On the Nonexistence of a Surface of Constant Mean Curvature with Finite Area and Prescribed Rectifiable Boundary

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Let Γ^* be a closed rectifiable Jordan curve in R^3 and let H be a real parameter. Consider the set $\mathfrak{Q}(\Gamma^*, H)$ of vector functions $\mathfrak{x}=\mathfrak{x}(w)=(x(w), y(w), z(w))$ $(w=u+iv)$ with the following properties:

(*a*) $\mathfrak{x}(w)$ belongs to $C^2(B)\cap C^0(\bar{B})$, where B is the unit disk $|w|<1$, and we have in B the equations

$$\Delta\mathfrak{x}=2H(\mathfrak{x}_u\times\mathfrak{x}_v), \tag{1}$$

and

$$\mathfrak{x}_u^2=\mathfrak{x}_v^2, \qquad \mathfrak{x}_u\mathfrak{x}_v=0. \tag{2}$$

(*b*) The mapping $\mathfrak{x}\colon \partial B\to\Gamma^*$ is topological.

(*c*) We have the inequality

$$A(\mathfrak{x})=\tfrac{1}{2}\iint_B(\mathfrak{x}_u^2+\mathfrak{x}_v^2)\,du\,dv<+\infty. \tag{3}$$

Geometrically speaking these conditions express the fact that $\mathfrak{x}=\mathfrak{x}(w)$ represents a surface in R^3 of constant mean curvature H, having finite area $A(\mathfrak{x})$ and an assigned boundary Γ^*. Under various conditions on Γ^* and H existence proofs for such surfaces have been given in the literature ([1], [5], [6], [8], [9], and [10]; see also [7] for a complete discussion of the non-parametric case). If one considers the totality of all boundary curves Γ^* lying in the unit ball $|\mathfrak{x}|\leqq 1$, the sharpest result hitherto obtained is due to HILDEBRANDT [6], who proved that in this case the class $\mathfrak{Q}(\Gamma^*, H)$ is nonempty, provided that $|H|\leqq 1$. On the other hand, from geometric considerations it is very plausible that for a given curve Γ^* the surfaces $\mathfrak{x}=\mathfrak{x}(w)$ should cease to exist, if $|H|$ exceeds a certain critical number $c(\Gamma^*)$.

The purpose of this note is to substantiate this fact by estimating the quantity $c(\Gamma^*)$ for a class of boundary curves Γ^*. Introduce the numbers

$$l(\Gamma^*)=\int_{\Gamma^*}|d\mathfrak{x}| \tag{4}$$

and

$$k(\Gamma^*)=\left|\int_{\Gamma^*}\mathfrak{x}\times d\mathfrak{x}\right|. \tag{5}$$

Here the integrals are to be taken in the sense of Stieltjes. Obviously they are independent of the particular representation of the curve Γ^*. Our result is the following

Theorem. *Assume that* $k(\Gamma^*)>0$. *Then the class* $\mathfrak{Q}(\Gamma^*, H)$ *is empty if* $|H|>\frac{l(\Gamma^*)}{k(\Gamma^*)}$.

This result is related to Bernstein's theorem that a nonparametric solution of the constant mean curvature equation cannot exist over a circle of radius larger than $1/|H|$ (for a simple proof see [2]).

For the circle $\Gamma^*=\{(\cos\varphi, \sin\varphi, 0)\colon 0\leqq\varphi\leqq 2\pi\}$ we have $l(\Gamma^*)=k(\Gamma^*)=2\pi$, which shows that HILDEBRANDT's result [6] mentioned above is the best possible one. The proof of our theorem is based upon the following lemma which is a specialization of Theorem 2 of [3].

Lemma 1. *Let* $\mathfrak{x}(w)\in\mathfrak{Q}(\Gamma^*, H)$ *and let* ε *be a positive number. Then we can determine a positive quantity* $\delta=\delta(\varepsilon)<1$, *a positive integer* $n=n(\varepsilon)$, *and* n *real numbers* $\varphi_k=\varphi_k(\varepsilon)$ $(k=1, \ldots, n(\varepsilon))$ *such that the following conditions are satisfied:*

(*a*) *We have the inequalities*

$$\varphi_1<\varphi_2<\cdots<\varphi_n<\varphi_{n+1}=\varphi_1+2\pi, \tag{6}$$

$$\varphi_{k+1}-\varphi_k<\varepsilon \quad (k=1, \ldots, n), \tag{7}$$

and

$$\int_{\varphi_k}^{\varphi_{k+1}} |d\mathfrak{x}(e^{i\varphi})|<\varepsilon \quad (k=1, \ldots, n). \tag{8}$$

(*b*) *For every* $\eta\in(0, \delta)$ *there exists a closed Jordan curve* $\Gamma\subset B$ *consisting of* n *piecewise analytic Jordan arcs* γ_k $(k=1, \ldots, n)$ *with endpoints* w_k *and* w_{k+1} $(w_k=(1-\eta)e^{i\varphi_k}, k=1, \ldots, n)$ *such that the estimates*

$$\int_{\gamma_k} |d\mathfrak{x}|<\varepsilon \tag{9}$$

and

$$\left|\int_\Gamma |d\mathfrak{x}|-l(\Gamma^*)\right|<\varepsilon \tag{10}$$

hold.

