Existence in the Large for Quasilinear Hyperbolic Conservation Laws

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1. Introduction

Let $g \in C^\infty(0,\infty)$. We consider systems of equations

$$(1.1)\qquad \begin{aligned} u_t - g(v)_x &= 0 \\ v_t - u_x &= 0 \end{aligned} \qquad -\infty < x < \infty,\ t > 0$$

which are strictly hyperbolic and genuinely nonlinear in the sense of LAX [1], that is, $g' > 0$, $g'' \neq 0$. We assume the reader is familiar with [1] and [2].

We seek a generalized solution of the Cauchy problem for (1.1) with initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x) > 0$, that is, a pair of bounded measurable functions $u(x,t)$, $v(x,t)$ which is defined for $-\infty < x < \infty$, $t > 0$ and which satisfies (1.1) in the sense of distributions, that is,

$$\iint_{t>0} \{u\phi_t - g(v)\phi_x\}\,dx\,dt + \int_{t=0} u_0(x)\,\phi(x,0)\,dx = 0$$

$$\iint_{t>0} \{v\phi_t - u\phi_x\}\,dx\,dt + \int_{t=0} v_0(x)\,\phi(x,0)\,dx = 0$$

for all $\phi \in C^\infty$ with compact support.

An example of (1.1) is given by the equations which govern the motion of an isentropic gas. For gas dynamics, v is the specific volume of a fluid element, u the velocity and $-g$ the pressure. For a polytropic gas, the equation of state is given by $g(v) = -a^2 v^{-\gamma}$, $\gamma \geqq 1$.

We study systems (1.1) for which

$$(1.2)\qquad \lim_{v\to 0} v^{\alpha+5} \frac{d^5}{dv^5} g(v) \neq 0, \qquad \lim_{v\to\infty} v^{\beta+5} \frac{d^5}{dv^5} g(v) \neq 0 \qquad \alpha, \beta > 1.$$

We note that the equation of state of a polytropic gas satisfies (1.2). For simplicity we assume

$$(1.3)\qquad \lim_{x\to-\infty} u_0(x) = \lim_{x\to\infty} u_0(x), \qquad \lim_{x\to-\infty} v_0(x) = \lim_{v\to\infty} v_0(x) > 0.$$

In order to state the main existence theorem, consider the classical Riemann invariants of (1.1)

$$z = u - \int_v^\infty \sqrt{g'(s)}\,ds, \qquad w = u + \int_v^\infty \sqrt{g'(s)}\,ds \tag{1.4}$$

and let

$$\eta_0(x) \equiv w_0(x) - z_0(x) = 2 \int_{v_0(x)}^\infty \sqrt{g'(s)}\,ds.$$

Theorem 1.5. *There exists a solution of the Cauchy problem for* (1.1) *if* (1.2) *and* (1.3) *are satisfied and if*

$$TVz_0 + TWw_0 \leqq \text{const.} \lim_{x \to \pm\infty} \eta_0(x). \tag{1.6}$$

Here, the constant depends only on the nonlinear function g in (1.1).

In the case where g is the equation of state for a polytropic gas, that is, $g(v) = -a^2 v^{-\gamma}$, $\gamma \geqq 1$, condition (1.6) reduces to the requirement that $TVu_0 + TV\rho_0 \leqq \frac{\text{const.}}{\gamma - 1} \lim_{x \to \pm\infty} \rho_0^{(\gamma-1)/2}(x)$ where $\rho_0(x) = 1/v_0(x)$ is the density of the gas at $t = 0$. A similar result for gas dynamics is given in [9] by different means.

Theorem 1.5 is proved using a result of [6]. In that paper existence of solutions of the Cauchy problem is established for a class of strictly hyperbolic, genuinely nonlinear systems, $u_t + f_1(u, v)_x = 0$, $v_t + f_2(u, v)_x = 0$. In terms of a pair of Riemann invariants z', w', membership in the class is determined by restrictions on the global geometry in the z'-w'-plane of the shock curves for the system. A statement of these restriction is given in Section 2.

Solutions are constructed in [6] as the pointwise limit almost everywhere of a sequence of GLIMM difference approximations $U_h = (u_h(x, t), v_h(x, t))$, $h > 0$. Convergence of the approximations follows from a bound on the total x-variation of $U_h(x, t)$ which is uniform in h and t. The bound is obtained by estimating the x-variation of the composite functions $z'(u_h(x, t), v_h(x, t))$ and $w'(u_h(x, t), v_h(x, t))$.

It is not difficult to show that, in general, the conditions for membership in the above class are not satisfied when the shock curves of (1.1) are expressed in terms of the classical Riemann invariants (1.4). For example, one condition requires, roughly, that the strength of the nonlinear coupling of the two equations decrease under translation of the shock curves into the bounded region of an invariant quadrant for solutions to the Riemann problem. This condition does not hold for the shock curves of gas dynamics with $g = -a^2 v^{-\gamma}$, $\gamma > 1$, and classical Riemann invariants (1.4).

We shall show, however, that there exists a map $T: (z, w) \to (\phi(z), \psi(w))$ which transforms the shock curves of (1.1) in the plane of classical Riemann invariants z, w into shock curves which satisfy the conditions of [6] when expressed in terms of new Riemann invariants $z' = \phi(z)$, $w' = \psi(w)$. It then follows that the x-variations of $z'(u_h(x, t), v_h(x, t))$ and $w'(u_h(x, t), v_h(x, t))$ are uniformly bounded in h and t, a condition sufficient for existence of solutions [3]. The construction of T will depend upon certain properties of the global geometry of the shock curves which will be discussed in Section 3. The geometric role of shock curves for hyperbolic systems is further illustrated in [4], [7], [8] and [9].

