An Existence Theorem for a Problem from Boundary Layer Theory

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Communicated by J. SERRIN

The main object of this paper is to prove that the boundary value problem

$$\left.\begin{aligned} f'''+f''(f+cg)+(1-f'^2)=0 \\ g'''+g''(f+cg)+c(1-g'^2)=0 \end{aligned}\right\} \tag{1}$$

$$f(0)=f'(0)=g(0)=g'(0)=0, \tag{2}$$

$$f'(\infty)=g'(\infty)=1 \tag{3}$$

has at least one solution for any $c \geqq 0$. We also consider the asymptotic behavior of solutions as the independent variable becomes large. These equations arise in the theory of three dimensional boundary layers and describe incompressible flow near a stagnation point. They were derived by HOWARTH in 1951 and their significance is discussed in his article [6]. Negative values of c are also of interest physically, but, in common with similar problems which have been treated before, this case seems more difficult than the case $c \geqq 0$. In fact DAVEY has shown in [2] that the problem has no solution for $c<-1$; fortunately, however, physically meaningful situations can always be reduced to the case $c \geqq -1$.

References [13], [8], [9], [2], [7] and [10] contain earlier theoretical work in the same general area as the present paper. The first four of these references deal with the well-known Falkner-Skan equation

$$f'''+f''f+\lambda(1-f'^2)=0, \tag{4}$$

a full discussion of which appears in [3]. Note that when $c=0$ equations (1) reduce to (4) plus an easily solved equation for g. It follows from known results about (4) that the problem (1)–(3) therefore has a solution when $c=0$; hence we shall consider only $c>0$.

The proof of our result will be based on a discussion of the topology of the space of initial conditions. It will be clear that the method applies to more general problems, but for simplicity we shall restrict our discussion to the problem (1)–(3).

We begin with several topological propositions concerning the Euclidean plane E^2.

Lemma 1 [11, pg. 73]. *Let γ be a connected subset of E^2 which intersects both U and E^2-U,* where U is a subset of E^2. *Then γ intersects ∂U, the boundary of U.*

The next lemma may be found in a dual form involving closed sets in reference [11, pg. 137]. The present form is an obvious consequence.

Lemma 2. *Let A and B be open subsets of E^2 with components A^* and B^* such that $A^* \cap B^*$ is disconnected. Then $A \cup B \neq E^2$.*

Lemma 3. *Let S_1, S_2, T_1, T_2 be open subsets of E^2, with $S_1 \cap T_1$ and $S_2 \cap T_2$ empty. Suppose there are components Q_1, Q_2, R_1, R_2 of S_1, S_2, T_1, T_2 respectively, such that $Q_1 \cap Q_2$, $Q_1 \cap R_2$, $Q_2 \cap R_1$, and $R_1 \cap R_2$ are all non-empty. Then $S_1 \cup S_2 \cup T_1 \cup T_2 \neq E^2$.*

Proof. In Lemma 2, let $A = S_1 \cup S_2$ and $B = T_1 \cup T_2$. Let A^* be the component of A containing $Q_1 \cup Q_2$ and let B^* be the component of B containing $R_1 \cup R_2$. The hypotheses imply that $Q_1 \cap R_2$ and $Q_2 \cap R_1$ are non-empty and disjoint; thus to apply Lemma 2 it is sufficient to show that at least one of these sets is a union of components of $A^* \cap B^*$ (in fact, they both are).

