On the Spatial Analyticity of Solutions of the Navier-Stokes Equations

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Introduction

The regularity of weak solutions of the Navier-Stokes equations has been investigated by a number of writers. In particular, SERRIN [8] showed that under rather moderate assumptions, weak solutions in the case of a conservative external force are C^∞ in the space variables. He further conjectured that these solutions are actually analytic in the space variables and it is the purpose of this paper to demonstrate this.

As a consequence of the analyticity, unique continuation properties for solutions of the Navier-Stokes equations will follow. Recently, DYER & EDMUNDS [2] have obtained such properties directly without recourse to analyticity.

Although the analyticity in space and time does not appear to be valid in general, MASUDA [7] has established it in an interesting special context. He considers solutions of the Navier-Stokes equations which satisfy the condition of adherence on a spatial boundary with the external force analytic. In turn, he obtains a corresponding unique continuation property as a corollary of his result.

Our proof of the spatial analyticity leans heavily on certain representation formulas employed by SERRIN in [8]. We shall make use of these formulas to estimate the successive spatial derivatives of the weak solutions of the Navier-Stokes equations that we will consider. This method of establishing analyticity was developed by GEVREY [5] and has been most recently used to great advantage by FRIEDMAN [4].

The arrangement of the paper is as follows. In Section 1 we state our principal result; its proof is given in Section 4. Sections 2 and 3 contain preliminary material needed for the proof.

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1. The Main Theorem

We will consider solutions of the Navier-Stokes equations

$$v_t - \Delta v + v \cdot \operatorname{grad} v = f - \operatorname{grad} p$$

$$\operatorname{div} v = 0$$

is some open region R of space-time. Here $v=v(x,t)$ denotes the velocity vector and p the pressure. By a weak solution of these equations in R, we will mean a vector v which is weakly divergence-free and which satisfies the condition

$$\iint [(v,\phi_t)+(v,\Delta\phi)+(v, v\cdot \operatorname{grad}\phi)]\,dx\,dt=-\iint (f,\phi)\,dx\,dt$$

for all C^∞, divergence-free vectors ϕ with compact support in R.

For such weak solutions SERRIN has obtained a regularity theorem under the assumption that v and v_x belong to certain Lebesgue spaces. These are the spaces, denoted by $L^{r,s}(R)$, consisting of all functions $g(x,t)$ for which the norm

$$\|g\|_{r,s}=\left(\int_{T_1}^{T_2}\left(\int_G |g(x,t)|^r\,dx\right)^{s/r}dt\right)^{1/s}$$

is finite, $G\times(T_1,T_2)$ being an arbitrary cylinder with compact closure in R. SERRIN's regularity theorem may then be stated as follows:

Theorem 1.1. (SERRIN [8]). *Let v be a weak solution of the Navier-Stokes equations with $v\in L^{2,\infty}(R)$ and $v_x\in L^{2,2}(R)$. Assume that the external force $f\in L^{1,1}(R)$ and is conservative. Suppose further that $v\in L^{r,s}(R)$ where*

$$\frac{n}{r}+\frac{2}{s}<1,$$

n denoting the dimension in the space variables. Then v is of class C^∞ in the space variables, and each derivative is bounded in compact subregions of R.

Our objective will be to establish

Theorem 1.2. *Under the same assumptions as in Theorem 1.1 the solution v is analytic in the space variables.*

We remark that the solutions to the initial value problem constructed by KISELEV and LADYZHENSKAYA [6] for $n=2$ and $n=3$ with f conservative, satisfy the hypotheses of Theorem 1.2; and so it will follow from this theorem that they are spatially analytic.

In the case of a non-conservative external force $f(x,t)$, the conclusion of Theorem 1.2 still applies if we assume that f is spatially analytic in the sense that the k^{th} order derivatives, $\partial_x^k f(x,t)$, of f satisfy estimates of the form

$$|\partial_x^k f(x,t)|\leqq \text{const}\, k^k (\text{const.})^k \qquad (k=0,1,2,\ldots)$$

in any subset having compact closure in R.

In the proof of Theorem 1.1 a key role is played by a pair of representation formulas. These formulas will also serve as the foundation of the proof of Theorem 1.2 and we wish to quote them here.

The first of these is formally a consequence of the vorticity equation. In the three dimensional case, letting $\omega=\operatorname{curl} v$ and considering any cylinder $G\times(T_1,T_2)$ having compact closure in R, G being a domain in E^3, the formula in question is

$$\omega(x,t)=\int_{T_1}^{t}\int_G (\operatorname{grad}_x h(x-y,t-\tau), Q(v(y,\tau),\omega(y,\tau)))\,dy\,d\tau+b(x,t) \tag{1.1}$$

for $(x, t) \in G \times (T_1, T_2)$. Here $h(x, t)$ denotes the heat kernel:

$$h(x,t)=\begin{cases}\text{const. } t^{-\frac{3}{2}} e^{-|x|^2/4t} & t>0\\ 0 & t\leqq 0,\end{cases}$$

$Q(v, \omega)$ is a bilinear vector funtion of v and ω, and $b(x, t)$ is a solution of the heat equation in $G \times (T_1, T_2)$. The common hypotheses of Theorems 1.1 and 1.2 are enough to assure the validity of (1.1).

Together with (1.1) use will also be made of the formula for recovering the vector v from its curl, ω:

$$v(x,t)=\int_G \operatorname{grad}(|x-y|)^{-1}) \times \omega(y,t)\,dy + a(x,t), \tag{1.2}$$

where for fixed $t \in (T_1, T_2)$, $a(x, t)$ is a harmonic function in x for $x \in G$.

In the higher dimensional case formulas similar to (1.1) and (1.2) are given in [8], with ω replaced by an appropriate tensor.

Our point of departure in proving Theorem 1.2 will be the representation formulas (1.1) and (1.2). In fact, we will assume that v and ω are solutions of (1.1) and (1.2) in the cylinder $G \times (T_1, T_2)$, and we will further suppose that they are bounded in this cylinder as is assured by Theorem 1.1. It then follows from (1.2) that $a(x, t)$ is also bounded over this cylinder. Hence the successive derivatives, $\partial_x^k a(x, t)$, of the harmonic function $a(x, t)$ will satisfy estimates of the form

$$|\partial_x^k a(x,t)| \leqq \text{const. } k^k (\text{const.})^k, \tag{1.3}$$

when (x, t) is restricted to a subset with compact closure in $G \times (T_1, T_2)$. Since the function $b(x, t)$ on the right of (1.1) is a solution of the heat equation in $G \times (T_1, T_2)$, its spatial derivatives will also satisfy estimates of the same type.

The main features of the situation just described are contained in the hypotheses of Theorem 1.3 below. Hence we will be able to regard Theorem 1.2 as a consequence of Theorem 1.3.

Theorem 1.3. *Let $u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$ be a bounded solution of*

$$\begin{aligned} u(x,t) = &\int_{T_1}^{t} \int_{|y|<L} \mathscr{A}(x-y, t-\tau) f(u(y,\tau))\,dy\,d\tau \\ &+ \int_{|y|<L} \mathscr{B}(x-y)\, g(u(y,t))\,dy + \varphi(x,t) \end{aligned} \tag{1.4}$$

in the cylinder $[|x|<L] \times (T_1, T_2)$. Here $\mathscr{A}(x, t)$ and $\mathscr{B}(x)$ are $m \times m$ matrices, the entries of $\mathscr{A}(x, t)$ being constant multiplies of first order spatial derivatives of the heat kernel

$$h(x,t)=\begin{cases}(4\pi t)^{-n/2} e^{-|x|^2/4t} & t>0\\ 0 & t\leqq 0,\end{cases}$$

while those of $\mathscr{B}(x)$ are constant multiples of first order derivatives of the potential kernel

$$p(x)=\begin{cases}\text{const. } |x|^{-n+2} & n>2\\ \text{const. } \log|x| & n=2.\end{cases}$$

Finally, f and g denote entire analytic mappings from E^m into E^m.

