DAVID S. FIELKER

A CALCULATOR, A TAPE RECORDER, AND THOU

ABSTRACT. Examples are given of transcripts made from tape recordings of the author working with groups of 10-year-old children on problems to do with calculators. Attention is focussed on the teacher's role in the interactions which take place. Suggestions are offered for the possibility of a structure of the strategies available to a teacher.

EXPLANATIONS

I should first explain the 'thou' in the title. 'I-thou-it' was the title of an article by David Hawkins (1969) in which he beautifully analogised the relationship between teacher, pupil, and a third necessary thing which gave some meaning and purpose to the discourse between the first two.

The function of the teacher receives what he calls an 'electronic analogy':

> Think of circuits that have to be completed. Signals go out along one bundle of channels, something happens, and signals come back along another bundle of channels; and there's some sort of feedback involved. Children are not always able to sort out all of this feedback for themselves. The adult's function, in the child's learning, is to provide a kind of external loop, to provide a selective feedback from the child's own choice and action. The child's involvement gets some response from an adult and this in turn is made available to the child. The child is learning about himself through his joint effects on the non-human *and* the human world around him.

and then, more prosaically:

> The function of the teacher, then, is to respond diagnostically and helpfully to a child's behaviour, to make what he considers to be an appropriate response, a response which the child needs to complete the process he's engaged in at a given moment.

It is this particular function of the teacher that I wish to discuss, but before doing so it is worth a brief explanation of the 'it' of David Hawkins' title. He goes on to say:

> I remember being very impressed by the way some people, in an encounter with a young child, would seem automatically to gain acceptance while other people, in apparently very friendly encounters with the same child, would produce real withdrawal and, if they persisted, fear and even terror. Such was the well-meaning adult who wanted to befriend the child – I and Thou – in a vacuum. It's traumatic, and I think we all know what it feels like. I came to realize (I learned with a good teacher) that one of the very important factors in this kind of situation is that there be some third thing which is of interest to the child *and* to the adult, in which they can join in outward projection. Only this creates a possible stable bond of communication, of shared concern.

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Now I think the 'it' is very important, and it goes a long way towards giving a rationale for the use of apparatus in the learning of mathematics. But that is beyond the scope of this article. Unfortunately neither the tape recorder, nor even the calculator, really serves as 'it' in what follows. The tape recorder is merely there to record what happens, and it was used to record most of nearly 20 sessions, each lasting about 45 minutes, in which groups of children round about the age of ten worked at problems with four-function electronic calculators.

The techniques of using a tape recorder are fairly simple. No attempt was made to hide it. Any mechanics of where to put it or plug it in were discussed with the children. They usually drew my attention to the flashing light which indicated the tape had run out. Apart from that they appeared quickly to become oblivious of its presence.

The recordings were not always perfect, especially with larger groups. Most of the time I remembered to say the name of any child talking, and to try to repeat what each said (in as natural a way as possible), so that enough information was recorded to facilitate the making of an accurate transcript. Even so, transcribing took a great deal of time, and many short passages had to be played back a large number of times in order to identify what was being said or who was saying it.

One of the surprising things was how often my supposed repetition of what a child had said was rephrased, sometimes altering the sense slightly or including more information that the child had given, or even misinterpreting it. This makes the process of tape recording a salutary experience, and if more teachers occasionally recorded themselves in action it would no doubt increase their awareness of how they listen to children.

Wherever possible my repetitions were edited out of the transcripts; that is, when they were nothing more than repetitions. They were kept in where a question from a child was put back to the group, or where a statement was being queried, or where the repetition formed a natural part of the conversation.

Karen: What about 90?
DF: What about 90?

Rachel: It's half way.
DF: Half way, is it?

Alan: We've got a one there.
DF: We've got a one there but I want one exactly.

The calculator is nearer to being David Hawkins' 'it' because without the calculator we would have had no conversations. (Everything could have taken place just as easily without the tape recorder, but of course there would have been no record of it!) But the conversations were about problems that arose in conversation with the calculator rather than the calculator itself. The *mathematics* was the 'it'.

All the same, it seemed that calculator, tape recorder, children and teacher combined to produce anecdotal material that provided a great deal of information about attitudes, problem-solving strategies, concepts, content and other ideas about number. The implications for the mathematics curriculum for younger children are described elsewhere (Fielker, 1985). Here I should like to discuss an alternative aspect of the study, related to the role of the teacher.

The ideas are first presented as they arise for each of the mathematical problems that the children were discussing. This way it will be easier to see the material in context.

SQUARE ROOTS

I ask the children to press 2, ×, =. A 4 appears in the display. I ask what has happened to the 2, i.e., what the effect of ×, = is.

It is a subtle choice on my part. It is reasonable to assume that the 2 is doubled, and on every occasion this is what the children have said. So I then ask what will appear in the display if we press 1, ×, =, and naturally they suggest 2. They try it, and get 1. Very often they then assume that the calculator has "gone wrong", and try again. Some suggest, using various forms of words, that the function is squaring. Usually it needs several more examples, and some discussion of the explanations, before everyone recognises what is happening.