Suppose that $Q_1 \cap R_2$ is not such a union. Then there is a component Ω of $A^* \cap B^*$ which intersects both $Q_1 \cap R_2$ and $E^2 - (Q_1 \cap R_2)$. By Lemma 1, Ω intersects $\partial(Q_1 \cap R_2)$ and so Ω intersects ∂Q_1 or ∂R_2. If, for example, there is a point $p \in \Omega \cap \partial Q_1$, then we have first of all $p \in \Omega \Rightarrow p \in A^* \Rightarrow p \in S_1 \cup S_2$ and $p \in \Omega \Rightarrow p \in B^* \Rightarrow p \in T_1 \cup T_2$. Then secondly we have also $p \in \partial Q_1 \Rightarrow p \notin S_1 \Rightarrow p \in S_2 \Rightarrow p \notin T_2 \Rightarrow p \in T_1$. But $p \in T_1 \cap \partial Q_1$ is impossible because T_1 and Q_1 are disjoint open sets. Other cases being handled similarly, this completes the proof.

Remark. These results are also true in higher dimensions. In [4] we used them to discuss the existence of solutions which are bounded on $(-\infty, \infty)$ for systems of differential equations, and it seems likely that other problems can be found for which a method based on Lemma 3 would be useful. (The proof of the above lemma which we gave in [4] was incomplete.)

Lemma 4 [11, pg. 123]. *Let p and q be points of E^2 which are separated by a closed set K (that is, p and q lie in distinct components of $E^2 - K$). Then p and q are separated by some component of K.*

In order to prove the existence of a solution to (1)–(3), consider for each pair of real numbers (α, β) the solution f, g of (1), (2) such that $f''(0) = \alpha$, $g''(0) = \beta$. We now define sets S_1, S_2, T_1, T_2 as follows:

$$S_1 = \{(\alpha, \beta) \mid \text{there exists an } x_1^+ > 0 \text{ such that } f'(x_1^+) > 1, \text{ and } f'(x) \geqq 0 \text{ for } 0 \leqq x \leqq x_1^+\}$$

$$S_2 = \{(\alpha, \beta) \mid \text{there exists an } x_2^+ > 0 \text{ such that } g'(x_2^+) > 1, \text{ and } g'(x) \geqq 0 \text{ for } 0 \leqq x \leqq x_2^+\}$$

$$T_1 = \{(\alpha, \beta) \mid \text{there exists an } x_1^- > 0 \text{ such that } f'(x_1^-) < 0, \text{ and } f'(x) \leqq 1 \text{ for } 0 \leqq x \leqq x_1^-\}$$

$$T_2 = \{(\alpha, \beta) \mid \text{there exists an } x_2^- > 0 \text{ such that } g'(x_2^-) < 0, \text{ and } g'(x) \leqq 1 \text{ for } 0 \leqq x \leqq x_2^-\}.$$

Clearly $S_1 \cap T_1$ and $S_2 \cap T_2$ are empty.

Lemma 5. *The sets S_1, S_2, T_1, T_2 are open.*

(The precise definitions of S_1, S_2, T_1 and T_2, and the use of this lemma were suggested to the author by conversations with Professor J. SERRIN and Dr. J. B. MCLEOD about their work in this area. Also, we were introduced to this problem as a result of a series of lectures by Professor SERRIN and the ensuing notes [12]).

Proof of Lemma 5. Equations (1) can be written in the form

$$\begin{aligned}[f'' e^{H(x)}]' &= -(1-f'^2)\, e^{H(x)} \\ [g'' e^{H(x)}]' &= -c(1-g'^2)\, e^{H(x)}\end{aligned} \tag{5}$$

where

$$H(x)=\int_0^x (f(s)+c\,g(s))\,ds. \tag{6}$$

We first show T_1 and T_2 are open. Let (α_1, β_1) be a point in T_1 and let f_1, g_1 be the corresponding solution of (1)–(2). We shall show that x_1^- in the definition of T_1 can be chosen so that $f_1'(x)<1$ on $(0, x_1^-]$. This will imply that some neighborhood of (α_1, β_1) lies in T_1, because solutions of (1) are continuous with respect to initial conditions. Suppose x_1^- cannot be chosen as described. Then there must exist a $y_1>0$ with $f_1'(y_1)=1, f_1'(y)\leqq 0$ in a neighborhood of y_1. But then $f_1''(y_1)=0$, and by the uniqueness of solutions of (1) we must have $f_1'(x)\equiv 1$, contradicting (2). Thus T_1 is open, and similarly, T_2 is open.

