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A Two-Layer Fuzzy Pattern Matching Procedure for the Evaluation of Conditions Involving Vague Quantifiers

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Abstract. In this note, we are interested in the evaluation of conditions of the form 'The value of attribute **a** for Q items of X is in F' or more shortly 'Q items of X are F', where X is a set of items, Q denotes a possibly vague proportion (which may be linguistically expressed, e.g. 'most'), F is a (possibly fuzzy) subset of the attribute domain of **a**, and where the available knowledge about the value $\mathbf{a}(x)$ of the attribute **a** for any item x may be imprecise or even vague. The evaluation is based on a fuzzy pattern matching procedure repeated two times. Such conditions may be encountered in queries addressed to an incomplete information data base or in the 'if-part' of expert rules.

Key words. Vague quantifier, vague query, pattern matching, incomplete information, possibility theory.

1. Introduction

If they are allowed, vague predicates and vague quantifiers are often encountered in the linguistic expression of (i) requests addressed by a user to an information system, or of (ii) pieces of knowledge given by human experts. In the first case, it means that the user may be more or less flexible about the required properties of the items he/she is looking for, while some vagueness in the expression of the condition part of an expert rule may mean that the rule still more or less apply to borderline situations (in other words, the conditions of application of the rule are robust); if the vagueness appears in the conclusion part of the rule, it often indicates some uncertainty. It is then interesting when evaluating a request or a condition to estimate to what extent the request can be considered as satisfied by a given item (it enables the ordering of the items according to their agreement with what is required), or to what extent the condition is fulfilled in order to combine, in a suitable way, this estimate with the uncertainty bearing on the conclusion of the rule if any.

A so-called fuzzy pattern matching procedure has been developed and designed (see Cayrol *et al.*, 1980, 1982) in the framework of fuzzy set and possibility theory in order to deal with evaluation of patterns made of the logical combination of fuzzy predicates. Besides, this procedure still applies if the available information about the items or the facts is pervaded with uncertainty and imprecision and is represented in terms of possibility distributions. In the case of partial imperfect information, we can thus distinguish between the extent to which it is *possible* and the extent to which it is *certain* that a condition is satisfied. This procedure has been applied to the management of incomplete information databases as well as to the treatment of fuzzy conditions of 'If . . . then rules' in expert systems. See Dubois and Prade (1988).

In this short paper, we consider the case where requests or conditions do not only involve vague predicates but also vague quantifiers. An example of such a request is: 'To what extent is it true that *most* of the products made by a company are sold on markets which are in *fast* expansion?'. The following section offers a background on the two measures of possibility and certainty used in fuzzy pattern matching and discusses a modification of one of these measures in the case of a pattern with a fuzzy predicate. Then it is explained how the number of items satisfying a condition (possibly fuzzy) can be counted in the case of partial information. Finally, the proposed treatment of conditions involving vague quantifiers is presented.

2. Background

2.1. MEASURES USED IN FUZZY PATTERN MATCHING

Let $X = \{x_1, \ldots, x_n\}$ be a finite set of items. Let $A(x_i)$ denote the (possibly fuzzy) subset restricting the (more or less) possible values of the single-valued attribute **a** for x_i; thus $A(x_i)$ is a subset of mutually exclusive values; $A(x_i)$ represents the available knowledge; the information is not precise as soon as $A(x_i)$ is not a singleton. The membership function of a fuzzy subset A will be denoted by μ_A and will range in the interval [0, 1]. The function $\mu_{A(x_i)}$ is also called a possibility distribution since $\mu_{A(x_i)}(d)$ estimates the extent to which it is possible that **a** takes the value d for x_i. Since $D_{\mathbf{a}}$ is supposed to include all the possible values of **a**, for any x there should exist at least a value d which is completely possible for $\mathbf{a}(x)$, i.e. $\exists d, \mu_{A(x)}(d) = 1$; then the fuzzy set $A(x)$ is said to be normalized. Let F be a (possibly fuzzy) subset of the attribute domain $D_{\mathbf{a}}$ of **a**. F will be also supposed to be normalized when F is fuzzy. The extent to which it is possible (resp. necessary or certain) that the value $\mathbf{a}(x)$ of the attribute **a** for the item x, known to be restricted by $A(x)$, is compatible with F is given by the possibility measure $\Pi(F; A(x))$ and the necessity measure $N(F; A(x))$ (Zadeh, 1978; Dubois and Prade, 1988) which are defined, respectively, by