2. Preliminaries

We now state the existence theorem of [6]. Consider a strictly hyperbolic, genuinely nonlinear system

$$(2.1) \qquad u_t + f_1(u, v)_x = 0, \qquad v_t + f_2(u, v)_x = 0.$$

Let z, w be a pair of Riemann invariants for (2.1) and let the left and right shock curves of the first and second kind with initial point (z_0, w_0) be given respectively by

$$(2.2) \qquad \begin{array}{ll} z = R_1(w; z_0, w_0), \ w \leqq w_0; & z = L_1(w; z_0, w_0), \ w \geqq w_0 \\ z = R_2(w; z_0, w_0), \ w \leqq w_0; & z = L_2(w; z_0, w_0), \ w \geqq w_0. \end{array}$$

These curves give states (z, w) which can be connected on the left (L_i) and right (R_i) to (z_0, w_0) through a shock of the i^{th} kind.

Let $\lambda_1(z, w) < \lambda_2(z, w)$ be the characteristic speeds of (2.1) associated with z and w respectively. Let

$$\Omega = \{(z, w): z \geqq \inf_x z_0(x), w \leqq \sup_x w_0(x), w - z \leqq \sup_{x_1 \leqq x_2} (w_0(x_1) - z_0(x_2))\}.$$

The existence theorem is obtained under the following conditions:

A_1. $\sup_{i, \Omega} |\lambda_i(z, w)| < \infty$.

A_2. If $(z, w) \in \Omega$, then $1 < \dfrac{\partial R_1}{\partial w}, \dfrac{\partial L_1}{\partial w} < \infty$ for $w \neq w_0$, and $0 < \dfrac{\partial R_2}{\partial w}, \dfrac{\partial L_2}{\partial w} < 1$ for $w \neq w_0$.

A_3. If $z_r = R_i(w_r; z_l, w_l)$, then $z = R_i(w; z_l, w_l)$, $w \leqq w_l$ and $z = L_i(w; z_r, w_r)$, $w \geqq w_r$ have only the point (z_l, w_l) in common.

A_4. If four points (z_l, w_l), (z_r, w_r), (z_m, w_m) and $(\tilde{z}_m, \tilde{w}_m)$ satisfy $z_m = R_2(w_m; z_l, w_l)$, $z_r = R_1(w_r; z_m, w_m)$, $\tilde{z}_m = R_1(\tilde{w}_m; z_l, w_l)$, and $z_r = R_2(w_r; \tilde{z}_m, \tilde{w}_m)$, then $(z_l - \tilde{z}_m) + (\tilde{w}_m - w_r) \leqq (z_l - z_m) + (w_m - w_r)$.

Theorem 2.3. *If a strictly hyperbolic, genuinely nonlinear system* (2.1) *satisfies conditions* A_1 *through* A_4, *then the Cauchy problem has a solution for arbitrary initial data* $u_0(x)$, $v_0(x)$ *in* BV_{loc}.

Loosely stated, conditions A_2, A_3, A_4 imply that the strength of shocks of the first (second) kind, as measured by the jump $\Delta z(\Delta w)$ in the Riemann invariant $z(w)$, decreases under interaction with other shock and rarefaction waves. To be precise, let (z_l, w_l) be a state connected on the left to (z_m, w_m) by a shock or rarefaction wave of the i^{th} kind, V_i, and let (z_r, w_r) be a state connected on the right to (z_m, w_m) by a shock or rarefaction wave of the j^{th} kind, V_j. If we solve the Riemann problem with data (z_l, w_l) on the left and (z_r, w_r) on the right and denote the wave of the first kind in the solution by U_1 and the wave of the second kind by U_2, then conditions A_2, A_3, A_4 imply

$$(2.4) \qquad St(U_1) + St(U_2) \leqq St(V_i) + St(V_j)$$

where the strength $St(V)$ equals 0 if V is a rarefaction wave, equals Δz if V is a shock of the first kind, and equals Δw if V is a shock of the second kind. For

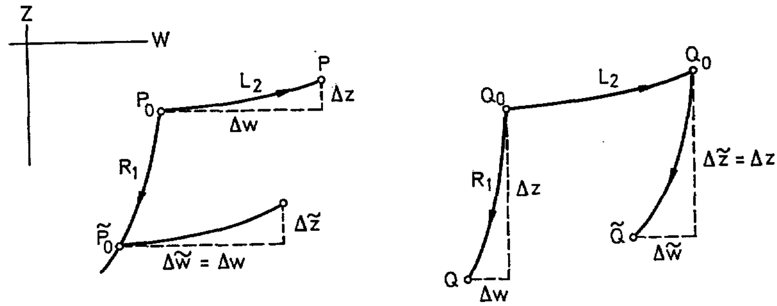

Fig. 1

example, if V_i are shocks of different kinds, that is, $i \neq j$, then it is easy to show that U_i are both shocks and A_4 is a necessary and sufficient condition for (2.4) to hold.

In the construction of the map T the following condition B_4, which is slightly stronger than A_4, will prove useful because of its simple geometric interpretation. Define

$$R_1(z_0, w_0) = \{(z, w): z = R_1(w; z_0, w_0), w \leqq w_0\}$$

$$L_2(z_0, w_0) = \{(z, w): z = L_2(w; z_0, w_0), w \geqq w_0\}$$

$$\Delta w = w - w_0, \quad \Delta \tilde{w} = \tilde{w} - \tilde{w}_0, \quad \Delta z = z - z_0, \quad \Delta \tilde{z} = \tilde{z} - \tilde{z}_0.$$

Then the condition in question may be stated as follows:

B_4. 1. Let $(\tilde{z}_0, \tilde{w}_0) \in R_1(z_0, w_0)$. If $z = L_2(w; z_0, w_0)$, $\tilde{z} = L_2(\tilde{w}; \tilde{z}_0, \tilde{w}_0)$ and $\Delta \tilde{w} = \Delta w$ then $\Delta \tilde{z} \geqq \Delta z$.

2. Let $(\tilde{z}_0, \tilde{w}_0) \in L_2(z_0, w_0)$. If $z = R_1(w; z_0, w_0)$, $\tilde{z} = R_1(\tilde{w}; \tilde{z}_0, \tilde{w}_0)$ and $\Delta \tilde{z} = \Delta z$ then $\Delta \tilde{w} \geqq \Delta w$.

