Bifurcations in the Presence of a Symmetry Group

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0. Introduction

The investigation of physical systems—in particular in hydrodynamics—often leads to the study of bifurcations of a vector field X or of a diffeomorphism f on a linear functional space E. Let the physical system be invariant under a group G. We assume that E is a Banach space, that G acts linearly and isometrically on E, and that X or f is equivariant with respect to this action. It is then an important practical problem to see how the equivariance affects the bifurcations. In the present paper we study the bifurcations of a fixed point located at the origin of E. The emphasis is on finding explicit procedures to deal with given group representations*. The problem is first reduced to a finite dimensional one by the use of center manifold theory (Section 1). A further basic simplification is generically possible and is achieved by putting X or f in normal form (Section 2). By studying an auxiliary *polynomial* vector field one determines invariant manifolds for the original X or f (Section 3). In simple cases this suffices to give a complete picture of the bifurcation. Finally, several examples are examined (Section 4).

We discuss now a remarkable feature emerging from the examples. For definiteness, let X_μ be an equivariant vector field depending on the real bifurcation parameter μ. We assume that the origin 0 of E is an attracting critical point of X_μ for $\mu<0$ and that it loses its attracting character for $\mu>0$. In general one finds that for $\mu>0$ attracting manifolds of critical points or closed orbits may be present. The remarkable thing is that for some group representations the bifurcation generically gives only periodic orbits, and no fixed points. The physical interpretation of this is that if a physical system has a certain type of symmetry and has a symmetric "stable steady state," loss of stability of this steady state will give rise to time-dependent behaviour, not to other steady states. An example will show what happens. Suppose our physical system is rotating with constant angular velocity and is initially in a rotationally symmetric state. If a bifurcation occurs, in which symmetry is lost, time independence will be replaced by time periodicity as a consequence of the rotation of the unsymmetric pattern of the system.

* A more general study of equivariant bifurcation theory, without linearity assumptions, is given in [3].

In general, notice also that for $\mu<0$ the origin 0 of E is invariant under $\Lambda_G \times \mathbb{R}$ (where $\mathbb{R}$ correspond to time evolution), but for $\mu>0$ the physical state of the system* is invariant only under a subgroup Σ of $\Lambda_G \times \mathbb{R}$. For instance if $G=\{1\}$ and an attracting periodic orbit appears for $\mu>0$, Σ is the discrete subgroup of $\mathbb{R}$ generated by a period. For other examples, see 4.4, 4.8, and 4.9. The fact that $\Sigma \neq \Lambda_G \times \mathbb{R}$ is an example of *symmetry breakdown*, a phenomenon of general occurrence and interest in physics.**

1. Use of Center Manifold Theory

Let Λ be a linear representation of the group G in the Banach space E. A C^k function $F: E \mapsto E$ is *Λ-equivariant* if

$$F(\Lambda_g x) = \Lambda_g F(x). \tag{1.1}$$

In particular, if $k \geqq 1$,

$$(DF(0))\Lambda_g = \Lambda_g(DF(0)), \tag{1.2}$$

and this is the only restriction on $DF(0)$ due to equivariance. We shall distinguish the case where F is considered as a transformation of E (denoted by f) or a vector field (denoted by X).

We study the bifurcation theory for f or X around the origin 0 of E. We thus assume that $f=f_\mu$ or $X=X_\mu$ depends smoothly on a real parameter μ and that 0 is a fixed point of f_μ, or a critical point of X_μ, and we investigate the change in the structure of the orbits of f_μ or X_μ near 0 as μ is varied. This problem is basically simplified by the use of center manifold theory, as shown by Theorem 1.1 (for a map f) and 1.2 (for a vector field X).

1.1. Theorem. *Let $1 \leqq k \leqq l < +\infty$, let E be a real Banach space with C^l norm***, and let Λ be a linear representation of a group G by isometries of E. Let $(x, \mu) \mapsto f_\mu(x)$ be a C^k function from $E \times \mathbb{R}$ to E such that, for each μ, f_μ is C^l: $E \to E$, f_μ is Λ equivariant, and $f_\mu(0)=0$.*

We assume that $Df_0(0)$ has a finite number of isolated eigenvalues of finite multiplicity on the unit circle $\{z: |z|=1\}$ and that the rest of the spectrum is disjoint from the unit circle.

Let E^0 be the finite dimensional subspace of E corresponding to the totality of the eigenvalues of $Df_0(0)$ which lie on the unit circle. We denote by Λ_g^0 the restriction of Λ_g to E^0, and we choose on E^0 a Hilbert norm such that Λ^0 is an orthogonal linear representation.

Under these conditions there exists in $E \times \mathbb{R}$ a manifold V with the following properties.

(a) *V is tangent to $E^0 \times \mathbb{R}$ at $(0, 0)$ and $V \supset \{0\} \times J$ where J is an open interval around* 0.

(b) *V is C^k and for each μ the intersection of V with $E \times \{\mu\}$ is C^l.*

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* Described by a point in an attracting invariant manifold bifurcating from 0.

** For a somewhat related study, see Michel [9].

*** That is, $x \mapsto \|x\|$ is a C^l function on $E \setminus \{0\}$.

(c) *V is invariant under* $\Lambda\times\mathbf{1}$ *and locally invariant* under the map* f: $(x,\mu)\mapsto(f_\mu x,\mu)$.

(d) *There is a neighbourhood U of* $(0,0)$ *in* $E\times\mathbb{R}$ *such that if* (x,μ) *is a local non-wandering point** (with respect to U), then* $(x,\mu)\in V$ *and is a local non-wandering point of f restricted to V.*

(e) *There is a chart* (V,φ) *such that* φ *maps V onto a neighbourhood of* $(0,0)$ *in* $E^0\times\mathbb{R}$ *and*

$$\varphi(x,\mu)=(\varphi_\mu x,\mu),\qquad \varphi_\mu(0=0,\qquad \varphi_\mu\Lambda_g=\Lambda_g^0\varphi_\mu. \tag{1.3}$$

(f) *Let* $h_\mu=\varphi_\mu\circ f_\mu\circ\varphi_\mu^{-1}$. *The map* $(x,\mu)\to h_\mu(x)$ *defined in a neighbourhood of* $(0,0)$ *in* $E^0\times\mathbb{R}$ *is* C^k. *Furthermore, for each* μ, h_μ *is a* C^l *diffeomorphism satisfying*

$$h_\mu(\Lambda_g^0 x)=\Lambda_g^0 h_\mu(x),\qquad h_\mu(0)=0.$$

First notice that, since E^0 is finite dimensional, the closure of Λ_G^0 is a compact group Γ. Let

$$(x,y)=\int_\Gamma d\gamma\langle\gamma x,\gamma y\rangle \tag{1.4}$$

where $d\gamma$ is Haar measure on Γ and $\langle\,,\rangle$ any Hilbert scalar product on E^0. Then (1.4) defines a Hilbert space scalar product with respect to which the Λ_g^0 are orthogonal.

V is a center manifold for the map $(x,\mu)\mapsto(f_\mu(x),\mu)$ at $(0,0)$. The existence of V follows from the work of HIRSCH, PUGH & SHUB [5]. Their construction can be done in a Λ-invariant manner. The properties (a), (b), (c), (d) then hold as discussed in [12], §5. Now let C be a closed curve around the eigenvalues of $Df_0(0)$ which lie on the unit circle, separating these eigenvalues from the rest of the spectrum. The operator

$$P=\frac{1}{2\pi i}\int_C\frac{dz}{z-Df_0(0)}$$

is a projection of E onto E_0 (see RIESZ & NAGY [11], §147), and (e) holds with

$$\varphi(x,\mu)=(Px,\mu).$$

In particular (1.3) follows from (1.2). (f) is a direct consequence of the previous results. In particular h_μ is a diffeomorphism because $Dh_\mu(0)$ is invertible.