Using Theorem 1 of [4] the preceding lemma can be generalized to arbitrary solutions of the differential inequality $|\Delta\mathfrak{x}|\leqq c|\mathfrak{x}_u\times\mathfrak{x}_v|$, where c is a positive constant. In this connection we may also point out that estimates of the type (10) play an important rôle in the investigation of the boundary regularity of surfaces of bounded mean curvature (see [4] for further details). Next we have

Lemma 2. *Let* $\mathfrak{x}\in\mathfrak{Q}(\Gamma^*, H)$. *Then for every* $\sigma>0$ *there exists a closed, piecewise analytic Jordan curve* $\Gamma\subset B$, *such that the inequalities*

$$\left|\int_\Gamma |d\mathfrak{x}|-l(\Gamma^*)\right|<\sigma \tag{11}$$

and

$$\left|\, \left|\int_{\Gamma} \mathfrak{x} \times d\mathfrak{x}\right| - k(\Gamma^*)\right| < \sigma \tag{12}$$

hold.

Proof. First choose ε such that

$$\varepsilon(1 + 2l(\Gamma^*)) + \varepsilon^2 < \sigma \tag{13}$$

and then determine the quantities

$$\delta = \delta(\varepsilon), \quad n = n(\varepsilon), \quad \text{and} \quad \varphi_k = \varphi_k(\varepsilon) \quad (k = 1, \ldots, n)$$

according to Lemma 1.

Since $\mathfrak{x}(w)$ is continuous in $\bar{B}$, we can choose $\eta \in (0, \delta)$ such that the estimate

$$\left| \sum_{k=1}^{n} \mathfrak{x}(w_k) \times (\mathfrak{x}(w_{k+1}) - \mathfrak{x}(w_k)) - \sum_{k=1}^{n} \mathfrak{x}(e^{i\varphi_k}) \times (\mathfrak{x}(e^{i\varphi_{k+1}}) - \mathfrak{x}(e^{i\varphi_k})) \right| < \varepsilon \tag{14}$$

holds, where $w_k = (1-\eta)e^{i\varphi_k}$ $(k = 1, \ldots, n)$. Furthermore, from (8) it follows that

$$\left| \sum_{k=1}^{n} \mathfrak{x}(e^{i\varphi_k}) \times (\mathfrak{x}(e^{i\varphi_{k+1}}) - \mathfrak{x}(e^{i\varphi_k})) - \int_{\partial B} \mathfrak{x} \times d\mathfrak{x} \right| < \varepsilon\, l(\Gamma^*). \tag{15}$$

Now let Γ be the Jordan curve constructed in Lemma 1, assertion (b). Then (9) and (10) entail

$$\left| \sum_{k=1}^{n} \mathfrak{x}(w_k) \times (\mathfrak{x}(w_{k+1}) - \mathfrak{x}(w_k)) - \int_{\Gamma} \mathfrak{x} \times d\mathfrak{x} \right| < \varepsilon \int_{\Gamma} |d\mathfrak{x}| < \varepsilon(l(\Gamma^*) + \varepsilon). \tag{16}$$

Now combining (14)–(16) we obtain

$$\left| \int_{\Gamma} \mathfrak{x} \times d\mathfrak{x} - \int_{\partial B} \mathfrak{x} \times d\mathfrak{x} \right| < \varepsilon(1 + 2l(\Gamma^*)) + \varepsilon^2 < \sigma. \tag{17}$$

In connection with (10) and (13) this implies the conclusions of the lemma.

Proof of the Theorem. Let $\mathfrak{x} \in \mathfrak{Q}(\Gamma^*, H)$ and let σ be a positive number. Then according to Lemma 2 there exists a piecewise analytic Jordan curve $\Gamma \subset B$ satisfying (11)–(12). Let G be the interior domain of Γ. Then $\bar{G} \subset B$. On integrating (1) over G and using Green's formula, we obtain

$$\int_{\Gamma} (-\mathfrak{x}_v\, du + \mathfrak{x}_u\, dv) = H \int_{\Gamma} \mathfrak{x} \times d\mathfrak{x}. \tag{18}$$

Now in virtue of (2) the equation

$$|-\mathfrak{x}_v\, du + \mathfrak{x}_u\, dv| = |d\mathfrak{x}| \tag{19}$$

holds on Γ. Hence we have

$$\left| \int_{\Gamma} (-\mathfrak{x}_v\, du + \mathfrak{x}_u\, dv) \right| \leqq \int_{\Gamma} |-\mathfrak{x}_v\, du + \mathfrak{x}_u\, dv| = \int_{\Gamma} |d\mathfrak{x}|. \tag{20}$$

Taking account of (11) and (12), we derive from (18) and (20) the inequality

$$|H|(k(\Gamma^*) - \sigma) \leqq l(\Gamma^*) + \sigma \tag{21}$$

for every $\sigma>0$; hence

$$|H| \leqq \frac{l(\Gamma^*)}{k(\Gamma^*)}. \tag{22}$$

Thus, if $|H| > \frac{l(\Gamma^*)}{k(\Gamma^*)}$, we arrive at a contradiction, which proves our theorem.

Added in Proof. From a general theorem on the boundary behavior of solutions of elliptic systems established recently (see HEINZ, E., Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern. Math. Zeitschr., to appear) it follows that in the definition of the class $\mathfrak{Q}(\Gamma^*, H)$ the inequality (3) can be dropped, if Γ^* is a regular closed Jordan curve of class C^2. It remains an open question, whether (3) is superfluous also in the general case of a rectifiable Jordan curve Γ^*.

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