Some important points can be made already. The first is about the business of forming and testing hypotheses. This is one of those cases which initially *invites* an *incorrect* hypothesis. It is interesting that such a hypothesis is difficult to relinquish, even in the face of contrary evidence. (Peter Wason (1968) has shown that even intelligent adults also have the same problem.) But in this case it rarely lasts long.

However, the wrong hypothesis has a certain effect of its own. Just *because* a lot of intellectual energy is needed to change it, the correct hypothesis somehow seems to have more sureness and power. This is part of a wider teaching strategy, the idea that children need to work at and organise information themselves, rather than have a teacher organise it for them, and therefore come to understand more about the situation. A relevant anecdote appeared in Fielker (1977a):

The student had been with us for a couple of weeks. He was keen and industrious, fitted well into the department, and seemed to understand something of what we were trying to do. When his tutor came to see him he was teaching a class of twelve-year-olds, and scattered over the blackboard were various number pairs:

$$(4, 13), (2, 7), (7, 22), (1, 4), \ldots .$$

After some incorrect hypotheses and no little discussion the class agreed that a suitable description of the set was

$$\{(x, y); y = 3x + 1\}.$$

Look, said the tutor at the end of the lesson; *they'll get it more easily if you set it out like this*, and he wrote:

(1,4)
(2,7)
(3,10)
(4,13)

But I don't want them to get it easily, said the student politely, *I want them to work at it and think about it.*

Now that the squaring function has been identified, I can establish it more firmly by asking the children to say in advance what will happen to numbers like, 5, 10, 7, 12. Then I can say, "What must I start with in order to get 64?"

Some know immediately. Some try 6, 7, 8. After another couple of examples I ask what, when squared, will give 225, and they find this fairly quickly, so we switch the calculators off and discuss it.

DF:	If you want to get 225, how do you know what to start with?
Cheryl:	A number that's got a 5 on the end.
Stephen:	A number that's got a 5 in it.
DF:	Like 53?
Stephen:	A 5 on the *end.*
Shelly:	A 5 or a nought on the end.
DF:	If we could start with a number with a nought on the end, that could give us 225, could it?
John:	10 doesn't go into 225.
DF:	So the other suggestion is, we start with a number with 5 on the end. How do you know it could give us 225?
Jonathan:	5-times-equals gives you 25.
DF:	That's obviously too small. Where do we go from there?
Edward:	Add another 5.
DF:	If we start with 5 and add 5 we get 10.
Edward:	Add *two* fives.
DF:	Then you get 15. Is it worth trying 10?
John:	10 doesn't go into 225.
DF:	If you start with a number with a nought on the end, what do you end up with?
Cheryl:	A nought.

DF: A nought? On the end you mean.
If you start with a number with a 5 on the end, what do you end up with?
Several: A nought or a 5.
DF: Can you give me an example of a number ending in 5 which when squared gives a nought on the end?

Here we see the 'it' in work, the 'it' in this case not just being a particular problem, but a problem they have just solved. This is usually where the teacher stops, and presents the next problem. But the ease with which the problem has been solved with the calculator may have put at least into the oblivion of intuition any strategies that the children used, and the discussion attempts to identify and make explicit those strategies.

The teacher's strategy here is very simple; it is merely to concentrate on the *incorrect* answers (not a common strategy among teachers who very often take the *correct* answers as their base for continuing the discussion) and to challenge them by presenting a counter-example, by straightforward querying, or by pointing out the consequence of a suggestion.

However, there is some hindsight built into a discussion of how you solve a problem when the problem is already solved. The trouble about the calculator is that its speed, which is its main advantage, drastically lessens the need for a systematic approach, and the individual concentration on trying out a large number of guesses certainly discourages discussion between children and teacher. So in the next session I tried a strategy which thereafter proved much more successful, after the first shock.

DF: I thought we'd do it differently this morning.
All: Without calculators?!!
DF: Yes, but I'll help you, because I'll do the working out. Now, I squared something, and got that (writes 15129 on the blackboard).
It's all right. You tell me what number you want to try, and I'll work it out for you. But you'll have to agree about which number you try.

We proceed as follows.

John: 71.
(General agreement.)

71 5041

Gary: 201.
Cheryl: 182. No, that's not even.
Jonathan: 233, because 5041 is about one third of the number on top (15129) and I timesed 71 by 3.

233 54289

Several: 139. 232. 137.
Paula: 217.
Shelley: I think that's too much.

Cheryl:	If it's over 232 it can't . . . that's too much . . . if it's less it might be able to . . . but if it's only just about, like, 232, that won't be it because that's too much too. 217 might give us a low number.
(Disagreement from others.)	
Cheryl:	It might give us a low number but it might give us a clue to what that number is, though. If it gives us a smaller number, 217, if it gives us a smaller number in the answer it means it's smaller than that, and if it's less than that . . . or more . . . wait a minute . . . if it's more than that . . . it's less than that, I think. (Pause.) If the answer comes out a big number, bigger than that number, it seems it's going to be less than 217.
John:	133, because it's an odd number.
Joshua:	93, because it gives us a small number.
Jonathan:	3 goes into 15129, so we're looking for a number that 3 goes into.
Joshua:	Like 93.
DF:	What sort of size number do we want?
Cheryl:	Somewhere around 200.
Jonathan:	It must be over 100, because a hundred hundreds is ten thousand, and that's less than that number.