$$\Pi(F; A(x)) = \sup_{d \in D_{\mathbf{a}}} \min (\mu_F(d), \mu_{A(x)}(d)) \tag{1}$$

and by

$$N(F; A(x)) = 1 - \Pi(F^c; A(x)) \tag{2}$$

which gives

$$N(F; A(x)) = \inf_{d \in D_{\mathbf{a}}} \max (\mu_F(d), 1 - \mu_{A(x)}(d)), \tag{3}$$

where the membership function of the complement F^c of F is defined by

$$\mu_{F^c}(d) = 1 - \mu_F(d). \tag{4}$$

The two quantities $\Pi(F; A(x))$ and $N(F; A(x))$ play a central role in the fuzzy pattern matching technique developed in Cayrol, *et al.* (1980, 1982), and are easy to compute in practice where it is sufficient to use membership functions with trapezoidal shape. The possibility measure $\Pi(F; A(x))$ estimates to what extent the intersection of F and $A(x)$ is not empty (i.e. there exists a value restricted by $A(x)$ which is (somewhat) compatible with F), while the necessity measure $N(F; A(x))$ estimates to what extent the (fuzzy) set of values restricted by $A(x)$ is included in F, which represents the requirement.

2.2. A NEW DEFINITION OF THE NECESSITY OF A FUZZY EVENT BASED ON A PARA-CONSISTENT COMPLEMENTATION

N is defined from Π by (2) using the fuzzy set complementation (4). The relationship between N and Π is in agreement with the idea that something is all the more certain as the opposite is impossible. When F is an ordinary subset, we always have

$$\Pi(F; A(x)) < 1 \Rightarrow N(F; A(x)) = 0. \tag{5}$$

It expresses that something should be fully possible before being somewhat certain. This no longer holds in the case of a fuzzy requirement F, as it can be checked. This is due to the complementation (4), taking for granted relation (2). Indeed with definition (4), the union of F and F^c (pointwisely defined on the membership functions by the max operation) does no perfectly cover the domain D (there exists some values at the border of F and F^c, which do not fully belong to F nor to F^c) and then it is not surprising that the possibility that the value $\mathbf{a}(x)$ of attribute $\mathbf{a}$ for x belongs to $F \cup F^c$ is less than 1 (although $\Pi(D_{\mathbf{a}}; A(x)) = 1$ since $\exists d, \mu_{A(x)}(d) = 1$, i.e. there exists at least a value in $D_{\mathbf{a}}$ completely possible for $\mathbf{a}(x)$). Indeed we have

$$\Pi(F \cup F^c; A(x)) = \max(\Pi(F; A(x)), \Pi(F^c; A(x))) < 1$$

if F is a fuzzy set, which leads to the violation of (5), taking (2) into account.

If we want to keep both (2) and (5) when F is fuzzy, since they are intuitively appealing (we shall see also that (5) plays an important role in our procedure for guaranteeing a normalization condition), we have thus to modify the definition of the complementation.

There are two other (extreme) ways of defining the complement A^c of a subset A of D by extending the following identities for classical sets

$$A^c = \bigcap\{S, A \cup S = D\} \tag{6}$$

or

$$A^c = \bigcup\{S, A \cap S = \emptyset\}. \tag{7}$$

This yields (see Dubois and Prade, (1983) for instance), using min and max, respectively, for defining the intersection and the union, to the following definitions in case of fuzzy sets

- *para-consistent complementation (from (6))*

$$\mu_{F^c}(d) = \inf\{s \in [0, 1], \max(\mu_F(d), s) = 1\} = \begin{cases} 1, & \text{if } \mu_F(d) < 1 \\ 0, & \text{if } \mu_F(d) = 1. \end{cases} \tag{8}$$

Note that $F^c = [\text{core}(F)]^c$ with $\text{core}(F) = \{d \in D,\ \mu_F(d) = 1\}$; $F^{cc} \subseteq F$ and $F \cup F^c = D$ (but we have *not* $F \cap F^c = \emptyset$).

