Stability Transitions for Periodic Orbits in Hamiltonian Systems

R. C. CHURCHILL, G. PECELLI & D. L. ROD

Communicated by C. DAFERMOS

Abstract

The paper considers one-parameter families of periodic solutions of real analytic Hamiltonian systems with two degrees of freedom, the parameter being the energy h. Conditions are given which guarantee that this family will undergo infinitely many changes in stability status as h tends to some finite value h_0. First considered is the case of a critical point (with eigenvalues $\pm\alpha$, $\pm i\beta$, α and $\beta>0$) of the Hamiltonian at energy h_0 with the property that the family limits to a homoclinic orbit asymptotic to this point. Some generalizations of this case are given, and applications are made to examples such as the Hénon-Heiles Hamiltonian. We obtain an infinite sequence of distinct energy intervals converging to h_0 on which the periodic orbits are elliptic. Requirements for the elliptic stability of the orbits are then given. The additional conditions for an infinite sequence of distinct energy intervals converging to h_0, on which the orbits are hyperbolic, involve the "coexistence problem" for an associated Hill's equation that appears when the relevant Poincaré maps along the orbits are computed in coordinates. The results are compared to the case where the critical point has eigenvalues $(\pm\alpha\pm i\beta)$, α and $\beta>0$, investigated by HENRARD and DEVANEY.

Contents

1. Introduction

This paper considers one-parameter families of periodic solutions of real analytic Hamiltonian systems with two degrees of freedom, the parameter being the energy h. We seek conditions which guarantee that this family will undergo infinitely many changes in stability status as h tends to some finite value h_0. We first consider the case where there is a critical point (with eigenvalues $\pm\alpha$, $\pm i\beta$, α and $\beta>0$) of the Hamiltonian at energy h_0, and the family limits to a homoclinic orbit asymptotic to this point. Some generalizations are given to the case where the family limits to several heteroclinic orbits connecting corresponding critical points. The eigenvalue condition on the critical points allows us to use MOSER coordinates [27] locally to analyze the flow. Provided the periodic orbits are "properly alligned," one can show that there are infinitely many distinct energy intervals converging to h_0 on which the corresponding orbit is elliptic (Theorem 2.1). Moreover, conditions for elliptic *stability* within these intervals are presented using the Moser-Rüssmann criterion. An example is given to show that this result is the best possible without further assumptions (Section 6, Example D).

To obtain corresponding energy intervals of hyperbolicity for the family of periodic orbits, assumptions are made which allow a reduction of the relevant Poincaré mappings to a family of Hill's equations (Theorem 3.1). Roughly, the energy intervals of hyperbolicity for the periodic orbits will correspond to the instability intervals of the associated Hill's equation (Theorem 3.3), thus bringing in the "coexistence problem" for such equations. Computable conditions are given in Section 4 for a special class of Hill's equations, appearing in the examples of Section 6, which show that there are infinitely many distinct energy intervals converging to h_0 on which the corresponding periodic orbit is hyperbolic.

We also consider contexts in which the limit of the family of periodic orbits is exactly one critical point, *i.e.*, one can consider the family as bifurcating from this critical point. Theorem 5.1 presents conditions on whether the periodic orbits "start out" elliptic or hyperbolic as they arise from this critical point as the energy is changed. These results use a theorem of E. HOEHN [20] on how stability boundaries of a 2-parameter Hill's equation meet the axes and involve detailed calculations based on an explicit solution for the periodic orbits.

The theory is applied to a variety of examples in Section 6, including the HÉNON-HEILES Hamiltonian [18]. Appendix A contains a technical argument necessary for the results in Section 2. Also, at the end of Section 3 these results for stability oscillation are compared with the results of HENRARD [19] and DEVANEY [12, 13, 14]. They consider the case where the critical point has eigenvalues $(\pm\alpha\pm i\beta)$, α and $\beta>0$, and has a *non-degenerate* homoclinic orbit asymptotic to it.

In Appendix B we relate Theorem 3.2 to classical Sturmian oscillation theory and sketch an alternative derivation of the results in Section 2.

For a short overview of the contents of this paper, we refer the reader to Section 6 of our survey [6], which also relates our results to other phenomena in the HÉNON-HEILES Hamiltonian and gives many references to related work.

2. Energy Intervals of Ellipticity

The purpose of this section is to prove Theorem 2.1 below; extensions and applications will be discussed following the proof. For terminology and general background we refer the reader to [31].

Points of R^4 will be written as pairs (x, y), where $x=(x_1, x_2)$, $y=(y_1, y_2)\in R^2$.

Theorem 2.1. *Let $U\subset R^4$ be an open neighborhood of the origin, and let $H:U\to R$ be a real analytic Hamiltonian of the form*

$$H(x, y)=\alpha x_1 y_1+(\beta/2)(x_2^2+y_2^2)+O_3(x, y), \tag{2.1}$$

where $\alpha>0$, $\beta>0$, and $O_3(x, y)$ denotes terms of order 3 and higher in the variables x_1, x_2, y_1, y_2. Note that 0 is a nondegenerate equilibrium solution, with eigenvalues $\pm\alpha$, $\pm i\beta$, of the associated differential equations

$$\dot{x}=H_y, \qquad \dot{y}=-H_x. \tag{2.2}$$

Assume:

(a) *There is a solution Π_0 of (2.2) homoclinic (i.e. doubly asymptotic) to 0; and*

(b) *$\Pi_0\cup\{0\}$ is the limit (as a point set) as $h\uparrow 0$ of a continuous family of periodic solutions Π_h of (2.2), with energies $h<0$, $|h|$ small, all of which project into the x_1-axis under the mapping $(x, y)\to x$.*

Then there is an $\varepsilon>0$, and four sequences $\{h_j\}$, $\{k_j\}$, $\{h'_j\}$, $\{k'_j\}\subset(-\varepsilon, 0)$ converging to 0, with $h_j<k_j\leqq k'_j<h'_j\leqq h_{j+1}$, $j=1,2,\ldots$, such that Π_h is elliptic when $h\in(h_j, k_j)\cup(k'_j, h'_j)$, parabolic with double eigenvalue -1 when $h=k_j$ or k'_j, and parabolic with double eigenvalue $+1$ when $h=h_j$ or h'_j. Moreover, the periodic orbits Π_h are elliptic stable for almost all $h\in(h_j, k_j)\cup(k'_j, h'_j)$.

Following the proof of Theorem 2.1, conditions for the hyperbolicity of Π_h over suitable subintervals of (k_j, k'_j) and/or (h'_j, h_{j+1}) will be indicated. These conditions, however, turn out to be impractical for computations, and in later sections of the paper we will develop an alternate approach to proving the existence of such intervals of hyperbolicity.

The proof of Theorem 2.1 depends heavily on a coordinate transformation due to MOSER [27], which in our context is best described by CONLEY [8]. We first review CONLEY's interpretation; for more details see [8].

For a real analytic Hamiltonian of the form (2.1), MOSER's theorem guarantees the existence of a (possibly non-canonical) analytic coordinate transformation

$$(u, v, w_1, w_2)\to(x_1, x_2, y_1, y_2), \tag{2.3}$$

defined near 0, under which solutions of (2.2) assume the following simple form:

$$\begin{aligned} u(t)&=u_0\exp(t\alpha^*),\\ v(t)&=v_0\exp(-t\alpha^*),\\ w(t)&=w_0\exp(t\beta^*),\\ \bar{w}(t)&=\bar{w}_0\exp(-t\beta^*), \end{aligned} \tag{2.4}$$

where

(a) $w = w_1 + i w_2$, $w_0 = w(0)$;
(b) the constants u_0, v_0, w_0 are determined by the initial conditions; and
(c) α^* and β^* are convergent power series, with real and purely imaginary coefficients respectively, in the "variables"

$$\chi = u v, \qquad \rho = |w|^2. \tag{2.5}$$

In fact,

$$\begin{aligned} \alpha^* &= \alpha + O_1(\chi, \rho), \\ \beta^* &= -i\beta + O_1(\chi, \rho). \end{aligned} \tag{2.6}$$

MOSER's transformation is given in terms of convergent power series expansions

$$\begin{aligned} x_1 &= u + O_2(u, v, w, \bar{w}), \\ y_1 &= v + O_2(u, v, w, \bar{w}), \\ z = x_2 + i y_2 &= w + O_2(u, v, w, \bar{w}) \end{aligned} \tag{2.7}$$

with Jacobian 1 at the origin. The transformed Hamiltonian, which may *not* govern the differential equations satisfied by (2.4), will take the form

$$H(x, y) = K(u, v, w, \bar{w}) = \alpha \chi + (\beta/2) \rho + O_2(\chi, \rho). \tag{2.8}$$

Notice that χ and ρ, which from (2.5) and (2.7) are given in terms of the original variables by

$$\begin{aligned} \chi &= u v = x_1 y_1 + O_3(x, y), \\ \rho &= |w|^2 = x_2^2 + y_2^2 + O_3(x, y), \end{aligned} \tag{2.9}$$

are local integrals of the flow (2.4). As a consequence, we see from (2.6) that α^* and β^* must also be local integrals of the flow.

Following CONLEY [8], we choose a ball B about the origin of (u, v, w_1, w_2)-space in which all series given by MOSER's theorem converge, and in which all these series, together with their first partial derivatives, are dominated by their lowest order terms. All subsequent discussion in (u, v, w_1, w_2)-coordinates will take place in B.

We first examine the surface $\{K = h\} \cap B$ topologically, and to this end we note from (2.8) that, for h fixed, the implicit function theorem gives ρ as a function of χ on $K = h$ with

$$\rho = \rho(\chi) \simeq (2/\beta)(h - \alpha \chi). \tag{2.10}$$

Since $\rho = |w|^2 \geqq 0$ and $\beta > 0$, we see that $\rho(\chi)$ is defined for

$$\chi \leqq \chi^* \simeq \frac{h}{\alpha}. \tag{2.11}$$

Moreover, $\chi^* = \chi^*(h)$ is a simple zero of $\rho(\chi)$, and is given by the implicit function theorem applied to $K = h$ in (2.8). Note that $\chi^*(h) \to 0$ as $h \uparrow 0$.

Now $\chi =$ constant defines a hyperbola in (u, v)-space, and from (2.11) we then conclude that the surfaces $K = h$ project into the (u, v)-plane as the shaded regions in Figure 1.

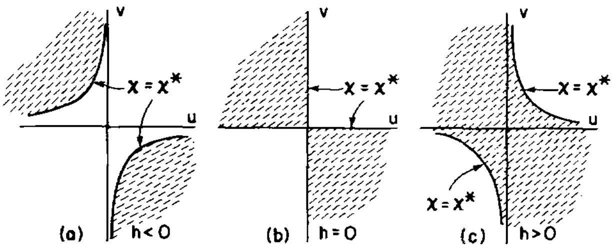

Fig. 1

Using Figure 1 we can give a description of $\{K = h\} \cap B$ itself. Indeed, if (u_0, v_0) is in one of these shaded regions, then by (2.10) we have $\rho = |w|^2 \geqq 0$, hence a circle of choices for w when $\chi < \chi^*$, and a single choice, namely $w = 0$, when $\chi = \chi^*$. $\{K = h\} \cap B$ is therefore the topological product of the corresponding shaded region with a circle, except that over χ^* each such circle is identified to a point. Moreover, since χ is constant along the solutions (2.4), all orbits move along cylinders $\rho = |w|^2 =$ constant lying above the hyperbolas $\chi < \chi^*$. Since $\alpha > 0$, the projected orbits move in the directions shown for various h in Figure 2. When $\chi = \chi^*$ and $h \neq 0$ these cylinders reduce to two orbit segments, one over each branch of $uv = \chi^*$, and when $h = 0$ these cylinders reduce to the one-dimensional stable and unstable manifolds of the equilibrium point at the origin.

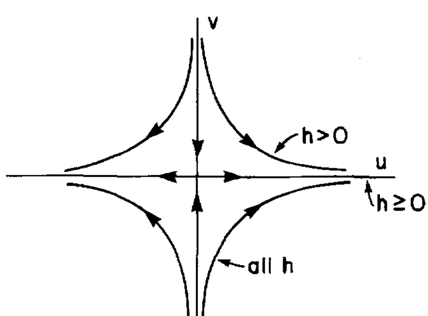

Fig. 2

For $s > 0$ and $|h|$ near 0 let $L(h, s)$ denote the collection of points in $K = h$ with $|u - v| \leqq s$. These sets project into the (u, v)-plane as the shaded regions shown in Figure 3. Letting $\Sigma_h(s)$ and $\Sigma'_h(s)$ denote, respectively, the subsets of $L(h, s)$ for which $u - v = s$ and $u - v = -s$, we use the following lemma from Conley [8, pp. 141–2]:

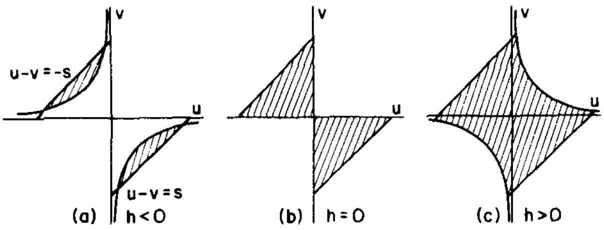

Fig. 3

Lemma 2.2. *For both $s>0$ and $|h|$ sufficiently small, $L(h, s)$ lies in the interior of B. If we fix $s=s_0>0$ in this range, $\Sigma_h=\Sigma_h(s_0)$ and $\Sigma'_h=\Sigma'_h(s_0)$ are 2-spheres forming the boundary of $L(h, s_0)$ in the energy surface $K=h$. If $h<0$, then $L(h, s_0)$ has two components, each a closed 3-cell, and if $h>0$, then $L(h, s_0)$ is topologically the product of a 2-sphere with a closed interval.*

For $h<0$ and $s_0>0$ as in Lemma 2.2, we view each component of $L(h, s_0)$ as the 3-cell shown in Figure 4, where the cylinders with $\rho=|w|^2=$constant project into the (u, v)-plane as the hyperbolas $\chi=$constant$<\chi^*$, and the center line, where $\rho=0$, projects into the boundary hyperbola given by $\chi=\chi^*$.