To prove that S_1 is open let (α_2, β_2) be a point in S_1 and note that (1) implies $\alpha_2>0$. Let f_2, g_2 be the corresponding solution of (1)–(2). We shall show that the point x_1^+ in the definition of S_1 can be chosen so that $f_2'(x)>0$ on $(0, x_1^+]$. This will imply that a neighborhood of (α_2, β_2) lies in S_1, again because of continuity with respect to initial conditions.

Suppose x_1^+ cannot be chosen as described. Then there must exist a value $y_2>0$ with $f_2'(y_2)=0, f_2''(y_2)=0$, and $0\leqq f_2'(x)\leqq 1$ for x in $[0, y_2]$. Also f_2'' must have at least two zeros in $[0, y_2]$. Just as in the proof that T_1 is open, it must in fact be true that

$$0\leqq f_2'(x)<1 \qquad \text{for } x \text{ in } [0, y_2]. \tag{7}$$

From (5) and (7) we see that $[f'' e^H]'<0$ on $[0, y_2]$. Therefore $f'' e^H$ can have at most one zero on $[0, y_2]$ which gives a contradiction. Therefore S_1, and similarly S_2, are open.

It is apparent that the half planes $\alpha<0$ and $\beta<0$ are subsets of T_1 and T_2 respectively. Let R_1 and R_2 be the components of T_1 and T_2 containing these half planes so that, in particular $R_1\cap R_2$ is non-empty. The definitions of Q_1 and Q_2 will depend on the next lemma.

Lemma 6. *There are continuous functions $M(\cdot)$ and $N(\cdot)$ defined on* $[0, \infty)$ *such that $\alpha\geqq M(|\beta|)$ implies $(\alpha, \beta)\in S_1$, while $\beta\geqq N(|\alpha|)$ implies $(\alpha, \beta)\in S_2$.*

Proof. Integrating the second equation of (1) and using (2) gives

$$\begin{aligned} g''(x) &= \beta-\int_0^x g''(s)(f(s)+c\,g(s))\,ds-c\int_0^x (1-g'(s)^2)\,ds \\ &= \beta-cx-g'(x)(f(x)+c\,g(x))+\int_0^x (f'(s)\,g'(s)+2c\,g'(s)^2)\,ds. \end{aligned} \tag{8}$$

For a given (α, β) suppose $\delta>0$ is chosen so that $f'(x)^2 \leqq 1$, $g'(x)^2 \leqq 1$ on $[0, \delta]$. Then on this interval $|f(x)| \leqq x$, $|g(x)| \leqq x$ and from (8) $-|\beta|-(2c+2)x \leqq g''(x) \leqq |\beta|+(3c+2)x$.

Now for any β let $\delta=\delta_\beta$ be the positive root of

$$(3c+2)\frac{\delta^2}{2}+|\beta|\delta-1.$$

If $f'(s)^2 \leqq 1$, $g'(s)^2 \leqq 1$ on $[0, x]$, and $0 \leqq x \leqq \delta_\beta$, we find

$$|g'(x)| \leqq \int_0^x |g''(s)|\,ds \leqq \int_0^{\delta_\beta} \big(|\beta|+(3c+2)s\big)\,ds=1.$$

It follows that $|g'(x)| \leqq 1$ and $|g(x)| \leqq x$, at least as long as $f'(x)^2 \leqq 1$ and $0 \leqq x \leqq \delta_\beta$. Integrating the first equation of (1) gives

$$\begin{aligned} f''(x)&=\alpha-f'(x)\big(f(x)+c\,g(x)\big)-x+\int_0^x \big(2f'(s)^2+c f'(s)\,g'(s)\big)\,ds \\ &\geqq \alpha-(2+2c)\,x. \end{aligned}$$

Therefore, for any $\alpha>0$, we have $f''(x) \geqq \alpha/2$ and $f'(x) \geqq \alpha x/2$ at least until $x=\alpha/(4c+4)$ or $x=\delta_\beta$ or $f'(x)>1$. Hence if we set $M(|\beta|)=2\sqrt{2+2c}+2/\delta_\beta$, then $\alpha \geqq M(|\beta|)$ implies $(\alpha, \beta) \in S_1$. The function $N(|\alpha|)$ can be defined similarly.