- *Intuitionistic complementation (from (7))*

$$\mu_{F^c}(d) = \sup\{s \in [0, 1], \min(\mu_F(d), s) = 0\} = \begin{cases} 1, & \text{if } \mu_F(d) = 0 \\ 0, & \text{if } \mu_F(d) > 0. \end{cases} \tag{9}$$

Note that $F^c = [\text{Support}(F)]^c$ with support $(F) = \{d \in D, \mu_F(d) > 0\}$; $F^{cc} \supseteq F$ and $F \cap F^c = \emptyset$ (but we have *not* $F \cup F^c = D$).

Using the para-consistent complementation for defining $N(F; A(x_i)) = 1 - \Pi(F^c; A(x_i))$ guarantees that $\Pi(F; A(x_i)) < 1 \Rightarrow \Pi(F^c; A(x_i)) = 1$ (provided that $A(x_i)$ is normalized), since $F \cup F^c = D$ and, consequently, $N(F; A(x_i)) = 0$ if $\Pi(F; A(x_i)) < 1$; then (5) is preserved. Thus, we keep the definition (1) of $\Pi(F; A(x))$, acknowledged by all authors in the fuzzy set literature, while $N(F; A(x))$ is now changed into

$$\inf_{d \in D_a} \max(\mu_{\text{core}(F)(d)}, 1 - \mu_{A(x)}(d)) = N(\text{core}(F); A(x)). \tag{10}$$

Clearly, $N(F; A(x)) = N(\text{core}(F); A(x))$ when F is not fuzzy, since then $F = \text{core}(F) = \text{support}(F)$, and $N(\text{core}(F); A(x)) \leqslant N(F; A(x))$ in the general case, which means that we are now more pessimistic in the expression of certainty; this is a cautious, and thus satisfying, behavior.

N.B.: Keeping the definition (3) of N and defining Π from N by $\Pi(F; A(x_i)) = 1 - N(F^c; A(x_i))$ using the intuitionistic complementation would preserve (5) also.

3. Fuzzy Evaluation of the Number of Items whose Attribute Value Belongs to *F*

3.1. CASE WHERE *F* IS AN ORDINARY SUBSET

In this section, F is assumed to be an *ordinary* subset of D_a.

For each $x_i \in X$ we have the pair $(\Pi(F; A(x_i)), N(F; A(x_i)))$ computed from (1) and (3). The $\Pi(F; A(x_i))$'s, for $i = 1, n$, define the fuzzy set of items whose attribute value possibly belongs to F. A fuzzy-valued cardinality of this fuzzy set is easily obtained using the following procedure (Prade, 1984; Dubois and Prade, 1985; Zadeh, 1983)

(1) rank the $\Pi(F; A(x_i))$'s in decreasing order. Let u_i be the value of rank i in this ordering, for $i = 1, n$ and let $u_0 = 1$.
(2) the fuzzy-valued cardinality we use is then the fuzzy subset of $\mathbb{N}$ defined by

$$u_0/0 + u_1/1 + \cdots + u_n/n,$$

where the membership grade is before the '/' and the corresponding element of $\mathbb{N}$ after; here + denotes the union of singletons.

EXAMPLE. $X = \{x_1, x_2, \ldots, x_6\}$ and we have the following values for the $\Pi(F; A(x_i))$'s:

	x_1	x_2	x_3	x_4	x_5	x_6
$\Pi(F; A(x_i))$	0.2	1	0.8	1	0.5	0.5

Then the fuzzy cardinality is given by

$$1/0 + 1/1 + 1/2 + 0.8/3 + 0.5/4 + 0.5/5 + 0.2/6$$

Note that this fuzzy integer is always normalized since we always have $u_0 = 1$. This fuzzy set represents the number of elements which may be found in the fuzzy set defined by the $\Pi(F; A(x_i))$'s. However, note that in case of an ordinary set with $p(\leqslant n)$ elements, it reduces to the subset of $\mathbb{N}$ $\{0, 1, \ldots, p\}$ and not to $\{p\}$. See Dubois and Prade (1985) for a discussion.