1.2. Theorem. *Let* $1\leqq k\leqq l<+\infty$, *let E be a real Banach space with* C^l *norm, and let* Λ *be a linear representation of a group G by isometries of E. Let* $(x,\mu)\mapsto X_\mu(x)$ *be a* C^k *function from* $E\times\mathbb{R}$ *to E such that, for each* μ, *the vector field* X_μ *is* C^l: $E\mapsto E$, X_μ *is* Λ*-equivariant, and* $X_\mu(0)=0$.

We assume that the Jacobian operator $DX_0(0)$ *has a finite number of isolated eigenvalues of finite multiplicity on the imaginary axis* $\{z\colon \operatorname{Re} z=0\}$ *and that the rest of the spectrum is disjoint from the imaginary axis.*

* Locally invariant means that there is a neighbourhood U of $(0,0)$ such that $U\cap V=U\cap fV$.

** Let M be a manifold, $U\subset M$ an open set, and $F\colon U\to M$ a continuous map (defined on U only!). We say that x is a local wandering point if, for every neighbourhood $\mathcal{N}$ of x, there exists an n_0 such that $\mathcal{N}\cap F^n\mathcal{N}=\emptyset$ for $n>n_0$.

Let E^0 be the finite dimensional subspace of E corresponding to the totality of the eigenvalues of $DX_0(0)$ which lie on the imaginary axis. We denote by Λ_g^0 the restriction of Λ_g to E^0, and we choose on E^0 a Hilbert norm such that Λ^0 is an orthogonal linear representation.

Under these conditions there exists in $E\times\mathbb{R}$ a manifold V with the following properties.

(a) *V is tangent to $E^0\times\mathbb{R}$ at $(0,0)$ and $V\supset\{0\}\times J$, where J is an open interval around* 0.

(b) *V is C^k and for each μ the intersection of V with $E\times\{\mu\}$ is C^l.*

(c) *V is invariant under $\Lambda\times\mathbf{1}$ and tangent to the vector field $\tilde{X}:(x,\mu)\mapsto(X_\mu(x),0)$.*

(d) *There is a neighbourhood U of $(0,0)$ in $E\times\mathbb{R}$ such that if (x,μ) is a local non-wandering point* (with respect to U), then $(x,\mu)\in V$ and is a local non-wandering point of $\tilde{X}$ restricted to V.*

(e) *There is a chart (V,φ) such that φ maps V onto a neighbourhood of $(0,0)$ in $E^0\times\mathbb{R}$ and*

$$\varphi(x,\mu)=(\varphi_\mu x,\mu),\qquad \varphi_\mu(0)=0,\qquad \varphi_\mu\Lambda_g=\Lambda_g^0\varphi_\mu.$$

Moreover φ is C^k and, for each μ, φ_μ is C^l.

(f) *Let Y_μ be the vector field on E^0 defined by*

$$Y_\mu(x)=D\varphi_\mu(\varphi_\mu^{-1}x)\left[X_\mu(\varphi_\mu^{-1}(x))\right].$$

The function $(x,\mu)\mapsto Y_\mu(x)$ defined in a neighbourhood of $(0,0)$ in $E^0\times\mathbb{R}$ is C^k. Furthermore, for each μ, Y_μ is C^l and satisfies

$$Y_\mu(\Lambda_g^0x)=\Lambda_g^0\,Y_\mu(x),\qquad Y_\mu(0)=0.$$

The proof of this theorem is similar to that of Theorem 1.1. Notice that, since φ is a linear projection, the appearance of $D\varphi_\mu$ in the definition of Y_μ does not bring any loss of derivative.

1.3. Remark on the Condition $f_\mu(0)=0$ or $X_\mu(0)=0$. If Λ is nondegenerate**, then (1.1) implies that $F(0)=0$. If Λ is degenerate, the property $f_\mu(0)=0$ is non-generic. Suppose however that x_0 is invariant under Λ_G and f_0 and that $1\notin$ spectrum $Df_0(x_0)$. The implicit function theorem yields then a C^k function $\mu\mapsto x_\mu$ such that $f_\mu(x_\mu)=x_\mu$. By uniqueness, x_μ is invariant under Λ_G, and we can apply Theorem 1.1 to the map f'_μ defined by

$$f'_\mu(x)=f_\mu(x+x_\mu)$$

(but remember that by assumption $1\notin$ spectrum $Df'_0(0)$).

Similarly, if Λ is degenerate, the property $X_\mu(0)=0$ is non-generic. Suppose however that x_0 is invariant under Λ_g, that $X_0(x_0)=0$, and that $0\notin$ spectrum

* Let M be a manifold, $U\subset M$ an open set and X a C^1 vector field on U. We say that x is a local wandering point if, for every neighbourhood $\mathcal{N}$ of x, there exists t_0 such that

$$\mathcal{N}\cap\mathcal{D}_{Xt}\mathcal{N}=\emptyset\quad\text{for } t>t_0$$

where $\mathcal{D}_{Xt}$ is the time t integral of X.

** That is, if $\Lambda_g x=x$ for all $g\in G$, then $x=0$.

$DX_0(x_0)$. The implicit function theorem yields a C^k function $\mu \mapsto x_\mu$ such that $X_\mu(x_\mu)=0$. By uniqueness, x_μ is invariant under Λ_g and we can apply Theorem 1.2 to the vector field X'_μ defined by

$$X'_\mu(x)=X_\mu(x+x_\mu).$$

1.4. Remark. We have assumed $k \leqq l < +\infty$ in Theorem 1.1 and Theorem 1.2 because the center manifold theorem does not hold in the C^∞ or real analytic case. If we start with C^∞ data, we obtain a C^k center manifold V for each finite k, but V will in general shrink as k is increased and tend to 0 when $k \to \infty$.

2. Reduction of the Bifurcation Problem by Use of Normal Forms

Theorem 1.1 (respectively 1.2) reduces the original bifurcation problem to a similar one for the diffeomorphism h_μ (respectively, the vector field Y_μ), equivariant with respect to the orthogonal representation Λ^0, in the finite dimensional real Hilbert space E^0. This problem is further simplified by discarding non-generic possibilities.

(a) If E^0 is written as a direct sum of subspaces corresponding to the inequivalent irreducible representations of G (over the reals), then Λ^0 and $Dh_0(0)$ (respectively $DY_0(0)$) appear as sums of diagonal blocks. The condition that the eigenvalues of $Dh_0(0)$ corresponding to different blocks have the same absolute value is non-generic (similarly, the condition that the eigenvalues of $DY_0(0)$ have the same real part is non-generic). Therefore Λ^0 is generically a multiple of an irreducible representation.

(b) Let A_μ denote $Dh_\mu(0)$ or $DY_\mu(0)$, as the case may be. Generically, for small $|\mu|$, either the spectrum of A_μ consists of a single real eigenvalue λ_μ with C^{k-1} dependence on μ (Case I) or the spectrum of A_μ consists of a pair of complex conjugate eigenvalues λ_μ, $\bar{\lambda}_\mu$ with C^{k-1} dependence on μ (Case II). We consider these possibilities separately*.