The geometric content of condition B_4 is described in Fig. 1. Here P, P_0, $\tilde{P}$, $\tilde{P}_0$ represent points (z, w), (z_0, w_0), $(\tilde{z}, \tilde{w})$, $(\tilde{z}_0, \tilde{w}_0)$, and similarly for Q.

Thus, B_4 implies that the strength of the nonlinear coupling of (1.1) increases under translation of the shock curves into the bounded region of the invariant quadrant.

3. Existence of Solutions

In this section, z and w denote the classical Riemann invariants (1.4). We begin by proving a mapping theorem for the shock curves of (1.1). For this purpose, it will be convenient to identify curves and regions in the z-w-plane with their images under the map $R_{-\pi/4}$: $(z, w) \to (\sigma, \eta)$, defined by

$$\sigma = w + z, \quad \eta = w - z.$$

For example, the equations of the shock curves of (1.1) are obtained by eliminating the shock speed s from the Rankine-Hugoniot relations

$$s(u - u_0) = g(v_0) - g(v), \quad s(v - v_0) = u_0 - u.$$

This yields the equation $(u-u_0)^2=\{g(v)-g(v_0)\}\{v-v_0\}$, which determines, in the σ-η-plane, the left and right shock curves of the first and second kinds with initial point (σ_0, η_0) as follows:

$$\text{(3.1)}\qquad \begin{aligned} &\sigma=L_1(\eta;\sigma_0,\eta_0),\quad \eta\leqq\eta_0; \qquad \sigma=L_2(\eta;\sigma_0,\eta_0),\quad \eta\geqq\eta_0\\ &\sigma=R_1(\eta;\sigma_0,\eta_0),\quad \eta\geqq\eta_0; \qquad \sigma=R_2(\eta;\sigma_0,\eta_0),\quad \eta\leqq\eta_0\end{aligned}$$

where

$$\text{(3.2)}\qquad L_i=\sigma_0+Q,\quad R_i=\sigma_0-Q,\quad Q^2=4\{g(v(\eta))-g(v(\eta_0))\}\{v(\eta)-v(\eta_0)\}$$

and $v(\eta)$ is the inverse function of $\eta(v)=2\int_v^\infty \sqrt{g'(v)}\,dv$.

These shock curves are identified with their counterimages under $R_{-\pi/4}$. They yield states (L_i, η) and (R_i, η) which can be connected to (σ_0, η_0) on the left and right, respectively, by a shock of the i^{th} kind. For convenience, the same functional notation, (2.2) and (3.2), is used to denote a shock curve and its image under $R_{-\pi/4}$, respectively.

In discussing T we shall say that a shock curve (2.2) or (3.2) lies in a region D if η and w are restricted so that (σ, η) and (z, w) lie in D. We shall call a region $W(a, k)$ of the form

$$W(a,k)=\{(\sigma,\eta): |\sigma-a|\leqq k\eta\},\qquad k\leqq 1,$$

a k-wedge with vertex $(a, 0)$.

We can now state the main mapping theorem.

Theorem 3.3. *There exists a 2-parameter family of transformations $T(a,\theta)$: $(z,w)\to(z',w')$, and constants k, c_1, $c_2(k)$ which have the following property. For sufficiently small k, $T(a,\theta)$ maps the shock curves of* (1.1) *in*

$$W(a,k)\cap\{c_1/\theta<\eta<c_2(k)/\theta\}$$

onto shock curves which satisfy A_i, $i=2, 3, 4$ in the $z'-w'$ variables. Furthermore, $\lim_{k\to 0} c_2(k)=\infty$.

In order to prove Theorem 3.3 some lemmas on the geometry of the shock curves of (1.1) will be needed. We define

$$\text{(3.3)}\qquad T(a,\theta): (z,w)\to(z',w')$$

by

$$\text{(3.4)}\qquad \begin{aligned} z'&=1-\exp\big(2\theta(a/2-z)\big)\equiv\phi(z;a,\theta),\qquad \theta>0\\ w'&=-1+\exp\big(2\theta(w-a/2)\big)\equiv\psi(z;a,\theta).\end{aligned}$$

We shall identify T with the map $R_{-\pi/4}\circ T\circ R_{\pi/4}^{-1}$; that is, using $z=(\sigma-\eta)/2$, $w=(\sigma+\eta)/2$, $\sigma'=w'+z'$ and $\eta'=w'-z'$, we shall consider T as a map of the σ-η-plane given by

$$T(a,\theta): (\sigma,\eta)\to(\sigma',\eta')$$

where

(3.5a) $$\sigma' = \exp\theta(\eta+\sigma-a) - \exp\theta(\eta-\sigma+a)$$

(3.5b) $$\eta' = \exp\theta(\eta+\sigma-a) + \exp\theta(\eta-\sigma+a) - 2.$$

The formulation (3.5) proves convenient for calculational purposes. The choice of a and θ will depend upon the initial data and the function g.

First, we shall discuss those aspects of the geometry of the shock curves of (1.1) relating to the requirement that their image under T satisfy A_2. To this end, we reformulate A_2 as follows.

Lemma 3.6. A_2 *is equivalent to the requirement that*

$$-\infty < \frac{\partial L_1}{\partial \eta}, \frac{\partial R_1}{\partial \eta} < -1, \quad 1 < \frac{\partial L_2}{\partial \eta}, \frac{\partial R_2}{\partial \eta} < \infty \quad \text{for } \eta \neq \eta_0.$$

Proof. The lemma follows immediately from the definition of σ and η.