Similarly, by ordering the $N(F; A(x_i))$'s in a decreasing order, we can compute a fuzzy subset of $\mathbb{N}$ which represents the number of items whose attribute value more or less certainly belongs to F. Let v_i be the value of rank i in the ordering of the $N(F; A(x_i))$'s. The fuzzy set restricting the possible values of the cardinality of the set of items x which are such that $\mathbf{a}(x) \in F$, is then given by

$$\sum_{i=0,n} \min(u_i, 1 - v_{i+1})/i, \tag{11}$$

where Σ stands for the repeated use of '+' and $v_{n+1} = 0$. (11) must be understood in the following way. We are certain at the degree v_i that there are at least i items which satisfy the condition, and then it is *possible* at the degree $1 - v_{i+1}$ that there are at *most* i items (due to (2) and since the negation of 'at least i' is 'at most $i - 1$'). Moreover, it is possible at the degree u_i that there are at least i items which satisfy the condition. Finally, $\min(u_i, 1 - v_{i+1})$ is the possibility that there are at least i and at most i items, in other words it is the possibility that there are exactly i items which satisfy the condition (Prade, 1984).

When F is an ordinary subset, as already said, we always have

$$\forall i,\ \Pi(F;\ A(x_i)) < 1 \Rightarrow N(F;\ A(x_i)) = 0.$$

Then the fuzzy subset of $\mathbb{N}$ defined by (11), is always normalized (i.e. there is at least one value with a membership grade equal to 1). In case of complete information (i.e. all the $A(x_i)$'s are singletons), this fuzzy set reduces to a singleton corresponding to a precise integer, which is the number (precisely known in that case) of items which satisfy the condition. Let K be the fuzzy proportion of items of X which satisfy the condition; we have

$$\forall i,\ 0 \leqslant i \leqslant n,\ \mu_K\left(\frac{i}{n}\right) = \min\,(u_i,\ 1 - v_{i+1}). \tag{12}$$

We are now in position for evaluating (in terms of possibility, and necessity) the condition 'The value of attribute **a** for Q items of X is in F' by computing (using (1) and (3) again) the quantities $\Pi(Q^*;\ K)$ and $N(Q^*;\ K)$ where μ_{Q^*} denotes the restriction of the membership function μ_Q (defined from [0, 1] to [0, 1], since it represents a vaguely specified proportion in the most general case) to the subset of rational number

$$\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}.$$

We have

$$\Pi(Q^*;\ K) = \max_{i=0,n} \min\left(\mu_Q\left(\frac{i}{n}\right),\ u_i,\ 1 - v_{i+1}\right), \tag{13}$$

$$N(Q^*;\ K) = \min_{i=0,n} \max\left(\mu_Q\left(\frac{i}{n}\right),\ 1 - u_i,\ v_{i+1}\right). \tag{14}$$

$\Pi(Q^*;\ K)$ (resp. $N(Q^*;\ K)$) estimates to what extent it is possible (resp. necessary or certain) that the condition 'The value of attribute **a** for Q items of X is in F' is satisfied when the available information is represented by the $A(x_i)$'s. Thus, by repeating the fuzzy pattern matching procedure two times, first in order to estimate the compatibility of each item with the requirement, second in order to compare the (fuzzily-known) relative number of items which satisfy this requirement with the requested proportion.

It can be checked that when Q corresponds to the existential quantifier '$\exists$' (i.e. $\mu_Q(0) = 0$ and $\forall i \geqslant 1$, $\mu_Q(i/n) = 1$), or to the universal quantifier '$\forall$' (i.e. $\mu_Q(1) = 1$ and $\forall i < n$, $\mu_Q(i/n) = 0$), then $\Pi(Q^*;\ K)$ reduces, respectively, to $\max_{i=1,n} \Pi(F;\ A(x_i))$ and to $\min_{i=1,n} \Pi(F;\ A(x_i))$, and similarly for $N(Q^*;\ K)$ changing Π into N in the preceding expressions.

N.B.: In case Q would restrict the possible value of an absolute number (rather than a relative number, i.e. a proportion), the above procedure applies as well changing $\mu_K(i/n)$ into $\mu_K(i)$ in (12) and $\mu_Q(i/n)$ into $\mu_Q(i)$ in (13)–(14).