Now define Q_1 to be the component of S_1 containing the set $\alpha \geqq M(|\beta|)$, and define Q_2 to be the component of S_2 containing the set $\beta \geqq N(|\alpha|)$. In order to complete the proof of the theorem we require the following lemma.

Lemma 7. *There exists a number m such that if $\alpha \geqq m$ and $\beta \geqq m$ then* $(\alpha, \beta) \in S_1 \cup S_2$.

Proof. Suppose $\alpha>0$, $\beta>0$ and $f'(x)^2 \leqq 1$, $g'(x)^2 \leqq 1$ on some interval $[0, x_1]$. Then on this interval we have $|f(x)| \leqq x$, $|g(x)| \leqq x$, and

$$\begin{aligned} f''(x)&\geqq \alpha-(2+2c)\,x \\ f'(x)&\geqq \alpha x-(2+2c)\,x^2/2 \\ g''(x)&\geqq \beta-(2+2c)\,x \\ g'(x)&\geqq \beta x-(2+2c)\,x^2/2. \end{aligned}$$

Therefore, for α and β sufficiently large at least one of the values $f'(x)$, $g'(x)$ must exceed 1 before becoming negative.

Proof of the Main Result. From Lemma 6 it follows that $Q_1 \cap R_2$ and $Q_2 \cap R_1$ are non-empty and, as we remarked above, $R_1 \cap R_2$ is also non-empty. If the same is true of $Q_1 \cap Q_2$, then Lemma 3 shows that there is a point (α, β) in the complement of $V=S_1 \cup S_2 \cup T_1 \cup T_2$. Suppose, on the other hand, that $Q_1 \cap Q_2$ is empty. Then let W denote the quadrant $\alpha>m$, $\beta>m$.

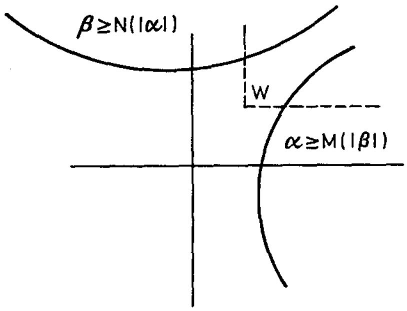

Fig. 1

Since Q_1 and Q_2 are disjoint, any point p in Q_2 is separated from any point q in Q_1 by the boundary ∂Q_2 of Q_2. By Lemma 4 some component γ of ∂Q_2 separates p from q. Also γ can be chosen independently of p and q because all points in the connected open set Q_2 lie in the same component of the complement of γ, and the same holds for all points of Q_1.

Since Q_2 is unbounded γ must also be unbounded. Furthermore, γ must intersect both W and R_1. Letting p_1 be a point on $\gamma \cap W$ we see from Lemma 7 that $p_1 \in S_1$. If S_1^* is the component of S_1 containing p_1, then $S_1^* \cap R_1$ is empty, since $S_1 \cap R_1$ is empty. It follows that γ intersects both S_1^* and $E^2 - S_1^*$; hence by Lemma 1 γ intersects ∂S_1^*. A point p_2 in $\gamma \cap \partial S_1^*$ lies in $\partial S_1 \cap \partial S_2$, and because $T_1 \cap S_1$ and $T_2 \cap S_2$ are empty, $p_2 \notin V$.

Thus, in any case there is at least one point $(\alpha, \beta) \notin V$. It remains to show that if f, g is the corresponding solution of (1), then $f'(\infty) = g'(\infty) = 1$.