Assume now that $\varphi(x, t)=(\varphi_1(x, t), \ldots, \varphi_m(x, t))$ *is analytic in* x *in the sense that for any subset* H *with compact closure in* $[|x|<L]\times(T_1, T_2)$, *there exist constants* A *and* a *so that*

$$(1.5) \qquad |\partial_x^k \varphi_j(x,t)| \leqq A\, k^k a^k \qquad (j=1,2,\ldots,m)$$

for $(x, t)\in H$, $k=0, 1, 2, \ldots$. *Then* $u(x, t)$ *is analytic in* x, *in the same sense.*

2. Preliminaries on Analytic Functions

In this section we will describe the class of real analytic functions that we will be dealing with.

It will be convenient to introduce the following norm and semi-norm for functions $\varphi(x)$ defined in the ball $|x|<r$:

$$|\varphi(x)|_r = \sup_{|x|<r} |\varphi(x)|$$

and

$$|\varphi(x)|_r^{(\mu)} = \sup_{|x|<r,\, |\bar{x}|<r} \left(\frac{|\varphi(x)-\varphi(\bar{x})|}{|x-\bar{x}|^{\mu}}\right) \qquad (0<\mu<1).$$

We note the following self-evident properties:

$$(2.1) \qquad |\varphi(x)\psi(x)|_r \leqq |\varphi(x)|_r\, |\psi(x)|_r$$

and

$$(2.2) \qquad |\varphi(x)\psi(x)|_r^{(\mu)} \leqq |\varphi(x)|_r\, |\psi(x)|_r^{(\mu)} + |\varphi(x)|_r^{(\mu)} |\psi(x)|_r\,.$$

Let α denote the multi-index $(\alpha_1, \ldots, \alpha_n)$. We shall use the notations

$$D_x^{\alpha}\varphi(x) = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\right)\varphi(x),$$

$$\alpha! = \alpha_1! \ldots \alpha_n!, \qquad |\alpha| = |\alpha_1| + \ldots + |\alpha_n|$$

and

$$x^{\alpha} = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

We will be concerned with the class of real analytic functions $\varphi(x)$ defined in $|x|<r$ whose derivatives satisfy inequalities of the form

$$(2.3) \qquad \begin{aligned} &|D_x^{\alpha}\varphi(x)|_r \leqq A|\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|} \\ &|D_x^{\alpha}\varphi(x)|_r^{(\mu)} \leqq A|\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1}. \end{aligned}$$

Here, when $\alpha=0$, $|\alpha|^{|\alpha|-\delta}$ and $|\alpha|^{|\alpha|+\nu}$ are to be interpreted as 1. In fact, in this section it will be convenient to adopt the convention: $k^p=1$ when $k=0$.

With δ and ν suitably restricted, the class of functions defined by (2.3) has a certain closure property under multiplication which is stated in Corollary 2.1 below. To establish this property we require the following.

Lemma 2.1. *Let* $\{s_\alpha\}$ *and* $\{t_\alpha\}$ *be multi-sequences and suppose that*

$$|s_\alpha| \leqq S|\alpha|^{|\alpha|+\delta} \quad \textit{and} \quad |t_\alpha| \leqq T|\alpha|^{|\alpha|+\varepsilon}.$$

Then if either δ or $\varepsilon < -\frac{1}{2}$

$$\left|\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} s_\beta t_\gamma\right| \leqq \lambda ST |\alpha|^{|\alpha|+\max(\delta,\varepsilon)},$$

where λ depends only on δ and ε.

Proof. We may assume $S=T=1$. Applying Stirling's formula

$$a^{-1} k!\, k^{-\frac{1}{2}} e^k \leqq k^k \leqq a\, k!\, k^{-\frac{1}{2}} e^k,$$

we find that

$$\begin{aligned}
\left|\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} s_\beta t_\gamma\right| &\leqq \sum_{b+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} |\beta|^{|\beta|+\delta} |\gamma|^{|\gamma|+\varepsilon} \\
&\leqq a^2 e^{|\alpha|} \alpha! \sum_{\beta+\gamma=\alpha} \frac{|\beta|!\,|\gamma|!}{\beta!\,\gamma!} |\beta|^{-\frac{1}{2}+\delta} |\gamma|^{-\frac{1}{2}+\varepsilon} \\
&= a^2 e^{|\alpha|} \alpha! \sum_{j=0}^{|\alpha|} j^{-\frac{1}{2}+\delta} (|\alpha|-j)^{-\frac{1}{2}+\varepsilon} \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|=j}} \frac{j!(|\alpha|-j)!}{\beta!\,\gamma!}.
\end{aligned}$$

The last sum on the right can be evaluated by making use of the multinormal theorem to expand each power in the identity

$$(x_1+\cdots+x_n)^j (x_1+\cdots+x_n)^{|\alpha|-j} = (x_1+\cdots+x_n)^{|\alpha|},$$

which yields

$$\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|=j}} \frac{j!(|\alpha|-j)!}{\beta!\,\gamma!} = \frac{|\alpha|!}{\alpha!}.$$

Hence, applying Stirling's formula once more, we obtain

$$\begin{aligned}
&\left|\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} s_\beta t_\gamma\right| \\
&\leqq a^3 |\alpha|^{|\alpha|+\frac{1}{2}} \sum_{j=0}^{|\alpha|} j^{-\frac{1}{2}+\delta} (|\alpha|-j)^{-\frac{1}{2}+\varepsilon} \\
&\leqq a^3 |\alpha|^{|\alpha|+\frac{1}{2}} \left(\sum_{j=0}^{|\alpha|/2} j^{-\frac{1}{2}+\delta} (|\alpha|-j)^{-\frac{1}{2}+\varepsilon} + \sum_{k=0}^{|\alpha|/2} (|\alpha|-k)^{-\frac{1}{2}+\delta} k^{-\frac{1}{2}+\varepsilon} \right) \\
&\leqq a^3 |\alpha^{|\alpha|+\frac{1}{2}} \left(\text{const.}\, |\alpha|^{-\frac{1}{2}+\varepsilon} \sum_{j=0}^{|\alpha|/2} j^{-\frac{1}{2}+\delta} + \text{const.}\, |\alpha|^{-\frac{1}{2}+\delta} \sum_{k=0}^{|\alpha|/2} k^{-\frac{1}{2}+\varepsilon} \right) \\
&= \text{const.}\, |\alpha|^{|\alpha|+\varepsilon} \sum_{j=0}^{|\alpha|/2} j^{-\frac{1}{2}+\delta} + \text{const.}\, |\alpha|^{|\alpha|+\delta} \sum_{k=0}^{|\alpha|/2} k^{-\frac{1}{2}+\varepsilon}.
\end{aligned}$$

The desired result then follows by separately considering the cases (*a*) $\varepsilon < -\frac{1}{2}$, $\delta < -\frac{1}{2}$, (*b*) $\varepsilon > -\frac{1}{2} > \delta$ and (*c*) $\varepsilon = -\frac{1}{2} > \delta$ with the aid of the elementary estimates

$$\sum_{l=1}^{m} l^p \leqq \begin{cases} \text{const.}\, m^{p+1} & \text{if } p > -1 \\ \text{const.}\, \log m & \text{if } p = -1 \\ \sum_{l=1}^{\infty} l^p = \text{const.} & \text{if } p < -1. \end{cases}$$

From Leibnitz's formula

$$D_x^\alpha[\varphi(x)\,\psi(x)] = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} D_x^\beta \varphi(x)\, D_x^\gamma \psi(x)$$

together with (2.1) and (2.2) we immediately obtain

Corollary 2.1. *Suppose that*

$$|D_x^\alpha \varphi(x)|_r \leqq A\,|\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|}$$
$$|D_x^\alpha \varphi(x)|_r^{(\mu)} \leqq A\,|\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1}$$

and

$$|D_x^\alpha \psi(x)|_r \leqq B\,|\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|}$$
$$|D_x^\alpha \psi(x)|_r^{(\mu)} \leqq B\,|\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1}$$

with $\delta > \frac{1}{2}$ and $\nu > -\frac{1}{2}$. Then

$$\left|D_x^\alpha(\varphi(x)\,\psi(x))\right|_r \leqq \kappa\, AB\,|\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|}$$
$$\left|D_x^\alpha(\varphi(x)\,\psi(x))\right|_r^{(\mu)} \leqq \kappa\, AB\,|\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1}$$

where the constant κ depends only on δ and ν.