The tape recorder picks up all the suggestions, even when the speakers are not identified. Jonathan is the most persistent about justifying his choices, and even his idea about proportion, incorrect though it is, is a reasonable hypothesis at this stage. It is interesting to speculate on possible reasons for some of the other suggestions. Cheryl raises an idea about the number they are looking for being odd, but even the 232 is at least less than 233.

Cheryl picks up the 232, and begins to think aloud, in a rambling sort of way. Her ramble is full of logical connectives, but the overall logic is interrupted by errors and corrections. It seems to be a superfluity of words in order to explain that it might be better to try 217 than 232. However, Cheryl is one of those unusual children who talk as they think, and although even she may not be saying everything that comes into her head, it is more than most children reveal, and it may be an indication of the circumventory way in which children come to a conclusion.

One of the important things about Cheryl's ramble is that it is *permitted* by the teacher, who allows it to proceed uninterrupted. She makes her point in the end, on her own terms and in her own way. Yet many teachers would have tried to 'help', by asking her to think carefully, or correcting her errors, or would even have stopped her altogether. Her classmates seem to understand enough to disagree halfway through, but permitting the ramble helps Cheryl to sort out her thoughts, so it is important for her; and it is important for the teacher, who may have a veiled glimpse into the way a child thinks.

John and Joshua were asked to give reasons for their respective suggestions of 133 and 93, but the requests were edited out of the transcript. This now seems unwise, because it hides the distinction between on the one hand

Jonathan and Cheryl, for whom justification is a more natural part of their responses, and on the other John and Joshua, who only offer justification when pressed to do so.

It becomes a different sort of problem to ask what has to be squared in order to produce 10. Practically all ten-year-olds can see that 3 is too small and 4 is too big, but then they range from those who think that there is no number between 3 and 4, to those who rapidly try successive approximations and reach 7 decimal places in about 3 minutes.

One group with whom I worked began with a hazy idea about decimals.

Grant: Three. That would give you nine.
Paul: Three and a quarter.
DF: Yes, but . . .
Grant: But you said we couldn't use the quarters.
DF: Did I?
Grant: Yes.
DF: When did I say that?
Grant: Last term.
DF: Oh, that was *last* term. This term we're allowed to use . . . well, the trouble about a quarter is we can't put the quarter on the calculator.
Grant: Point five.
DF: Is it?
Grant: Yes. No. Three point two point five.
DF: I don't think you can have two points! If you want to try three and a quarter . . .
Karen: Three point five is three and a half.
DF: Yes. What would three and a quarter be?
Karen: Three point two five.
Three point two point five.
DF: That's two points again. Can you remember what a quarter is?
Grant: Two? Point two? No, that would be a fifth, wouldn't it?
DF: Well, it doesn't matter, because if we're going to use three point something we can just think of three point something. We don't have to think about fractions at all, do we? Do you want to try three point five? Do you think that's about right?
Grant: Oh, no, it can't be.
DF: Why not?
Grant: Three and a half times three and a half. Three threes are nine. Plus three halves. Ten and a half. You've got a half extra. So it can't be three point five. Try three point two.

The interesting thing in the above extract is to see what the teacher chooses to correct and what errors he chooses to ignore. The idea of having two decimal points is not all that unreasonable when one thinks about it, but a long discussion about it would perhaps distract attention from the main problem far too much. It is not even necessary to the main theme to know what one quarter is as a decimal, and my intervention ("Well, it doesn't matter . . . ") is heavily directive towards getting the children to think in decimals rather than fractions.

This does not stop Grant using fractions to square $3\frac{1}{2}$. He gets it wrong, but I do not correct him. It is enough that 3.5 is too big. It is more important that he reverts immediately to decimals, and that we have *something* reasonable with which to continue. Besides, *I* know that 3.2 is also going to be too big.

The effect of all this careful playing down of fractions is that the children now begin to think purely in terms of decimals. I calculate for them the square of 3.2, and write it on the board.

$3.2 \rightarrow 10.24$

Rachel:	Oh. 3.1.
Karen:	What about 3.3?
DF:	What about 3.3?
Grant:	It'd be too much.
DF:	Karen, why not 3.3?
Karen:	I just thought of it.
DF:	All right. Grant, why is it too much?
Grant:	I don't know. Or it might, just about.
Rachel:	I think it would make it.
DF:	Alan, why not?
Alan:	Because if 3.2 equals 10.24, 3.3 will be more than that.
DF:	Do we want more or less?
All:	Less.
DF:	So we want something less than 3.2.
Grant:	3.1.
DF:	O.K. I happen to know that one.