Case I. We notice that the symmetric part S and the antisymmetric part T of A_μ each commute with Λ_G^0. If $T \neq 0$, one can find arbitrarily small real ε such that the spectrum of $S+(1+\varepsilon)T$ consists of more than one point (otherwise, since $\operatorname{Tr} T=0$, we would have

$$\det[z-(A_\mu+\varepsilon T)]=(z-\lambda_\mu)^{\dim E^0}$$

for small ε, hence identically in ε: this is seen to be false when $\varepsilon \to \infty$). Therefore in the generic situation we must have $T=0$ and S a multiple of $\mathbf{1}$, that is, $A_\mu=\lambda_\mu \mathbf{1}$. This situation is generic only if the commutant of Λ_G^0 consists of just the multiples of $\mathbf{1}$ (that is, if Λ_G^0 is irreducible of real type). Notice that $\lambda_0=\pm 1$ in the case of a map, and $\lambda_0=0$ in the case of a vector field.

Case II. Let $\Lambda^0 \otimes 1$ be the unitary extension of the representation Λ^0 to the complex Hilbert space $E^0 \otimes \mathbb{C}$. The complex subspaces F_μ and $\bar{F}_\mu$ of $E^0 \otimes \mathbb{C}$, corresponding to the points λ_μ and $\bar{\lambda}_\mu$ of the spectrum of $A_\mu \otimes 1$, are invariant under $\Lambda^0 \otimes 1$. In analogy with Case I, let $\lambda_\mu(S+T)$ be the restriction of $A_\mu \otimes 1$

* In the following discussion we could use the known form of the commutant of an orthogonal group representation; see H. WEYL [16].

to F_μ (S self-adjoint and T anti-self-adjoint). We see that $T=0$, $S=1$ in the generic situation, and the restriction of $\Lambda^0 \otimes 1$ to F_μ is irreducible.

If $x \in E^0$, let z_μ be the component along F_μ of $x \otimes 1$ in the decomposition $E^0 \otimes \mathbb{C} = F_\mu \oplus \bar{F}_\mu$. Now let z be the orthogonal projection of z_μ on $F=F_0$. For small $|\mu|$ the map $E^0 \to E^0 \otimes \mathbb{C} \to F_\mu \to F$ defined by $x \to z$ is a linear isomorphism (over $\mathbb{R}$) with C^{k-1} dependence on μ. Changing coordinates from x to z, we see that Λ goes over into an irreducible unitary representation Λ' (the restriction of $\Lambda^0 \otimes 1$ to F) and A_μ into complex multiplication by λ_μ.

Thus, changing coordinates from x to z, the map h_μ becomes h'_μ: $F \to F$ (or the vector field Y_μ becomes Y'_μ: $F \to F$) and we have

$$h'_\mu(z) = \lambda_\mu z + \text{higher order} \qquad (\text{or } Y'_\mu(z) = \lambda_\mu z + \text{higher order})$$
$$h'_\mu(\Lambda'_g z) = \Lambda'_g h'_\mu(z) \qquad (\text{or } Y'_\mu(\Lambda'_g z) = \Lambda'_g Y'_\mu(z)).$$

We now put h'_μ or Y'_μ in normal form* by a change of coordinates $z = \psi(z')$,

$$\psi(z') = z' + \sum_{m=2}^{3} \psi_m(z') \tag{2.1}$$

where ψ_m is a homogeneous polynomial of total degree m in z' and $\bar{z}'$. We suppose that $k \geqq 3$ and, in the case of a map h', make the generic assumption that $\lambda_0^3 \neq 1$ and $\lambda_0^4 \neq 1$. In the case of a vector field Y', we assume that $\lambda_0 \neq 0$. One can, by (2.1), put h' or Y' in the canonical form

$$\lambda z' + P(z') + Q(z')$$

where P is a homogeneous polynomial of degree 2 in z' and 1 in $\bar{z}'$, and Q is $o_3(|z|)$. Term by term identification in the relations

$$\psi(\lambda z' + P(z') + Q(z')) = h'(\psi(z'))$$

or

$$D\psi(z')[\lambda z' + P(z') + Q(z')] = Y'(\psi(z'))$$

uniquely determines P and ψ, except for the terms of degree 2 in z and 1 in $\bar{z}$ of ψ, which can be chosen to vanish. By uniqueness, the Λ' invariance is preserved:

$$P(\Lambda'_g z') = \Lambda'_g P(z').$$

ψ and P are determined by the derivatives up to order 3 of h'_μ or Y'_μ, and these are determined by the derivatives up to order 3 of h_μ or Y_μ. Therefore ψ and P are C^{k-3} functions of μ. Similarly, the derivatives up to order k of $Q(z')$ with respect to z', $\bar{z}'$ are determined by the derivatives of h_μ or Y_μ and are thus continuous with respect to μ, z'. In particular there exists a function $c(\cdot)$ independent of μ (for small $|\mu|$) such that $c(\cdot) \geqq 0$, $\lim_{u \to 0} c(u) = 0$, and

$$|Q(z)| \leqq c(|z|)\,|z|^3, \qquad |DQ(z)| \leqq c(|z|)\,|z|^2.$$

Summary: Under the conditions of Theorem 1.1 *or* 1.2, *the following two cases are generic (for Case II, assume that* $k \geqq 3$).

* See SIEGEL [14] for the use of normal forms for a simplectic map. The use of normal forms in the present context was suggested by JOST & ZEHNDER [7].

I. Λ_G^0 *is irreducible of real type (its commutant is trivial), and* $Dh_\mu(0)=\lambda_\mu 1$ (respectively $DY_\mu(0)=\lambda_\mu 1$) *where* λ_μ *is real with* C^{k-1} *dependence on* μ *and* $\lambda_0=\pm 1$ (respectively $\lambda_0=0$).

II. *There is a finite dimensional complex Hilbert space F and an irreducible unitary representation* Λ' *of G in F with the following properties.*

(e′) *There is a chart* (V, φ') *such that* φ' *maps V onto a neighbourhood of* $(0, 0)$ *in* $F\times\mathbb{R}$ *and*

$$\varphi'(x, \mu)=(\varphi'_\mu x, \mu), \qquad \varphi'_\mu(0)=0, \qquad \varphi'_\mu \Lambda_g=\Lambda'_g \varphi'_\mu.$$

Moreover, φ' *is* C^{k-3} *and, for each* μ, φ'_μ *is* C^l.

We write the next property separately for maps and for vector fields.

(f′-map). *Let* $h'_\mu=\varphi'_\mu\circ f_\mu\circ\varphi'^{-1}_\mu$. *The function* $(z, \mu)\mapsto h'_\mu(z)$ *defined in a neighbourhood of* $(0, 0)$ *in* $F\times\mathbb{R}$ *is* C^{k-3}, *and the mapping* $\mu\mapsto h'_\mu$ *is continuous from* $\mathbb{R}$ *to* C^k. *Furthermore, for each* μ, h'_μ *is a* C^l *diffeomorphism satisfying*

$$h'_\mu(\Lambda'_g z)=\Lambda'_g h'_\mu(z), \qquad h'_\mu(z)=\lambda_\mu z+P_\mu(z)+Q_\mu(z).$$