In the next corollary we want to consider vector-valued functions. If $v(x) = (v_1(x), \ldots, v_k(x))$, we will use the notations

$$|v(x)|_r = \max_{1 \leqq j \leqq k} |v_j(x)|_r$$

and

$$|v(x)|_r^{(\mu)} = \max_{1 \leqq j \leqq k} |v_j(x)|_r^{(\mu)}.$$

Corollary 2.2. *Assume $u(x)$ is a vector-valued function taking values is E^m whose derivatives satisfy*

$$|D_x^\alpha u(x)|_r \leqq A\,|\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|}$$
$$|D_x^\alpha u(x)|_r^{(\mu)} \leqq A\,|\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1} \tag{2.4}$$

with

$$\delta > \tfrac{1}{2} \quad and \quad \nu > -\tfrac{1}{2}.$$

Suppose further that $f(w)$ is an entire analytic mapping from E^m into E^l. Then

$$\left|D_x^\alpha f(u(x))\right|_r \leqq \gamma\,|\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|}$$
$$\left|D_x^\alpha f(u(x))\right|_r^{(\mu)} \leqq \gamma\,|\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1}, \tag{2.5}$$

where γ depends on f, δ, ν and A but is independent of α, ρ and σ.

Proof. Without loss of generality we may assume that $f(w)$ is a single, entire analytic function with the power series expansion

$$f(w) = \sum a_\beta\, w^\beta.$$

Repeated application of Corollary 2.1 yields

$$|D_x^\alpha u^\beta(x)|_r \leqq \kappa^{|\beta|-1} A^{|\beta|} |\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|}$$

$$|D_x^\alpha u^\beta(x)|_r^{(\mu)} \leqq \kappa^{|\beta|-1} A^{|\beta|} |\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1}$$

for $\beta > 0$. Hence

$$\left|D_x^\alpha f(u(x))\right|_r \leqq \gamma |\alpha|^{|\alpha|-\delta} \rho^{|\alpha|} \sigma^{|\alpha|}$$

$$\left|D_x^\alpha f(u(x))\right|_r^{(\mu)} \leqq \gamma |\alpha|^{|\alpha|+\nu} \rho^{|\alpha|} \sigma^{|\alpha|+1}$$

where

$$\gamma = a_0 + \sum_{\beta>0} a_\beta \kappa^{|\beta|-1} A^{|\beta|}.$$

Remark. For the application we have in mind, the estimates (2.4) will be known to hold only for the derivatives of u of order $\leqq p$. It is clear that the conclusion (2.5) will still be valid for the derivatives of $f(u)$ of order $\leqq p$.

3. Estimates for Potentials

In this section we will derive some estimates for potentials which will be used in the proof of our main result.

Let $h(x, t)$ and $p(x)$ denote the heat and potential kernels respectively:

$$h(x,t) = \begin{cases} (4\pi t)^{-n/2} e^{-|x|^2/4t} & \text{for } t>0 \\ 0 & \text{for } t \leqq 0 \end{cases}$$

and

$$p(x) = \begin{cases} \text{const.}\, |x|^{-n+2} & \text{for } n>2 \\ \text{const.} \log|x| & \text{for } n=2. \end{cases}$$

We will require estimates for higher order derivatives of these kernels. In the case of the heat kernel, these will be based on some known properties of the Hermite polynomials, $H_j(s)$, $s \in (-\infty, +\infty)$, $j = 0, 1, 2, \ldots$. They are defined through the orthogonality conditions

$$\int_{-\infty}^{+\infty} H_j(s) H_k(s) e^{-s^2} ds = \begin{cases} \sqrt{\pi}\, 2^j j! & \text{for } j=k \\ 0 & \text{for } j \neq k, \end{cases}$$

and we quote from [9] the following properties

$$\left(\frac{d}{ds}\right)^j e^{-s^2} = (-1)^{j+1} H_j(s) e^{-s^2} \tag{3.1}$$

and

$$|H_j(s)| \leqq \text{const.}\, 2^{j/2} (j!)^{1/2} e^{+s^2/2}. \tag{3.2}$$

In what follows b will denote a positive constant, although not necessarily the same one each time it occurs.

If we differentiate the heat kernel repreatedly we obtain, for $t > 0$

$$D_x^\alpha h(x,t) = \text{const.}\, (4t)^{-(|\alpha|+n)/2} e^{-|x|^2/4t} Q_\alpha\left((4t)^{-\frac{1}{2}} x\right),$$

where by (3.1)

$$Q_\alpha(y)=(-1)^{|\alpha|+n} H_{\alpha_1}(y_1)\, H_{\alpha_2}(y_2)\dots H_{\alpha_n}(y_n).$$

In view of (3.2)

$$|Q_\alpha(y)|\leqq \text{const. } b^{|\alpha|}\, |\alpha|^{|\alpha|/2}\, e^{|y|^2/2}.$$

This implies the following pair of estimates for $D_x^\alpha h(x,t)$:

$$|D_x^\alpha h(x,t)|\leqq \text{const. } b^{|\alpha|}\, |\alpha|^{|\alpha|/2} (4t)^{-(|\alpha|+n)/2}\, e^{-\frac{1}{2}(|x|^2/4t)} \tag{3.3}$$

and

$$|D_x^\alpha h(x,t)|\leqq \text{const. } \frac{b^{|\alpha|}\, |\alpha|^{|\alpha|}}{|x|^{n-2\delta+|\alpha|}\, t^\delta} \qquad (0<\delta<1), \tag{3.4}$$

for $t>0$, with the constant in the last estimate depending on δ.

From this point on we will use the notation $\partial_x^k v(x)$ to denote a derivative of $v(x)$ of order k.

Lemma 3.1. *Assuming f to be bounded and $S\geqq 0$, we set*

$$\varphi(x,t)=\int_S^t \int_{A<|y|<B} \partial_k^{k+2} h(x-y,t-\tau)\, f(y,\tau)\, dy\, d\tau \tag{3.5a}$$

and

$$\psi(x,t)=\int_S^t \int_{|y|=A} \partial_x^{k+1}\, h(x-y,t-\tau)\, f(y,\tau)\, \eta\, d\sigma(y)\, d\tau$$

where $\eta=\eta(y)$ is a direction cosine and $d\sigma(y)$ denotes the element of area on the sphere $|y|=A$.

Then for $a<A$ and any $\varepsilon>0$,

$$|\varphi(x,t)|_a\leqq \text{const. } \frac{b^k\, k^k}{(A-a)^{k+\varepsilon}} \Big(\sup_{\substack{A<|y|<B\\ S<\tau<t}} |f(y,\tau)|\Big), \tag{3.5}$$

and

$$|\varphi(x,t)|_a^{(\mu)}\leqq \text{const. } \frac{b^k\, k^k}{(A-a)^{k+1+\varepsilon}} \Big(\sup_{\substack{A<|y|<B\\ S<\tau<t}} |f(y,\tau)|\Big) \tag{3.6}$$

for $k=0, 1, 2, \dots$, with the constants depending on ε and upper bounds for B and t. Similarly,

$$|\psi(x,t)|_a\leqq \text{const. } \frac{b^k\, k^k}{(A-a)^{k+\varepsilon}} \Big(\sup_{\substack{|y|=A\\ S<\tau<t}} |f(y,\tau)|\Big) \tag{3.7}$$

and

$$|\psi(x,t)|_a^{(\mu)}\leqq \text{const. } \frac{b^k\, k^k}{(A-a)^{k+1+\varepsilon}} \Big(\sup_{\substack{|y|=A\\ S<\tau<t}} |f(y,\tau)|\Big) \tag{3.8}$$

for $k=0, 1, 2, \dots$, with the constants depending on ε and upper bounds for A and t.