$3.1 \rightarrow 9.61$

Grant:	3.15.
DF:	Why?
Grant:	Well, 3.1 isn't enough. It must be more than one, so try 3.15.

Grant instinctively chooses a number halfway between 3.1 and 3.2, but does not seem able to explain this precisely.

DF: All right.

$3.15 \rightarrow 9.9225$

What the teacher has begun to do now is to press for explanations and reasons. It is not easy, and it seems to be a careful blend of pressing, relaxing, redistributing the pressure, or even waiting while a pattern of more logical thinking gradually establishes itself.

Note that Rachel's reasonable 3.1 is deliberately ignored while I focus on Karen's 3.3, using again the strategy of asking explanations of the wrong answers rather than the correct ones.

It is difficult to know how to respond to the apparently evasive replies ("I just thought of it", "I don't know"), because one cannot tell whether these are genuine, or are ways of avoiding having to put reasons into words.

Perhaps it is a feeling that I am pressing a little too much which makes me accept Grant's 3.1 and his possibly intuitive 3.15 without challenge. If so, my instincts were right, because after further discussion, and my calculation of

$$3.16 \rightarrow 9.9856$$

and

$$3.17 \rightarrow 10.0489$$

the conversation proceeds thus:

Grant: Ah! 3.165.
DF: Now why that?
Grant: I don't really know. It's just a guess!
DF: 3.165 is a guess?
Grant: Yes. Well no-one else has thought of one.
Karen: You're too quick!
DF: Well, it depends what you mean by a guess. It's not just a wild guess, is it?
Grant: No.
DF: What made you say 3.165?
Grant: Well, three one seven is too much, and three one six is too little. It must be . . .
Paul: . . . in the middle . . .
Grant: . . . which is 3.165.

A little more pressure about the guess, and Paul has put into words what Grant presumably is thinking.

There is also a slightly different way of looking at the teacher's actions. It is not just a question of pressing for reasons. Although attention is often focussed on incorrect answers, as has already been pointed out a couple of times, attention can also be given purposefully to correct ones. The important thing is that both are done, and they are done impartially. Otherwise children quickly learn that with certain teachers only wrong answers are challenged, and right ones accepted. (A typical instance is given by Burrell *et al.* (1975) who relate how, when a bright 15-year-old was asked to check a measurement, she said, "Do you mean I've got it wrong?" And other examples are given by Holt (1964).) Hence my strong challenge to Grant's 3.165.

THE RECIPROCAL FUNCTION

Tristan was one of a bright group who, using calculators themselves, were able to find the square root of 10 very quickly. Tristan then had further ideas, which I later asked him to explain to six of the others.

Tristan: Well . . . you told us about pressing 'times-equals' which meant multiply by itself, and I tried to do it with dividing, but it didn't come out. I did '3-divide-equals' and it came out to nought point three three recurring . . . into infinity.
DF: Infinity?
Tristan: Into infinity.
DF: How do you know?
Tristan: That's what Mr. B__ told us.
DF: What do you get on the calculator?
Tristan: Six threes.

It did not seem appropriate for the moment to query further the idea of infinity that seemed to come from Mr. B__, their class teacher. My last question above establishes that we are going to be interested in what we obtain on the calculator, and I invite the group to find out what 'divide-equals' does. (Incidentally, it will only give the reciprocal if the calculator has an automatic constant on division.)

They are the sort of group who do not always need reaction from me, because they provide it for each other.

Jamie: 4-divide-equals means 4 divided by 4.
Tristan: But it doesn't work like that because 3 divided by 3 is 1.
Susan: If you do 5-divide-equals, it goes out to 0.2.
Jamie: It should come out to 0.25.
Susan: 4-divide-equals comes out to 0.25.

On the other hand, when no further ideas are forthcoming it seems that some reaction from the teacher is needed in order to provoke more thought or activity. But it is too easy to ask leading questions (something I had pointed out to Mr. B__ in the previous session), and if one believes that as many ideas as possible should come from the children, then one has to find ways of facilitating that rather than provide the ideas oneself.

One idea *I* have is that they need more information. It would be easy, and perhaps harmless, merely to say so. I prefer to be more subtle, and I suggest it would help if they wrote down the information they already had.

It works, for a moment! They write down the results in order for 3, 4 and 5, Tristan suggests trying 6, and they work up to 10, with an unvoiced agreement about stopping there. Another pause! It needs something less subtle from me!!

DF: Well the number you start with has been getting bigger, hasn't it? What happens to the number that comes out?
Tristan: It's getting smaller.
DF: So if you put in a very large number what would you expect to get out?
All: A very small number.
DF: Could you try a very large number?
Simon: 100-divide-equals gives 0.01.

DF: And what's 0.01?
Simon: One-hundredth.
Tristan: Ah, that's interesting.
Simon: And ten comes out as one-tenth.
Tristan: And 100 comes out as one-hundredth.

And now it looks as if I have pushed hard enough. A few more leading questions like "What happens to 1000?" and we should be there. But I restrain myself. *They* have to make a generalisation.