λ_μ *is a complex* C^{k-1} *function of* μ *such that* $|\lambda_0|=1$. $P_\mu(z)$ *is a homogeneous polynomial of degree* 2 *in z and* 1 *in* $\bar{z}$; *its coefficients are complex* C^{k-3} *functions of* μ. *There exists a function* $c(\cdot)$ *independent of* μ *such that* $c(\cdot)\geqq 0$, $\lim_{u\to 0} c(u)=0$, *and*

$$|Q_\mu(z)|\leqq c(|z|)\,|z|^3, \qquad |DQ_\mu(z)|\leqq c(|z|)\,|z|^2.$$

(f′-vector field). *Let* $Y'_\mu(z)=D\varphi'_\mu(\varphi'^{-1}_\mu z)\,[X_\mu(\varphi'^{-1}_\mu z)]$. *The function* $(z, \mu)\mapsto Y'_\mu(z)$ *defined in a neighbourhood of* $(0, 0)$ *in* $F\times\mathbb{R}$ *is* C^{k-3}, *and the mapping* $\mu\to Y'_\mu$ *is continuous from* $\mathbb{R}$ *to* C^k. *Furthermore, for each* μ, Y'_μ *is* C^l *and satisfies*

$$Y'_\mu(\Lambda'_g z)=\Lambda'_g Y'_\mu(z), \qquad Y'_\mu(z)=\lambda_\mu z+P_\mu(z)+Q_\mu(z).$$

λ_μ *is a complex* C^{k-1} *function of* μ *such that* $\operatorname{Re}\lambda_0=0$. $P_\mu(z)$ *is a homogeneous polynomial of degree* 2 *in z and* 1 *in* $\bar{z}$; *its coefficients are complex* C^{k-3} *functions of* μ. *There exists a function* $c(\cdot)$ *independent of* μ *such that* $c(\cdot)\geqq 0$, $\lim_{u\to 0} c(u)=0$, *and*

$$|Q_\mu(z)|\leqq c(|z|)\,|z|^3, \qquad |DQ_\mu(z)|\leqq c(|z|)\,|z|^2.$$

In specific applications one should, of course, check whether the generic assumptions of this section hold.

3. Invariant Manifolds

In Case II of Section 2, further analysis of the bifurcation is possible, using Theorem 3.1 or 3.2 below, with $h'_\mu=\Phi_\mu$, $Y'_\mu=Z_\mu$.

3.1. Theorem. *Let* $\Phi_\mu\colon \mathbb{C}^n\mapsto\mathbb{C}^n$ *be a* C^l *diffeomorphism* $(1\leqq l<+\infty)$ *depending on a real parameter* μ *varying in an interval around* 0:

$$\Phi_\mu(z)=\lambda_\mu z+P_\mu(z)+Q_\mu(z) \tag{3.1}$$

where $\mu\mapsto\lambda_\mu$ *is a continuous complex function, and* P_μ *is a homogeneous polynomial of degree* 2 *in z and* 1 *in* $\bar{z}$ *with coefficients continuous in* μ. *We assume that there* *exists a function* $c(\cdot)$ *independent of* μ *such that* $c(\cdot)\geqq 0$, $\lim_{u\to 0} c(u)=0$, *and*

$$|Q_\mu(z)|\leqq c(|z|)\,|z|^3, \qquad |DQ_\mu(z)|\leqq c(|z|)\,|z|^2.$$

We also assume that $|\lambda_0|=1$ *and* $|\lambda_\mu|>1$ *for* $\mu>0$. *Let the vector field*

$$z\to z+\lambda_0^{-1}P_0(z) \tag{3.2}$$

*be normally hyperbolic** *to the compact invariant manifold S, and let its flow restricted to S be isometric. Suppose also that S is invariant under the transformations* $z\to z e^{i\sigma}$ *(all real* σ*).*

Then, for small $\mu>0$, *there exist* $\Theta_\mu\in C^l(S,\mathbb{C}^n)$ *and* $S_\mu\subset\mathbb{C}^n$ *such that*

(a) Θ_μ *is a diffeomorphism of S onto* S_μ.

(b) S_μ *is invariant under* Φ_μ *and* Φ_μ *is l-normally hyperbolic to* S_μ.

(c) *Write* $\Theta_\mu(z)=(\log|\lambda_\mu|)^{1/2}\,\theta_\mu(z)$. *When* $\mu\to 0$, θ_μ *tends in* $C^1(S,\mathbb{C}^n)$ *to the inclusion map* $\theta_0\colon S\mapsto\mathbb{C}^n$. *In particular* $\lim S_\mu=\{0\}$.

(d) *If* $\{\Lambda\}$ *is a group of unitary transformations of* $\mathbb{C}^n$ *such that* $\Lambda\Phi_\mu=\Phi_\mu\Lambda$ *and* $\Lambda S=S$, *then* $\Theta_\mu\Lambda=\Lambda\Theta_\mu$.

(e) *If* $\mu\mapsto\Phi_\mu$ *is continuous from* $\mathbb{R}$ *to* C^k, $k\leqq l$, *then* $\mu\to\theta_\mu$ *is continuous from* $\{\mu\colon\mu>0\}$ *to* $C^k(S,\mathbb{C}^n)$.

We divide the proof into a number of steps.

(I) Let $A\colon\mathbb{C}^n\mapsto\mathbb{C}^n$ be a homogeneous polynomial of degree 2 in z and 1 in $\bar z$. Then the integral of the vector field $X(z)=\alpha z+A(z)$ is

$$\mathscr{D}_{Xt}\,z=e^{\alpha t}z+\frac{e^{(\alpha+\bar\alpha)t}-1}{\alpha+\bar\alpha}\,e^{\alpha t}A(z)+\cdots,$$

where terms of order $\geqq 5$ in $|z|$ have been omitted. In particular, let

$$X(z)=\alpha z+\frac{\alpha+\bar\alpha}{e^{\alpha+\bar\alpha}-1}\,e^{-\alpha}P(z).$$

Then, for sufficiently small a and $|z|<a$, we have

$$\begin{aligned}|e^\alpha z+P(z)-\mathscr{D}_X z|&<C_1\,|z|^5\\ |D_z[e^\alpha z+P(z)-\mathscr{D}_X z]|&<C_1\,|z|^4\end{aligned} \tag{3.3}$$

where C_1 depends on a, but may be chosen independent if α, P if $\alpha+\bar\alpha$ and the coefficients of P remain bounded.

(II) Choosing $e^\alpha=\lambda_\mu$, $-\pi<\operatorname{Im}\alpha<\pi$, $P=P_\mu$, we obtain, for $|z|<a$,

$$\begin{aligned}|\Phi_\mu(z)-\mathscr{D}_X z|&<c'(|z|)\,|z|^3\\ |D_z[\Phi_\mu(z)-\mathscr{D}_X z]|&<c'(|z|)\,|z|^2\end{aligned} \tag{3.4}$$

where D_z denotes differentiation with respect to $z,\bar z$, $c'(\cdot)$ is independent of μ and decreasing, $c'(\cdot)\geqq 0$, and $\lim_{u\to 0} c'(u)=0$.

* See Hirsch, Pugh, & Shub [4]. If the vector field (3.2) is 1-normally hyperbolic to S, it is l-normally hyperbolic for all $l<+\infty$ because the flow on S is isometric.