Proof. We will only prove (3.5) and (3.6), the proof of (3.7) and (3.8) being quite similar.

Applying (3.4), we obtain

$$|\varphi(x,t)| \leqq \text{const.}\, b^k k^k \Big(\sup_{\substack{A<|y|<B\\ S<\tau<t}} |f(y,\tau)|\Big) \int_{A<|y|<B} \frac{dy}{|x-y|^{n-2\delta+k+2}} \int_S^t \frac{d\tau}{(t-\tau)^\delta}$$

with $\delta \in (0,1)$ at our disposal. For $|x|<a$ the first integral on the right is estimated as follows:

$$\int_{A<|y|<B} \frac{dy}{|x-y|^{n-2\delta+k+2}} \leqq \frac{1}{(A-a)^{k+\varepsilon}} \int_{A<|y|<B} \frac{dy}{|x-y|^{n-2\left(\frac{\varepsilon}{2}+\delta-1\right)}}$$

where δ is so chosen that $2\left(\frac{\varepsilon}{2}+\delta-1\right)>0$, thus ensuring the existence of the last integral. The desired result (3.5) then follows immediately.

The second estimate (3.6) is easily seen to be a consequence of (3.5) by noting that

$$|\varphi(x,t)|_a^{(\mu)} \leqq \text{const.}\, |\text{grad}_x\, \varphi(x,t)|_a$$

and that each component of $\text{grad}_x\, \varphi(x,t)$ is of the form (3.5a) with k replaced by $k+1$.

Lemma 3.2. *Suppose that f is bounded and that $P>Q\geqq 0$. Set*

$$\varphi(x,t)=\int_Q^P \int_{|y|<B} \partial_x^{k+2} h(x-y,t-\tau) f(y,\tau)\, dy\, d\tau.$$

Then for $t>P$

$$\text{(3.9)} \qquad |\varphi(x,t)|_a \leqq \text{const.}\, \frac{b^k k^{k/2}}{(t-P)^{k/2}} \Big(\sup_{Q<\tau<P} |f(y,\tau)|_B\Big) \qquad (k\geqq 1),$$

and

$$\text{(3.10)} \qquad |\varphi(x,t)|_a^{(\mu)} \leqq \text{const.}\, \frac{b^k k^{k/2}}{(t-P)^{(k+1)/2}} \Big(\sup_{Q<\tau<P} |f(y,\tau)|_B\Big) \qquad (k\geqq 0).$$

Remark. When $k=0$, (3.9) is not valid. It is replaced in this case by

$$\text{(3.11)} \qquad |\varphi(x,t)|_a \leqq (\text{const.}+\text{const.}\,|\log(t-P)|) \Big(\sup_{Q<\tau<P} |f(y,\tau)|_B\Big)$$

for $t>P$, where the first constant depends on P, Q and an upper bound for t.

Proof of Lemma 3.2. Application of (3.3) yields

$$|\varphi(x,t)|_a \leqq \text{const.}\, b^k k^{k/2} \Big(\sup_{Q<\tau<P} |f(y,\tau)|_B\Big) \int_Q^P (t-\tau)^{-(k+2)/2}\, d\tau \cdot \int_{E^n} e^{-\frac{1}{2}(|x-y|^2/4(t-\tau))} [4(t-\tau)]^{-n/2}\, dy.$$

By a simple change of variable the last integral is easily seen to be a constant. Hence from

$$\int_Q^P (t-\tau)^{-(k+2)/2}\, d\tau \leqq \text{const.}\, (t-P)^{-k/2} \qquad (k\geqq 1)$$

for $t>P$, we obtain (3.9).

The estimate (3.10) is derived from (3.9) by exactly the same argument as in the proof of Lemma 3.1.

The estimates which appear in the next lemma are the analogues of those in Lemma 3.1 for integrals involving the potential kernel. Their proofs are based on the estimate

$$|\partial_x^k p(x-y)| \leqq \text{const.} \frac{b^k k^k}{|x-y|^{n-2+k}} \quad (k \geqq 0),$$

which is easily established inductively and parallel the proofs in Lemma 3.1.

Lemma 3.3. *For fixed t, assume $f(y,t)$ to be bounded as a function of y. Set*

$$\varphi(x,t) = \int_{A<|y|<B} \partial_x^{k+2} p(x-y) f(y,t)\, dy$$

and

$$\psi(x,t) = \int_{|y|=A} \partial_x^{k+1} p(x-y) f(y,t) \eta\, d\sigma(y)$$

where $\eta = \eta(y)$ is a direction cosine. Then for $a<A$ and any $\varepsilon>0$

$$|\varphi(x,t)|_a \leqq \text{const.} \frac{b^k k^k}{(A-a)^{k+\varepsilon}} \big(\sup_{A<|y|<B} |f(y,t)|\big), \tag{3.12}$$

and

$$|\varphi(x,t)|_a^{(\mu)} \leqq \text{const.} \frac{b^k k^k}{(A-a)^{k+1+\varepsilon}} \big(\sup_{A<|y|<B} |f(y,t)|\big) \tag{3.13}$$

for $k=0,1,2,\dots$, with the constants depending on ε and an upper bound for B. Similarly,

$$|\psi(x,t)|_a \leqq \text{const.} \frac{b^k k^k}{(A-a)^{k+\varepsilon}} \big(\sup_{|y|=A} |f(y,t)|\big), \tag{3.14}$$

and

$$|\psi(x,t)|_a^{(\mu)} \leqq \text{const.} \frac{b^k k^k}{(A-a)^{k+1+\varepsilon}} \big(\sup_{|y|=A} |f(y,t)|\big) \tag{3.15}$$

for $k=0,1,2,\dots$, with the constants depending on ε and an upper bound for A.

The principal estimates that we will need concern the preservation of Hölder continuity under the application of singular integral operators whose kernels are either second spatial derivatives of the heat kernel or second derivatives of the potential kernel. In the case of the heat kernel our estimates will be a consequence of the following.

Theorem 3.1. *Assume $u(x,t)$ is a bounded measurable function of (x,t) in $E^n \times (S,T)$, $S \geqq 0$. Suppose further that $u(x,t)$ has compact support in x which is independent of t, and that*

$$|u(x,t) - u(\bar{x},t)| \leqq A|x-\bar{x}|^\mu$$

for $x \in E^n$, $\bar{x} \in E^n$ and $S<t<T$. Then if

$$v(x,t) = \int_S^t \int_{E^n} \partial_x^2 h(x-y, t-\tau) u(y,\tau)\, dy\, d\tau$$

we have

$$|v(x,t)-v(\bar{x},t)| \leqq \text{const.}\, A|x-\bar{x}|^{\mu} \tag{3.16}$$

for $x \in E^n$, $\bar{x} \in E^n$ and $S<t<T$, with the constant independent of t. Moreover,

$$\sup_{\substack{x\in E^n\\ S<t<T}} |v(x,t)| \leqq \text{const.}\, A \tag{3.17}$$

where the constant depends on T.

Proof. The proof of (3.16) can be patterned on the proof of a closely related result of FABES [3] (see pp. 109–111), and we shall not include the details.