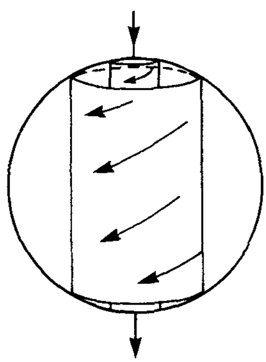

Fig. 4

We shall indicate projections of sets and orbits in phase space into the (u, v)-plane by bars. For example $\bar{L}(h, s_0)$, $h<0$, consists of the two shaded regions in Figure 3(a). Also, $\bar{\Sigma}_h$ and $\bar{\Sigma}'_h$ are the line segments in the boundary of $\bar{L}(h, s_0)$ along which $u-v=+s_0$ and $u-v=-s_0$, respectively (see Fig. 3(a)). We now fix $s_0>0$ and $h<0$ so that Lemma 2.2 applies.

Assumption (b) of Theorem 2.1 guarantees that $x_2=y_2=0$ along Π_h. A consequence of this assumption, proved in Appendix A, is that in Moser's coordinates $\rho=|w|^2=0$ along the intersection of Π_h with $L(h, s_0)$. Assuming Π_h projects into the fourth quadrant of Figure 3(a), we can then view $\Pi_h \cap L(h, s_0)$ as the vertical axis of the 3-cell of Figure 4. It will be important to note, however, that the results of Lemma 2.3 below hold *verbatim* when Π_h projects into the second quadrant of Figure 3(a).

The line $u+v=0$ intersects $\bar{\Sigma}_h$ at a point which is the projection of an equator on Σ_h that separates Σ_h into two hemispheres. Let Σ_h^+ be the "upper" hemisphere, where $u+v<0$, and Σ_h^- the "lower" hemisphere, where $u+v>0$. Note from (2.4) that the flow is transverse to these two hemispheres.

From $K=h$ in (2.8) we obtain the equation for Σ_h as

$$(\alpha/4)(u+v)^2+(\beta/2)|w|^2+O_2(\chi,\rho)=h+(\alpha/4)(u-v)^2=h+(\alpha/4)s_0^2, \quad (2.12)$$

where $u>0$, $v<0$, $u=s_0+v$, $\chi=uv=(s_0+v)v$, and $h\leqq 0$ has been chosen small enough in modulus so that $h+(\alpha/4)s_0^2>0$. Recall that the ball B in (u, v, w_1, w_2)-space was chosen small enough so that all series mentioned in Moser's theorem, together with their first partial derivatives, converged in B and were dominated there by their lowest order terms. Since $s_0+2v=u+v\neq 0$ on $\Sigma_h^\pm$, on setting

$$M(v, w_1, w_2)=(\alpha/4)(s_0+2v)^2+(\beta/2)|w|^2+O_2(\chi,\rho), \quad (2.13)$$

we see from (2.12), with $\chi=uv=(s_0+v)v$, that

$$\frac{\partial M}{\partial v}(v, w_1, w_2)\neq 0 \quad \text{on } \Sigma_h^\pm. \quad (2.14)$$

Thus $u=s_0+v$ and v can be expressed as analytic functions of w_1 and w_2 on $\Sigma_h^\pm$, and therefore w_1 and w_2 serve as analytic coordinates on $\Sigma_h^\pm$ for all sufficiently small energy values, including $h=0$.

We can now introduce "polar coordinates" on $\Sigma_h^\pm$, representing each point other than $w_1=w_2=0$ by the pair $\rho=|w|^2$, $\arg(w)$, where $w=w_1+iw_2$, of course respecting the multi-valued nature of $\arg(w)$. Notice that the points $w=0$ on $\Sigma_h^\pm$ are precisely the intersections of Π_h with $\Sigma_h^\pm$. The coordinates ρ and $\arg(w)$ are clearly analytic in w_1 and w_2 on their domain of definition.

Using (2.4) we define an analytic diffeomorphism ϕ_h: $\Sigma_h^+\to\Sigma_h^-$ by following points in Σ_h^+ in forward time until they first intersect Σ_h^-; ϕ_h is depicted in Figure 4. Since w_1 and w_2 serve as coordinates on both surfaces, we can regard ϕ_h as an analytic mapping of the (w_1, w_2)-plane into itself which fixes the origin.

The following lemma is a version of [8, Lemma 3.1, p. 144]:

Lemma 2.3. (1) *The flow mapping* ϕ_h: $\Sigma_h^+\to\Sigma_h^-$ *is given in* $\rho=|w|^2$, $\arg(w)$ *coordinates by* $\rho\to\rho$ *and* $\arg(w)=\theta\to\theta+\Delta\theta$, *where* $\Delta\theta$ *is the change in* $\arg(w)$.

(2) $\Delta\theta$ *and its derivatives can be continuously extended in* (w_1, w_2)*-coordinates to* $\rho=0$.

(3) *In* (w_1, w_2)*-coordinates, the Jacobian matrix of* ϕ_h *at* $\Pi_h\cap\Sigma_h^+$ *(where* $\rho=0$*) is given by*

$$D\phi_h=\begin{pmatrix}\cos(\Delta\theta) & \sin(\Delta\theta)\\ -\sin(\Delta\theta) & \cos(\Delta\theta)\end{pmatrix}_{\rho=0},$$

and $(\Delta\theta)_{\rho=0}\downarrow-\infty$ *as* $h\uparrow 0$.

Proof. (1) This follows from the fact that, by (2.4), $\rho=|w|^2$ is a local integral of the flow.

(2) Recall that β^* in (2.6) is purely imaginary and is a local integral of the flow. But (2.6) and (2.4) then imply that along any orbit (with $\rho\neq 0$) we have

$(d/dt)\arg(w)=d\theta/dt=-i\beta^*$. If τ is the time for some orbit segment (with $\rho\neq 0$) to reach Σ_h^- from Σ_h^+, then for this orbit segment we have $\Delta\theta=-i\beta^*\tau$. The transversality of the flow with the surfaces Σ_h^+ and Σ_h^- yields the smoothness of τ in (w_1,w_2)-coordinates, even at $w_1=w_2=0$. That β^* is smooth follows from (2.6), and thus (2) is proven since β^* and τ, together with their derivatives, can all be continuously extended to the origin $w_1=w_2=0$.

(3) We first prove that the time τ_h which Π_h consumes in going from Σ_h^+ to Σ_h^- increases to $+\infty$ as $h\uparrow 0$. Let $(u_0(h),v_0(h))$ be the point on $\chi^*(h)=uv$ where $\bar{\Pi}_h$ intersects $\bar{\Sigma}_h^+$. Using (2.6) at $\rho=0$, we define

$$\begin{aligned}\alpha_0^*(h)&=\alpha+O_1(\chi^*(h),0),\\ \beta_0^*(h)&=-i\beta+O_1(\chi^*(h),0).\end{aligned}\tag{2.15}$$

By (2.4) the orbit Π_h satisfies the following constraints at $t=0$, $t=\tau_h$:

(a) $u_0(h)-v_0(h)=s_0$ on $\bar{\Sigma}_h^+$,

(b) $u_0(h)\exp(\alpha_0^*(h)\tau_h)-v_0(h)\exp(-\alpha_0^*(h)\tau_h)=s_0$ on $\bar{\Sigma}_h^-$.

Recall that $\chi^*(h)\to 0$ as $h\uparrow 0$. From (a) and (b) we determine

$$\chi^*(h)=-s_0^2\left[\frac{\sinh(\alpha_0^*(h)\,\tau_h/2)}{\sinh(\alpha_0^*(h)\,\tau_h)}\right]^2.$$

Since $s_0>0$ is constant and $\tau_h>0$ for all h, we must have

$$\liminf_{h\uparrow 0}(\tau_h)\geqq 0.$$

If $\liminf(\tau_h)=0$, then an application of L'Hospital's rule yields $\chi^*(h)\to -(s_0^2/4)\neq 0$ as $h\uparrow 0$, a contradiction. If $\liminf(\tau_h)<\infty$, then there exists a $T>0$ and a sequence $\{h_n\}$, $h_n\uparrow 0$ as $n\to\infty$, such that $\tau_{h_n}\to T$ as $n\to\infty$. But then $\chi^*(h_n)\not\to 0$; again a contradiction. Thus $\liminf\limits_{h\uparrow 0}(\tau_h)=+\infty$.

By the proof of (2) we have $\Delta\theta=-i\beta_0^*\tau_h$ on Π_h, and coupled with (2.15) this yields the last statement in (3). The first part of (3) follows by observing that in (w_1,w_2)-coordinates

$$\phi_h(w_1,w_2)=\begin{pmatrix}\cos(\Delta\theta) & \sin(\Delta\theta)\\ -\sin(\Delta\theta) & \cos(\Delta\theta)\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix},$$

and that $\Delta\theta$ is differentiable with respect to w_1 and w_2, even at the origin, by (2). Q.E.D.

We now proceed to complete the proof of Theorem 2.1. Let ψ_h be the flow mapping obtained by following points in a suitably small neighborhood U_h of $\rho=0$ in Σ_h^- forward in time until their first intersection with Σ_h^+. Since Σ_h^- is transverse to the flow (2.4), the map $\Phi_h=\phi_h\circ\psi_h\colon U_h\to\Sigma_h^-$ is a Poincaré mapping along Π_h. From Assumption (a) of Theorem 2.1 we see that ψ_h is defined when $h=0$, although ϕ_h is not. As a consequence, ψ_h and ψ_0 differ only slightly when $|h|$ is sufficiently small.

For $h\leqq 0$ let $T\Sigma_h^\pm$ denote the tangent planes to $\Sigma_h^\pm$ at the poles, where $\rho=0$ (corresponding to the intersections of Π_h with $\Sigma_h^\pm$). Choose any $v\in T\Sigma_0^-$, and let

$\eta = D\psi_0(v) \in T\Sigma_0^+$. Using w_1 and w_2 as coordinates on $\Sigma_h^\pm$, we can identify $T\Sigma_h^\pm$ with the (w_1, w_2)-plane, and thus measure angles in these tangent spaces. Moreover, there are natural identifications for Euclidean space between all the $T\Sigma_h^\pm$ for those $h \leqq 0$ of interest, and so we can regard the fixed vector v as being in $T\Sigma_h^-$ for all such h.

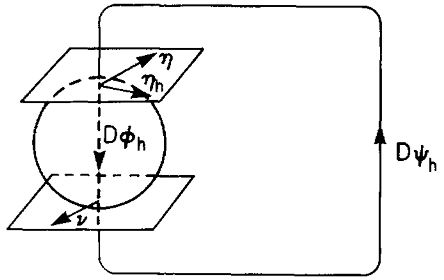

Fig. 5

For $h \leqq 0$ as above, let $\eta_h = D\psi_h(v) \in \Sigma_h^+$, and choose $\delta > 0$ such that $|\arg(\eta) - \arg(\eta_h)| < \pi/4$ when $-\delta < h \leqq 0$. By Lemma 2.3, $D\phi_h(\eta_h)$ can be considered a rotation of η_h through an angle $\Delta\theta(h)$ which tends to $-\infty$ as $h \uparrow 0$. It follows that the rotation of v under $D\Phi_h = D\phi_h \circ D\psi_h$ tends to $-\infty$ as $h \uparrow 0$. In particular, $D\Phi_h(v)$ must lie in the subspace of $T\Sigma_h^-$ generated by v for infinitely many $h \uparrow 0$, *i.e.* there must be sequences $l_j \uparrow 0$ and $\lambda_j > 0$ such that $D\Phi_{l_j}(v) = (-1)^j \lambda_j v$, $j = 1, 2, \ldots$. But Φ_h is symplectic; hence $D\Phi_h$ admits either real reciprocal eigenvalues or else complex conjugate eigenvalues of unit modulus. In particular, as h progresses from l_j to l_{j+1}, the eigenvalues of Φ_h must move around the unit circle from $+1$ to -1 if j is even, and from -1 to $+1$ if j is odd. This immediately implies the existence of the intervals given in the statement of Theorem 2.1.

In Section 3 (Remark (2) following Theorem 3.1) we shall see that the eigenvalues $\mu_1(h)$, $\mu_2(h)$ of $D\Phi_h$ are analytic functions of $h < 0$ whenever Π_h is elliptic, and thus they cannot be constant over any corresponding energy interval. This implies that $\gamma(h)$ in $\mu_1(h) = \overline{\mu_2(h)} = \exp(2\pi i \gamma(h))$ is badly approximated by rationals for almost all values of h for which Π_h is elliptic. For h in this set the mapping $\Phi_h: U_h \to \Sigma_h^-$ can be transformed by a formal power series $C_h = C_h(w_1, w_2)$ into Birkhoff normal form [31, Section 23]. If this resultant normal form is linear, then C_h is a convergent power series by a result of RÜSSMANN [30]; hence the Poincaré mapping is conjugate to a rotation, from which stability is immediate. Alternatively, if the normal form is not linear, then elliptic stability of Π_h is a consequence of the first nonvanishing exponent in the normal form [28, Theorem 2.13, p. 56]. In either case the set of invariant tori in $H = h$ encasing Π_h will have nonzero measure. The proof of Theorem 2.1 is now complete.