From the definitions of S_1, S_2, T_1, and T_2 we see that $0 \leqq f'(x) \leqq 1$ and $0 \leqq g'(x) \leqq 1$, as long as $f(x)$ and $g(x)$ exist. It easily follows from (1) that f and g can be extended to $[0, \infty)$. From (5) we conclude that f'' and g'' are eventually of one sign; hence f' and g' are eventually monotone. Let $\omega_1 = \lim\limits_{x \to \infty} f'(x) = f'(\infty)$, $\omega_2 = g'(\infty)$. Define φ and ψ by $f'(x) = \omega_1 + \varphi(x)$, $g'(x) = \omega_2 + \psi(x)$. Substituting these in (1) and integrating the first equation of (1) yields

$$\begin{aligned}\varphi'(x) - \varphi'(0) + \varphi(x)(\omega_1 + \omega_2 c)\, x + \varphi(x) \int_0^x (\varphi(u) + c\,\psi(u))\, du \\ + (1 - \omega_1^2)\, x - \int_0^x \varphi(s)(3\omega_1 + \omega_2 c + 2\varphi(s) + c\,\psi(s))\, ds = 0.\end{aligned}$$

The conditions $\varphi(\infty) = \psi(\infty) = 0$ show that $\varphi'(x) = (1 - \omega_1^2) x + o(x)$ as $x \to \infty$. Since $0 \leqq \omega_1 \leqq 1$ we must have $\omega_1 = 1$, and similarly $\omega_2 = 1$, completing the proof.

We finish with a few remarks about the behavior of solutions of (1)–(3). A physically meaningful solution f, g must satisfy $f' \geqq 0$, $g' \geqq 0$, and it is clear that these conditions hold for the solution obtained in our proof. Using (5) we can show further that $0 < f' < 1$, $0 < g' < 1$, $f'' > 0$, $g'' > 0$, $f''' < 0$, $g''' < 0$ on $(0, \infty)$.

The asymptotic behavior of f, g is more complicated. We are interested in the rates at which $f' \to 1$, $g' \to 1$. Fortunately the main part of the necessary work can be carried over from the discussion of the Falkner-Skan equation in [3], (see also [1]). We shall outline the procedure briefly; the reader is referred to [3, pgs. 534, 536] for the details.

Letting $\varphi(x)=f'(x)-1$, $\psi(x)=g'(x)-1$, and $u(x)=f(x)+cg(x)$, it follows from (1) that

$$\varphi''+u\varphi'-(1+f')\varphi=0$$

$$\psi''+u\psi'-c(1+g')\psi=0.$$

Eliminating the middle terms in these equations by setting

$$\Phi(x)=e^{\frac{1}{2}\int_0^x u(s)\,ds}\varphi(x), \qquad \Psi(x)=e^{\frac{1}{2}\int_0^x u(s)\,ds}\psi(x),$$

we obtain

$$\Phi''-q_1\Phi=0, \qquad \Psi''-q_2\Psi=0$$

where $q_1=1+f'+\frac{1}{4}u^2+\frac{1}{2}u'$ and $q_2=c(1+g')+\frac{1}{4}u^2+\frac{1}{2}u'$. We know $f'\sim 1$, $g'\sim 1$, $u'\sim 1+c$, and by carrying out the procedure in [3] on f and g in parallel we obtain

$$1-f'(x)\sim a_1 x^{-1-\frac{2}{1-c}}e^{-\frac{1}{2}(1+c)x^2-a_2x}$$

$$f''(x)\sim(1+c)x(1-f'(x))$$

and

$$1-g'(x)\sim b_1 x^{-1-\frac{2c}{1+c}}e^{-\frac{1}{2}(1+c)x^2-b_2x}$$

$$g''(x)\sim(1+c)x(1-g'(x))$$

for some constants a_1, a_2, b_1, b_2.

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(Received December 4, 1968)