To prove (3.17) we use the fact that

$$\int_{E^n} \partial_y^2 h(y,\tau)\, dy = 0 \qquad (\tau>0).$$

Hence

$$v(x,t) = \int_0^{t-S} \int_{E^n} \partial_y^2 h(y,\tau)[u(x-y,t-\tau)-u(x,t-\tau)]\, dy\, d\tau$$

so that by the assumed Hölder continuity of u

$$|v(x,t)| \leqq A \int_0^T \int_{E^n} |\partial_y^2 h(y,\tau)|\, |y|^{\mu}\, dy\, d\tau.$$

Inserting the estimate (3.3) for $\partial_y^2 h(y,\tau)$ into the integrand, the desired result then follows after a brief calculation.

Lemma 3.4. *Suppose that for each fixed $\tau \in (S,T)$, $S \geqq 0$, the function $f(y,\tau)$ is Hölder continuous in y for $|y|<A$ with exponent μ. Assume, further, that*

$$\sup_{S<\tau<T} |f(y,\tau)|_A < \infty \quad \textit{and} \quad \sup_{S<\tau<T} |f(y,\tau)|_A^{(\mu)} < \infty.$$

Then for each $t \in (S,T)$ the function

$$g(x,t) = \int_S^t \int_{|y|<A} \partial_x^2 h(x-y,t-\tau) f(y,\tau)\, dy\, d\tau$$

is Hölder continuous in x for $|x|<a<A$ with the same exponent. Moreover, for any $\lambda \in (0,2)$

$$\left.\begin{matrix} |g(x,t)|_a^{(\mu)} \\ |g(x,t)|_a \end{matrix}\right\} \leqq \text{const.} \left[\sup_{S<\tau<T} |f(y,\tau)|_A^{(\mu)} + \frac{1}{(A-a)^{\mu+\lambda}} \sup_{S<\tau<T} |f(y,\tau)|_A\right] \tag{3.18}$$

with the constant depending on λ, A and T.

Proof. We shall first construct a special cut-off function $\varphi(y)$ as follows: let $\zeta(s)$, $s \in (-\infty, +\infty)$, be a fixed C^{μ} function, say,

$$|\zeta(s)-\zeta(\bar{s})| \leqq M\,|s-\bar{s}|^{\mu}$$

with $\varsigma(s)\equiv 1$ for $s\leqq 0$, $\varsigma(s)\equiv 0$ for $s\geqq 1$ and $0\leqq\varsigma(s)\leqq 1$ for $0<s<1$. Now for $a<A$, let $\delta=A-a$ and set

$$\varphi(y)=\zeta\left(\frac{|y|-(a+\delta/3)}{\delta/3}\right).$$

Then $\varphi(y)\equiv 1$ for $|y|<a+\delta/3$, $\varphi(y)\equiv 0$ for $|y|>a+2\delta/3$, $0\leqq\varphi(y)\leqq 1$ for $a+\delta/3<|y|<a+2\delta/3$ and

$$|\varphi(y)-\varphi(\bar{y})|\leqq\frac{M\,3^{\mu}}{\delta^{\mu}}|y-\bar{y}|^{\mu}. \tag{3.19}$$

With the aid of φ we decompose f into a sum of two function by writing

$$f=\varphi f+(1-\varphi)f=f_1+f_2.$$

This leads to a corresponding decomposition for g:

$$g=g_1+g_2 \tag{3.20}$$

where

$$g_i(x,t)=\int_S^t\int_{|y|<A}\partial_x^2 h(x-y,t-\tau)f_i(y,\tau)\,dy\,d\tau,\qquad i=1,2.$$

Since $f_1(y,\tau)$ has compact support contained in $|y|<A$ we may apply (3.16) of Theorem 3.1 in conjunction with (2.2) and (3.19) to obtain

$$\begin{aligned}|g_1(x,t)|_a^{(\mu)}&\leqq\text{const.}\sup_{S<\tau<T}|\varphi(y)f(y,\tau)|_A^{(\mu)}\\&\leqq\text{const.}\left(\sup_{S<\tau<T}|f(y,\tau)|_A^{(\mu)}+\frac{1}{(A-a)^{\mu}}\sup_{S<\tau<T}|f(y,\tau)|_A\right).\end{aligned}\tag{3.21}$$

To estimate the Hölder constant for $g_2(x,t)$ we make use of the inequality

$$|\partial_x^2 h(x,t)-\partial_x^2 h(\bar{x},t)|\leqq\text{const.}\,|x-\bar{x}|^{\mu}\left(\frac{1}{|x|^{n+2(1-\gamma)+\mu}}+\frac{1}{|\bar{x}|^{n+2(1-\gamma)+\mu}}\right)\frac{1}{t^{\gamma}}$$

for $t>0$, $x\neq 0$, $\bar{x}\neq 0$, where $\gamma\in(0,1)$ with the constant depending on γ. The inequality is established by a suitable application of the mean value theorem together with the estimate (3.4).

Since $f_2(y,\tau)$ vanishes for $|y|<a+\delta/3$, the above inequality yields, for $|x|<a$ and $|\bar{x}|<a$,

$$\begin{aligned}&|g_2(x,t)-g_2(\bar{x},t)|\\&\quad=\left|\int_S^t\int_{a+\delta/3<|y|<A}[\partial_x^2 h(x-y,t-\tau)-\partial_x^2 h(\bar{x}-y,t-\tau)]f_2(y,\tau)\,dy\,d\tau\right|\\&\quad\leqq\text{const.}\,|x-\bar{x}|^{\mu}\sup_{S<\tau<T}\left|(1-\varphi(y))f(y,\tau)\right|_A\\&\qquad\cdot\left[\int_{|y-x|>\delta/3}\frac{dy}{|x-y|^{n+2(1-\gamma)+\mu}}+\int_{|y-\bar{x}|>\delta/3}\frac{dy}{|\bar{x}-y|^{n+2(1-\gamma)+\mu}}\right]\int_S^t\frac{d\tau}{(t-\tau)^{\gamma}}\\&\quad\leqq\text{const.}\,|x-\bar{x}|^{\mu}\frac{1}{\delta^{\mu+2(1-\gamma)}}\sup_{S<\tau<T}|f(y,\tau)|_A.\end{aligned}$$

Hence

$$|g_2(x,t)|_a^{(\mu)} \leqq \text{const.}\ \frac{1}{(A-a)^{\mu+\lambda}} \sup_{S<\tau<T} |f(y,\tau)|_A \tag{3.22}$$

where $\lambda = 2(1-\gamma)$.

Combining (3.20), (3.21) and (3.22) we obtain (3.18) for $|g(x,t)_a^{(\mu)}$. The proof of (3.18) for $|g(x,t)|_a$ runs along the same lines; accordingly the details are omitted.

Finally we need estimates of the same type as in the last lemma for singular integral operators whose kernels are second derivatives of the potential kernel. It is well-known that such operators map Hölder continuous functions with compact support into Hölder continuous functions of the same order (see e.g. [1]). With this as our basis we can establish the estimates stated in the lemma below, by paralleling the steps used in proving Lemma 3.4.

Lemma 3.5. *Suppose that for a fixed t, $f(y,t)$ is Hölder continuous in y for $|y|<A$ with exponent μ. Assume, further, that*

$$|f(y,t)|_A^{(\mu)} < \infty \quad \textit{and} \quad |f(y,t)|_A < \infty .$$

Then the function

$$g(x,t) = \int_{|y|<A} \partial_x^2 p(x-y) f(y,t)\, dy$$

is Hölder continuous in x for $|x|<a<A$ with the same exponent. Furthermore

$$\left.\begin{matrix} |g(x,t)|_a^{(\mu)} \\ |g(x,t)|_a \end{matrix}\right\} \leqq \text{const.} \left[|f(y,t)|_A^{(\mu)} + \frac{1}{(A-a)^{\mu}} |f(y,t)|_A \right], \tag{3.23}$$

with the constant depending on A.