Concerning the comments immediately following the statement of Theorem 2.1, assume that the mapping $D\psi_0$ of Figure 5 is hyperbolic, where both ϕ_h and ψ_h are regarded as mappings of the (w_1, w_2)-plane into itself. Then $D\psi_h$ must be hyperbolic for all small $-h$. Since $D\phi_h$ is a pure rotation, $D\Phi_h = D\phi_h \circ D\psi_h$ must then be hyperbolic for all h for which $D\phi_h = \pm id$. Under these circumstances we could obtain the hyperbolicity of Π_h over suitable subintervals of the (k_j, k_j') and (h_j', h_{j+1}). Unfortunately, the "hyperbolicity" of $D\psi_0$ would depend on the "size" of the 3-cell in Figure 4, and thus verification of this property would require a precise knowledge of the domains of convergence involved in Moser's coordinate transformation, or else some geometrical peculiarity of the problem being considered.

On the other hand, the hyperbolicity of the mapping $D\psi_h$ is not a necessary condition for the hyperbolicity of $D\Phi_h$, and hence of Π_h. Indeed, consider the one-parameter family of symplectic mappings of R^2 into R^2 given by

$$D\Phi_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & \cos^2(\theta/2) \\ 0 & 1 \end{pmatrix}, \tag{2.16}$$

and note that each mapping is the composition of a parabolic symplectic mapping and a pure rotation. The eigenvalues of $D\Phi_\theta$ are given by

$$\lambda = (1/2)(b \pm \sqrt{b^2 - 4}),$$

where the 2π-periodic function $b(\theta) = 2\cos(\theta) + \sin(\theta) \cdot \cos^2(\theta/2)$ is shown in the graph of Figure 6. When $b > 2$ we see that $D\Phi_\theta$ is hyperbolic, while for $|b| < 2$ the mapping is elliptic. Notice that $D\Phi_\theta$ is never hyperbolic with negative real reciprocal eigenvalues. We shall see a concrete example of this phenomenon in Example C of Section 6. Note how $D\Phi_\theta$ "locks in" to being hyperbolic over non-trivial θ-intervals despite the continuing presence of the rotation matrix. This helps explain the nature of the examples in Section 6.

The following formulation of Theorem 2.1 will be useful.

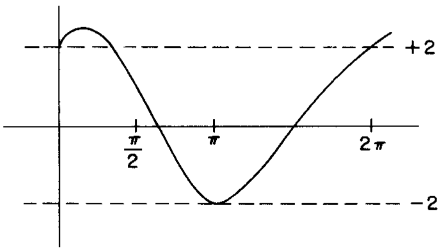

Fig. 6

Theorem 2.4. *Consider an analytic Hamiltonian $H: R^4 \to R$ of the form*

$$H(x, y) = \tfrac{1}{2}|y|^2 + W(x). \tag{2.17}$$

Let p be a saddle point of the potential $W: R^2 \to R$ with eigenvalues $-\alpha^2$, β^2, where α and $\beta > 0$, let $h_0 = W(p)$, and assume:

(a) *There is a solution Π_0 of the differential equations (2.2) homoclinic to $(p, 0)$; and*

(b) $\Pi_0 \cup \{(p, 0)\}$ *is the limit of a continuous family of periodic solutions* Π_h *of (2.2) with energies* $h < h_0$, $h_0 - h$ *small, all of which project into some fixed line in the plane under the mapping* $(x, y) \to x$.

Then the conclusions of Theorem 2.1 hold, except that $(-\varepsilon, 0)$ *is replaced by* $(h_0 - \varepsilon, h_0)$, *and the four sequences now converge to* h_0.

Proof. By translations and rotations in the x-plane, which induce canonical transformations preserving the form of (2.17), we can assume (on dropping constants) that p is at the origin, $h_0 = 0$, and all the Π_h project into the x_1-axis. Assumption (b) then forces the potential to have the form

$$W(x) = -(\alpha^2/2)x_1^2 + (\beta^2/2)x_2^2 + O_3(x).$$

The canonical transformation

$$\begin{aligned} x_1 &= (1/\sqrt{2\alpha})(\tilde{x}_1 - \tilde{y}_1), & x_2 &= (1/\sqrt{\beta})\tilde{x}_2, \\ y_1 &= \sqrt{\alpha/2}(\tilde{x}_1 + \tilde{y}_1), & y_2 &= \sqrt{\beta}\,\tilde{y}_2, \end{aligned} \tag{2.18}$$

then brings (2.17) into the form (2.1), and all Π_h now project into the $\tilde{x}_1$-axis. The assertions of the theorem now follow from Theorem 2.1. Q.E.D.

Theorem 2.4 can be modified to cover the case in which the periodic orbits Π_h converge as $h \uparrow h_0$ to the union of two critical points together with two orbits heteroclinic to (*i.e.* connecting) these two points.

Theorem 2.5. *Consider an analytic Hamiltonian* $H: R^4 \to R$ *of the form*

$$H(x, y) = \tfrac{1}{2}|y|^2 + W(x),$$

and let p_1, p_2 *be saddle points of the potential* $W: R^2 \to R$ *with respective eigenvalues* $-\alpha_j^2$, β_j^2, *where* α_j *and* $\beta_j > 0$, $j = 1, 2$, *and* $h_0 = W(p_1) = W(p_2)$. *Assume:*

(a) *There are two solutions* Π_1, Π_2 *of the associated system of differential equations (2.2) that are heteroclinic to* $(p_1, 0)$ *and* $(p_2, 0)$; *and*

(b) $\Pi_1 \cup \Pi_2 \cup \{(p_1, 0), (p_2, 0)\}$ *is the limit of a continuous family of periodic solutions* Π_h *of (2.2), with energies* $h < h_0$, $h_0 - h$ *small, all of which project into the line segment in the x-plane connecting* p_1 *and* p_2.

Then the conclusions of Theorem 2.1 hold, except that $(-\varepsilon, 0)$ *is replaced by* $(h_0 - \varepsilon, h_0)$, *and the four sequences now converge to* h_0.

Proof. As in the proof of Theorem 2.4, we can assume that $h_0 = 0$, and that p_1 and p_2 lie on the x_1-axis. After p_j has been translated along the x_1-axis to the origin, (b) then forces $W(x)$ to take the form

$$W(x) = -(\alpha_j^2/2)x_1^2 + (\beta_j^2/2)x_2^2 + O_3(x), \quad j = 1, 2.$$

The transformation (2.18) then brings H into the form (2.1), and the previous analysis of the flow can be applied to each one of these critical points in turn.

In that analysis we used w_1 and w_2 as coordinates on the hemispheres $\Sigma_h^{\pm}$, and for the Jacobian matrix of the mapping ϕ_h obtained a pure rotation R_h through an angle $\Delta\theta$ with limit $-\infty$ as $h \uparrow 0$. But now observe from (2.7) that we can also use x_2 and y_2 as analytic coordinates on these hemispheres since u and v are analytic functions of w_1 and w_2 on $\Sigma_h^\pm$ by (2.14). (Note that this construction may require a further shrinking of the ball B in order to have the series expressing x_2 and y_2 in terms of w_1 and w_2 on $\Sigma_h^\pm$, and also the first partials of these series, dominated by lowest order terms. For the present this means picking s_0, v and $u=s_0+v$ sufficiently small; see Lemma 2.2.) In these (x_2, y_2)-coordinates the Jacobian matrix of $D\phi_h$ must then take the form

$$T_h = P_h^{-1} \circ R_h \circ P_h, \tag{2.19}$$

where the matrix P_h is a slight perturbation of the identity (as can be observed from (2.7) and the smallness conditions mentioned above). Though T_h is no longer a pure rotation, it is clear that the total change in the argument of any vector in the (x_2, y_2)-plane after an application of T_h must still approach $-\infty$ as $h\uparrow 0$, just as in the proof of Theorem 2.1.

Observe that the same (x_2, y_2)-coordinates can be used on the spheres relating to both critical points. This is immediate from the fact that translation of the critical points along the x_1-axis to the origin does not affect these two coordinates.

Now consider the Poincaré mapping $\Phi_h: U_h \to \Sigma_h^-(1)$ along Π_h, where U_h is a suitably small compact neighborhood in $\Sigma_h^-(1)$ of $\Pi_h \cap \Sigma_h^-(1)$, with obvious meaning for the notation given by Figure 7.

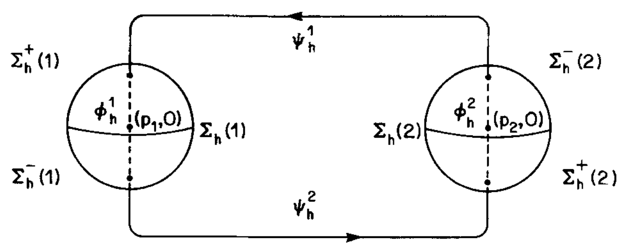

Fig. 7

By analogy with Figure 5 this can be expressed as the composition of four mappings $\Phi_h = \phi_h^1 \circ \psi_h^1 \circ \phi_h^2 \circ \psi_h^2$, each of which can be considered as a mapping of the (x_2, y_2)-plane into itself which fixes the origin. Notice that ψ_h^1 and ψ_h^2 are small perturbations of ψ_0^1 and ψ_0^2, the analogues at $h=0$ of the mapping ψ_0 in the proof of Theorem 2.1, while *both* ϕ_h^1 and ϕ_h^2 can be viewed in (x_2, y_2)-coordinates as distorted clockwise rotations which decrease in argument to $-\infty$ as $h\uparrow 0$. (Here we have used the fact that the computations in Lemma 2.3 are independent of $\Pi_h \cap L(h, s_0)$ projecting to the second or fourth quadrants in Figure 3(a), since by (2.4) we have $d\theta_j/dt = -i\beta_j^* = -\beta_j + O_1(\chi, \rho)$, with $\beta_j > 0$, $j=1,2$, for both quadrants. Otherwise ϕ_h^1 and ϕ_h^2 might conceivably rotate in opposite directions.) In (x_2, y_2)-coordinates $D\Phi_h$ therefore rotates vectors $v \in T\Sigma_h^-(1)$ clockwise (with possible scaling) through angles $\Delta\theta \to -\infty$ as $h\uparrow 0$. The conclusion now follows just as in the proof of Theorem 2.1. Q.E.D.

Analogues of Theorems 2.1, 2.4 and 2.5 also hold for families of periodic orbits Π_h with h *above* the energy of the critical point(s). The proofs are easy modifications of those already given by means of computations found in [8, pp. 141–5], where the case $h>0$ is explicitly treated. We state these results for future reference.

Theorem 2.6. *Suppose assumptions* (a) *and* (b) *of Theorem 2.1 are modified as follows:*

(a) *There are two distinct solutions Π_0 and Π_1 of (2.2) each homoclinic to 0; and*

(b) *$\Pi_0 \cup \Pi_1 \cup \{0\}$ is the limit as $h \downarrow 0$ of a continuous family Π_h of periodic solutions of (2.2) with energies $h>0, h$ small, all of which project into the x_1-axis under the mapping $(x, y) \to x$.*

Then the conclusions of Theorem 2.1. hold as $h \downarrow 0$, except that $(-\varepsilon, 0)$ is replaced by $(0, \varepsilon)$, and all inequalities and endpoints for the energy intervals are reversed.

Theorem 2.7. *In Theorem 2.4 replace assumptions* (a) *and* (b) *by the corresponding assumptions in Theorem 2.6, where in* (b) *we require only that the Π_h all project into some fixed line segment. Then the conclusions of Theorem 2.6 hold as $h \downarrow h_0 = W(p)$.*

An example of a potential W satisfying the hypotheses of Theorem 2.7 would be a "double bowl" potential; the critical point p sits in the "pass," which at energies $h>h_0$ connects the two bowls.

Theorem 2.8. *Consider an analytic Hamiltonian $H: R^4 \to R$ of the form*

$$H(x, y)=\tfrac{1}{2}|y|^2+W(x),$$

let p_j be saddle points of $W: R^2 \to R$ with eigenvalues $-\alpha_j^2, \beta_j^2$, where α_j and $\beta_j>0$, $j=1, \ldots, n$, and suppose $W(p_j)=h_0$ for all j. Also, assume

(a) *There are $2n$ solutions Π_1, $\Pi_j^{\pm}$, Π_{2n} of the associated system* (2.2), *$j=2, \ldots, n$, with Π_1 and Π_{2n} homoclinic to $(p_1, 0)$ and $(p_n, 0)$ respectively, and for $j=2, \ldots, n$, Π_j^+ is heteroclinic from $(p_{j-1}, 0)$ to $(p_j, 0)$, while Π_j^- is heteroclinic from $(p_j, 0)$ to $(p_{j-1}, 0)$; and*

(b) $\Pi_1 \cup \Pi_{2n} \cup \left\{\bigcup_{j=2}^{n} \Pi_j^{\pm}\right\} \cup \left\{\bigcup_{j=1}^{n} (p_j, 0)\right\}$ *is the limit as $h \downarrow h_0=W(p_j)$, $j=1, \ldots, n$, of a continuous family of periodic solutions Π_h of (2.2) with energies $h>h_0$, $h-h_0$ small, all of which project into a fixed line segment in the x-plane (which must therefore contain all the p_j, $j=1, \ldots, n$).*

Then all conclusions of Theorem 2.6 hold as $h \downarrow h_0$.

Figure 8 depicts a typical graph for a potential which satisfies the hypotheses of Theorem 2.8, with the critical points all lying on the x_1-axis. Note the many possibilities for applying the previous theorems as $h \uparrow h_0$, and also at critical energies other than $h=h_0$.