From Lemma 3.6 and [5], it follows that the shock curves of (1.1) satisfy A_2 when expressed in terms of the classical Riemann invariants (1.4). We also have

Lemma 3.7. *The image of a shock curve* $\sigma = R_i(\eta; \sigma_0, \eta_0)$ *or* $\sigma = L_i(\eta; \sigma_0, \eta_0)$, $i = 1, 2$, *under* $T(a, \theta)$ *satisfies* A_2 *if*

$$|\sigma - a| \leqq \frac{1}{2\theta} |\log\{(\sigma_\eta - 1)/(\sigma_\eta + 1)\}|.$$

Proof. Without loss of generality set $a = 0$. For concreteness, consider the curve $\sigma = R_1(\eta; \sigma_0, \eta_0)$. Setting $\sigma = R_1$ in (3.5a) and (3.5b) with $a = 0$, we respectively obtain functions $\sigma' = N(\eta; \sigma_0, \eta_0)$ and $\eta' = D(\eta; \sigma_0, \eta_0)$. Since

$$\frac{\partial \sigma'}{\partial \eta'} = \frac{\partial N}{\partial \eta} \Big/ \frac{\partial D}{\partial \eta},$$

it follows from Lemma 3.6 that A_2 is equivalent to the condition

$$\frac{\partial N}{\partial \eta} < -\frac{\partial D}{\partial \eta} \leqq 0 \quad \text{for } \eta \neq \eta_0.$$

But

$$\begin{aligned}\frac{\partial N}{\partial \eta} &= \theta\left\{\left(\frac{\partial R_1}{\partial \eta} + 1\right)\exp\theta(\eta + R_1) + \left(\frac{\partial R_1}{\partial \eta} - 1\right)\exp\theta(\eta - R_1)\right\} \\ &= \theta\exp\{\theta(\eta - R_1)\}\left[\left(\frac{\partial R_1}{\partial \eta} + 1\right)\exp 2\theta R_1 + \left(\frac{\partial R_1}{\partial \eta} - 1\right)\right]\end{aligned}$$

and

$$\frac{\partial D}{\partial \eta} = \theta\exp\{\theta(\eta - R_1)\}\left[\left(\frac{\partial R_1}{\partial \eta} + 1\right)\exp 2\theta R_1 - \left(\frac{\partial R_1}{\partial \eta} - 1\right)\right].$$

Therefore, $\dfrac{\partial N}{\partial \eta} < -\dfrac{\partial D}{\partial \eta}$ for $\eta \neq \eta_0$ since $\dfrac{\partial R_1}{\partial \eta} < -1$ for $\eta \neq \eta_0$.

Since $\frac{\partial D}{\partial \eta} \geqq 0$ if and only if

$$\left(\frac{\partial R_1}{\partial \eta}+1\right) \exp 2\theta R_1 - \left(\frac{\partial R_1}{\partial \eta}-1\right) \geqq 0,$$

it is clear that A_2 holds if

$$|R_1| \leqq \frac{1}{2\theta} \left| \log \left(\frac{\partial R_1}{\partial \eta}-1\right) \Big/ \left(\frac{\partial R_1}{\partial \eta}+1\right) \right|.$$

In conjunction with the inequality appearing in Lemma 3.7, we prove

Lemma 3.8. *There exists a constant $M(k)$, independent of a, with the property that*

$$\left|\frac{\partial R_i}{\partial \eta}\right| \leqq M(k) \quad \text{and} \quad \left|\frac{\partial L_i}{\partial \eta}\right| \leqq M(k), \qquad i=1,2,$$

if $\sigma = R_i(\eta; \sigma_0, \eta)$ and $\sigma = L_i(\eta; \sigma_0, \eta_0)$ lie in $W(a,k)$. Also $\lim_{k\to 0} M(k)=1$.

Proof. Without loss of generality we may assume that $a=0$. Since the shock curves $\sigma=R_i(\eta; \sigma_0, \eta_0)$, $\sigma=L_i(\eta; \sigma_0, \eta_0)$ satisfy

$$(\sigma-\sigma_0)^2 = 4\{g(v(\eta)) - g(v(\eta_0))\}\{v(\eta)-v(\eta_0)\}$$

we have

$$\frac{1}{2}(\sigma-\sigma_0)\frac{d\sigma}{d\eta} = g'(v(\eta))\, v'(\eta)\{v(\eta)-v(\eta_0)\} + \{g(v(\eta)) - g(v(\eta_0))\}\, v'(\eta).$$

Now fix σ_0, η_0 and define ε, δ by $\sigma-\sigma_0=\delta\eta_0$ and $\eta=\varepsilon\eta_0$. Then we have

$$\begin{aligned}\frac{1}{2}\left|\frac{d\sigma}{d\eta}\right| \leqq\; & |g'(v(\varepsilon\eta_0))\, v'(\varepsilon\eta_0)\{v(\varepsilon\eta_0)-v(\eta_0)\}|/\delta\eta_0 \\ & + |\{g(v(\varepsilon\eta_0)) - g(v(\eta_0))\}\, v'(\varepsilon\eta_0)|/\delta\eta_0.\end{aligned}$$

For concreteness, we prove the lemma for R_1 and L_2, that is, for $\varepsilon>1$. By use of (1.2) and $g'>0$, it follows that $g''<0$ and therefore that $v'(\eta)<0$ and $v''(\eta)>0$. Thus from the concavity of g, the convexity of $v(\eta)$, and the mean value theorem, we have

$$\left|\frac{d\sigma}{d\eta}\right| \leqq 4\, \frac{(\varepsilon-1)}{\delta}\, g'(v(\varepsilon\eta_0))\, v'(\eta_0)\, v'(\varepsilon\eta_0) \equiv \frac{(\varepsilon-1)}{\delta}\, h(\varepsilon, \eta_0).$$

Now, from Lemma 3.6, we have $|\eta-\eta_0|/|\sigma-\sigma_0| \leqq 1$ and hence $(\varepsilon-1)/\delta \leqq 1$. Therefore, since $\lim_{\eta\to\eta_0}\left|\frac{d\sigma}{d\eta}\right|=1$, we need only show that there exists a constant $M(k)$ such that $\lim_{\eta\to 0,\infty} h(\varepsilon,\eta) \leqq M(k)$ and $\lim_{k\to 0} M(k)=1$.

