-1}[j] \vdash \alpha$.

But $\|x_j\|^{\mathfrak{P}} = \{\alpha \models^{\mathfrak{P}}_{x_j} \alpha\}$ is a classical theory. Therefore, $\vdash_{x_j} \alpha$.

($\Leftarrow$) Suppose that $\not\models^{\mathfrak{P}}_{u} \Box\alpha$. Then $\alpha \notin \Box(u)^{\mathfrak{P}}$.

Therefore, $\Box(u)^{\mathfrak{P}}$ $[\not\vdash \alpha(\Box(u)^{\mathfrak{P}}$ is an n-theory).
Therefore, $\exists \delta \in \Delta(u)$: $\forall i(1 \leqslant i \leqslant n), \delta^{-1}[i] \vdash \alpha$.

By the fullness of $\mathfrak{P}$, $\exists x_1 \ldots x_n \in U$: $uRx_1 \ldots x_n$ & $\models^{\mathfrak{P}}_{x_1} \alpha$ & ... & $\models^{\mathfrak{P}}_{x_n} \alpha$.

This completes the proof of the theorem.

The constructions with respect to which natural relations are defined give the appearance of excessive particularity. We will show, however, that the class of structures so generated determine the modal logic, K_n whose canonical model is a full $PC(\Box)$ premodel, satisfying the closure restriction of the theorem.

III.3. The logic K_n

The weakest of the standard modal logics (usually called K) may be axiomatized by a single rule (following Dana Scott) in addition to some collection of principles adequate for classical sentence logic. The rule is:

$$[RK] \qquad \frac{\Gamma \vdash \beta}{\Box[\Gamma] \vdash \Box\beta} \quad \text{where } \Box[\Gamma] = \{\Box\alpha \mid \alpha \in \Gamma\}.$$

The rule may be read as stating that if Γ (where $c(\Gamma) = 1$) forces β, then $\Box[\Gamma] \vdash \Box\beta$. Semantically this amounts to stipulating that $\Box(u)^{\mathfrak{M}}$ is a 1-theory. If we stipulate instead that $\Box(u)^{\mathfrak{M}}$ is an n-theory, the resulting rule is the rule:

$$[RK_n] \qquad \frac{\Gamma\ [\vdash \beta}{\Box[\Gamma] \vdash \Box\beta}\, c(\Gamma) = n.$$

The system which results when this rule replaces $[RK]$ axiomatizes the logic K_n.[5]

A remark about $[RK_n]$
The rule illustrates the importance of the guarding function of the $\Box$ operator. For the set Γ may well be classically inconsistent and therefore so far as classical logic is concerned inferentially useless. But as we have noted above, if a classically inconsistent set has a finite coherence level it does have a non-trivial inferential role. By guarding each member of Γ with $\Box$, we salvage this role at the level of classical inference.

III.4. n-ary relational semantics

We now deploy the structures (U, R), $U \neq \phi$, $R \subseteq U^n$ together with the truth-condition derived above in order to provide an adequate semantical analysis of the K_n logics. In the account of arbitrary models $\mathfrak{M}$ for the logic K_n we require then:

$$\models^{\mathfrak{M}}_{u} \Box\alpha \Leftrightarrow \forall x_1, \ldots, x_n\colon uRx_1, \ldots, x_n \Rightarrow \models^{\mathfrak{M}}_{x_1} \alpha$$
$$\text{or} \ldots \text{or} \models^{\mathfrak{M}}_{x_n} \alpha.$$

That this account *is* adequate follows as usual from:

[*Theorem*] $\Gamma \vdash_{K_n} \alpha \Leftrightarrow \Gamma \models_{K_n} \alpha.$

When the right side of this means as usual:

$$\forall \mathfrak{M}, \forall u, u \in \|\Gamma\|^{\mathfrak{M}} \Rightarrow u \in \|\alpha\|^{\mathfrak{M}}.$$

The only if direction (soundness) may be obtained by routine calculations. The if direction (completeness) is somewhat more difficult.

Using the Henkin strategy we construct the K_n canonical model $\mathfrak{M}_{K_n} = (U_{K_n}, R_{K_n}, V_{K_n})$ where:

U_{K_n} is the set of K_n maximal 1-theories.

$uR_{K_n}x_1 \ldots x_n \Leftrightarrow \forall \alpha,$

$\Box\alpha \in u \Rightarrow \alpha \in x_1$ or ... or $\alpha \in x_n$.