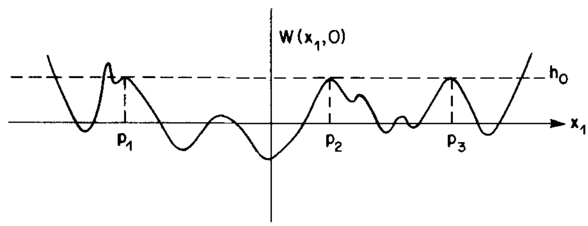

Fig. 8

3. Energy Intervals of Hyperbolicity and Hill's Equations

In this section we give conditions for the existence of energy intervals over which the families of periodic orbits Π_h considered in the theorems of Section 2 are hyperbolic. This will involve the study of a Hill's equation associated with the Poincaré map along the periodic orbit. Henceforth we assume the reader's elementary acquaintance with the theory of Hill's equation [16, 23], and in particular the definition and properties of the Hill discriminant.

Let $H: R^4 \to R$ be a C^2-Hamiltonian whose associated differential equations (2.2) admit a non-trivial periodic solution $\Pi(t)$. Denote the linearized equations along $\Pi(t)$ by

$$\dot{Z} = JH_{**}(\Pi(t))Z, \qquad Z(0) = I_4, \quad Z = (z_{ij}), \tag{3.1}$$

where H_{**} denotes the Hessian matrix of second partials of H, and for $0 < T$ = minimal period of $\Pi(t)$, set

$$d = z_{22}(T) + z_{44}(T). \tag{3.2}$$

Theorem 3.1. *Let the Hamiltonian H be C^2 smooth. Assume $\Pi(t)$ is a nontrivial periodic solution of* (2.2), *with minimal period $T > 0$, that projects to the x_1-axis under the mapping $(x, y) \to x$. Then we have:*

(a) *$\Pi(t)$ is hyperbolic when $|d| > 2$, parabolic when $|d| = 2$, and elliptic when $|d| < 2$.*

(b) *Assume in addition that $H_{x_2 y_2} \equiv 0$ and $H_{y_2 y_2} \equiv 1$ along $\Pi(t)$. Then the Poincaré map along $\Pi(t)$ of the (x_2, y_2)-plane (based at $\Pi(0)$) into itself will have Jacobian matrix*

$$P(T) = \begin{pmatrix} z_{22} & z_{24} \\ \dot{z}_{22} & \dot{z}_{24} \end{pmatrix}(T), \tag{3.3}$$

and $d = \operatorname{trace} P(T)$, where the $z_{22}(t)$ and $z_{24}(t)$ of (3.1) *are the "normalized" solutions $(z_{22}(0) = 1 = \dot{z}_{24}(0),\ \dot{z}_{22}(0) = 0 = z_{24}(0))$ of the single Hill's equation*

$$\ddot{z} + H_{x_2 x_2}(\Pi(t))\, z = 0. \tag{3.4}$$

Proof. (a) The assumption that $\Pi(t)$ projects to the x_1-axis implies $H_{x_1 x_2} = H_{y_1 x_2} = H_{x_1 y_2} = H_{y_1 y_2} = 0$ along $\Pi(t)$. This forces the $Z(t)$ of (3.1), with $Z(0) = I_4$, to have the form

$$Z(t)=\begin{pmatrix} z_{11} & 0 & z_{13} & 0 \\ 0 & z_{22} & 0 & z_{24} \\ z_{31} & 0 & z_{33} & 0 \\ 0 & z_{42} & 0 & z_{44} \end{pmatrix}(t). \tag{3.5}$$

Setting $E=\dot{\Pi}(0)=(H_{y_1}, 0, -H_{x_1}, 0)$ and $N=\operatorname{grad} H(\Pi(0))=(H_{x_1}, 0, H_{y_1}, 0)$ gives

$$\begin{aligned} Z(T)E&=E, \\ Z^*(T)N&=N, \end{aligned} \tag{3.6}$$

where the $*$ denotes transpose. It follows from (3.6) that $+1$ is a double eigenvalue of $Z(T)$. The Hamiltonian character of the flow implies $\det Z(T)=+1$, so that the remaining two eigenvalues μ_1 and μ_2 satisfy $\mu_1\cdot\mu_2=+1$. Letting $D=\operatorname{trace} Z(T)$ and comparing $\det(Z(T)-\lambda I_4)=\lambda^4-D\lambda^3+\dots$ with $(\lambda-1)^2(\lambda-\mu_1)(\lambda-\mu_2)=\lambda^4-(2+\mu_1+\mu_2)\lambda^3+\dots$, we find that

$$\mu_1, \mu_2=(\tfrac{1}{2})\left[(D-2)\pm(D(D-4))^{\frac{1}{2}}\right]. \tag{3.7}$$

When written out in coordinates, the relations (3.6) imply $z_{11}(T)+z_{33}(T)=+2$, so that $D=d+2$ and (3.7) reduces to

$$\mu_1, \mu_2=(\tfrac{1}{2})\left[d\pm(d^2-4)^{\frac{1}{2}}\right]. \tag{3.8}$$

Since μ_1 and μ_2 are the eigenvalues of the planar Poincaré maps of Section 2, part (a) follows.

(b) This follows directly from the stated hypotheses and the initial conditions $Z(0)=I_4$, since using the form (3.5) in (3.1) gives

$$\begin{aligned} \dot{z}_{2j}&=H_{x_2y_2}(\Pi(t))\,z_{2j}+H_{y_2y_2}(\Pi(t))\,z_{4j}, \\ \dot{z}_{4j}&=-H_{x_2x_2}(\Pi(t))\,z_{2j}-H_{x_2y_2}(\Pi(t))\,z_{4j}, \end{aligned} \tag{3.9}$$

for $j=1,2,3,4$. One can check that the eigenvalues of $P(T)$ are those given by (3.8). Q.E.D.

Remarks. (1) The hypotheses on the second partials of H in part (b) of Theorem 3.1 are automatically satisfied when $H(x,y)=\frac{1}{2}|y|^2+W(x)$.

(2) If H is analytic then the component functions and periods of the Π_h are analytic in h for $h\in(h_j, k_j)\cup(k'_j, h'_j)$; this follows by Poincaré continuation [31; §21]. (3.8) then implies that $\mu_1=\mu_1(h)$ and $\mu_2=\mu_2(h)$ depend analytically on h for such h. Recall that this was the only unfinished item in the proof of Theorem 2.1.

Under the hypotheses of Theorem 3.1(b), if $Q(t)=H_{x_2x_2}(\Pi(t))$ has the same minimal period T as $\Pi(t)$, then d equals the Hill discriminant Δ of (3.4). If $Q(t)$ is even in t, then standard identities in [23, Chapter 1] imply:

(C1) If $Q(t)$ has minimal period T, then $d=\Delta=2z_{22}(T)$ and Π is hyperbolic when $|z_{22}(T)|>1$, and elliptic when $|z_{22}(T)|<1$.

(C2) If $Q(t)$ has minimal period $T/2$, then $d=\Delta^2-2$ and $\Delta=2z_{22}(T/2)$. Then Π is elliptic if and only if $0<|\Delta|<2$. Moreover, Π can never be hyperbolic with negative eigenvalues since $d\geqq -2$. (This case occurs in Example C of Section 6.)

Theorem 3.2. *Let $H: R^4 \to R$ be real analytic and satisfy the hypotheses of Theorems 2.1 and 3.1(b), where $\Pi_h(t)$ of period T_h is substituted for $\Pi(t)$ in (3.4). Then any nontrivial solution $z(t; h)$ of (3.4) has a number of zeros in the interval $0\leqq t\leqq T_h$ that tend to $+\infty$ as $h\uparrow 0$. (The same conclusions hold for the other variants of Theorem 2.1 in Section 2 as $h\to h_0$.)*

Proof. Recall from Section 2 that the argument of $v_h=D\,\Phi_h(v)$ tends to $-\infty$ as $h\uparrow 0$ for any vector v in the (w_1, w_2)-plane. Since all planar Poincaré maps along $\Pi_h(t)$ are conjugate, for any vector $e\neq 0$ in the (x_2, y_2)-plane the argument of $P_h(T_h)\,e$ must also tend to $-\infty$ as $h\uparrow 0$. This implies that the curve $\begin{pmatrix} z(t;h) \\ \dot z(t;h)\end{pmatrix}$ $=P_h(t)\,e$ wraps clockwise around the origin in the (x_2, y_2)-plane adding (at least) two zeros of $z(t; h)$ in the interval $0\leqq t\leqq T_h$ for each lessening by -2π in the argument of $P_h(T_h)\,e$ as $h\uparrow 0$. Q.E.D.

Consider the family of Hill's equations parametrized by h,

$$\ddot z+(\lambda+Q(t;h))\,z=0, \tag{3.10}$$

where $Q(t;h)$ is C^0 smooth. The "Oscillation Theorem" [23, Theorem 2.1] applies at each fixed value of h to imply that (3.10) has a real periodic solution of the same minimal period as Q, or twice this period, if and only if λ is on a *stability boundary* at which the Hill discriminant $\Delta=\Delta(\lambda, h)$ of (3.10) satisfies $\Delta=\pm 2$. By a result of HAUPT [23, Theorem 2.14] the number of zeros such periodic solutions may have in one period is fixed on each stability boundary, with the "higher order" stability boundaries allowing more zeros.

Theorem 3.3. *Let $H: R^4\to R$ be real analytic and satisfy the hypotheses of Theorems 2.1 and 3.1(b). Assume (3.4), with $\Pi_h(t)$ substituted for $\Pi(t)$, is of the form (3.10), with $\lambda=\lambda(h)$ and $Q(t;h)$ both C^1 smooth and Q even in t. Assume that for all energies h sufficiently near 0 the Hill's equation (3.10) has for each h all intervals of instability above a certain fixed level (independent of h) not collapsing. Then under conditions (C1) or (C2) above, as $h\uparrow 0$, the periodic orbits Π_h will have an infinite number of transitions to hyperbolicity over distinct nontrivial energy intervals converging to 0. When (C1) holds these transitions occur at both the eigenvalues $\mu_1=1=\mu_2$ and $\mu_1=-1=\mu_2$; when (C2) holds they occur only at $\mu_1=1=\mu_2$. (The same conclusions hold for the other variants of Theorem 2.1 in Section 2 as $h\to h_0$.)*

Proof. The hypotheses imply that $\Delta(\lambda, h)$ is C^1 smooth in λ and h, and hence the stability boundaries in the (λ, h)-plane where $\Delta=\pm 2$ are graphs of functions continuous in h (Lemma 4.2 gives an analytic analogue of this fact). The parameter arc $\gamma(h)=(\lambda(h), h)$ is also continuous in h. Theorem 2.1 implies that the eigenvalues μ_1 and μ_2 repeatedly take the values ± 1 as $h\uparrow 0$. Since $d=\mu_1+\mu_2$, the relation between d and Δ given in conditions (C1) and (C2) implies that

$(\Delta \circ \gamma)(h)$ repeatedly takes the values $+2$ and/or -2 as $h\uparrow 0$. At these values of Δ the Oscillation Theorem implies that (3.10) has periodic solutions to which Theorem 3.2 and Haupt's Theorem (discussed above) apply, showing that $\gamma(h)$ intersects all stability boundaries above some level as $h\uparrow 0$. The hypothesis that the regions of instability above a certain level do not collapse and another application of the Oscillation Theorem imply that $(\Delta \circ \gamma)(h)$ will repeatedly take on values outside of $[-2, 2]$ as $h\uparrow 0$. The remainder of the proof then follows from the relation between d and Δ in conditions (C1) and (C2) and the formula (3.8) for the eigenvalues $\mu_1(h)$ and $\mu_2(h)$ of the Poincaré map along Π_h. Q.E.D.

We conclude this section with some remarks on Theorem 3.3 and a general observation about Hill's equations.

(A) The hypothesis in Theorem 3.3 on the failure of intervals of instability to collapse cannot be eliminated. Example *D* of Section 6 shows that the twisting in Moser coordinates, although sufficient to give the energy intervals of ellipticity, is not sufficient for the transitions to hyperbolicity.

Variations of Theorem 3.3 can be easily formulated for the case when only an infinite number of fixed (not necessarily successive) intervals of instability do not collapse. It is an open question whether such a case can occur under condition (C1) with, for example, transitions to hyperbolicity only at $\mu_1 = 1 = \mu_2$.

(B) DEVANEY [12, 13, 14] and HENRARD [19] consider a real analytic Hamiltonian flow with two degrees of freedom near a critical point P of energy 0 with eigenvalues $(\pm\alpha \pm i\beta)$, α and β positive. If the flow admits a *nondegenerate* homoclinic orbit asymptotic to P as time $t\to\pm\infty$, then using Moser coordinates about P they prove the *existence* of a one-parameter family of periodic orbits Π_h which goes through an infinite sequence of oscillations between ellipticity and hyperbolicity as $h\to 0$. Furthermore, the eigenvalues become *unbounded* in the positive *and* negative directions as $h\to 0$ (see [13, Theorem B] or [14, Theorem E]). In our case this latter result is not generally attainable, as Example D of Section 6 illustrates for the "double twist" case of Theorem 2.5.

We note that under both the Devaney-Henrard eigenvalue condition on P and the one we assume, results of MEYER [25, 26] can be applied. These state that *generically* one or two additional families of periodic orbits branch off from Π_h at those values of h for which the eigenvalues of the Poincaré mapping are n^{th} roots of unity.