The existence of $M(k)$ follows from the asymptotic behavior of $g(v)$, $\eta(v) = 2\int_v^\infty \sqrt{g'(v)}\, dv$, and its inverse $v(\eta)$. From (1.2), it is not difficult to show that there exists a constant $K \neq 0$ such that $\lim_{v\to 0} v^{\alpha+1} g'(v) = K^2$ and hence that

$$\text{(3.9)}\qquad \begin{aligned} &\lim_{\eta\to\infty}\left(\frac{\alpha-1}{4K}\eta\right)^{(\alpha+1)/(\alpha-1)} v'(\eta) = -1/2K, \\ &\lim_{\eta\to\infty}\left(\frac{\alpha-1}{4K}\eta\right)^{2(\alpha+1)/(1-\alpha)} g'(v(\eta)) = K^2. \end{aligned}$$

Note that the limiting values obtained in (3.9) are precisely those obtained in the special case of gas dynamics where $g(v) \equiv \frac{-K^2}{\alpha} v^{-\alpha}$. Applying (3.9), we have

$$h(\varepsilon, \eta) \approx (\varepsilon\eta)^{2(\alpha+1)/(\alpha-1)} \eta^{(\alpha+1)/(1-\alpha)} (\varepsilon\eta)^{(\alpha+1)/(1-\alpha)} = \varepsilon^{(\alpha+1)/(\alpha-1)}$$

near $\eta = \infty$. A similar result holds for η near 0. Then, since $(\sigma, \eta) \in W(a, k)$, we have $\varepsilon \leqq (k+1)/(1-k)$ and the existence of $M(k)$ follows. The proof is complete.

Next, we shall discuss those aspects of the geometry of the shock curves of (1.1) which relate to the requirement that their image under T satisfy B_4. (Recall that B_4 implies A_4.) To do this, we need some notation and terminology. Let ϕ and f be real valued functions of a real variable and let $[a, b]$ denote a closed interval on the real line.

We define

$$\phi[a, b] = [\phi(a), \phi(b)]$$

and shall say that ϕ expands a family of intervals $K_1(s) = [s, s+f(s)]$ or $K_2(s) = [s-f(s), s]$ as s increases (decreases) if the length of $\phi K_1(s)$ or $\phi K_2(s)$ is an increasing (decreasing) function of s.

Given $\varepsilon > 0$, z_0 and w_0, we define the family of intervals I and J by

$$\text{(3.10)}\qquad I(z) = I(z, \varepsilon; z_0, w_0) = [z, z + Z(z, \varepsilon; z_0, w_0)]$$

$$\text{(3.11)}\qquad J(w) = J(w, \varepsilon; z_0, w_0) = [w - W(w, \varepsilon; z_0, w_0), w]$$

where

$$\text{(3.12)}\qquad \begin{aligned} &W = w - R_1^{-1}(z-\varepsilon; z, w) \quad \text{with} \quad z = L_2(w; z_0, w_0), \quad w \geqq w_0 \\ &Z = L_2(w+\varepsilon; z, w) - z \quad \text{with} \quad w = R_1^{-1}(z; z_0, w_0), \quad z \leqq z_0. \end{aligned}$$

Here $w = R_1^{-1}(z; z_0, w_0)$ is the inverse of $z = R_1(w; z_0, w_0)$. See Fig. 2.

Using Fig. 2, we can recast condition B_4 as follows:

Lemma 3.13. B_4 *is equivalent to the requirement that* $\frac{\partial Z}{\partial z} \leqq 0$ *and* $\frac{\partial W}{\partial w} \leqq 0$, *that is, to the requirement that I and J expand as z decreases and w increases, respectively.*

We note that not all systems satisfy B_4. For example, the equations of gas dynamics, with classical Riemann invariants (1.4) and with $g(v) = -a^2 v^{-\gamma}$, $\gamma > 1$, satisfy $\frac{\partial Z}{\partial z} > 0$, $\frac{\partial W}{\partial w} < 0$. However, it will be proved that the images of the shock curves in a suitable region under $T(a, \theta)$ do satisfy B_4. To this end, we give a

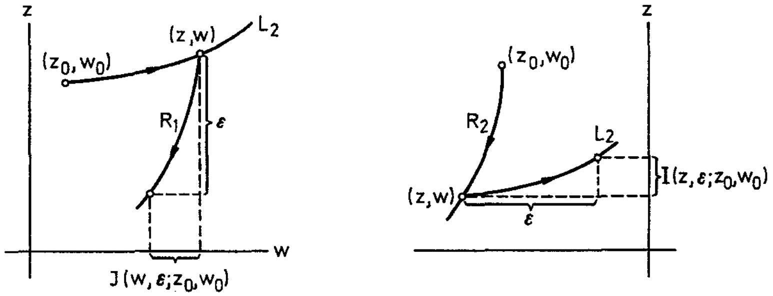

Fig. 2

general mapping lemma. Let ϕ and ψ be smooth monotone functions of a real variable.

Lemma 3.14. *If S: $(z, w) \to (\phi(z), \psi(w))$ has the property that ϕ expands $I(z)$ as z decreases and ψ expands $J(w)$ as w increases, then S maps shock curves $z=R_1(w; z_0, w_0)$, $z=L_2(w_0; z_0, w_0)$ onto shock curves which satisfy* B_4 *as functions of $z'=\phi(z)$, $w'=\psi(w)$.*

Proof. Write $T=T_1 \circ T_2 = T_2 \circ T_1$, where

$$T_1: (z, w) \to (z_1, w_1) \equiv (z, \psi(w)),$$

$$T_2: (z, w) \to (z_2, w_2) \equiv (\phi(z), w).$$

Since ϕ expands $I(z)$ as z decreases, the image of $z=L_2(w; z_0, w_0)$ under T_2 satisfies $B_{4,1}$. Since ψ expands $J(w)$ as w increases, the image of $z=R_1(w; z_0, w_0)$ under T_1 satisfies $B_{4,2}$. Now, it is easy to see that property $B_{4,i}$ is invariant under T_i. Therefore $T(R_1)=T_2(T_1(R_1))$ satisfies $B_{4,2}$ since $T_1(R_1)$ does, and $T(L_2)=T_1(T_2(L_2))$ satisfies $B_{4,1}$ since $T_2(L_2)$ does. The proof is complete.