3.2. CASE WHERE F IS A FUZZY SET

When F is a fuzzy set, (5) no longer holds and we only have $\Pi(F; A(x_i)) \geqslant N(F; A(x_i))$ (provided that $A(x_i)$ is normalized). Then, the fuzzy set of $\mathbb{N}$ defined by (11) and the fuzzy set K may be subnormalized if we use definition (3). This is due to the fact that with a fuzzy set we cannot definitely say if a precise attribute value satisfy or not the corresponding requirement. This uncertainty is echoed by the subnormalization of K. However, this situation is undesirable here, since this subnormalization would blur the computation of $\Pi(Q^*; K)$ and $N(Q^*; K)$.

One way of escaping this problem would be to renormalize K by dividing each membership grade by the height of K. A more satisfying way to cope with this difficulty is to modify the definition of N in order to preserve (5). This may be done in the following way.

When F is a fuzzy set, we compute the $\Pi(F; A(x_i))$'s by (1) and the $N(F; A(x_i))$'s are replaced by the $N(\text{core}(F); A(x_i))$'s given by (10), which is based on (2) and the para-consistent complementation (8). Then (5) is preserved, K remains normalized and (13)–(14) still apply without any particular problem.

Let us examine the particular case where F is a fuzzy set and the pieces of information $A(x_j)$ are precise; i.e. $\forall j$, $A(x_j) = \{d_j\}$, $d_j \in D$. Then

$$\Pi(F; A(x_j)) = \mu_F(d_j)$$

and

$$N(\text{core}(F); A(x_j)) = 1 - \Pi([\text{core}(F)]^c; A(x_j)) = \mu_{\text{core}(F)}(d_j)$$

(which contrasts with the situation where the complementation (4) is used, leading to $\Pi(F; A(x_j)) = N(F; A(x_j)) = \mu_F(d_j)$ in case of precise information). Then the v_i's introduced in Section 3.1 are such that $\exists k$, $v_i = 1$ for $i = 0, k$ and $v_i = 0$ for $i = k + 1, n$. Moreover, $u_k = 1$ (due to (5)). Then the subset of $\mathbb{N}$ defined by (11) can now be written

$$\sum_{i=k,n} u_i/i, \tag{15}$$

since $\forall i$, $k \leqslant i \leqslant n - 1$, $v_{i+1} = 0$ and $\forall i$, $0 \leqslant i \leqslant k - 1$, $v_{i+1} = 1$. Besides, since $\mu_{\text{core}(F)}(d_j) = 0 \Rightarrow \mu_F(d_j) < 1$, then $v_j = 0 \Rightarrow u_j < 1$ and in particular $u_{k+1} < 1$. Thus, in case F would be an ordinary set, k would be the precise value of the number of times satisfying the requirement.

Yager (1984) has proposed the following estimation of the extent to which the condition 'Q items of X are F' is satisfied, where Q and/or F are fuzzy, Q being a relative quantifier, in *case of precise information*

$$\max_{C \subseteq X} \min \left(\mu_Q \left(\frac{|C|}{n} \right), \min_{d_j \in C} {}_j\, \mu_F(d_j) \right) \tag{16}$$

where $|C|$ denotes cardinality of C. Introducing the rank k defined above, (16) is still equal to

$$\max\left[\max_{|C|\leqslant k}\min\left(\mu_Q\left(\frac{|C|}{n}\right), \min_{d_j\in C}{}_j\,\mu_F(d_j)\right), \max_{|C|\geqslant k+1}\min\left(\mu_Q\left(\frac{|C|}{n}\right), \min_{d_j\in C}{}_j\,\mu_F(d_j)\right)\right]$$

$$= \max\left[\max_{j\leqslant k}\mu_Q\left(\frac{j}{n}\right), \max_{j\geqslant k+1}\min\left(\mu_Q\left(\frac{j}{n}\right), u_j\right)\right]$$

$$= \max_{0\leqslant j\leqslant n}\min\left(\mu_Q\left(\frac{j}{n}\right), u_j\right). \tag{17}$$