(C) For $Q_1(x_1) = \sum_{i=2}^{k_1} a_{i0}(x_1)^i$ assume that the relation $t = \int_0^{g_h(t)} \frac{dx_1}{\sqrt{2(h - Q_1(x_1))}}$ gives a well-defined periodic Abelian function $g_h(t)$. Given another polynomial $Q_2(x_1) = \sum_{i=0}^{k_2} 2a_{i2}(x_1)^i$, define a potential

$$W(x_1, x_2) = \sum_{i,j} a_{ij}(x_1)^i (x_2)^j, \tag{3.11}$$

where the double sum is over those indices $0 \leqq i \leqq k_1$ and $0 \leqq j \leqq k_2$ satisfying $j \neq 1$ and $i+j \geqq 2$. The coefficients a_{i0} and a_{i2} in (3.11) are specified by the polynomials Q_1 and Q_2, and the other a_{ij} are arbitrary. Then $g_h(t)$ is the periodic solution of $\ddot{x} = -W_x$ along the x_1-axis with energy h satisfying the initial conditions $g_h(0)=0$, $\dot{g}_h(0)=\sqrt{2h}$. Moreover, the Hill's equation (3.4) associated with the Poincaré map along $g_h(t)$ takes the form

$$\ddot{z} + Q_2(g_h(t))\, z = 0. \tag{3.12}$$

Thus any Hill's equation of the form (3.12) can be "embedded" into the Poincaré map along a periodic orbit in a Hamiltonian system where $H(x, y) = \frac{1}{2}|y|^2 + W(x)$ for a suitable potential given by (3.11). The previous results on the status of stability of the periodic orbit g_h can then be applied at critical points $(p, 0)$ on the x_1-axis of W for which

$$\begin{aligned} &W(p, 0) = Q_1(p) = h_0, \\ &(\partial W/\partial x_1)(p, 0) = (\partial Q_1/\partial x_1)(p) = 0, \\ &(\partial^2 W/\partial x_1^2)(p, 0) = (\partial^2 Q_1/\partial x_1^2)(p) = -\alpha^2, \\ &(\partial^2 W/\partial x_2^2)(p, 0) = Q_2(p) = \beta^2, \end{aligned} \tag{3.13}$$

with α and $\beta > 0$.

4. A Special Class of Hill's Equations

In order to apply the results of the previous sections to the examples of Section 6, we need to present some general properties of the Hill's equation

$$\ddot{z} + (a + b k^2 \cdot sn^2(t; k^2))\, z = 0, \tag{4.1}$$

where a and b are real parameters and k is the modulus of the Jacobi elliptic function sn. Note that we write $sn(t; k^2)$ rather than $sn(t; k)$, using the fact that the Taylor series for $sn(t; k)$ contains only even powers of k. The remaining notation will be standard and we refer to [4, Chapters 1 and 2] and [23] respectively for definitions and notations relating to elliptic functions and Hill's equations.

Lemma 4.1. *For $b > 0$ and $0 < c = k^2 < 1$ both fixed, the Hill discriminant $\Delta = \Delta(a)$ of* (4.1) *has only simple roots, and no interval of instability collapses.*

Proof. The transformation on pp. 103-4 of [23] will bring (4.1) into the form of Ince's equation, with the relevant polynomials $Q(\mu)$ and $Q^*(\mu)$, used in applying [23, Theorem 7.1], being

$$Q(\mu) = \frac{c}{2(2-c)} (4\mu^2 + 2\mu + b),$$

$$Q^*(\mu) = \frac{c}{2(2-c)} (4\mu^2 - 2\mu + b).$$

For $b>0$ these cannot have integral roots, and thus the result follows (see the discussion on pp. 90–93 [23]). Q.E.D.

The result in Lemma 4.1 contrasts with the case of the classical Lamé equation where $b=-n(n+1)$ in (4.1) (and n is a positive integer) and all but $n+1$ intervals of instability collapse [23, Theorem 7.8].

To write (4.1) in a more convenient form let $c=k^2$, and [4, p. 10 (19) and p. 17 (11), (12)]

$$K(c)=\int_0^1 (1-x^2)^{-\frac{1}{2}}(1-cx^2)^{-\frac{1}{2}}dx, \tag{4.2a}$$

$$E(c)=\int_0^1 (1-x^2)^{-\frac{1}{2}}(1-cx^2)^{\frac{1}{2}}dx, \tag{4.2b}$$

$$K'(c)=\int_0^1 (1-x^2)^{-\frac{1}{2}}(1-(1-c)x^2)^{-\frac{1}{2}}dx, \tag{4.2c}$$

for $0<c<1$. In fact K is real analytic at each $0\leqq c<1$ [11, p. 409], and is the real quarter-period of $sn(t;c)$. We refer the reader to [4, p. 10] for the appropriate asymptotic properties as $c\downarrow 0$ and $c\uparrow 1$.

We shall make use of the function

$$q(c)=\exp[-\pi(K'(c)/K(c))], \tag{4.3}$$

which is real analytic at each $c\in[0,1)$ [10, p. 413] with

$$q(c)\to 0^+ \quad \text{as } c\downarrow 0, \qquad q(c)\to 1 \quad \text{as } c\uparrow 1. \tag{4.4}$$

Using the identity $c\cdot sn^2(t;c)=1-dn^2(t;c)$ and the Fourier expansion of $dn^2(t;c)$ [10; p. 419] we have

$$c\cdot sn^2(t;c)=\left[1-(E/K)+q\sum_1^\infty a_n\cos(n\pi t/K)\right](c), \tag{4.5}$$

where

$$a_n(c)=\left[\frac{-2\pi^2 nq^{n-1}}{K^2(1-q^{2n})}\right](c). \tag{4.6}$$

The series (4.5) is convergent for all real t and $0\leqq c<1$, and it can be shown that $K^2(1-q^{2n})=K^2[1-\exp(-2n\pi K'/K)]\to+\infty$ as $c\uparrow 1$ for all $n\geqq 1$. (To see this write out the series expansion for exp and look at the leading terms, using $K(c)\to+\infty$ and $K'(c)\to\pi/2$ as $c\uparrow 1$.) Hence

$$\begin{aligned} &a_n(c)\to 0 \quad \text{as } c\uparrow 1, n\geqq 1,\\ &a_1(c)\to -8 \quad \text{as } c\downarrow 0,\\ &a_n(c)\to 0 \quad \text{as } c\downarrow 0, n\geqq 2. \end{aligned} \tag{4.7}$$

Using (4.6) we can now rewrite (4.1) as

$$\ddot{z}+(A+BQ(t;c))z=0, \tag{4.8}$$

where

$$A = a + b[1 - E/K], \tag{4.9a}$$

$$B = bq, \tag{4.9b}$$

$$Q(t;c) = \sum_1^\infty a_n(c)\cos(n\pi t/K), \tag{4.9c}$$

and the functions E, K, q are evaluated at c. Observe that $Q(t;c)$ is even and periodic in t, with minimal period $2K(c)$ and mean value 0 over one period.

Lemma 4.2. *Let A, B, and c be independent parameters in* (4.8), *with $Q(t;c)$ specified by* (4.9c). *For $r \in R$ consider the set S_r of points (A,B,c) where the Hill discriminant satisfies*

$$\Delta(A,B,c) = r. \tag{4.10}$$

Then B and c may be used as real analytic coordinates on S_r near each $(A,B,c) \in S_r$ satisfying $0 \leqq c < 1$ provided $|r| < 2$. Moreover, if $|r| = 2$, then B and c may be used as real analytic coordinates on S_r near each $(A,B,c) \in S_r$ satisfying $0 \leqq c < 1$ provided $B > 0$ (recall that if $B = B(c)$ is given by (4.9b), *then $B(c) \to 0$ as $c \downarrow 0$ by* (4.4)).

Proof. Since $K(c)$ and $q(c)$ are real analytic at each $0 \leqq c < 1$, the function $Q(t;c)$ is analytic for all t and $0 \leqq c < 1$. Thus (4.8) is an analytic differential equation whose solutions are analytic at these (t,c) and at all values of the parameters A and B [15, p. 299]. Setting $t = 2K(c)$ implies that the Hill discriminant Δ is a real analytic function of A, B, and c. If (A,B,c) satisfies (4.10), and if $|r| < 2$, then $\partial\Delta/\partial A \neq 0$ by [16, p. 27(b)]. Also, if (A,B,c) satisfies (4.10) and $|r| = 2$, then Lemma 4.1 shows that for fixed (B,c) the equation (4.10) has a simple root in A provided $B > 0$; thus again $\partial\Delta/\partial A \neq 0$. The analytic implicit function theorem [15, p. 272] now implies the result. Q.E.D.

To cover the case $|r| = 2$ and $B = 0$ not considered in Lemma 4.2, we now state a 1-parameter version of a result of E. HOEHN [20], using the notation of LOUD [21, Theorem 1].

Theorem 4.3 (HOEHN). *In* (4.8) *let $Q(t;c)$ be even and $2K(c)$-periodic in t, of mean value zero, differentiable, and have convergent Fourier series $\sum_1^\infty b_n \cos(n\pi t/K(c))$ for all $c \in [0,1)$. Then for each fixed $c \in [0,1)$ (hence fixed period $2K(c)$), the following results hold in the plane determined by $c =$ constant in (A,B,c)-space (hereafter called the (A,B,c)-plane):*

(a) *The stability boundary $A = A_0(B)$ of the zeroth instability region U_0 is tangent to the line $A = 0$ at $(0,0,c) = M_0(c)$.*

(b) *For $n \geqq 1$ and $b_n = 0$, the two stability boundaries of the n^{th} instability region U_n are both tangent to the line $A = (n\pi/2K(c))^2$ at $([n\pi/2K(c)]^2, 0, c) = M_n(c)$.*

(c) *For $n \geqq 1$ and $b_n \neq 0$, the two stability boundaries of U_n are respectively tangent to the lines $2(A - [n\pi/2K(c)]^2) \pm Bb_n = 0$ at $M_n(c)$.*

(In our applications the b_n will equal the $a_n(c)$ of (4.6).)

Proof. Allowing for the period $2K(c)$ instead of π, the proof is exactly as in [21, pp. 229–230], although that proof contains additional information on the stability boundaries, not needed here. Q.E.D.

We conclude this section with some information about the stability boundaries of (4.8), (4.9c) in (A,B,c)-space.

Theorem 4.4. *The stability boundaries* $|\Delta(A,B,c)|=2$ *for* (4.8), (4.9c), *with* $B\geqq 0$ *and* $0\leqq c<1$, *intersect the A-axis at* $\{(n^2,0,0),\ n=0,1,\dots\}$, *and at these points their tangent planes are spanned by the vectors*

$$\begin{aligned}&(-n^2/2,0,1),(0,1,0) && \text{for } n\neq 1, \text{ and}\\ &(-1/2,0,1),(\pm 4,1,0) && \text{for } n=1.\end{aligned}$$

Thus only the two boundaries for the first region of instability meet non-tangentially on the A-axis (at $(1,0,0)$*).*

Proof. For fixed $c\in[0,1)$, the stability boundaries for the n^{th} region of instability U_n in the (A,B,c)-plane, $n\geqq 0$, issue from $M_n(c)$ where, for $K(c)$ as in (4.2a), one can show

$$A_n(c)=[n\pi/2K(c)]^2\to\begin{cases}n^2 & \text{as } c\downarrow 0\\ 0 & \text{as } c\uparrow 1.\end{cases}$$

From (4.2a) we have $(dK/dc)(0)=\pi/8$, thus giving

$$(dA_n/dc)(0)=-n^2/2.$$

Hence the stability curves $M_n(c)$, $n\geqq 0$, appear in the $B=0$ plane as in Figure 9, with tangents

$$(dM_n/dc)(0)=M_n'(0)=(-n^2/2,0,1). \tag{4.11}$$

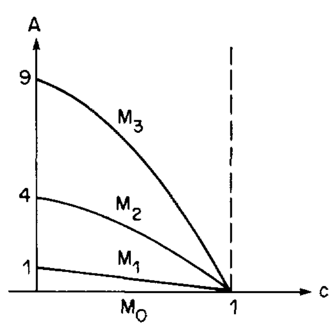

Fig. 9

After the definition of $a_0(c)=0$, Theorem 4.3 states, for fixed c and with $K(c)$ as in 4.2(a), that the stability boundaries intersect the A-axis in the (A,B,c)-plane at $A_n(c)$ with slopes which, by (4.7), have the following limiting behavior as $c\downarrow 0$:

$$(dA/dB)_{B=0}=\pm a_n(c)/2\to\begin{cases}0 & \text{for } n\neq 1,\\ \mp 4 & \text{for } n=1.\end{cases}$$

Thus in the $c=0$ plane the stability curves, obtained from $A=A(B,0)$ at $|\Delta|=2$ in Theorem 4.3 by using the differentiable analogue of Lemma 4.2, intersect the A-axis perpendicularly at $(n^2,0,0)$ at all levels except $n=1$, their tangent vectors being

$$\begin{cases} E_n=(0,1,0) & \text{for } n\neq 1, \\ E_1^{\pm}=(\pm 4,1,0) & \text{for } n=1. \quad \text{Q.E.D.} \end{cases} \tag{4.12}$$

5. A Criterion of Stability for Low Energies

Here we consider a C^3 smooth Hamiltonian $H:R^4\to R$, admitting 0 as a critical point, whose associated differential equations (2.2) admit a one-parameter family of periodic solutions Π_h, $0\leqq h<h_1$, h_1 small. We assume that the Π_h all project into the x_1-axis under the mapping $(x,y)\to x$, with $\Pi_0=(0,0,0,0)$. We further assume that for $h>0$ the linearized Poincaré map along Π_h can be reduced as in Theorem 3.1(b) to the single Hill's equation (3.4) in the specific form (4.1), with the coefficients a and b functionally dependent on $c=k^2$ with $c=c(h)$. Finally, we assume that

(i) $b(c(h))$ and $c(h)$ are positive for $h>0$ small;
(ii) $c(h)\to 0^+$ as $h\downarrow 0$;
(iii) The derivatives da/dc, db/dc, dc/dh exist for $c>0$, $h>0$, and possess limits as $c\downarrow 0$, $h\downarrow 0$; and
(iv) The limits $\lim_{c\downarrow 0} a(c)=a(0)$ and $\lim_{c\downarrow 0} b(c)=b(0)\geqq 0$ both exist.