In the proof of the existence Theorem 1.4, the parameter a and the total variation of the initial data will be restricted so that the approximate solutions lie in a region $a-z/2>0$, $w/2-a>0$. In such a region, we shall show that the larger the value of $\eta=w-z$, the smaller the first derivative of ϕ and ψ (and hence θ) can be taken and still have T expand I and J. The bound on the total variation of the data will roughly be of the order $1/\theta$.

We proceed with the proof of Theorem 3.3 by first applying Lemma 3.14 to the map $T(a, \theta)$.

Lemma 3.15. *There exist constants $k_0>0$, $c_1>0$, depending only on g, such that $T(a, \theta)$ maps the shock curves $z=R_1(w; z_0, w_0)$, $z=L_2(w; z_0, w_0)$ in $W(a, k) \cap \{\eta \geq c_1/\theta\}$ onto shock curves which satisfy* B_4 *in the z', w' variables if $k \leq k_0$.*

Proof. We must find k_0 and c_1 such that $z'=\phi(z; a, \theta)$ and $w'=\psi(w; a, \theta)$ respectively expand the intervals $I(z)=I(z, \varepsilon; z_0, w_0)$ as z decreases and $J(w)=J(w, \varepsilon; z_0, w_0)$ as w increases provided z, w, z_0, w_0 are restricted to lie in $W(a,k_0) \cap \{\eta \geq c_1/\theta\}$.

To do this, we consider the auxiliary family of intervals

$$K(\eta, \varepsilon)=[\eta-S(\eta, \varepsilon), \eta], \qquad \varepsilon>0$$

where

$$S(\eta, \varepsilon)=\{4[g(v(\eta+\varepsilon))-g(v(\eta))][v(\eta+\varepsilon)-v(\eta)]\}^{1/2}-\varepsilon. \tag{3.16}$$

In the σ-η-plane, $S(\eta_0, \varepsilon)$ equals the distance, at the point $\eta=\eta_0+\varepsilon$, that the shock curve $\sigma=L_2(\eta; \sigma_0, \eta_0)$ lies above the line through (σ_0, η_0) with slope $+1$; and by symmetry this equals the distance that $\sigma=R_1(\eta; \sigma_0, \eta_0)$ lies below the line through (σ_0, η_0) with slope -1.

Since the z-w-plane is the image of the σ-η-plane under $R^{-1}_{\pi/4}$, to prove the lemma it is sufficient to find constants k and c'_1 such that $\exp\theta\eta$ expands the family $K(\eta, \varepsilon)$ as η increases when $\eta \geqq c'_1/\theta$ and $\varepsilon \leqq k\eta$. The expansion of $K(\eta, \varepsilon)$ by $\exp\theta\eta$ follows directly from the following two propositions.

Proposition 1. *If* $\left|\frac{d}{dx}\log f\right| \leqq \theta$, *then* $\exp\theta x$ *expands the family of intervals* $[x-f(x), x]$ *as* x *increases.*

Proof. $\frac{d}{dx}\{\exp\theta x-\exp\theta(x-f)\}=\{\theta\exp\theta x\}\{1-(1-f')\exp(-\theta f)\}\geqq 0$, since the hypothesis of the proposition implies that $1-f'\leqq\exp\theta f$.

Proposition 2. *There exist constants* k, c'_1 *depending only on* g *in* (1.1) *such that* $\left|\frac{\partial}{\partial\eta}\log S(\eta, \varepsilon)\right| \leqq c'_1/\eta$ *if* $\varepsilon\leqq k\eta$.

Proof. First, we prove the case where $g=-K^2 v^{-\gamma}/\gamma$, $\gamma>1$. Here

$$z=u-\frac{2K}{\gamma-1}v^{(1-\gamma)/2}, \qquad w=u+\frac{2K}{\gamma-1}v^{(1-\gamma)/2}, \qquad \sigma=2u$$

$$\eta(v)=\frac{4K}{\gamma-1}v^{(1-\gamma)/2}, \qquad v(\eta)=[(\gamma-1)\eta/4K]^{2/(1-\gamma)}$$

$$S(\eta, \varepsilon)=c\{[(\eta+\varepsilon)^a-\eta][\eta-(\eta+\varepsilon)^b]\}^{1/2}-\varepsilon$$

where $a=2\gamma/(\gamma-1)$, $b=2/(1-\gamma)$, $c=(\gamma-1)/2\gamma^{1/2}$. Setting $\varepsilon=(\tau-1)\eta$ in $S(\eta, \varepsilon)$ and $S_\eta(\eta, \varepsilon)$, we obtain

$$S=\eta f_1(\tau, \gamma) \quad\text{and}\quad S_\eta=f_2(\tau, \gamma)$$

where

$$f_1(\tau, \gamma)=c\{(\tau^a-1)(1-\tau^b)\}^{1/2}-(\tau-1) \tag{3.17}$$

$$f_2(\tau, \gamma)=\left\{\frac{c}{2}a(\tau^{a-1}-1)(1-\tau^b)+\frac{cb}{2}(\tau^a-1)(1-\tau^{b-1})\right\}\{(\tau^a-1)(1-\tau^b)\}^{-1/2}.$$

Now it follows from a straightforward calculation that for each γ, the functions f_1 and f_2 have precisely third order zeros at $\tau=1$ $(\varepsilon=0)$. Therefore, there exist constants k, c'_1 such that $|f_2|/|f_1|\leqq c'_1$ for $\tau\leqq 1+k$, that is, for $(\tau-1)\eta=\varepsilon\leqq k\eta$, and the lemma is proved.