But they do not. Susan asks if it is divided by itself, but immediately rejects this hypothesis. Some try other large numbers. Tristan experiments with "10 divided by something". Jamie tries 11.

One of the things that helps the teacher is knowing what mathematical ideas are involved in the situation. I realise that one of the difficulties lies in the inability to recognise certain decimals as fractions. So for the moment the least I can do is to encourage them to try 2, which is missing from their list, and even 1.

DF: Let's put the idea the other way round. If the number you put in gets bigger, the answer gets smaller. If the number you put in gets smaller, the answer . . .
All: Bigger.
DF: How about putting in a smaller number then?

Indeed, 1 and 2 are suggested, but before anyone can try them Tristan sends them off on a different tack.

Tristan: Can we put in decimal points? Hey. I did '0.5-divide-equals' and it came out as 2. And then you do '5-divide-equals' and it comes out as *nought*-point-two.
DF: What next, then?
Elizabeth: 50. You get 0.02.
DF: So what would you expect if you tried 500?
Elizabeth: 0.005. Um . . . 0.002. And 5000 would give 0.0002.
Tristan: 0.4 comes out as 2.5. 0.7 would come out as 1.42 . . . 8 . . . 5 . . . 7 . . . 1 . . . 0, I think. Well, it might or it might not. I'll try it. (Does so.) You get a 4 on the end. That's interesting.

My interventions here are solely to encourage more of a generalisation about this particular aspect of reciprocals, and both Tristan and Elizabeth take on the idea. But Tristan has tested his hypothesis on 0.7 and found a spurious 4.

Similar things happen elsewhere.

Simon: That's funny. I punched in 0.090909, and I got 11.000011!
DF: Which is roughly what?
Simon: 11.
Tristan: If you put 11.000011-divide-equals it comes out as 0.090909.
Jamie: If you do 11 you get that answer.

I am not sure if my question has established anything about the rounding off of the calculator. And maybe I become a little impatient, because there is now a pause, and I take the opportunity to remind them of the original question.

DF:	The thing is, you are able to tell what 'divide-equals' does to a number.
Elizabeth:	It divides. It divides and timeses.
Tristan:	It divides twice. Four divide . . . no. Hang on. Four divided by . . . equals 0.25. And then divide 0.25 by itself . . .
Simon:	0.25 is a quarter.
DF:	So if you put in 4, what do you get out?
(A pause.)	
Tristan:	A sixteenth! Divided by sixteen. Oh! It divides by four and then four again. Because you divide four by four, and it comes out as one, and you divide that . . .
Jamie:	Yeah! And you divide that one by four and you get a quarter of it.
DF:	So you're telling me that 'divide-equals' divides by 4 and then divides by 4 again?
Jamie:	It depends what you divide it by.
DF:	So if you put 2 in?
Tristan:	It would divide by 2 and then divide by 2 again.
Jamie:	Your answer would be 0.5.

My questions are purely for clarification, and it looks as if we have a perfectly good hypothesis from Tristan, which just needs bullying into a more straightforward form. Again, a few leading questions would do it, but yet again I seem hesitant to do this, and in any case the hypothesis now becomes lost in some more inconsistencies which are due to rounding errors.

So, at my instigation we then spent a long time on a discussion of how we changed fractions to decimals, before we went back to the main problem. One of the principles, after all, of teaching mathematics is that "it is better to travel hopefully than to arrive"; and in some senses the importance of the problem was that it was created by Tristan, rather than that there were important conclusions about it for ten-year-olds, but the various things that happened on the journey were all appropriate to these particular children, especially in the age of the calculator. The overall strategy of the teacher therefore seemed to be investigate anything that cropped up on the way, and be in no hurry to reach the end. (We did, and a summary is in Fielker (1985)!)

'THE BIG ONE'

'The Big One' is the last of a sequence of three games developed by Meissner (1980).

I program a calculator to divide by 31 and give it to each child in turn, asking them to press a number, then the equals button, then state the result. We obtain:

$$5 \rightarrow 0.1612903$$
$$9 \rightarrow 0.2903225$$
$$6 \rightarrow 0.1935483$$
$$2 \rightarrow 0.0645161$$

DF:	O.K. What can you say about the numbers we're getting.
Karen:	They've all got nought at the beginning.
DF:	Do you think you could get something which didn't have a nought at the beginning? How would you do that?
Helen & Karen:	Go into double figures.
DF:	Try a double figure one. What are you trying Karen?
Karen:	Ten. 0.3225806.

$$10 \rightarrow 0.3225806$$

DF:	It still starts with a nought, doesn't it? Do you think we could get something starting with something else? Helen?
Helen:	Hundred.

$$100 \rightarrow 3.2258064$$

DF:	O.K. So this time we start with . . . three. So we don't start with a zero. Do you think you could start with something that would end up with one?
Rachel:	It's got to be between ten and a hundred.
DF:	What do you fancy?
Rachel:	51?
DF:	All right. Try 51.

$$51 \rightarrow 1.6451612$$

	Any better?
Alan:	We've got a one there.
DF:	We've got a one there but I want one exactly.