Theorem 5.1. *Under the above hypotheses, set $\Gamma(c)=(A(c),B(c),c)$ (see (4.9)), and define $\Gamma'(0)=\lim_{c\downarrow 0}(d\Gamma/dc)$. We then have:*

(1) *Under condition (C1) of Section 3, if $a(0)\geqq 0$ and $a(0)\neq n^2$ for any integer n, then there is a $\delta>0$ such that Π_h is elliptic for $0<h<\delta$. The same conclusion holds under condition (C2) provided $a(0)\geqq 0$ and $a(0)\neq n^2$ or $(n+\frac{1}{2})^2$ for any integer n. (Under either condition (C1) or (C2), if $a(0)<0$, then there is a $\delta>0$ such that Π_h is hyperbolic for $0<h<\delta$.)*

(2) *Under condition (C1) or (C2), if $a(0)=n^2$ for some integer $n\geqq 2$, then there is a $\delta>0$ such that Π_h is elliptic for $0<h<\delta$ provided $\langle\Gamma'(0),E_n\times M_n'(0)\rangle\neq 0$, where $M_n'(0)$ and E_n are given by (4.11) and (4.12). The same conclusion holds for $n=0$ provided $\langle\Gamma'(0),E_0\times M_0'(0)\rangle>0$.*

(3) *Under condition (C1) or (C2), if $a(0)=1$, then there exists a $\delta>0$ such that Π_h is elliptic for $0<h<\delta$ provided either $\langle\Gamma'(0),E_1^+\times M_1'(0)\rangle>0$ or $\langle\Gamma'(0),E_1^-\times M_1'(0)\rangle<0$.*

Moreover, if the Hamiltonian H is real analytic, then Π_h is elliptic stable at values of h in a set of full measure in the interval $(0,\delta)$ for cases (2) and (3) above. A corresponding result holds in case (1) provided Π_h is known to become hyperbolic as h increases (as in Theorem 3.3).

Remark. Geometrically, the conditions in (2) for ellipticity simply state that $\Gamma(c)$ "approaches" $a(0)=n^2=A$ in the $(A,B,0)$-plane as in Figure 10. Analogous con-

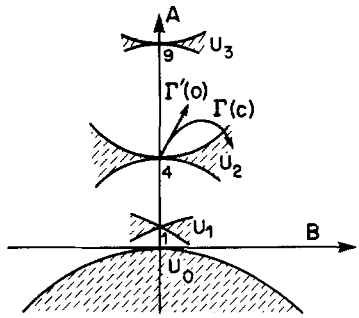

Fig. 10

ditions obviously exist in (3) to guarantee hyperbolicity on some energy interval $(0, \delta)$ when $\Gamma(0) = (1, 0, 0)$.

Proof. (1) From general results on Hill's equation [2, p. 129], the point $(a(0), 0, 0)$ is in the interior of a stability region in the $(A, B, 0)$-plane. Since $c(h) \to 0^+$ as $h \downarrow 0$, and $A(c) \to a(0)$, $B(c) = b(c)q(c) \to 0^+$ as $c \downarrow 0$, the parameter arc $\Gamma(c(h))$ remains interior to the stability region for small h. The requirement that $a(0) \neq (n+\frac{1}{2})^2$ when condition (C2) holds comes from the need to exclude the surfaces $\Delta = 0$ and the observation [2, p. 129] that the curves $\Delta(A, B, 0) = 0$ intersect the A-axis at $A = (n+\frac{1}{2})^2$.

(For $a(0) < 0$ the result follows since all the regions of stability and their boundaries intersect the *non-negative* A-axis.)

(2) From Theorem 4.4 and its proof we see that for $n \geq 2$ the two stability boundaries of the n^{th} region of instability U_n meet at $(n^2, 0, 0)$ with common tangent plane having normal $E_n \times M_n'(0)$. The hypotheses guarantee that the parameter arc $\Gamma(c)$ enters, for small positive c, one or the other of the two regions of stability adjacent to U_n, since the boundaries are C^1 surfaces for $B > 0$, by the differentiable analogue of Lemma 4.2, and C^1 at $B = 0$ by hypothesis. The case $n = 0$ is clear.

(3) This follows by the same type of reasoning as in (2).

The last statements of the theorem follow by observing that, under the hypotheses that imply ellipticity, the function $(\Delta \circ \Gamma)(c)$ is not constant over any non-trivial c-interval (h-interval) in case (1), and in cases (2) and (3) strictly monotone in c for $c > 0$ small by Lemma 4.2. Thus the Moser-Rüssmann criterion, as presented in the proof of Theorem 2.1, implies the conclusion via the relation between the eigenvalues of the Poincaré map and Δ given by (3.8) and conditions (C1) and (C2). (See Remark (2) following Theorem 3.1.) Q.E.D.

Remark. This theorem reflects the need to obtain a criterion of stability for *specific* orbits in *specific* Hamiltonians. The examples in Section 6 show that its application, although limited by reliance on "first order" approximations, can provide useful information.

6. The Examples

Details concerning the applicability of the theorems in Sections 2–5 will only be sketched for the examples; any gaps can easily be filled.

(A) *The Hénon-Heiles Hamiltonian.* We consider the Hamiltonian

$$H(x,y)=\tfrac{1}{2}|y|^2+\tfrac{1}{2}|x|^2+\varepsilon x_1^3-x_1x_2^2, \tag{6.1}$$

with potential

$$W(x)=\tfrac{1}{2}|x|^2+\varepsilon x_1^3-x_1x_2^2, \tag{6.2}$$

and parameter $\varepsilon>0$. When $\varepsilon=\frac{1}{3}$ the Hamiltonian H is equivalent to the HÉNON-HEILES Hamiltonian [18] via the canonical transformation $(q_1,q_2,p_1,p_2)\Leftrightarrow -(x_2,x_1,y_2,y_1)$, and the potential W is invariant under rotation through an angle $\pm(2\pi/3)$. A sketch of the level curves of W, when $\varepsilon=\frac{1}{3}$, is given in Figure 11.

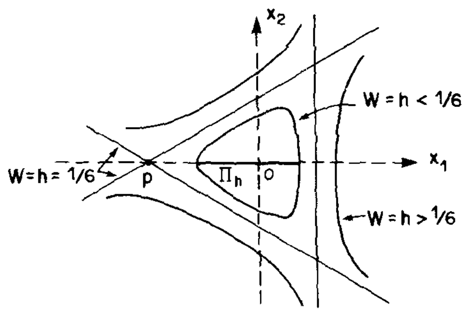

Fig. 11

The potential (6.2) admits a critical point $p=(-(3\varepsilon)^{-1},0)$ on the x_1-axis at energy $h_0=W(p)=(54\varepsilon^2)^{-1}$, with eigenvalues -1 and $1+2/(3\varepsilon)$. Moreover, the symmetry in the x_1-axis for the potential easily implies that Hypotheses (a) and (b) of Theorem 2.4 hold, and thus that result can be applied. The periodic orbits Π_h, which project to the x_1-axis with endpoints on the level curve $W=h$, exist for all $0<h<h_0$. They collapse to the origin, which is also a critical point of (6.1), as $h\downarrow 0$. This suggests that Theorem 5.1 might also be applicable, but we shall find that this is not the case.

In order to study the Hill's equation (3.4) governing the linearized Poincaré mapping we need to compute Π_h explicitly. But along this solution (6.1) becomes

$$\tfrac{1}{2}[(\dot{x}_1)^2+x_1^2]+\varepsilon x_1^3=h, \tag{6.3}$$

leading to the second-order equation

$$\ddot{x}_1+x_1+3\varepsilon x_1^2=0. \tag{6.4}$$

We choose as trial solution

$$x_1(t)=\delta+\alpha\cdot sn^2(\beta t;c) \tag{6.5}$$

satisfying the initial conditions $x_1(0)=\delta>0$, $\dot{x}_1(0)=0$, corresponding to the point on the positive x_1-axis where $x_1(\Pi_h)$ intersects the level curve $W=h$. (Recall from Section 4 that we are using $c=k^2$ where k is the modulus.) We first observe that (6.3) implies that $\delta=\delta(\varepsilon,h)$ is the unique positive solution to $\frac{1}{2}\delta^2+\varepsilon\delta^3=h$. Since $0<h<(54\varepsilon^2)^{-1}$, we have $0<\delta(\varepsilon,h)<1/(6\varepsilon)$, where for fixed ε the quantity $\delta(\varepsilon,h)\downarrow 0$ if and only if $h\downarrow 0$. Substituting (6.5) into (6.4) gives the algebraic system

$$2\beta^2 c+\varepsilon\alpha=0, \tag{6.6a}$$

$$4\beta^2(1+c)-(1+6\varepsilon\delta)=0, \tag{6.6b}$$

$$2\alpha\beta^2+\delta+3\varepsilon\delta^2=0. \tag{6.6c}$$

The requirements that $\lim_{h\downarrow 0}\alpha=0$ and that $\lim_{\varepsilon\downarrow 0}\alpha$ be bounded give the unique solution

$$\alpha=-(4\varepsilon)^{-1}[1+6\varepsilon\delta-\sigma], \tag{6.7a}$$

$$\beta^2=[2\varepsilon\delta(1+3\varepsilon\delta)][(1+6\varepsilon\delta)-\sigma]^{-1}, \tag{6.7b}$$

$$c=[(1+6\varepsilon\delta)-\sigma]^2[16\varepsilon\delta(1+3\varepsilon\delta)]^{-1}, \tag{6.7c}$$

where $\sigma=[1-4\varepsilon\delta-12(\varepsilon\delta)^2]^{\frac{1}{2}}$. The following observations concerning these solutions will prove useful:

(i) For a fixed $\varepsilon>0$, $c\downarrow 0$ if and only if $h\downarrow 0$;
(ii) since $0<\varepsilon\delta<1/6$, α is real and negative;
(iii) $\lim_{\varepsilon\delta\downarrow 0}\beta^2=1/4=\lim_{\varepsilon\delta\uparrow 1/6}\beta^2$;
(iv) $\lim_{\varepsilon\delta\uparrow 1/6} c=1$;
(v) for a fixed $\varepsilon>0$, $\delta=\delta(c)$ is a C^1-invertible function of c from (6.7c), and $c=c(\delta(h))$ is a C^1-function of h.

Now fix $\varepsilon>0$. The Hill's equation (3.4) then becomes

$$\ddot{z}+(1-2x_1(t))z=\ddot{z}+(1-2\delta-2\alpha\cdot sn^2(\beta t;c))z=0, \tag{6.8}$$

which, after making the change of variable $\beta t=\mu$ and using (6.6), can be written in the form (4.1) as

$$z''+[a(c)+b(c)\cdot c\cdot sn^2(\mu;c)]\,z=0, \tag{6.9}$$

where $'=d/d\mu$ and

$$a(c)=(1-2\delta(c))(4c+4)(1+6\varepsilon\delta(c))^{-1}, \tag{6.10a}$$

$$b(c)=4/\varepsilon. \tag{6.10b}$$

Furthermore, (6.9) can be seen to be in the form (4.8), (4.9), with

$$\begin{aligned} A(c)&=a(c)+b(c)[1-E(c)/K(c)],\\ B(c)&=b(c)\,q(c)=(4/\varepsilon)\,q(c), \end{aligned} \tag{6.11}$$

where $q(c)$ is defined in (4.3), and $Q(t;c)$ is given by (4.9c) and (4.6).

We note that the period of $\Pi_h(t)$ is the same as that of $sn^2(\beta t; c)$, and thus condition (C1) of Section 3 is satisfied. By use of Lemma 4.1, the hypotheses of Theorem 3.3 can easily be seen to hold, and as Theorem 2.4 has already been applied, we conclude that there are two increasing sequences of energy intervals, converging to h_0, on which Π_h is respectively elliptic and hyperbolic. On the elliptic energy intervals the Moser-Rüssmann criterion applies, and the transition to hyperbolicity takes place at both $+1$ and -1 eigenvalues.

On the other hand, we now show that Theorem 5.1 does not apply; we present the calculations so as to motivate the limit hypotheses of Theorem 5.1. From [10, p. 413] we have

$$[2K(c)/\pi]^{\frac{1}{2}} = 1 + 2q(c) + O_4(q(c)). \tag{6.12}$$

Since $q(c) \to 0^+$ as $c \downarrow 0$, and $(dK/dc)(0) = \pi/8$ with $K(0) = \pi/2$, we have from (6.12)

$$\lim_{c \downarrow 0} (dq/dc) = \tfrac{1}{16}. \tag{6.13}$$

It is easy to check that $(dE/dc)(0) = -\pi/8$, and hence

$$(d/dc)(-E(c)/K(c))(0) = \tfrac{1}{2}. \tag{6.14}$$

For the parameter arc $\Gamma(c) = (A(c), B(c), c)$ of Theorem 5.1, where $A(c)$ and $B(c)$ are given by (6.11), the fact that $[1 - E(c)/K(c)] \to 0$ as $c \downarrow 0$ implies

$$\Gamma(0) = \lim_{c \downarrow 0} \Gamma(c) = (4, 0, 0), \tag{6.15}$$

independent of the parameter $\varepsilon > 0$ in (6.1).