4. Proof of the Main Theorem

In order to prove the desired analyticity of a bounded solution $u(x,t)$ of

$$\begin{aligned} u(x,t) = &\int_{T_1}^{t} \int_{|y|<L} \mathscr{A}(x-y, t-\tau) f(u(y,\tau))\, dy\, d\tau \\ &+ \int_{|y|<L} \mathscr{B}(x-y)\, g(u(y,t))\, dy + \varphi(x,t) \end{aligned} \tag{4.1}$$

in the cylinder $[|x|<L] \times (T_1, T_2)$, it will be sufficient to prove the analyticity in x over any sub-cylinder $[|x|<R] \times (S,T)$ where $T_1<S<T<T_2$ and $R<L$. Accordingly, we replace (4.1) which u satisfies in the given cylinder, by the equation

$$\begin{aligned} u(x,t) = &\int_{S}^{t} \int_{|y|<R} \mathscr{A}(x-y, t-\tau) f(u(y,\tau))\, dy\, d\tau \\ &+ \int_{|y|<R} \mathscr{B}(x-y)\, g(u(y,t))\, dy + \psi(x,t) \end{aligned} \tag{4.2}$$

which u satisfies over the sub-cylinder; the function $\psi(x,t)$ being defined through

$$\psi(x,t) = \varphi(x,t) + \theta(x,t),$$

where

$$\theta(x,t)=\int_S^t \int_{R<|y|<L} \mathscr{A}(x-y,t-\tau)f(u(y,\tau))\,dy\,d\tau$$
$$+\int_{T_1}^S \int_{|y|<L} \mathscr{A}(x-y,t-\tau)f(u(y,\tau))\,dy\,d\tau+\int_{R<|y|<L} \mathscr{B}(x-y)\,g(u(y,t))\,dy.$$

From the estimates (3.5), (3.9), (3.11) and (3.12) and the analyticity hypothesis (1.5) on $\varphi(x,t)$, it follows that

$$|\partial_x^k \psi(x,t)|_r \leqq A\,\frac{k^k a^k}{(R-r)^k (t-S)^{k/2}}$$

for $0<r<R$, $S<t<T$ and $k=0,1,2,\ldots$, A and a being suitable constants. We may, therefore, view Theorem 1.3 as a corollary result of the following theorem. (For convenience, we assume that $S=0$ in (4.2).)

Theorem 4.1. *Let $u(x,t)$ be a bounded solution of*

$$u(x,t)=\int_0^t \int_{|y|<R} \mathscr{A}(x-y,t-\tau)f(u(y,\tau))\,dy\,d\tau + \int_{|y|<R} \mathscr{B}(x-y)\,g(u(y,t))\,dy+\psi(x,t) \tag{4.3}$$

in the cylinder $[|x|<R]\times(0,T)$, with $\mathscr{A}$, $\mathscr{B}$, f and g having the same description as in the statement of Theorem 1.3. Assume that

$$|\partial_x^k \psi(x,t)|_r \leqq A\,\frac{k^{k-\delta} a^k}{(R-r)^k (t^{\frac12})^k}, \qquad |\partial_x^k \psi(x,t)|_r^{(\mu)} \leqq A\,\frac{k^{k+\nu} a^k}{(R-r)^{k+1} (t^{\frac12})^{k+1}} \tag{4.4}$$

or $0<r<R$, $0<t<T$, $k=0,1,2,\ldots$, with

$$\tfrac12<\delta<1 \quad and \quad 0<\nu\leqq 1-\delta. \tag{4.5}$$

Then there exist constants M and c such that

$$|\partial_x^k u(x,t)|_r \leqq M\,\frac{k^{k-\delta} c^k}{(R-r)^k (t^{\frac12})^k}, \qquad |\partial_x^k u(x,t)|_r^{(\mu)} \leqq M\,\frac{k^{k+\nu} c^k}{(R-r)^{k+1} (t^{\frac12})^{k+1}} \tag{4.6}$$

for $0<r<R$, $0<t<T$, $k=0,1,2,\ldots$.

I. Our first step in proving Theorem 4.1 will be to develop a formula for the spatial derivatives of u of order $m+1$ in terms of lower order spatial derivatives of u.

Assume that $u(x, t)$, the solution of (4.3), has spatial derivatives of order $\leqq m$ which are bounded in any set with compact closure in the cylinder $[|x|<R]\times(0,T)$. Suppose further that the m^{th} order derivatives are Hölder continuous in the space variables with a uniform Hölder constant for any set with compact closure in $[|x|<R]\times(0,T)$. Then the spatial derivatives order $m+1$, $\partial_x^{m+1}u(x,t)$, exist in this cylinder and they may be obtained in the following manner. Let

$$|x|<R_m<R_{m-1}<\cdots<R_1<R_0=R \tag{4.7}$$

and

$$t>T_m>T_{m-1}>\cdots>T_1>T_0=0; \tag{4.8}$$

set

$$v(x,t)=f\big(u(x,t)\big) \quad \text{and} \quad w(x,t)=g\big(u(x,t)\big).$$

The derivative $\partial_x^{m+1}u(x,t)$ is then given by the following formula which is established inductively through integration by parts.

$$\partial_x^{m+1}u(x,t)=\sum_{j=1}^{m}(I_j+\bar{I}_j-J_j-\bar{J}_j+E_j)+L+\bar{L}+\partial_x^{m+1}\psi(x,t), \tag{4.9}$$

where

$$I_j(x,t)=\int_{T_{m-j+1}}^{t}\int_{R_{m-j+1}<|y|<R_{m-j}}\partial_x^{j+1}\mathscr{A}(x-y,t-\tau)\,\partial_y^{m-j}v(y,\tau)\,dy\,d\tau \tag{4.10}$$

$$\bar{I}_j(x,t)=\int_{R_{m-j+1}<|y|<R_{m-j}}\partial_x^{j+1}\mathscr{B}(x-y)\,\partial_y^{m-j}w(y,t)\,dy \tag{4.11}$$

$$J_j(x,t)=\int_{T_{m-j+1}}^{t}\int_{|y|=R_{m-j+1}}\partial_x^{j}\mathscr{A}(x-y,t-\tau)\,\partial_y^{m-j}v(y,\tau)\,\eta\,d\sigma(y)\,d\tau \tag{4.12}$$

$$\bar{J}_j(x,t)=\int_{|y|=R_{m-j+1}}\partial_x^{j}\mathscr{B}(x-y)\,\partial_y^{m-j}w(y,t)\,\eta\,d\sigma(y) \tag{4.13}$$

$$E_j(x,t)=\int_{T_{m-j}}^{T_{m-j+1}}\int_{|y|<R_{m-j}}\partial_x^{j+1}\mathscr{A}(x-y,t-\tau)\,\partial_y^{m-j}v(y,\tau)\,dy\,d\tau \tag{4.14}$$

$$L(x,t)=\int_{T_m}^{t}\int_{|y|<R_m}\partial_x\mathscr{A}(x-y,t-\tau)\,\partial_y^{m}v(y,\tau)\,dy\,d\tau \tag{4.15}$$

$$\bar{L}(x,t)=\int_{|y|<R_m}\partial_x\mathscr{B}(x-y)\,\partial_y^{m}w(y,t)\,dy. \tag{4.16}$$

In (4.12) and (4.13) $\eta=\eta(y)$ denotes appropriate direction cosines and $d\sigma(y)$ signifies the element of area on the sphere $|y|=R_{m-j+1}$.

II. We now enter into the proof of the existence of M and c for which the estimates (4.6) hold. We choose M so that (4.6) holds for $k=0$. The hypothesis that u is bounded allows us to do this.

Suppose now that for suitable large c (4.6) has been established for $k\leqq m$. We will show that (4.6) also holds for $k=m+1$ by estimating $\partial_x^{m+1}u(x,t)$ through (4.9).