The teacher's first question is completely open, but Karen's response immediately gives a way towards the rules of the game, which gradually emerge as a result of the teacher's further interventions. It is all fairly simple anyway, but this strategy is quite a contrast to those situations where any rules or conditions are given all at the same time, either spoken by the teacher or written on a workcard, when very often the children cannot take everything in and the teacher needs to go over the details more slowly again anyway.

There is more to it than this, however. If the rules can be established as a result of dialogue between teacher and children then sometimes it is the *children* who can make decisions about what the rules are. This means that the problem or game becomes theirs, rather than the teacher's, with a consequent increase in interest and motivation. But it also means that the children begin to see that such decisions *can* be arbitrary, that *they* are entitled to make them, and as a corollary that if things do not work out then they can change their minds.

So, on other occasions (alas, unrecorded!) I have at some appropriate point asked, "What would you like to make the answer come to?" and "One" has often been a popular response. (And if it is not, then the spirit of the game is not really affected by choosing an alternative.)

ONE WHOLE NUMBER DIVIDED BY ANOTHER . . .

It seems a perfectly reasonable thing to divide one number by another on a calculator, obtain an answer on the display, and then forget what the two numbers were.

I do this for one group of children, obtaining 0.4705882, and invite them to find the two numbers I used. A first suggestion is $8 \div 9$, justified by Paul.

Paul: Can we just do that one, and then we know something to start off with, and then we can starting working it out.
DF: That's a good idea. O.K.

$8 \div 9 \quad 0.8888888$

The results are written on the board. Only the teacher has a calculator.

Immediately the children focus on the problem of how to obtain the 4 after the decimal point. In a discussion far too long to reproduce here, they argue about, and ask me to calculate, $4 \div 4.5$, $2 \div 2.25$, obtaining 0.8888888 each time, and finally reject a suggestion of trying $1 \div 1.125$. (Again, see Fielker (1985) for the details.)

During the discussion the teacher's participation can be typified by the following extract.

Grant: Four divided by five should give you four.
DF: Do the rest of you agree with that? (Pause.)
Paul: All right, then.
DF: So you think if eight divided by nine gives you a row of eights, four divided by five is going to start off with a four. Yes?
Karen: Possibly. It might be all fours.
Grant: Yes, but then you're still getting somewhere, aren't you?
DF: Yes. How would you get all fours?
Grant: Four divided by five.
DF: That would be all fours, would it? (Pause.)
Paul: If it's half that eight and it's four . . . 'cos that's what you've done, isn't it? . . . and he said . . . if he halves the nine . . . like . . . he wanted to do, didn't you?
Grant: Yes.
Paul: Agreed. It wouldn't be five, would it?
DF: What would it be?
Paul: That'd be . . . four and a half . . .

DF: So you think to be exactly all fours you ought to divide four by four and a half? (Pause.)
I could try that if you want to, just to see what happens. I mean, I did say there were two *whole* numbers to start with, but if you want to try four divided by four and a half I can do that, can't I?
Grant: Four divided by four point five.
DF: That's right. Do you want to try that?
Grant: Yes, just to see what happens.

I ask for possible agreement, check that *I* have understood, check that they are sure about what they want me to do. Generally I am trying to be as passive as possible, but at the same time acting as a sounding board for their ideas, and otherwise the only steering I do is to answer Paul's "It wouldn't be five" with "What would it be?"

Later in the discussion it looks as if I am becoming concerned about the persistence of equivalent divisions. As the argument becomes quite heated, I begin to press for a better prediction.

DF: That's true. What about two divided by two point two five? Is that going to give the same again?
Karen: Yes.
Paul: Can you just try it?
DF: Well, do you know what you're going to get this time?
Grant: Nought point a row of eights.
DF: You know you're going to get a row of eights, do you?
Paul: You don't *know*; you're just guessing.
DF: Is it just a guess, or do you know?
Paul: I don't know. I don't think it *will* do it again, though.
Grant: It could give us a row of fours.
Paul: It could.
DF: Well, let's try it then.

$2 \div 2.25 \quad 0.8888888$

Grant: One point . . .
Paul: No, don't do that again.
Grant: Three shared by five.
Paul: What did you say that for?
Grant: I don't know. Because it can't keep on going half of that, half of this, half of that again, because it'll always be nought point a row of eights.

So finally Paul and Grant share the responsibility for discarding the hypothesis, and perhaps my concern was unnecessary.

However, the mathematical problem itself can be quite a difficult one. This particular one was once solved very quickly by a 9-year-old working on his own, but with a choice of 3-digit numbers for the input mathematics teachers have often given up after half an hour. The difficulty at any level is due mainly to the absence of an obvious algorithm (unless one happens to be known already), but also to the discontinuous nature of the function of two variables that the problem involves. One can get closer and closer to the quotient without necessarily coming any nearer to the dividend and divisor.

With these particular children I later began to encourage a more structured approach, but it took a very long time and a lot of heart-searching in my unaccustomed ambivalent position. (For details the interested reader is once more referred to Fielker (1985).)