The motivation for performing all differentiations with respect to the variable c rather than h can be seen from the relation $\frac{1}{2}\delta^2 + \varepsilon\delta^3 = h$ which implies $\lim_{h \downarrow 0} (d\delta/dh) = +\infty$. To avoid such singularities we use L'Hospital's Rule and formulate the hypotheses on the coefficients in Theorem 5.1 in terms of limits. For any C^1 function $f(c)$ we now use the notation $f'(0) = \lim_{c \downarrow 0} (df/dc)$. Using implicit differentiation in (6.6) and $\beta^2 \to 1/4$ and $\alpha \to 0$ as $c \downarrow 0$, one can calculate $\delta'(0) = 1/(4\varepsilon)$. Since $a'(0) = 4(1 - 2\delta'(0) - 6\varepsilon\delta'(0))$ from (6.10a), on using $\delta(c) \to 0$ as $c \downarrow 0$, we have

$$A'(0) = a'(0) + 2/\varepsilon = -2. \tag{6.16}$$

Since $B'(0) = 1/4\varepsilon$ by (6.13), we have, in the notation of Theorem 5.1,

$$\begin{aligned} \Gamma'(0) &= (-2, 1/(4\varepsilon), 1), \\ M_2'(0) &= (-2, 0, 1), \\ E_2 &= (0, 1, 0), \end{aligned} \tag{6.17}$$

and thus

$$\langle \Gamma'(0), E_2 \times M_2'(0) \rangle = 0, \tag{6.18}$$

independent of the parameter $\varepsilon > 0$ in (6.1). That is, the parameter arc $\Gamma(c)$ is tangent to the stability boundaries at $\Gamma(0) = (4, 0, 0)$ as $c \downarrow 0$ (equivalently, $h \downarrow 0$), and Theorem 5.1 cannot be applied.

We have already remarked that when $\varepsilon=\frac{1}{3}$ the potential (6.2) is invariant under rotation through an angle $\pm(2\pi/3)$. Thus the results above actually apply in this case to three symmetrically placed families of periodic orbits which exist for each $0<h<\frac{1}{6}$ and link one another within the associated energy manifold. Even though Theorem 5.1 cannot be invoked, preliminary numerical work does indicate that the three families of periodic orbits are elliptic for $0<h<h'\approx 0.13$, at which point they appear to become hyperbolic through the eigenvalues $\mu_1=-1=\mu_2$. Note that this happens above the energy $h\approx 0.11$ at which the transition to large scale, observable stochastic behavior of the flow is reported to take place [18, p. 77]. The relation of these stability transitions to other phenomena reported by HÉNON & HEILES [18] is explored in the survey [6], which contains references to other work. The existence of invariant tori at a number of energies in the interval $0<h<1/6$ confirms in part some conjectures of LUNSFORD & FORD [22].

(B) We consider the Hamiltonian

$$H(x,y)=\tfrac{1}{2}|y|^2+\tfrac{1}{2}|x|^2-x_1x_2^2, \tag{6.19}$$

and in Figure 12 we sketch the level curves of the potential $W=\frac{1}{2}|x|^2-x_1x_2^2$. W admits two critical points $p^\pm=(\frac{1}{2})(1,\pm\sqrt{2})$ at energy $h_0=1/8$, each with eigenvalues $-1, 2$, and Hypotheses (a) and (b) of Theorem 2.4 hold for both $p^\pm$, the relevant lines in Hypothesis (b) being $x_2=\pm\sqrt{2}x_1$.

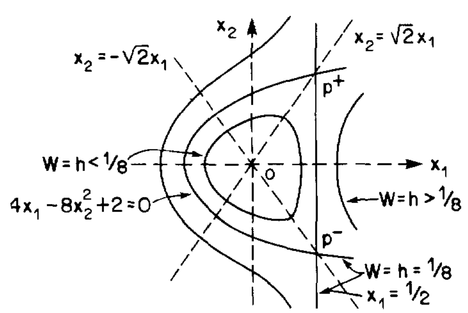

Fig. 12

To apply the results of Sections 3–5, one first rotates the line $x_2=\sqrt{2}x_1$ into the x_1-axis and then reflects through the x_2-axis to obtain

$$W^*(x)=\tfrac{1}{2}|x|^2+(\sqrt{3}/3)[(2/3)x_1^3-x_1x_2^2-(\sqrt{2}/3)x_2^3]. \tag{6.20}$$

The rescaling $x=\omega\tilde{x}$, $y=\omega\tilde{y}$ in the resulting Hamiltonian, for appropriate choice of ω, then achieves an equivalent Hamiltonian

$$H(x,y)=\tfrac{1}{2}|y|^2+\tfrac{1}{2}|x|^2+(\tfrac{2}{3})x_1^3-x_1x_2^2-\left(\frac{\sqrt{2}}{3}\right)x_2^3. \tag{6.21}$$

The single second order differential equation for $x_1(\Pi_h)$ is the same as (6.4) with $\varepsilon=\frac{2}{3}$, leading to the Hill's equation (6.8). Thus Theorem 3.4 applies with condition (C1). By analyticity the Moser-Rüssmann criterion also applies on the energy intervals of ellipticity, but Theorem 5.1 is still not applicable. Note by symmetry that the two families of periodic orbits that respectively project to the lines $x_2=\pm\sqrt{2}x_1$ of Figure 12 must undergo identical stability transitions as h changes.

There is a third family of periodic orbits that exist for all $h>0$ and project to the x_1-axis in Figure 12. By a direct calculation of the Poincaré map along these orbits, one can show that they will be hyperbolic for $0<h<h_1\approx 8$, and thereafter alternate between being elliptic and hyperbolic infinitely often as $h\uparrow+\infty$. At a fixed h all three of the families of periodic orbits discussed above link one another in the energy manifold $H=h$.

For some numerical work on variants of (6.19), with various parameters, we refer to [3] and [9].

(C) The Hamiltonian

$$H(x,y)=\tfrac{1}{2}|y|^2+\tfrac{1}{2}|x|^2-\tfrac{1}{2}x_1^2x_2^2 \tag{6.22}$$

was considered by HÉNON & HEILES [18], and special properties of solutions were shown in [7]. For a comparison of this Hamiltonian with that of Example (A), see [6].

The level lines of the potential $W=\frac{1}{2}|x|^2-\frac{1}{2}x_1^2x_2^2$ are sketched in Figure 13. Note that W is invariant under rotation through an angle $\pi/2$ and admits the four symmetrically placed critical points $(\pm 1,\pm 1)$ at energy $\frac{1}{2}$, each with eigenvalues -2, 2. It is a simple matter to see that Theorem 2.5 can be applied if we choose critical points in opposite corners of the square of Figure 13, with the relevant line segment in (b) of that theorem being the connecting diagonal through the origin.

To apply the results of Sections 3-5, we rotate the potential clockwise by $\pi/4$ radians, obtaining

$$W^*(x)=(1/2)|x|^2-(1/8)(x_1^4-2x_1^2x_2^2+x_2^4). \tag{6.23}$$

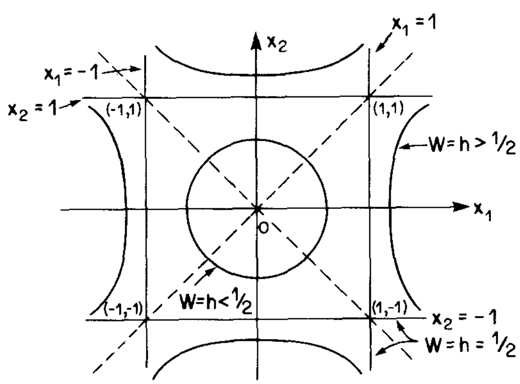

Fig. 13

The differential equation for the periodic orbit, which now lies along the x_1-axis, is seen to be

$$\ddot{x}_1 + x_1 - \tfrac{1}{2}x_1^3 = 0, \tag{6.24}$$

subject to the constraint

$$(\tfrac{1}{2})(\dot{x}_1)^2 + x_1^2 - \tfrac{1}{2}x_1^4 = h. \tag{6.25}$$

It is then a simple matter to check that (6.24) is satisfied by

$$x_1(t) = \delta \cdot sn(\beta t; c), \tag{6.26}$$

where $\delta > 0$, $\beta > 0$, and $0 < c < 1$ are defined in terms of the parameter $0 < \sigma = (1 - 2h)^{\frac{1}{2}}$ (where $0 < h < \frac{1}{2}$) by

$$\begin{aligned} &(a)\ \ \delta^2 = 2(1-\sigma), \\ &(b)\ \ \beta^2 = (\tfrac{1}{2})(1+\sigma), \\ &(c)\ \ c = (1-\sigma)/(1+\sigma). \end{aligned} \tag{6.27}$$

Notice that

$$\begin{aligned} &(a)\ \ 2c\beta^2 = \delta^2/2, \\ &(b)\ \ (1+c)\beta^2 = 1, \\ &(c)\ \ \delta^2\beta^2 = 2h. \end{aligned} \tag{6.28}$$

Here the Hill's equation (3.4) is

$$\ddot{z} + (1 + \tfrac{1}{2}\delta^2 sn^2(\beta t; c))z = 0. \tag{6.29}$$

Setting $\beta t = \mu$ and writing $(d/d\mu) = '$, in (6.29) we transform it into

$$z'' + [(1/\beta^2) + (\delta^2/(2\beta^2))\, sn^2(\mu; c)]z = 0,$$

which, by using the identities (6.28), reduces to

$$z'' + [a(c) + b(c) \cdot c \cdot sn^2(\mu; c)]z = 0, \tag{6.30}$$

where

$$a(c) = 1 + c, \qquad b(c) = 2. \tag{6.31}$$

For the parameter arc $\Gamma(c) = (A(c), B(c), c)$, with $A(c)$ and $B(c)$ given by (4.9), we have

$$\begin{aligned} &(a)\ \ \Gamma(0) = (1, 0, 0), \\ &(b)\ \ \Gamma'(0) = (2, 1/8, 1). \end{aligned} \tag{6.32}$$

Also, from Theorems 4.4 and 5.1 the relevant tangent vectors to the stability surface at $(1, 0, 0)$ are

$$\begin{aligned} &(a)\ \ M_1'(0) = (-\tfrac{1}{2}, 0, 1), \\ &(b)\ \ E_1^{\pm} = (\pm 4, 1, 0). \end{aligned} \tag{6.33}$$

Thus $N^{\pm} = E_1^{\mp} \times M_1'(0) = (1, \mp 4, \frac{1}{2})$, with

$$\text{(a)}\ \langle \Gamma'(0), N^+ \rangle = 2,$$
$$\text{(b)}\ \langle \Gamma'(0), N^- \rangle = 3.$$

Theorem 5.1(3) therefore applies, proving that the two families of periodic orbits projecting into the diagonals $x_1 = \pm x_2$ of Figure 13 must be elliptic for $h > 0$ small, and elliptic stable for almost all such h by the Moser-Rüssmann criterion. For each fixed h these two families are linked in the energy manifold $H = h$.

Moreover, Theorem 3.3 also applies, as $h \uparrow \frac{1}{2}$, but now condition (C2) obtains. Therefore the transitions to hyperbolicity can only take place at $\mu_1 = \mu_2 = +1$, just as for the mapping (2.16). This result could, of course, have been anticipated from the symmetry of the problem.

We note that the two periodic orbits that project to the axes in Figure 13 were shown in [7] to be elliptic for energies $0 < h < h_1 \approx 1.15$, and hyperbolic for all $h > h_1$. In $H = h$ they link one another and also link the previous periodic orbits that project to the diagonals in Figure 13.

(D) The differential equations associated with the Hamiltonian

$$H(x, y) = \tfrac{1}{2}|y|^2 - (\cos(x_1) + \cos(x_2)) \tag{6.34}$$

form a completely integrable system. At energies $-2 < h < 0$ there are two families of periodic orbits that project to the two axes, and which satisfy the hypotheses of Theorem 2.5 as $h \uparrow 0$. (The relevant critical points of the potential are $p_1^{\pm} = (\pm\pi, 0)$ and $p_2^{\pm} = (0, \pm\pi)$ and have eigenvalues ± 1.) Here the associated Hill's equation (3.4) is simply

$$\ddot{z} + z = 0, \tag{6.35}$$

for which *all* intervals of instability collapse. The periodic orbits are therefore repeatedly elliptic and parabolic as $h \uparrow 0$, but never hyperbolic. This can be seen directly from the geometry of the completely integrable system and shows the necessity of the hypothesis of non-collapsing regions of instability for the Hill's equation as given in Theorem 3.3. All other periodic solutions of this system at energies $-2 < h < 0$ are known to be parabolic [29, Corollary 2.8].

Appendix A

In this appendix we show that Hypothesis (b) of Theorem 2.1 implies that every monomial in the series represented by the term O_2 in the transformation $x_2 + i y_2 = w + O_2(u, v, w, \bar{w})$ of (2.7) contains a factor of w or $\bar{w}$. For $w = w_1 + i w_2$ this will imply that $w_1 = 0 = w_2$ if and only if $x_2 = 0 = y_2$.

First observe that if we write $z = x_2 + i y_2$ as in (2.7), then the differential equations associated with the Hamiltonian (2.1) will take the form

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 + O_2,\\ \dot{y}_1 &= -\alpha y_1 + O_2,\\ \dot{z} &= -i\beta z + O_2,\\ \dot{\bar{z}} &= i\beta \bar{z} + O_2.\end{aligned} \tag{A.1}$$

Moreover, by virtue of Hypothesis (b) of Theorem 2.1, each monomial in the O_2 terms for $\dot{z}$ and $\dot{\bar{z}}$ must contain a factor of z or $\bar{z}$. If we let $\alpha_1 = \alpha$, $\alpha_2 = -i\beta$, and relabel z and $\bar{z}$ by x_2 and y_2 (these are not the x_2 and y_2 above), then (A.1) can be written in the more convenient form

$$\begin{aligned}\dot{x}_i &= \alpha_i x_i + f_i(x, y),\\ \dot{y}_i &= -\alpha_i y_i + g_i(x, y),\end{aligned} \qquad i = 1, 2, \tag{A.2}$$

where each monomial in f_2 and g_2 now contains a factor of x_2 or y_2. The series f_i and g_i begin with terms of order two or more.