In fact, we will prove that

$$\sum_{j=1}^{m}|I_j|_r+\sum_{j=1}^{m}|\bar{I}_j|_r+\sum_{j=1}^{m}|J_j|_r+\sum_{j=1}^{m}|\bar{J}_j|_r+\sum_{j=1}^{m}|E_j|_r + |L|_r+|\bar{L}|_r \leqq \text{const.}\frac{(m+1)^{m+1-\delta}c^m}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}} \tag{4.17}$$

and

$$\sum_{j=1}^{m}|I_j|_r^{(\mu)}+\sum_{j=1}^{m}|\bar{I}_j|_r^{(\mu)}+\sum_{j=1}^{m}|J_j|_r^{(\mu)}+\sum_{j=1}^{m}|\bar{J}_j|_r^{(\mu)}+\sum_{j=1}^{m}|E_j|_r^{(\mu)} + |L|_r^{(\mu)}+|L|_r^{(\mu)} \leqq \text{const.}\frac{(m+1)^{m+1+\nu}c^m}{(R-r)^{m+2}(t^{\frac{1}{2}})^{m+2}} \tag{4.18}$$

for c sufficiently large, by establishing the corresponding estimates for each term in the left sides. A simple argument given at the end will then yield the desired estimates for $\partial_x^{m+1} u(x, t)$.

The proof of (4.17) and (4.18) requires estimates for the derivatives of the functions $v=f(u)$ and $w=g(u)$. These are provided by Corollary 2.2. Since the derivatives $\partial_x^k u(x, t)$ satisfy (4.6) for $k \leqq m$, by the remark made after Corollary 2.2, it follows that

$$\left.\begin{matrix}|\partial_x^k v(x,t)|_r\\ |\partial_x^k w(x,t)|_r\end{matrix}\right\} \leqq \gamma\frac{k^{k-\delta}c^k}{(R-r)^k(t^{\frac{1}{2}})^k} \tag{4.19}$$

and

$$\left.\begin{matrix}|\partial_x^k v(x,t)|_r^{(\mu)}\\ |\partial_x^k w(x,t)|_r^{(\mu)}\end{matrix}\right\} \leqq \gamma\frac{k^{k+\nu}c^k}{(R-r)^{k+1}(t^{\frac{1}{2}})^{k+1}}, \tag{4.20}$$

for $0<r<R$, $0<t<T$ and $k=0, 1, \dots, m$, with the constant γ depending only on M, f, g, δ and ν.

Now suppose that we wish to estimate $|\partial_x^{m+1}u(x, t)|_r$ and $|\partial_x^{m+1}u(x, t)|_r^{(\mu)}$ for given $r\in(0, R)$ and $t\in(0, T)$. We choose the R_j and T_j in (4.7) and (4.8), respectively, as follows

$$R_j=R-\frac{j}{m+1}(R-r) \quad\text{and}\quad T_j=\frac{j}{m+1}t \quad (j=0,1,\dots,m). \tag{4.21}$$

We are now prepared to carry out the estimations of the various terms occurring on the left of (4.17) and (4.18). In what follows it will be convenient to consider the largest of the several constants b appearing and Lemmas 3.1, 3.2, and 3.3; in the sequel b will be used to denote that largest constant.

III. We begin with $\sum_{j=1}^{m}|I_j(x, t)|_r$. Applying (3.5) we have

$$|I_j(x,t)|_r \leqq \text{const.}\frac{b^j j^j}{(R_{m-j+1}-r)^{j+\varepsilon}}\Big(\sup_{T_{m-j+1}<\tau<t}|\partial_y^{m-j}v(y,\tau)|_{R_{m-j}}\Big). \tag{4.22}$$

By the induction hypothesis and (4.19)

$$\sup_{T_{m-j+1}<\tau<t}|\partial_y^{m-j}v(y,\tau)|_{R_{m-j}} \leqq \gamma\frac{(m-j)^{m-j-\delta}c^{m-j}}{(R-R_{m-j})^{m-j}(T_{m-j+1}^{\frac{1}{2}})^{m-j}}.$$

Inserting the last estimate into (4.22), we find, in view of (4.21), that

$$|I_j(x,t)|_r \leqq \text{const.} \frac{b^j j^j}{\left[\frac{j}{m+1}(R-r)\right]^{j+\varepsilon}} \frac{(m-j)^{m-j-\delta} c^{m-j}}{\left[\frac{m-j}{m+1}(R-r)\right]^{m-j} \left[\frac{m-j+1}{m+1} t\right]^{(m-j)/2}}$$

$$= \text{const.} \frac{(m+1)^{m+\varepsilon} c^m}{(R-r)^{m+\varepsilon}(t^{\frac{1}{2}})^m} \left(\frac{b t^{\frac{1}{2}}}{c}\right)^j \left(\frac{m+1}{m-j+1}\right)^{\frac{m-j}{2}} \frac{1}{j^\varepsilon (m-j)^\delta}$$

$$\leqq \text{const.} \frac{(m+1)^{m+\varepsilon} c^m}{(R-r)^{m+\varepsilon}(t^{\frac{1}{2}})^m} \left(\frac{b T^{\frac{1}{2}}}{c}\right)^j e^{j/2}.$$

Hence

$$\sum_{j=1}^{m} |I_j(x,t)|_r \leqq \text{const.} \frac{(m+1)^{m+\varepsilon} c^m}{(R-r)^{m+\varepsilon}(t^{\frac{1}{2}})^m} \sum_{j=1}^{\infty} 2^{-j} \tag{4.23}$$

provided that

$$c \geqq 2 b T^{\frac{1}{2}} e^{\frac{1}{2}}. \tag{4.24}$$

Choosing the ε in (4.23), which is at our disposal, so that $\varepsilon < 1-\delta$, we thus obtain

$$\sum_{j=1}^{m} |I_j(x,t)|_r \leqq \text{const.} \frac{(m+1)^{m+1-\delta} c^m}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}} \tag{4.25}$$

as desired, provided that (4.24) is satisfied.

Proceeding in the same way we obtain the estimates

$$\sum_{j=1}^{m} |I_j(x,t)|_r^{(\mu)} \leqq \text{const.} \frac{(m+1)^{m+1+\nu} c^m}{(R-r)^{m+2}(t^{\frac{1}{2}})^{m+2}} \tag{4.26}$$

$$\sum_{j=1}^{m} |J_j(x,t)|_r \leqq \text{const.} \frac{(m+1)^{m+1-\delta} c^m}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}} \tag{4.27}$$

and

$$\sum_{j=1}^{m} |J_j(x,t)|_r^{(\mu)} \leqq \text{const.} \frac{(m+1)^{m+1+\nu} c^m}{(R-r)^{m+2}(t^{\frac{1}{2}})^{m+2}} \tag{4.28}$$

for c satisfying (4.24).

In like fashion, based on Lemma 3.3 we obtain the following analogous estimates for the corresponding terms of (4.17) and (4.18) which involve the potential kernel:

$$\sum_{j=1}^{m} |\bar{I}_j(x,t)|_r \leqq \text{const.} \frac{(m+1)^{m+1-\delta} c^m}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}}, \tag{4.29}$$

$$\sum_{j=1}^{m} |\bar{I}_j(x,t)|_r^{(\mu)} \leqq \text{const.} \frac{(m+1)^{m+1+\nu} c^m}{(R-r)^{m+2}(t^{\frac{1}{2}})^{m+2}}, \tag{4.30}$$

$$\sum_{j=1}^{m} |\bar{J}_j(x,t)|_r \leqq \text{const.} \frac{(m+1)^{m+1-\delta} c^m}{(R-r)^{m+1} (t^{\frac{1}{2}})^{m+1}} \tag{4.31}$$

and

$$\sum_{j=1}^{m} |\bar{J}_j(x,t)|_r^{(\mu)} \leqq \text{const.} \frac{(m+1)^{m+1+\nu} c^m}{(R-r)^{m+2} (t^{\frac{1}{2}})^{m+2}} \tag{4.32}$$

provided that

$$c \geqq 2b\,T^{\frac{1}{2}}. \tag{4.33}$$

IV. The estimation of the sums $\sum_{j=1}^{m} |E_j(x,t)|_r$ and $\sum_{j=1}^{m} |E_j(x,t)|_r^{(\mu)}$ is somewhat different from the preceding. Accordingly, we will furnish the details.