(III) Let $\lambda_\mu > 1$ and make the change of coordinates

$$z = (\log|\lambda_\mu|)^{1/2}\zeta = \left(\frac{\alpha+\bar{\alpha}}{2}\right)^{1/2}\zeta.$$

The transformation Φ_μ becomes Ψ_μ and $\mathscr{D}_X$ becomes $\mathscr{D}_Y$ where

$$Y(\zeta) = \alpha\zeta + \frac{(\alpha+\bar{\alpha})^2}{e^{\alpha+\bar{\alpha}}-1}\,\frac{e^{-\alpha}}{2}\,P(\zeta).$$

If we write

$$\Psi_\mu(\zeta) = \mathscr{D}_Y\zeta + R(\zeta),$$

(3.4) yields

$$\begin{aligned} |R(\zeta)| &\leqq c'\big((\log|\lambda_\mu|)^{1/2}\zeta\big)\times\log|\lambda_\mu|\times|\zeta|^3 \\ |D_\zeta R(\zeta)| &\leqq c'\big((\log|\lambda_\mu|)^{1/2}\zeta\big)\times\log|\lambda_\mu|\times|\zeta|^2 \end{aligned} \tag{3.5}$$

for $|\zeta| < (\log|\lambda_\mu|)^{-1/2}a$. Choosing $b>0$, we shall henceforth assume that $|\zeta|<2b$ and $\log|\lambda_\mu| < a^2/4b^2$. If $c''(\mu) = c'\big(2b(\log|\lambda_\mu|)^{1/2}\big)$, (3.5) yields

$$\begin{aligned} |R(\zeta)| &\leqq c''(\mu)\times\log|\lambda_\mu|\times|\zeta|^3 \\ |D_\zeta R(\zeta)| &\leqq c''(\mu)\times\log|\lambda_\mu|\times|\zeta|^2. \end{aligned} \tag{3.6}$$

Furthermore, using (3.3) and (3.6) we see that there exists an $m>0$, independent of μ, such that

$$|\Psi_\mu(\zeta)| \leqq |\lambda_\mu|^m\,|\zeta|, \qquad |D_\zeta\Psi_\mu(\zeta)| \leqq |\lambda_\mu|^m. \tag{3.7}$$

(IV) Iterating Ψ_μ and $\mathscr{D}_Y$, we can write

$$\Psi_\mu^n(\zeta) = \mathscr{D}_{nY}\zeta + R_n(\zeta).$$

Then

$$\begin{aligned} R_{n+1}(\zeta) &= [\mathscr{D}_{nY}\Psi_\mu(\zeta) - \mathscr{D}_{(n+1)Y}\zeta] + R_n(\Psi_\mu(\zeta)) \\ &= [\mathscr{D}_{nY}\Psi_\mu(\zeta) - \mathscr{D}_{nY}(\Psi_\mu(\zeta) - R(\zeta))] + R_n(\Psi_\mu(\zeta)). \end{aligned} \tag{3.8}$$

(V) Notice that, by our assumptions on P,

$$\mathscr{D}_{nY}\zeta = \left(\exp n\,\frac{\alpha-\bar{\alpha}}{2}\right)\mathscr{D}_{nY'}\zeta \tag{3.9}$$

where

$$Y'(\zeta) = \frac{\alpha+\bar{\alpha}}{2}\zeta + \frac{(\alpha+\bar{\alpha})^2}{e^{\alpha+\bar{\alpha}}-1}\,\frac{e^{-\alpha}}{2}\,P(\zeta).$$

Therefore, one can choose $\beta>0$ and $C>0$ such that if $|\lambda_\mu|^n < e^{2\beta}$ $\left(\text{that is, } n\,\frac{\alpha+\bar{\alpha}}{2} < 2\beta\right)$, then

$$|D_\zeta\mathscr{D}_{nY}\zeta| < C, \qquad |D_\zeta^2\mathscr{D}_{nY}\zeta| < C \tag{3.10}$$

for $|\zeta|<2b$.

(VI) Using (3.6), (3.7), (3.8), and (3.10), one verifies that, if μ is small enough, $|\zeta|<b$, and $|\lambda_\mu|^n < e^{2\beta}$, then

$$|R_n(\zeta)| \leqq K_n|\zeta|^3, \qquad |D_\zeta R_n(\zeta)| \leqq K_n|\zeta|^2$$

provided

$$K_1 \geqq c''(\mu)\cdot\log|\lambda_\mu|, \qquad K_{k+1} \geqq C(1+|\lambda_\mu|^m)K_1 + K_k|\lambda_\mu|^{3m}.$$

If $A \geqq 1$, $A \geqq C(1+|\lambda_\mu|^m)$, we may thus take

$$\begin{aligned} K_n &= A c''(\mu)\log|\lambda_\mu|(1+|\lambda_\mu|^{3m}+\cdots+|\lambda_\mu|^{3m(n-1)}) \\ &\leqq A c''(\mu)\log|\lambda_\mu|\cdot n|\lambda_\mu|^{3mn} \leqq 2\beta e^{6m\beta} A c''(\mu). \end{aligned}$$

(VII) We have shown that for $|\zeta|<b$, $|\lambda_\mu|^n<e^{2\beta}$, and μ small, Ψ_μ^n is C^1 close to $\mathscr{D}_{nY}$. Let

$$Z(\sigma)(z) = (\beta+i\sigma)z + \beta\lambda_0^{-1}P_0(z)$$

where $-\pi<\sigma\leqq\pi$. For small μ we can find an n such that $n\dfrac{\alpha+\bar\alpha}{2}$ and $(n+1)\cdot\dfrac{\alpha+\bar\alpha}{2}$ are close to β. From (3.9) it follows then that, for $|\zeta|<b$, $\mathscr{D}_{nY}$ is C^1 close to $\mathscr{D}_{Z(\sigma')}$ and $\mathscr{D}_{(n+1)Y}$ is C^1 close to $\mathscr{D}_{Z(\sigma'')}$ if σ', σ'' are suitably chosen. From Hirsch, Pugh, & Shub [4] it then follows that Ψ_μ^n and Ψ_μ^{n+1} have invariant manifolds S' and S'', respectively, close to S. But the construction of S' and S'' (taking the stable and unstable manifolds of S, iterating the action of Ψ_μ^n or Ψ_μ^{n+1} on these, etc.) shows that they are identical. Therefore $\Psi_\mu^n S' = \Psi_\mu^{n+1} S'$, and hence $S' = \Psi_\mu S'$. We define S_μ to be the image of S' when the coordinates are changed back from ζ to z. The properties listed in the theorem then follow directly from [4].

3.2. Theorem. *Let Z_μ: $\mathbb{C}^n \to \mathbb{C}^n$ be a C^l vector field $(1\leqq l<+\infty)$ depending on a real parameter μ varying in an interval around* 0:

$$Z_\mu(z) = \lambda_\mu z + P_\mu(z) + Q_\mu(z)$$

where $\mu \mapsto \lambda_\mu$ is a continuous complex function, and P_μ is a homogeneous polynomial of degree 2 in z and 1 in $\bar z$ with coefficients continuous in μ. We assume that there exists a function $c(\cdot)$ independent of μ such that $c(\cdot)\geqq 0$, $\lim_{n\to 0} c(u)=0$, and

$$|Q_\mu(z)| \leqq c(|z|)|z|^3, \qquad |DQ_\mu(z)| \leqq c(|z|)|z|^3.$$

We also assume that $\operatorname{Re}\lambda_0=0$ *and* $\operatorname{Re}\lambda_\mu>0$ *for $\mu>0$. Let the vector field*

$$z \mapsto z + P_0(z)$$

be normally hyperbolic to the compact invariant manifold S, and let its flow restricted to S be isometric. Suppose also that S is invariant under the transformations $z\mapsto ze^{i\sigma}$ (all real σ).

Then, for small $\mu>0$, there exist $\Theta_\mu \in C^l(S, \mathbb{C}^n)$ and $S_\mu \subset \mathbb{C}^n$ such that

(a) *Θ_μ is a diffeomorphism of S onto S_μ.*

(b) *S_μ is invariant under Z_μ and Z_μ is l-normally hyperbolic to S_μ.*

(c) *Write $\Theta_\mu(z) = (\operatorname{Re}\lambda_\mu)^{1/2}\,\theta_\mu(z)$. When $\mu\to 0$, θ_μ tends in $C^1(S, \mathbb{C}^n)$ to the inclusion map θ_0: $S \mapsto \mathbb{C}^n$. In particular* $\lim S_\mu = \{0\}$.