$V_{K_n}(\alpha) = \{u \in U_{K_n} \mid \alpha \in u\}$ if $\alpha \in At$.

In order to satisfy all the K_n maximal 1-theories we must prove:

$$\models_u^{\mathfrak{M}K_n} \alpha \Leftrightarrow \alpha \in u \quad \text{[Fundamental Theorem].}^6$$

This is proved by induction on the structure of α. We sketch here the 'hard' direction of the induction step for α of the form $\Box\beta$:

Suppose $\Box\beta \notin u$.
Decompose $\Box(u)^{\mathfrak{M}K_n}$ into n subsets $\delta_1, \ldots, \delta_n$ such that $\delta_i \nvdash \beta \, (1 \leqslant i \leqslant n)$. There must be at least one such decomposition for otherwise there would be some finite subset Γ of $\Box(u)^{\mathfrak{M}K_n}$ such that Γ $[\vdash \beta$. By $[RK_n]$ $\Box[\Gamma] \vdash \Box\beta$ and thus $u \vdash \Box\beta$ and since u is a 1-theory $\Box\beta \in u$ contrary to hypothesis. Next expand $\delta_i \cup \{\neg\beta\}$ into a maximal 1-theory for each i, let these theories be $x_1, \ldots, x_n$. Clearly $uR_{K_n}x_1 \ldots x_n$ and by hypothesis $\models_{x_i}^{\mathfrak{M}K_n} \neg\beta$; thus by the truth-condition $\not\models_u^{\mathfrak{M}K_n} \Box\beta$.

Related results

(1) $\Box[\Gamma] \vdash_{K_n} \Box\alpha \Rightarrow \Gamma \; [\vdash \alpha \, (c(\Gamma) = n)$.

The converse of $[RK_n]$ follows from the soundness result together with

(2) $\Box[\Gamma] \models_{K_n} \Box\alpha \Rightarrow \Gamma \; [\vdash \alpha$,

the proof of which is trivial.

(3) $\forall u$: $\Box(u)^{\mathfrak{M}K_n}$ is an n-theory.

By the same sort of argument used to prove the fundamental theorem one easily shows:

(4) $\Box(u)^{\mathfrak{M}K_n} \; [\vdash \alpha \Rightarrow \models_u^{\mathfrak{M}K_n} \Box\alpha$

from which the result follows immediately.

With these results we have made good our earlier promise. The derivation of $\Box$ from $[\vdash$ has been shown to be adequate. Not only can a modal logic be derived from a notion of consequence but within the logic which results the originating theory of inference may be recovered.

IV. CONCLUDING REMARKS

In Schotch and Jennings (1979) we propose a revision of the classification of modal logics which Segerberg introduces (Segerberg, 1971). That proposal leaves untouched the classification of extensions of E as classical modal logics. In Segerberg's scheme E is the smallest modal logic which includes PC and is closed under $[RE]$:

$$\vdash \alpha \leftrightarrow \beta \Leftrightarrow \vdash \Box\alpha \leftrightarrow \Box\beta.$$

If we consider the connexions between modal logics and inference, it is clear from the above that the logic which answers to the classical theory of inference is not E but K. We would suggest, moreover, that the claims of this connexion are more puissant than the fact that truth sets of modalized formulae lie in 2^U.

It is sometimes suggested that the era of philosophically significant work in propositional modal logic is drawing to a close, and that the remaining problems are of purely formal interest. We take this essay to have refuted that suggestion. For from it may be glimpsed a programme which will draw out more and more the connexions between modal logics and systems of inference. Divergences from classical inference are almost always grounded in deeply philosophical concerns.

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NOTES

[1] Partially supported by SSHRC grant 410-78-0084. Section II of this paper was presented to the 9th International Symposium on Multiple-Valued Logic under the title 'Multiple-valued consistency'.

[2] See Schotch and Jennings (1979).

[3] No doubt some apology is required here for our lifting of this word from the work of Cohen. In our view however this is too good a name to be left inviolate. In this

respect it is like "entailment" which has been employed by a number of authors to their own ends.

[4] See Scott (1972, pp. 414–416).

[5] Described in Schotch and Jennings (1979) in a somewhat different way.

[6] In the terminology of Schotch and Jennings (1979) this is the fundamental theorem for all normal regular quasi aggregative logics.

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