Making use of (3.9) in Lemma 3.2 we have

$$|E_j(x,t)|_r \leqq \text{const.} \frac{b^j\, j^{j/2}}{(t-T_{m-j+1})^{j/2}} \Big(\sup_{T_{m-j}<\tau<T_{m-j+1}} |\partial_y^{m-j} v(y,\tau)|_{R_{m-j}} \Big). \tag{4.34}$$

By the induction hypothesis and (4.19)

$$\sup_{T_{m-j}<\tau<T_{m-j+1}} |\partial_y^{m-j} v(y,\tau)|_{R_{m-j}} \leqq \gamma \frac{(m-j)^{m-j-\delta} c^{m-j}}{(R-R_{m-j})^{m-j} (T_{m-j})^{\frac{1}{2}}}.$$

Inserting this into (4.34) taking (4.21) into account, we find that

$$\begin{aligned}
|E_j(x,t)|_r &\leqq \text{const.} \frac{b^j\, j^{j/2}}{\left[\frac{j}{m+1} t\right]^{j/2}} \frac{(m-j)^{m-j-\delta} c^{m-j}}{\left[\frac{m-j}{m+1} t\right]^{(m-j)/2} \left[\frac{m-j}{m+1}(R-r)\right]^{m-j}} \\
&= \text{const.} \frac{(m+1)^{m/2} (m-j)^{\frac{m-j}{2}-\delta}}{(t^{\frac{1}{2}})^m} \frac{c^m}{(R-r)^m} \left(\frac{b[R-r]}{c}\right)^j \left(\frac{m+1}{m-j}\right)^{m-j} \\
&\leqq \text{const.} \frac{(m+1)^m}{(t^{\frac{1}{2}})^m} \frac{c^m}{(R-r)^m} \left(\frac{b\,R}{c}\right)^j e^{j+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=1}^{m} |E_j(x,t)|_r &\leqq \text{const.} \frac{(m+1)^m c^m}{(R-r)^m (t^{\frac{1}{2}})^m} \sum_{j=1}^{\infty} 2^{-j} \\
&\leqq \text{const.} \frac{(m+1)^{m+1-\delta} c^m}{(R-r)^{m+1} (t^{\frac{1}{2}})^{m+1}}
\end{aligned} \tag{4.35}$$

provided that

$$c \geqq 2b\,e\,R. \tag{4.36}$$

In exactly the same way starting from (3.10), we find that

$$\sum_{j=1}^{m} |E_j(x,t)|_r^{(\mu)} \leqq \text{const.} \frac{(m+1)^{m+1+\nu} c^m}{(R-r)^{m+2} (t^{\frac{1}{2}})^{m+2}} \tag{4.37}$$

provided that (4.36) holds.

V. For the estimation of $|L(x,t)|_r$ and $|L(x,t)|_r^{(\mu)}$ we employ (3.18) of Lemma 3.4, which gives us

$$\left.\begin{matrix}|L(x,t)|_r\\ |L(x,t)|_r^{(\mu)}\end{matrix}\right\}\leqq \text{const.}\left(\sup_{T_m<\tau<T}|\partial_y^m v(y,\tau)|_{R_m}^{(\mu)}+\frac{1}{(R_m-r)^{\mu+\lambda}}\sup_{T_m<\tau<T}|\partial_y^m v(y,\tau)|_{R_m}\right)$$

with $\lambda\in(0,2)$, the constant depending on λ.

Applying the induction hypothesis, (4.19), (4.20) and (4.21) we obtain

$$\left.\begin{matrix}|L(x,t)|_r\\ |L(x,t)|_r^{(\mu)}\end{matrix}\right\}\leqq \text{const.}\left[\frac{m^{m+\nu}c^m}{(R-R_m)^{m+1}(T_m^{\frac{1}{2}})^{m+1}}+\frac{m^{m-\delta}c^m}{(R_m-r)^{\mu+\lambda}(R-R_m)^m(T_m^{\frac{1}{2}})^m}\right]$$

$$=\text{const.}\left[\frac{m^{m+\nu}c^m}{\left(\frac{m}{m+1}[R-r]\right)^{m+1}\left(\frac{m}{m+1}t\right)^{(m+1)/2}}\right.$$

$$\left.+\frac{m^{m-\delta}(m+1)^{\mu+\lambda}c^m}{\left(\frac{R-r}{m+1}\right)^{\mu+\lambda}\left(\frac{m}{m+1}[R-r]\right)^m\left(\frac{m}{m+1}t\right)^{m/2}}\right]$$

$$\leqq\text{const.}\left(\frac{m+1}{m}\right)^{\frac{3(m+1)}{2}}\left[\frac{m^{m+\nu}c^m}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}}+\frac{m^{m-\delta}(m+1)^{\mu+\lambda}c^m}{(R-r)^{m+\mu+\lambda}(t^{\frac{1}{2}})^m}\right].$$

By the hypothesis (4.5), $\nu\leqq 1-\delta$. If we now select $\lambda\in(0,2)$ so that $\mu+\lambda<1$, then from the foregoing inequality it will follow that

$$|L(x,t)|_r\leqq\text{const.}\frac{(m+1)^{m+1-\delta}c^m}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}} \tag{4.38}$$

and that

$$|L(x,t)|_r^{(\mu)}\leqq\text{const.}\frac{(m+1)^{m+1+\nu}c^m}{(R-r)^{m+2}(t^{\frac{1}{2}})^{m+2}}. \tag{4.39}$$

A similar argument based on Lemma 3.5 yields

$$|\bar{L}(x,t)|_r\leqq\text{const.}\frac{(m+1)^{m+1-\delta}c^m}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}} \tag{4.40}$$

and

$$|\bar{L}(x,t)|_r^{(\mu)}\leqq\text{const.}\frac{(m+1)^{m+1+\nu}c^m}{(R-r)^{m+2}(t^{\frac{1}{2}})^{m+2}}. \tag{4.41}$$

VI. The results of the preceding subsections establish the estimates (4.17) and (4.18) for $c\geqq K$, K denoting the largest of the constants on the right of the conditions (4.24), (4.33) and (4.36). Inserting (4.17) and (4.18) into (4.9) and making use of (4.4) we find that

$$|\partial_x^{m+1}u(x,t)|_r\leqq\frac{(m+1)^{m+1-\delta}}{(R-r)^{m+1}(t^{\frac{1}{2}})^{m+1}}[\text{const.}\,c^m+A\,a^{m+1}]$$

and

$$|\partial_x^{m+1}u(x,t)|_r^{(\mu)}\leqq\frac{(m+1)^{m+1+\nu}}{(R-r)^{m+2}(t^{\frac{1}{2}})^{m+2}}[\text{const.}\,c^m+A\,a^{m+1}]$$

for $c\geqq K$.

Let B denote the larger of the two constants appearing on the right of the last two inequalities. In order to complete the induction it will, therefore, be sufficient to show that $Bc^m + Aa^{m+1} \leqq Mc^{m+1}$, i.e.

$$B/M + (Aa/M)(a/c)^m \leqq c \qquad (m \geqq 0)$$

for a suitable choice of c. This may be accomplished by first selecting $c \geqq a$, and then taking $c \geqq (B/M + Aa/M)$. The proof of Theorem 4.1 is thereby completed.

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