(d) *If $\{A\}$ is a group of unitary transformations of $\mathbb{C}^n$ such that $AZ_\mu = Z_\mu A$ and $AS=S$, then $\Theta_\mu A = A\Theta_\mu$.*

(e) *If* $\mu \mapsto Z_\mu$ *is continuous from* $\mathbb{R}$ *to* C^k, $k \leq l$, *then* $\mu \mapsto \theta_\mu$ *is continuous from* $\{\mu: \mu > 0\}$ *to* $C^k(S, \mathbb{C}^n)$.

One could write a direct proof of this theorem, but it is more convenient to apply Theorem 3.1 to the time t integral of Z_μ:

$$\Phi_\mu(z) = \mathscr{D}_{Z_\mu t} z = e^{\lambda_\mu t} z + \frac{e^{(\lambda_\mu + \bar{\lambda}_\mu)t} - 1}{\lambda_\mu + \bar{\lambda}_\mu} e^{\lambda_\mu t} P_\mu(z) + \tilde{Q}_\mu(z).$$

Notice that the change of coordinates $z \mapsto \zeta$, used in the proof of Theorem 3.1, becomes $z = (t \operatorname{Re} \lambda_\mu)^{1/2} \zeta$ (if $t > 0$) and depends on t. Here we make the choice $z = (\operatorname{Re} \lambda_\mu)^{1/2} \zeta$.

3.3. Remarks on Theorems 3.1 and 3.2. (a) Under the assumptions of Section 2, $\mu \mapsto \lambda_\mu$ is C^{k-1}. Therefore $d\lambda/d\mu$ is continuous and, for a map, generically, $d|\lambda|/d\mu \neq 0$ at $\mu = 0$. For a vector field, generically, $d \operatorname{Re} \lambda/d\mu \neq 0$ at $\mu = 0$. Theorem 3.1 covers the case of a map with $d|\lambda|/d\mu > 0$ and Theorem 3.2 the case of a vector field with $d \operatorname{Re} \lambda/d\mu > 0$. Notice that with these assumptions the diameter of the invariant manifold S_μ tends to zero like $\sqrt{\mu}$ when $\mu \to 0$.

The case $d|\lambda|/d\mu < 0$ (respectively $d \operatorname{Re} \lambda/d\mu < 0$) is dealt with by changing μ to $-\mu$.

(b) Theorems 3.1 and 3.2 give information on invariant manifolds for $\mu > 0$. Similar information for $\mu < 0$ is obtained by applying the theorems to $\Phi_{-\mu}^{-1}$ or $-Z_{-\mu}$. Since

$$\Phi_{-\mu}^{-1}(z) = \lambda_{-\mu}^{-1} z - \lambda_{-\mu}^{-3} \bar{\lambda}_{-\mu}^{-1} P_{-\mu}(z) + Q'_{-\mu}(z),$$

one has to look for invariant manifolds of

$$\frac{dz}{dt} = z - \lambda_0^{-1} P_0(z)$$

in the case of a map, and of

$$\frac{dz}{dt} = z - P_0(z)$$

in the case of a vector field.

(c) From the equation

$$\frac{dz}{dt} = z \pm \lambda_0^{-1} P_0(z) \qquad \left(\text{resp.} \frac{dz}{dt} = z \pm P_0(z)\right)$$

for z one can deduce a system of equations for the polynomials in z, $\bar{z}$ which are invariant under the transformations Λ'_g and $M_\sigma z = z e^{i\sigma}$. This permits the practical determination of manifolds S to which the above theorems apply.

(d) Theorems 3.1 and 3.2 say nothing about the nature of Φ_μ or Z_μ and their non-wandering points outside of a small neighbourhood of S_μ. In simple cases one can complete this information and obtain a picture of Φ or Z in a full neighbourhood of $(0, 0) \in E \times \mathbb{R}$.

(e) It would be interesting to prove some differentiability in the dependence of S_μ with respect to μ.

(f) If we start with C^∞ data, Theorems 3.1 and 3.2 will yield a C^l invariant manifold S_μ for each finite l. The set of values of μ for which S_μ is C^l will, however, in general shrink as l increases and tend to $\emptyset$ when $l \to \infty$ (cf. Remark 1.4). This means that, as μ increases, S_μ becomes less and less differentiable (a late stage of "shriveling" of this sort is described in [8]). This loss of differentiability is not, however, completely general; it does not occur for the Hopf bifurcation for a vector field (see Remark 4.3 below).

4. Examples

We assume that the conditions of Theorem 1.1 or 1.2 are satisfied and we examine various special cases.

*4.1. The Hopf Bifurcation for a Vector Field**. Let $G=\{1\}$, that is, no symmetry is assumed. We suppose that $DX_0(0)$ has exactly two simple complex conjugate eigenvalues $\lambda_0, \bar{\lambda}_0 \neq 0$ on the imaginary axis and that $d\,\mathrm{Re}\,\lambda_\mu / d\mu > 0$. According to Section 2 we have to study the vector field

$$Y'_\mu(z) = \lambda_\mu z + a_\mu z^2 \bar{z} + Q_\mu(z)$$

for $z \in \mathbb{C}$. In view of Theorem 3.2 we consider the differential equation

$$\frac{dz}{dt} = z + a_0 z^2 \bar{z}$$

and its consequence

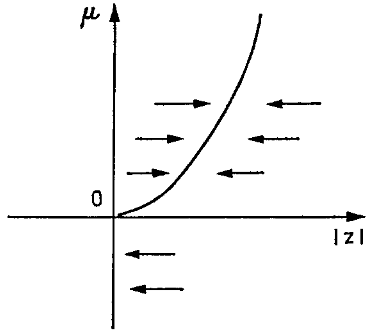
$$\frac{1}{2} \frac{d}{dt}(z\bar{z}) = (z\bar{z}) + [\mathrm{Re}\,a_0](z\bar{z})^2.$$

We can take $S^{(1)} = \{0\}$ and $S^{(2)} = \{z : |z|^2 = -\mathrm{Re}\,a_0\}$ if $\mathrm{Re}\,a_0 < 0$ (the case $\mu < 0$ is treated similarly).

For definiteness, suppose that the spectrum of $DX_\mu(0)$ is in $\{z : \mathrm{Re}\,z < 0\}$ except for the eigenvalues λ_μ, $\bar{\lambda}_\mu$. Also let $\mathrm{Re}\,a_0 < 0$ (that is, $\{0\}$ is a "weak attractor" for X_0). We have the following C^l invariant manifolds: $S_\mu^{(1)} = \{0\}$ which consists of a fixed point, attracting for $\mu < 0$, and $S_\mu^{(2)}$ which is a one-parameter family of attracting closed orbits, present for $\mu > 0$. These manifolds contain all the non-wandering points of Z_μ with orbit remaining in a neighbourhood $\{z : |z| < a\}$ of 0 in E. This last point follows from an analysis of the (simple) structure of orbits of Y'_μ outside $S_\mu^{(1)}$ and $S_\mu^{(2)}$. In a neighbourhood of $S_\mu^{(1)}$ and $S_\mu^{(2)}$ we have the hyperbolicity required by Theorem 3.2. In the rest of the region $|z| \leq (\mathrm{Re}\,\lambda_\mu)^{1/2} b$, we know from the proof of Theorems 3.1 and 3.2 that $\mathcal{D}_{nY'_\mu}$ is close to $\mathcal{D}_{Z(\theta')}$. In the region $(\mathrm{Re}\,\lambda_\mu)^{1/2} b < z < a$, for sufficiently large b, $Y'_\mu(z)$ is close to $a_\mu z^2 \bar{z}$. We therefore have the situation shown in Figure 1. We summarize the results obtained.