FACTORS

My last example is extremely chastening. It reveals the frustrations of a teacher who has an idea in his head, and is trying to educe this from the children, thinking that he is doing this in the best possible way. As a consequence he is ignoring all the other ways of solving the problem that the children are suggesting.

DF: Can you tell me what that was about?
John: A prime number is . . . a number which only one and the number itself goes into.
DF: Aha. So, is 52 prime?
All: No.
DF: Why not?
(Pause. They are not sure.)
Jonathan: 2 goes into it.
Cheryl: There's another number as well.
Several: 4 . . . 5 . . . 52.
DF: If 2 goes into it, what else do you know goes into it?
Paula: One.
DF: Cheryl said something about another number as well. Does the calculator help?
Cheryl: Yes.
DF: How does it help?
Cheryl: You can use it to times all different numbers to see if you can get 52.
DF: If you *know* that 2 goes into 52, *how* do you know that 2 goes into 52?
Cheryl: 'Cos it end in 2.
Others: Or 4. Or 6. Or 8. Or nought.

(We discuss even numbers, and the idea that 2 divides into them, but it still seems difficult to educe the idea of *pairs* of factors.)

DF: So, if 2 goes into 52, what else will go into it?
Edward: 26.
DF: How do you know?
Edward: Because if you add 26 and 26 you get 52.
DF: Well, if 2 goes into 52 exactly, how many times does it go in?
Steven: 26.
DF: How did you know that?
Steven: 'Cos two twenty fives are 50, and another one makes 52, and if you share out the other 2 you get 26.
DF: Yes. Is there any other way you could work out how many twos in 52?
Cheryl: Use a calculator.
DF: Yes. How?

Jonathan: You would times the thing you thought by 2.
DF: Is that what you did?
Jonathan: Yes.
DF: What did you start off with?
Jonathan: 26.

Oh dear! There are so many different ways of showing that 2 is a factor of 52, yet the children valiantly refuse to divide!

In fact a moment later Cheryl did suggest division. But what if she had not? What on earth can a teacher do, if he feels that the idea should come from the children themselves, and it does not come? How long can he wait? What other action can he take? Should he leave it for another time, or just tell them?

My instincts are always never to tell, to wait a reasonable time, and then leave it if necessary. However the point about this extract is that the frustration on my part at not getting the answer I wanted made me ignore one of the principles mentioned earlier, about "travelling hopefully". And the point about the frustration is that it makes one oblivious to the fact that one *is* ignoring a principle. But for the tape recorder, I might never have known.

TOWARDS A STRUCTURE

It would be very premature to present some sort of taxonomy of teacher strategies at this stage, but the sort of material described above could form the basis for such a taxonomy. At the moment it may help to suggest tentatively a possible classification of the types of action a teacher can take when interacting with children.

1. *Introducing an Activity*

The initial contact with any group of children is fairly critical. One obviously wishes to motivate the maximum amount of interest, and thereafter sustain it.

One way is to create some sort of tension. In the case of the squaring function this was done by 'inviting' an incorrect hypothesis. Tension can also be created by presenting an initial problem which has just enough difficulty to be intriguing, yet not so difficult as to be offputting. Or it can be created by leaving children to collect and organise their own information rather than structuring it for them.

If a situation, an activity or a game is fairly complicated then it helps to introduce it gradually, rather than present the conditions and rules all at once. 'The Big One' was introduced in a fairly general way (yet so that the results were sufficiently complicated to create some interest) and then, one step at a time, the rules were increased so that the children were eventually invited to home in on 1 as a target. Obviously this type of strategy makes it very difficult to rely on texts for the presentation of activities.

A further advantage of an oral approach is that the clarification and establishment of the situation can develop through a dialogue between teacher and children. This way there is every possibility that the children themselves can make suggestions about the nature of the problem. This relies far more on the skills of the teacher in giving opportunities for the children to do this, and this is a subtle business which requires more analysis. Demands are also made on the skills of the teacher who decides to give up predetermined ideas in favour of the children's suggestions about what the problem is, firstly because this procedure is itself a departure from normal practice, and secondly because the teacher has quickly to recognise what value there may be in the children's suggestions, even in the case where no value can be seen but the teacher is still willing to go along with the suggestions so that the children can find that out for themselves! Whatever else happens, a dialogue certainly creates more easily the opportunity for the problem to be one that children are more fully involved in right from the beginning, and one in whose construction they can play an active role, rather than the passive one of listening to the teacher's description.

2. *Responses to Children*

This category of teacher strategies is probably the most difficult to classify.

One of the difficulties is that, given any action, statement or question by a child, a teacher can immediately think of several different ways of responding, and generally he or she has to choose just one of them, in about 3 seconds! In this short time, responses are likely to be instinctive and intuitive, but like all instinct and intuition it is based both on experience and on philosophy, which for each individual teacher have together provided a personal structure of strategies, that may be loosely or precisely formulated, and may continually change – in the long term because of the teacher's own development, and in the short term according to circumstances.