For a series f we denote by f_N the homogeneous polynomial of all terms of degree N in f. Also, we change notation so that $(u, v, w, \bar{w})$ of (2.7) is now denoted by (u_1, v_1, u_2, v_2). As described in [27], the transformation (2.7) will convert (A.2) into the form

$$\begin{aligned}\dot{u}_i &= a_i(uv)u_i,\\ \dot{v}_i &= b_i(uv)v_i,\end{aligned} \qquad i = 1, 2, \tag{A.3}$$

where

$$\begin{aligned}a_i(uv) &= \alpha_i + \sum_{N=1}^{\infty} a_{i,N}(uv),\\ b_i(uv) &= -\alpha_i + \sum_{N=1}^{\infty} b_{i,N}(uv),\end{aligned} \tag{A.4}$$

are convergent power series in $u_i v_i$, $i = 1, 2$, with $a_{i,N} = b_{i,N} = 0$ for N odd ($a_{i,N}$ is a homogeneous polynomial of degree N, *etc.*). Moreover, the actual form of (2.7), in the present notation, is for $i = 1, 2$

$$\begin{aligned}x_i &= \phi_i(u_1, u_2, v_1, v_2) = \sum_{j=1}^{2} \delta_{ij} u_j + \sum_{N=2}^{\infty} \phi_{i,N}(u_1, u_2, v_1, v_2),\\ y_i &= \psi_i(u_1, u_2, v_1, v_2) = \sum_{j=1}^{2} \delta_{ij} v_j + \sum_{N=2}^{\infty} \psi_{i,N}(u_1, u_2, v_1, v_2).\end{aligned} \tag{A.5}$$

Thus we need to show that if each monomial in f_2 and g_2 of (A.2) contains a factor of x_2 or y_2, then each monomial in ϕ_2 and ψ_2 of (A.5) must contain a factor of u_2 or v_2. To prove this we use induction on N.

Since $\phi_{2,1} = u_2$ and $\psi_{2,1} = v_2$ in (A.5), the result is true for $N = 1$. We therefore assume the result holds for all monomials in the series for ϕ_2 and ψ_2 of degree $\leqq (M-1)$.

We need two remarks. First, the Hamiltonian character of the equations (A.2) implies that $b_i = -a_i$ in (A.4) [27, p. 262]. Also, the series (A.5) are not uniquely determined, but uniqueness can be achieved provided all terms of the form

$$c_i u_i (u_1 v_1)^{\sigma_1} (u_2 v_2)^{\sigma_2}, \\ c_i v_i (u_1 v_1)^{\sigma_1} (u_2 v_2)^{\sigma_2}, \tag{A.6}$$

in the respective series for ϕ_i and ψ_i, are normalized in a certain way [27, p. 266], with the specific normalization being unimportant for our purposes.

Define the differential operators $D_j = u_j(\partial/\partial u_j) - v_j(\partial/\partial v_j)$, $j = 1, 2$, and note that D_j either annihilates a monomial, or else preserves its degree and the property of its admitting a u_j or a v_j. Next let $u^\rho v^\sigma$ denote $\prod_{j=1}^{2} u^{\rho_j} v^{\sigma_j}$, define $D = \alpha_1 D_1 + \alpha_2 D_2$, and note that for any constant c we have

$$D(c u^\rho v^\sigma) = \left(\sum_{j=1}^{2} \alpha_j (\rho_j - \sigma_j) \right) c u^\rho v^\sigma. \tag{A.7}$$

Substituting (A.5) into (A.2) and recalling that $a_i = -b_i$, we obtain

$$D\phi_i + \sum_{j=1}^{2} \sum_{N=1}^{\infty} a_{j,N} D_j \phi_i = \alpha_i \phi_i + f_i(\phi, \psi), \\ D\psi_i + \sum_{j=1}^{2} \sum_{N=1}^{\infty} a_{j,N} D_j \psi_i = -\alpha_i \psi_i + g_i(\phi, \psi). \tag{A.8}$$

Comparing terms of order M then gives

$$\begin{aligned} &(D - \alpha_i)\phi_{i,M} + \sum_{j=1}^{2} \delta_{ij} u_j a_{j,M-1} \\ &\quad = -\sum_{j=1}^{2} \sum_{N=1}^{M-2} a_{j,N} D_j \phi_{i,M-N} + (f_i(\phi, \psi))_M, \\ &(D + \alpha_i)\psi_{i,M} + \sum_{j=1}^{2} \delta_{ij} v_j a_{j,M-1} \\ &\quad = -\sum_{j=1}^{2} \sum_{N=1}^{M-2} a_{j,N} D_j \psi_{i,M-N} + (g_i(\phi, \psi))_M. \end{aligned} \tag{A.9}$$

The induction hypothesis assures that each monomial in $\phi_{2,N}$ and $\psi_{2,N}$ for $1 \leqq N < M$ contains a factor of u_2 or v_2. Also, since the expressions $(f_i(\phi, \psi))_M$ and $(g_i(\phi, \psi))_M$ involve only terms of the form $\phi_{j,N}$ and $\psi_{j,N}$, $1 \leqq N < M$, we see that the right-hand side of (A.9) for $i = 2$ must have each monomial containing a factor of either u_2 or v_2. Finally, since the summands on the left-hand side of (A.9) reduce when $i = 2$ to terms always involving u_2 or v_2, we see that if $(D - \alpha_2)$ does not annihilate a term of $\phi_{2,M}$, then that term must contain u_2 or v_2, with a similar statement holding for $\psi_{2,M}$. However, from (A.7) it is a simple matter to see that the only terms annihilated by $(D \pm \alpha_2)$ are those of the form (A.6) with $i = 2$. Thus each monomial of $\phi_{2,M}$ and $\psi_{2,M}$ must contain a u_2 or v_2, and by induction the same is true for the series ϕ_2 and ψ_2 in (A.5).

Appendix B

Here we indicate the relation of Theorem 3.2 to classical Sturmian oscillation theory and sketch a proof of that result under the additional assumptions of Theorem 2.4 (*i.e.*, the Hamiltonian has the form $H(x, y)=\frac{1}{2}|y|^2+W(x)$ and $p\in R^2$ is a saddle point of the potential W with eigenvalues $-\alpha^2$, β^2, where α and $\beta>0$). As $h\uparrow 0=h_0$ equation (3.4) attains the limit

$$\ddot{z}+H_{x_2 x_2}(\Pi_0(t))\, z=0, \tag{B.1}$$

where $\Pi_0(t)$ is an orbit homoclinic to the critical point $(p, 0)\in R^4$. But $\Pi_0(t)$ is near $(p, 0)$ for "most" t, and thus $H_{x_2 x_2}(\Pi_0(t))$ must be well-approximated for such t by $W_{x_2 x_2}(p)=\beta^2$ (see formulas (3.13), in which p is denoted by $(p, 0)\in R^2$). It follows that (B.1) must be well-approximated over long periods of time by the equation

$$\ddot{z}+\beta^2 z=0. \tag{B.2}$$

Since solutions of (B.2) have infinitely many zeros in $[0, \infty)$, the Sturm comparison and separation theorems (see [17, pp. 333–337]) show that (B.1) must be oscillatory at $+\infty$. This gives the content of Theorem 3.2 without the use of Moser coordinates (and perhaps, therefore, with less geometrical insight).

Such techniques can also be applied to give an alternative proof of Theorem 2.4. Moreover, in the case of "multiple twisting" past several critical points (Theorems 2.5–2.8), these approximation procedures can be employed at each of the critical points in each of the relevant directions taken by the orbit $\Pi_h(t)$ in passing near each critical point. Of course one must verify that the zeros from the respective systems do not cancel, i.e. that the rotations discussed in the proof of Theorem 2.5 are in the same direction. With our techniques this was easily seen from the Moser coordinates (2.4) about each critical point as shown in the proof of Theorem 2.5; any other approach must also account for the "additivity" of the twistings to obtain Theorems 2.5–2.8.

For more general systems of the form (3.9), with Hamiltonians of the form (2.1), the standard Sturmian oscillation theory would have to be replaced by matrix oscillation theory (as in [32, pp. 328–336]) if one wished to achieve analogous alternative proofs of results such as Theorem 2.1.

Acknowledgements. G. PECELLI's research was partially supported by the U.S. National Science Foundation under grant MGS 7801504. D.L. ROD's research was supported in part by the National Research Council of Canada, Grant A8507. They both gratefully acknowledge the hospitality of the State University of New York at Albany during the preparation of this paper.

References

1. V. I. ARNOLD & A. AVEZ, "Ergodic Problems of Classical Mechanics," Benjamin, New York, 1968.
2. F. M. ARSCOTT, "Periodic Differential Equations," Pergamon Press, Oxford, 1964.
3. B. BARBANIS, On the isolating character of the "third" integral in a resonance case, *Astronom. J.* **71** (1966), 415–424.

4. F. BOWMAN, "Introduction to Elliptic Functions with Applications," Dover Publications, New York, 1961.
5. M. BRAUN, Particle motions in a magnetic field, *J. Differential Equations* **8** (1970), 294–332.
6. R. C. CHURCHILL, G. PECELLI & D. L. ROD, A survey of the Hénon-Heiles Hamiltonian with applications to related examples, in "Como Conference Proceedings on Stochastic Behavior in Classical and Quantum Hamiltonian Systems" (G. CASATI & T. FORD, Eds.), Lecture Notes in Physics, vol. 93, Springer-Verlag, New York, 1979.
7. R. C. CHURCHILL, G. PECELLI, S. SACOLICK & D. L. ROD, Coexistence of stable and random motion, *Rocky Mountain J. of Math.* **7** (1977), 445–456.
8. C. CONLEY, On the ultimate behavior of orbits with respect to an unstable critical point (I), *J. Differential Equations* **5** (1969), 136–158.
9. G. CONTOPOULOS, Orbits in highly perturbed dynamical systems, *Astronom. J.* (I) **75** (1970), 96–107. (II) **75** (1970), 108–130. (III) **76** (1971), 147–156.
10. E. T. COPSON, "An Introduction to the Theory of Functions of a Complex Variable," Oxford Univ. Press, Oxford, 1957.
11. R. COURANT, "Differential and Integral Calculus," vol. 1, second edition, Interscience Publishers, New York, 1970.
12. R. L. DEVANEY, Homoclinic orbits in Hamiltonian systems, *J. Differential Equations* **21** (1976), 431–438.
13. R. L. DEVANEY, Blue sky catastrophes in reversible and Hamiltonian systems, *Indiana Univ. Math. Jour.* **26**, No. 2, (1977), 247–263.
14. R. L. DEVANEY, Homoclinic orbits to hyperbolic equilibria, in "Bifurcation Theory and Applications in Scientific Disciplines" (O. GRUEL & O. RÖSSLER, Eds.), *Annals of the New York Academy of Sciences*, **316**, New York, 1979.
15. J. DIEUDONNÉ, "Foundations of Modern Analysis," second edition, Academic Press, New York, 1969.
16. M. S. P. EASTHAM, "The Spectral Theory of Periodic Differential Equations," Scottish Academic Press, Edinburgh, 1973.
17. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.
18. M. HÉNON & C. HEILES, The applicability of the third integral of motion; some numerical experiments, *Astronom. J.* **69** (1964), 73–79.
19. J. HENRARD, Proof of a conjecture of E. Strömgren, *Celestial Mechanics* **7** (1970), 449–457.
20. E. HOEHN, Untersuchungen über die Stabilität von Hillschen Differentialgleichungen, *ZAMM*, Sonderheft (GAMM-Tagung) Band **45** (1965), T39–T41.
21. W. S. LOUD, Stability regions for Hill's equations, *J. Differential Equations* **19** (1975), 226–241.
22. G. H. LUNSFORD & J. FORD, On the stability of periodic orbits for nonlinear oscillator systems in regions exhibiting stochastic behavior, *J. Math. Phys.* **13** (1972), 700–705.
23. W. MAGNUS & S. WINKLER, "Hill's Equation," Wiley-Interscience, New York, 1966.
24. J. MEIXNER & F. W. SCHAEFKE, Mathieusche Funktionen and Sphaeroid Funktionen, Springer-Verlag, Berlin, 1954.
25. K. MEYER, Generic bifurcation of periodic points, *Trans. Amer. Math. Soc.* **149** (1970), 95–107.
26. K. MEYER, Generic bifurcation in Hamiltonian systems, in "Dynamical Systems – Warwick 1974" (ANTHONY MANNING, Ed.), 62–70, Lecture Notes in Mathematics, vol. 468, Springer-Verlag, New York, 1975.
27. J. MOSER, On the generalization of a theorem of A. Liapounoff, *Commun. Pure Appl. Math.* **11** (1958), 257–271.

28. J. MOSER, "Stable and Random Motions in Dynamical Systems," Annals of Mathematics Studies No. 77, Princeton Univ. Press, Princeton, N.J., 1973.
29. G. PECELLI & E. S. THOMAS, Normal modes, uncoupling, and stability for a class of nonlinear oscillators, *Quart. of Applied Math.* **37** (1979), 281–301.
30. H. RÜSSMANN, Über die Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung, *Math. Annalen* **169** (1967), 55–72.
31. C. L. SIEGEL & J. MOSER, "Lectures on Celestial Mechanics," Springer-Verlag, New York, 1971.
32. F. V. ATKINSON, "Discrete and Continuous Boundary Problems," Academic Press, New York, 1964.

Department of Mathematics
Hunter College
City University of New York
and
Department of Mathematics and Statistics
University of Calgary
Alberta, Canada

(Received August 24, 1979)