4.2. Theorem. *Let E be a Banach space with C^l norm and X_μ a vector field on E depending on the real parameter μ. We assume that $(x, \mu) \mapsto X_\mu(x)$ is C^k and that, for each μ, X_μ is C^l for $3 \geq k \leq l < +\infty$. The function $\mu \mapsto \xi_\mu$ is assumed to be C^l, $X_\mu(\xi_\mu) = 0$, and the spectrum of $DX_\mu(\xi_\mu)$ is assumed to lie in* $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$

* See Hopf [6] and also [10], [1], [2], [15].



Fig, 1

except for two eigenvalues λ_μ *and* $\bar{\lambda}_\mu$ *such that*

$$\text{(a)}\ \operatorname{Re}\lambda_{\mu_1}=0, \quad \operatorname{Im}\lambda_{\mu_1}>0,$$

and

$$\text{(b)}\ \frac{d}{d\mu}\operatorname{Re}\lambda_\mu>0.$$

Under these conditions there are two generic possibilities depending on the sign of some coefficient computed from third order derivatives of X_{μ_1}. *In both cases there exist constants* $a>0$ *and* $\delta>0$ *such that* X_μ *is qualitatively described for* $|x-\xi_\mu|<a$ *and* $|\mu-\mu_1|<\delta$. *Also in both cases the critical point* ξ_μ *is attracting for* $\mu<\mu_1$ *and non-attracting for* $\mu>\mu_1$.

(a) *In the first case (if* ξ_{μ_1} *is a "weak attractor") there is an attracting* C^1 *closed orbit for* $\mu>\mu_1$. *Its diameter is* $\sim\sqrt{\mu-\mu_1}$ *and its period tends to* $2\pi/\operatorname{Im}\lambda_{\mu_1}$ *as* $\mu\to\mu_1$.

(b) *In the second case there is a non-attracting* C^1 *closed orbit for* $\mu<\mu_1$. *Its diameter is* $\sim\sqrt{\mu_1-\mu}$ *and its period tends to* $2\pi/\operatorname{Im}\lambda_{\mu_1}$ *as* $\mu\to\mu_1$.

There are no local non-wandering points for $|x-\xi_\mu|<a$ *and* $|\mu-\mu_1|<\delta$ *other than those indicated above.*

4.3. Remark. In the Hopf bifurcation for a vector field, if X_μ is C^∞, so is S_μ. Indeed, S_μ is here a periodic orbit of the vector field, and therefore the image of a C^∞ integral curve $t\to x(t)$ with $dx(t)/dt\neq 0$. HOPF's original proof [6] deals with the real analytic case for finite dimensional E.

4.4. The Case $G=\mathbb{R}/\mathbb{Z}$. Suppose that Λ is a nondegenerate representation of $\mathbb{R}/\mathbb{Z}=SO(2)$ in E. Then 0 is a critical point of X_μ (see Remark 1.3). The irreducible unitary representations of $\mathbb{R}/\mathbb{Z}$ are of complex (not of real) type; therefore Case I of Section 2 does not arise. The eigenvalues of $DX_\mu(0)$ thus go by complex conjugate pairs, and the results of 4.1 (or 4.2) apply. *Let* $t\mapsto x(t)$ *be the motion on the closed orbit* $S_\mu^{(2)}$ (we assume for definiteness that 0 is a weak attractor for X_0 and that $\mu>0$). *Then there exists a real continuous function* ν_μ *of* μ *such that*

$$x(t)=\Lambda(\nu_\mu t)\,x(0) \tag{4.1}$$

and, when $\mu \to 0$, $v_\mu \to \pm \frac{\operatorname{Im}\lambda_0}{2\pi n}$, *where n is an integer* >0.

To see this notice that the irreducible unitary representation Λ' of $\mathbb{R}/\mathbb{Z}$ in $F = \mathbb{C}$ is of the form

$$\Lambda'(u)\,z = e^{\pm 2\pi i n u} z.$$

Since Y' is Λ' equivariant, the motion on the closed orbit $S_\mu^{(2)}$ is given by

$$y(t) = e^{2\pi i \omega_\mu t}\, y(0)$$

where ω_μ is a real continuous function of μ and

$$\lim_{\mu \to 0} \omega_\mu = \frac{\operatorname{Im}\lambda_0}{2\pi}.$$

Therefore

$$y(t) = \Lambda'\left(\pm \frac{\omega_\mu t}{n}\right) y(0).$$

If X_μ describes a physical system with rotational invariance, (4.1) means that the motion on $S_\mu^{(2)}$ becomes time-independent if observed in a coordinate frame rotating with a suitable constant angular velocity.

*4.5. The Hopf Bifurcation for a map**. The situation is similar to that of a vector field. Again we let $G = \{1\}$, that is, we assume no symmetry. We suppose that $Df_0(0)$ has exactly two simple complex conjugate eigenvalues λ_0, $\bar{\lambda}_0$ of modulus 1 and that $\lambda_0^3 \neq 1$, $\lambda_0^4 \neq 1$, $d|\lambda_\mu|/d\mu > 0$.

4.6. Theorem. *Let E be a Banach space with* C^l *norm and* $f_\mu: E \mapsto E$ *a map depending on the real parameter* μ. *We assume that* $(x, \mu) \mapsto f_\mu(x)$ *is* C^k *and that, for each* μ, f_μ *is* C^l *where* $3 \leqq k \leqq l < +\infty$. *We also assume that* $f_\mu(0) = 0$ *and that the spectrum of* $Df_\mu(0)$ *lies in* $\{z: |z| < 1\}$ *except for two eigenvalues* λ_μ *and* $\bar{\lambda}_\mu$ *such that*

$$\text{(a)}\quad |\lambda_0| = 1, \qquad \lambda_0^3 \neq 1, \qquad \lambda_0^4 \neq 1,$$

$$\text{(b)}\quad \frac{d}{d\mu}|\lambda_\mu| > 0.$$

Under these conditions there are two generic possibilities depending on the sign of some coefficient computed from third order derivatives of f_0. *In both cases there exist constants* $a > 0$ *and* $\delta > 0$ *such that* f_μ *is qualitatively described for* $|x| < a$ *and* $|\mu| < \delta$. *In both cases the origin* 0 *is attracting for* $\mu < 0$ *and non-attracting for* $\mu > 0$.

(a) *In the first case (if* 0 *is a "weak attractor" for* f_0*) there is an attracting curve for* $\mu > 0$ *which is* C^1 *diffeomorphic to a circle, and with diameter* $\sim\sqrt{\mu}$.

(b) *In the second case there is an invariant non-attracting curve for* $\mu < 0$ *which is* C^1 *diffeomorphic to a circle, and with diameter* $\sim\sqrt{-\mu}$.

There are no local non-wandering points for $|x| < a$ *and* $|\mu| < \delta$ *other than those indicated above.*

* See [10], [13], [12].

4.7. Invariance under $O(n)$. If Λ^0 is a representation of the full orthogonal group $O(n)$, both Case I and Case II of Section 2 can appear. We discuss an example of each in 4.8 and 4.9, respectively. For definiteness we consider maps, and we assume that the conditions of Theorem 1.1 are satisfied.