The general case follows from the asymptotic behavior of g given by (1.2). We sketch the proof as follows. Consider

$$S(\eta, \varepsilon)=\{4[g(v(\eta+\varepsilon)-g(v(\eta))][v(\eta+\varepsilon)-v(\eta)]\}^{1/2}-\varepsilon.$$

We define $S_1(\eta, \tau)=S(\eta,(\tau-1)\eta)$ and $S_2=S_\eta(\eta,(\tau-1)\eta)$; for example,

$$S_1(\eta, \tau)=\{4[g(v(\tau\eta)-g(v\eta))][v(\tau\eta)-v(\eta)]\}^{1/2}-(\tau-1)\eta. \tag{3.18}$$

By use of (1.2), it follows that

$$\begin{aligned}&\lim_{\eta\to 0} S_1(\eta, \tau)/(\tau-1)^3\eta=f_1(\tau, \beta)/(\tau-1)^3,\\ &\lim_{\eta\to\infty} S_1(\eta, \tau)/(\tau-1)^3\eta=f_1(\tau, \alpha)/(\tau-1)^3,\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\lim_{\eta\to 0} S_2(\eta, \tau)/(\tau-1)^3=f_2(\tau, \beta)/(\tau-1)^3,\\ &\lim_{\eta\to\infty} S_2(\eta, \tau)/(\tau-1)^3=f_2(\tau, \alpha)/(\tau-1)^3\end{aligned} \tag{3.20}$$

uniformly in $\tau\in[0, \tau_0)$ for some small τ_0. The lemma follows from the continuity of $\eta S_2/S_1$ and its boundedness for η near 0 and ∞ (implied by (3.19), (3.20)).

We shall establish (3.19); the proof of (3.20) is similar. To do this, define $s(\tau, \eta)$ by

$$S_1(\eta, \tau)=s(\eta, \tau)^{1/2}-(\tau-1)\eta.$$

Since $S_1(\eta, \tau)=0(\tau-1)^3$ from [1], we have $\dfrac{\partial^3}{\partial\tau^3}s(\eta, 1)=0$ and therefore

$$s(\eta, \tau)=(\tau-1)^2\eta^2+\frac{1}{3!}\int_1^\tau(\tau-x)^3\frac{\partial^4}{\partial x^4}s(\eta, x)\,dx\equiv(\tau-1)^2\eta^2+s_0(\eta, \tau). \tag{3.21}$$

Next, let $\sqrt{1+y}=1+\frac{1}{2}y+p(y)$. Then

$$S_1(\eta, \tau)/(\tau-1)\eta=\{1+s_0/(\tau-1)^2\eta^2\}^{1/2}-1$$

and

$$S_1/(\tau-1)^3\eta=\tfrac{1}{2}s_0(\tau, \eta)/(\tau-1)^4\eta^2+p(s_0(\tau, \eta)/(\tau-1)^2\eta^2)/(\tau-1)^2. \tag{3.22}$$

Using the definition of s_0 given in (3.21), we see that the behavior of the right hand side of (3.22) for η near zero or infinity is determined by the behavior of

$$\frac{\partial^4}{\partial\tau^4}s(\eta, \tau)=\frac{\partial^4}{\partial\tau^4}\{[g(v(\tau\eta))-g(v(\eta))][v(\tau\eta)-v(\eta)]\}. \tag{3.23}$$

By use of standard differentiation formulae, it follows that (3.23) is the sum of terms T_i each involving derivatives of at most fourth order. The asymptotic behavior of each factor of T_i is determined by (1.2); for example, if

$$\lim_{v\to\infty} v^{\beta+5}\frac{d^5}{dv^5}g=L,$$

then

$$\lim_{v\to\infty} v^{\beta+j} \frac{d^j}{dv^j} g = (-1)^{j+1} L/(\beta+j)(\beta+j+1)\dots(\beta+4) \quad \text{for } 0 \leqq j \leqq 4,$$

that is, the limit of $v^{\beta+j} \frac{d^j}{dv^j} g$ is precisely the value obtained in the special case where $g = Lv^{-\beta}/\beta(\beta+1)\dots(\beta+4)$. Using (1.2), it is not difficult to show that $\lim_{\eta\to 0} T_i/\eta^2$ exists uniformly in τ and equals $\lim_{\eta\to 0} G_i/\eta^2$, where G_i is the term in the special case of gas dynamics which corresponds to T_i. Similar results hold for limits as η approaches ∞. Therefore $\lim_{\eta\to 0} \frac{d^4}{d\tau^4} s(\tau, \eta)$ exists uniformly in τ and equals the value obtained in the special case of gas dynamics; (3.19) follows at once.

Combining the previous lemmas, we now prove the main mapping theorem.

Proof of Theorem 3.3. Fix a. Let $0<k<1$. Consider the wedge $W(a,k)$ and the constant $M(k)$ given by Lemma 3.8. From Lemma 3.7, it follows that the image of a shock curve in $W(a,k)$ under $T(a,\theta)$ satisfies A_2 provided that $|\sigma - a| \leqq c(k)/\theta$, where $c = \log(M(k)+1)/(M(k)-1)$. Therefore, there exists a value $c_2(k)$ such that the image of a shock curve in $W(a,k) \cap \{\eta < c_2(k)/\theta\}$ satisfies A_2. Also $\lim_{k\to 0} c_2(k) = \infty$ since $\lim_{k\to 0} M(k) = 1$.

Consider the constants c_1 and k_0 of Lemma 3.15. For $k \leqq k_0$ sufficiently small, we have from the above and from Lemma 3.15 that the images of shock curves in $W(a,k) \cap \{c_1/\theta < \eta < c_2(k)/\theta\}$ under $T(a,\theta)$ satisfy A_2 and A_4 and that $\lim_{k\to 0} c_2(k) = \infty$.

The validity of A_3 in the z', w' variables follows from the starlike property of the shock curves proved in [5] and from the fact that T is one to one.

In order to prove Theorem 1.5 we need an additional lemma. Put

$$W'(a,k) = \{(\sigma', \eta') : |\sigma'| \leqq k\eta'\}.$$

Lemma 3.23. *Let c_1 and $c_2(k)$ be the constants of Theorem 3.3. Then, there exist constants d_1 and $d_2(k)$, independent of θ, such that for small k*

$$\begin{aligned} T^{-1}(a,\theta)[W'(0,k) \cap \{(\sigma',\eta') : d_1 < \eta' < d_2(k)\}] \\ \subset W(a,k) \cap \{c_1/\theta < \eta < c_2(k)/\theta\} \end{aligned} \tag{3.24}$$

for all a and θ.