Expression (16) estimates to what extent it is possible to find a relative number, compatible with Q, of items which are F. In other words, it corresponds to the possibility that there is a proportion 'at least Q' of items which are F. Indeed, (17) is equal to the expression of $\Pi(Q^*;\,K)$ taking into account (15), i.e. to

$$\Pi(Q^*;\,K) = \max_{k\leqslant i\leqslant n}\min\left(\mu_Q\left(\frac{i}{n}\right), u_i\right)$$

provided that 'Q' and 'at least Q' are identical, i.e. μ_Q is non-decreasing (which implies that $\max_{0\leqslant j\leqslant k}\min(\mu_Q(j/n), u_j) = \mu_Q(k/n)$). Thus, the approach presented here and Yager's proposal are in agreement in the particular situations where both apply. See also Dubois *et al.* (1988) for the relation with the evaluation of conditions of the kind 'at least Q criteria (among n) are satisfied; (here, the n criteria are 'the value of attribute **a** for x_i is in F' for $i = 1, n$) viewed as a special case of weighted pattern matching; an expression similar to (17) is then obtained.

4. Concluding Remarks

The proposed approach which repeatedly make use of the pattern matching procedure first described in Cayrol *et al.* (1980) can be easily implemented and is computationally simple using trapezoidal possibility distributions. Moreover, the condition 'the value of attribute **a** for the item x is in F' can be replaced more generally in the approach by a compound condition involving several attributes.

Possibility theory thus offers a powerful and natural framework for the representation of vague quantifiers and vague predicates and their evaluation in presence of incomplete information.

The procedure which has been presented here is implemented in the inference engine TAIGER (Wyss, 1988; Farreny *et al.*, 1986), which has been applied to expert knowledge in financial analysis.

References

Cayrol, M., Farreny, H. and Prade, H. (1980) Possibility and necessity in a pattern matching process, *Proc. IXth. Inter. Cong. on Cybernetics*, Namur, Belgium, pp. 53–65.

Cayrol, M., Farreny, H. and Prade, H. (1982) Fuzzy pattern matching, *Kybernetes*, **11**, 103–116.

Dubois, D and Prade, H. (1983) Fuzzy set-theoretic differences and inclusions and their use in fuzzy arithmetics and analysis. Presented at the 5th Inter. Seminar on Fuzzy Set Theory, J. Kepler Univ., Linz (Austria), 5–9 Sept., 1983. In: Tech. Rep. *'Ensembles Flous-83'*, No 191 (L.S.I., Univ. P. Sabatier, Toulouse), pp. 107–129, April 1984.

Dubois, D. and Prade, H. (1985) Fuzzy cardinality and the modeling of imprecise quantification, *Fuzzy Sets and Systems* **16**, 199–230.

Dubois, D. and Prade, H. (1988) (with the collaboration of H. Farreny, R. Martin-Clouaire, C. Testemale), *Possibility Theory. An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.

Dubois, D., Prade, H. and Testemale, C. (1988) Weighted fuzzy pattern matching, *Fuzzy Sets and Systems* **28**(3), 313–331.

Farreny, H., Prade, H. and Wyss, E. (1986) Approximate reasoning in a rule-based expert system using possibility theory: a case study, *Proc. 10th IFIP World Congress*, Dublin, Sept. 1–5, 1986, *Information Processing '86* (H.J. Kugler, ed.), North-Holland, Amsterdam, pp. 407–413.

Prade, H. (1984) Lipski's approach to incomplete information data bases restated and generalized in the setting of Zadeh's possibility theory, *Information Systems* **9**(1), 27–42.

Wyss, E. (1988) TAIGER, un générateur de systèmes experts adapté au traitement de données incertaines et imprécises, Thèse I.N.P., Université Paul Sabatier, Toulouse.

Yager, R.R. (1984) General multiple objective decision functions and linguistically quantified statements, *Int. J. Man-Machine Studies* **21**, 389–400.

Zadeh, L.A. (1978) Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems* **1**(1), 3–28.

Zadeh, L.A. (1983) A computational approach to fuzzy quantifiers in natural languages, *Computer and Mathematics with Applications* **9**(1), 149–184.