4.8. Let $\lambda_0=1$ be the only eigenvalue of $Df_0(0)$ on the unit circle, and let Λ_G^0 be the full orthogonal group of $E^0=\mathbb{R}^n$, $n\geqq 1$. We are thus in Case I of Section 2. The map h_μ of Theorem 1.1 is of the form

$$h_\mu(x)=\lambda_\mu x+p_\mu(x)x$$

where $x\mapsto p_\mu(x)$ is C^{k-1} and depends on $|x|$ only. Assuming $k\geqq 3$, we have $p_\mu(x)=a_\mu|x|^2+O(|x|^2)$. Generically $a_\mu\neq 0$, and the non-wandering set of h_μ in a neighbourhood of the origin consists of $\{0\}$ and $\{x: p_\mu(x)=1-\lambda_\mu\}$.

Suppose that the spectrum of $Df_\mu(0)$ is contained inside the unit circle, except for the real eigenvalue λ_μ, and that $k\geqq 3$. Also let

$$\frac{d}{d\mu}\lambda_\mu>0$$

and

$$a_0=\frac{1}{2}\left.\frac{d^2(x,h_0(x))}{d(|x|^2)^2}\right|_{|x_0|=0}<0$$

(that is, 0 is a "weak attractor" for f_0).

We have the following invariant manifolds: $S_\mu^{(1)}=\{0\}$ which consists of a fixed point, invariant under Λ_G, attracting for $\mu<0$ and non-attracting for $\mu>0$; and $S_\mu^{(2)}$ which is an attracting manifold, C^1 diffeomorphic to a $(n-1)$-sphere, present for $\mu>0$, and with diameter $\sim\sqrt{\mu}$. $S_\mu^{(2)}$ consists of fixed points for h_μ which are permuted by Λ_G. The union of the $S_\mu^{(2)}$ is a C^k submanifold of $E\times\mathbb{R}$.

The manifolds $S_\mu^{(1)}$ and $S_\mu^{(2)}$ contain all the non-wandering points of f_μ with orbit remaining in a neighbourhood of 0 in E.

4.9. Let Λ_G^0 be a multiple (multiplicity 2) of the full orthogonal group in two dimensions $O(2)$, and let $Df_0(0)$ have exactly two (double) complex conjugate eigenvalues λ_0, $\bar\lambda_0$ on the unit circle. Let $\lambda_0^3\neq 1$, $\lambda_0^4\neq 1$. We are thus in Case II of Section 2. Here $F=\mathbb{C}^2$ and*

$$h'_\mu(z)=\lambda_\mu z+a_\mu(z,\bar z)z+b_\mu(z,z)\bar z+Q_\mu(z).$$

According to Theorem 3.1 we are led to study the differential equation

$$\frac{d}{dt}z=z\pm\lambda_0^{-1}[a_0(z,\bar z)z+b_0(z,z)\bar z].$$

* To find the form of the polynomial $P_\mu(z)$, let z', z'' be the two components of z, and introduce J such that $J^2=-1$. We write $z=z'+Jz''$, $z^*=z'-Jz''$, $\bar z=\bar z'+J\bar z''$, $\bar z^*=\bar z'-J\bar z''$. By invariance we find

$$\begin{aligned}P_\mu(z)&=A_\mu z\bar z z^*+B_\mu z\bar z^*\\&=(A_\mu-B_\mu)(zz^*)\bar z+B_\mu(z\bar z^*+\bar z z^*)z\\&=(A_\mu-B_\mu)(z,z)\bar z+2B_\mu(z,\bar z)z.\end{aligned}$$

A_μ and B_μ are ordinary complex numbers because $O(2)$ contains the reflexions.

Let

$$s=(z,\bar{z})=|x_1|^2+|x_2|^2, \qquad d=i\det(z,\bar{z})=2\det(x_1,x_2)$$

$$\alpha=\mathrm{Re}(\lambda_0^{-1}a_0), \qquad \beta=\mathrm{Re}(\lambda_0^{-1}b_0).$$

We find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}s&=s\pm[\alpha s^2+\beta(s^2-d^2)]\\ \frac{1}{2}\frac{d}{dt}d&=d\pm\alpha s d.\end{aligned} \tag{4.2}$$

We make the generic assumption that α, β, $\alpha+\beta$ are all nonzero. The right-hand sides of (4.2) then vanish only in the following cases:

(1) $s=d=0$, (2) $s=\mp(\alpha+\beta)^{-1}$, $d=0$,

(3) $s=d=\mp\alpha^{-1}$, (4) $s=-d=\mp\alpha^{-1}$.

The vector field $z\mapsto z\pm\lambda_0^{-1}[a_0(z,\bar{z})\,z+b_0(z,z)\,\bar{z}]$ is normally hyperbolic to the following manifolds:

$$S^{(1)}=\{0\}.$$

[The Jacobian of the field is 1.]

$$S^{(2)}=\{z\colon (z,\bar{z})=\mp(\alpha+\beta)^{-1},\ \det(z,\bar{z})=0\}.$$

[The Jacobian of (4.2) with respect to the variables s, d is $\begin{pmatrix}-2 & \\ & -2\beta/(\alpha+\beta)\end{pmatrix}$.]

$$S^{(3)}\cup S^{(4)} \text{ where } S^{(3)}=\{z\colon (z,\bar{z})=i\det(z,\bar{z})=\mp\alpha^{-1}\}.$$

[The Jacobian of (4.2) with respect to the variables* s, $\delta=(s-d)^{\frac{1}{2}}$ is $\begin{pmatrix}-2 & \\ & -2\beta/\alpha\end{pmatrix}$.]

$$S^{(4)}=\{z\colon (z,\bar{z})=-i\det(z,\bar{z})=\mp\alpha^{-1}\}.$$

We are thus in position to apply Theorem 3.1. *For definiteness, suppose that the spectrum of $Df_\mu(0)$ is contained inside the unit circle, except for the eigenvalues λ_μ, $\bar{\lambda}_\mu$. Let $|\lambda_\mu|>1$ for $\mu>0$ and $|\lambda_\mu|<1$ for $\mu<0$. Also let $\alpha=\mathrm{Re}(\lambda_0^{-1}a_0)<0$ and $\alpha+\beta=\mathrm{Re}(\lambda_0^{-1}a_0+\lambda_0^{-1}b_0)<0$ (that is, f_0 is a weak attractor). We have the following C^k invariant manifolds:*
*$S_\mu^{(1)}=\{0\}$ which consists of a fixed point, attracting for $\mu<0$, and $S_\mu^{(2)}$, $S_\mu^{(3)}$, $S_\mu^{(4)}$ which occur for $\mu>0$. $S_\mu^{(2)}$ is a union of curves invariant under f, diffeomorphic to circles, and interchanged by Λ_G^0**. $S_\mu^{(3)}$ and $S_\mu^{(4)}$ are two circles invariant under f and the connected component of the identity in Λ_G^0; they are interchanged by reflexions. If $\beta>0$ then $S_\mu^{(2)}$ is attracting and $S_\mu^{(3)}$, $S_\mu^{(4)}$ are non-attracting. If $\beta<0$ then $S_\mu^{(2)}$ is non-attracting and $S_\mu^{(3)}$, $S_\mu^{(4)}$ are attracting. These manifolds contain all the non-wandering points of f_μ with orbit remaining in a neighbourhood of 0 in E.*

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* If z', z'' are the components of z, we have $s-d=(z'+iz'')(\bar{z}'-i\bar{z}'')$, hence $\delta=|z'+iz''|$.

** To see this, let $\Pi_\zeta=\{z\colon z \text{ and } \bar{z} \text{ are parallel to } \zeta\}$; then $S^{(2)}$ is a union of circles $S^{(2)}\cap\Pi_\zeta$. Since Π_ζ is the subspace of $\mathbb{C}^2$ left invariant by the reflexions which preserve ζ ($\zeta\neq0$), we have $\Theta_\mu(S^{(2)}\cap\Pi_\zeta)\subset\Pi_\zeta$.

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