Proof. Without loss of generality, set $a=0$. First, we shall show that

$$T(0,\theta)[W(0,k)] \supset W'(0,k) \quad \text{for } \theta>0,\ 0<k<1. \tag{3.25}$$

To do this, recall that $T(0,\theta)$ is given by

$$\begin{aligned} \sigma' &= \exp\theta(\eta+\sigma) - \exp\theta(\eta-\sigma) \\ \eta' &= \exp\theta(\eta+\sigma) + \exp\theta(\eta-\sigma) - 2. \end{aligned}$$

It is not difficult to show that $T(0, \theta)$ maps the curves $\sigma = k\eta$ and $\sigma = -k\eta$ onto curves $\sigma' = p(\eta')$ and $\sigma' = -p(\eta')$, where p satisfies $p(0)=0$, $\dot{p}(0)=k$, $\ddot{p}(\eta)>0$. Since $\sigma = \pm k\eta$ form the boundary of $W(0, k)$ and since $T(0, \theta)$ maps $(0, 0)$ onto $(0, 0)$, we have (3.25).

Next, let $G(d_1, d_2)$ be the set on the left hand side in (3.24). We shall estimate the maximum and minimum value for η in $G(d_1, d_2)$. To do this, solve for η in terms of σ', η', obtaining

$$\eta = \frac{1}{2\theta} \log\left(\tfrac{1}{2}(\eta' + \sigma') + 1\right)\left(\tfrac{1}{2}(\eta' - \sigma') + 1\right).$$

Since $|\sigma'| < \eta'$ in $W'(0, k)$,

$$\eta' + 1 \leqq \left(\tfrac{1}{2}(\eta' + \sigma') + 1\right)\left(\tfrac{1}{2}(\eta' - \sigma') + 1\right) \leqq \eta'^2 + 2$$

and

$$\frac{1}{2\theta} \log(d_1 + 1) \leqq \min_G \eta \leqq \max_G \eta \leqq \frac{1}{2\theta} \log(d_2^2 + 2).$$

Therefore, for sufficiently small k, we have $G \subset W(0, k) \cap \{c_1/\theta < \eta < c_2(k)/\theta\}$ if d_1 and $d_2(k)$ are chosen to satisfy $\frac{1}{2}\log(d_1 + 1) = c_1$ and $\frac{1}{2}\log(d_2^2 + 2) = c_2(k)$.

Proof of Theorem 1.5. Consider initial data $u_0(x)$, $v_0(x)$ with finite total variation. Let $a = \lim_{x \to \pm\infty} \sigma_0(x)$, $\bar{\eta} = \lim_{x \to \pm\infty} \eta_0(x)$. Choose k sufficiently small so that the conclusions of Theorem 3.3 and Lemma 3.23 hold with constants $c_1, c_2(k), d_1, d_2(k)$. Let $\theta = (c_1 + c_2(k))/\bar{\eta}$.

We shall apply Theorem 2.3 in the variables $(z', w') = T(a, \theta)(z, w)$. Condition A_1 is satisfied since $\lambda_i = \pm\sqrt{g'(v)}$ and since (1.2) holds. By Theorem 3.3, $T(a, \theta)$ maps the shock curves of (1.1) in $W(a, k) \cap \{c_1/\theta < \eta < c_2(k)/\theta\} \equiv H$ onto shock curves which satisfy A_2, A_3, A_4 in the (z', w') variables. Therefore, to apply Theorem 2.3, we need only show that there exists a constant N such that

$$TVz_0 + TVw_0 \leqq N\bar{\eta} \tag{3.26}$$

implies that the Glimm difference approximations $U_h = (u_h(x, t), v_h(x, t))$ with initial data $u_0(x)$, $v_0(x)$ lie in H for all x and t (in the sense that $(z(U_h), w(U_h)) \in H$). Then it follows by [6] that the functional

$$F[U_h](t) = \sum_{1\text{-shocks}} \Delta z_h'(t) + \sum_{2\text{-shocks}} \Delta w_h'(t)$$

decreases as a function of t, a condition which is sufficient for existence of a sequence U_{h_i} converging pointwise almost everywhere to a solution of the Cauchy problem for (1.1).

In order to determine N, we need the inequality

$$\begin{aligned} TV_x z_h'(x, t) + TV_x w_h'(x, t) &\leqq 2F[U_h](t) \\ &\leqq 2TV_x z_h'(x, t) + 2TV_x w_h'(x, t), \end{aligned} \tag{3.27}$$

which follows from the facts that $\Delta w' \leqq \Delta z'$ for shocks for the first kind and $\Delta z' \leqq \Delta w'$ for shocks of the second kind.

Next, let D be one-fourth the minimum distance from $T(a, \theta)\{(a, \bar{\eta})\} \equiv P$ to the boundary of $W'(0, k) \cap \{d_1 < \eta' < d_2(k)\}$ and let S be the sphere of radius D centered at P. Note that P is independent of a, θ and $\bar{\eta}$. If the data satisfies

$$TVz_0' + TVw_0' \leqq D, \tag{3.28}$$

then (z_h, w_h) lies in $T^{-1}(a, \theta)(S) \subset H$ at $t=0$ and, therefore, by (3.27) and the decreasing property of the functional F, for all time t.

From (3.4) it follows that

$$TVz_0' + TVw_0' \leqq \theta M\{TVz_0 + TVw_0\}$$

where $M = 2 \sup_S \{|z' - 1| + |w' + 1|\}$. Hence, the difference approximations lie in H if

$$TVz_0 + TVw_0 \leqq D\bar{\eta}/M(c_1 + c_2(k)).$$

Therefore, we chose $N = D/M(c_1 + c_2(k))$ and the theorem is proved.

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