The choices seem very often to do with balance: how much to press for explanations and how much merely to accept answers; whether things need to be made verbally explicit or can be left implicitly understood; challenging both 'correct' and 'incorrect' answers; choosing how much pressure to apply; choosing what mistakes to correct and what mistakes to ignore; choosing when to intervene and when not to; keeping things moving or allowing them to relax; following up diversions or bringing children back to the main problem; intervening or not intervening; helping or not helping.

David Hawkins' notion of 'selective feedback' raises problems itself for the teacher. There is first the difficulty about what to select. Sometimes the choice is arbitrary; sometimes (like so many other things) it depends on personal taste and interest; sometimes it is only a choice in time, because other aspects can be selected later; and sometimes the selection can be left to the children themselves.

But the second difficulty is whether to select or not. This is part of the choice about helping or not helping, and it is also a feature of the strategy of presenting children with complexity in order to give them something to work at, for various reasons stated already. For selection in a way implies simplification, and simplifying is not always the best thing for a teacher to do.

3. *Methods of Control*

One important aspect of a teacher's responses is the extent to which that teacher wishes to control the direction in which things are going and the way things are done. To decide to control or not to control can be difficult enough, even for the teacher whose philosophy is to control but who occasionally decides not to, or for the teacher whose philosophy is not to control but who now and then decides to do so. However, if a decision to control in some way has been made, then there are choices about how this is done, varying from a straightforward instruction on the one hand, all the way across the spectrum to some of the subtleties that have already been described.

The strategies here may be even more difficult to classify. One can for the moment go over again some of the kinds of strategies that have been used: a carefully chosen particular question that encourages the posing of more questions; a suggestion to write down information that may result in that information being organised in a helpful way; an invitation to collect more information that may lead to hypotheses being formed and/or tested; incorrect hypotheses being called into question by asking for further examples, or by presenting counter-examples, or by taking them to their logical conclusions and thus arriving at contradictions; a decision to focus attention on a sub-problem which is blocking the way to a main problem, or to focus on the main problem when subsidiary problems have been suggested.

Focussing attention on particular examples is far more of a selection, of course, and comes under the heading of 'leading questions', but even leading questions are a legitimate strategy, provided they are the result of careful and deliberate choice rather than of sheer frustration. However, in a wide sense all the strategies mentioned above can be considered to be forms of 'selective feedback', since they all in some way focus attention on what the children have offered.

4. *The Consequences of One's Philosophy*

The strategies one uses will largely depend on the philosophy one has about the aims of teaching mathematics. Those that have been suggested are very much due to a personal philosophy which views children as mathematicians, doing mathematics in the same way as mathematicians do. This means, among other things, that they create problems as well as solve them, invent their own algorithms and methods of solution, and correct their own errors.

The simple consequences are, respectively, that a teacher not only presents problems, but also encourages children to create their own, by presenting situations from which questions can arise, by valuing any questions that children ask, and by allowing such questions to be followed up; that he does not teach algorithms or methods of solution, and perhaps does not even try to encourage the development of particular ones; and that he does not correct errors. I hope that the material presented above illustrates at least the first two categories here. More about not correcting errors is in Fielker (1977b).

The personal nature of a philosophy implies that there will also be something personal about any one teacher's set of strategies. I am not sure at this stage whether personal philosophy affects the *structure* of strategies. Such a structure could be different for each teacher, and perhaps the word *taxonomy* could be reserved for that overall structure of strategies from which personal structures are selected.

CONCLUSIONS . . .

. . . are difficult. And I am aware that this article may seem very much like that section of a thesis entitled 'topics for further research'! But I am not suggesting that this is a bad thing, only that, like a session with some children, this may ask more questions than it answers.

The questions are mainly about the teacher's role and the strategies used. A sort of 'answer' is that the form of evidence supplied by transcripts of tape recordings can be useful in helping to identify what actions a teacher takes.

In some ways a sound tape recorder captures things more globally than does a video camera, where the directional microphone and the restricted field of vision combine to give a very selective view of what is happening. What one misses on sound tape (apart from facial expressions!) is obviously any visual information, but when one works with calculators this is minimal, and whenever numerical information is written down this can easily be recorded on paper by the teacher and added to the transcript.

I have found both tape recordings and transcripts useful when working with teachers on inservice courses, with both advantages and disadvantages compared to using videos or having 'live' children. Advantages of transcripts are that they can be considered at one's own pace, selected from, referred to very easily, discussed in small groups.

It has also been possible on appropriate courses to encourage teachers to tape-record themselves and make transcripts, with the possibility of sharing them with other teachers for discussion, or commenting on them as a written exercise. The results are often salutary, as some of mine have been. But generally it is a useful way of reflecting on what one has done, of revealing what actually happened when sometimes one thought otherwise, of considering what might have happened if one had acted differently, and above all of asking the key question of cause and effect about the reaction between teacher and children: "What did the teacher do to enable that to happen?"

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Abbey Wood Mathematics Centre,
Hither Green Primary School,
Beacon Road,
London, SE13 6EH,
England.