Geometriae Dedicata **62:** 65–72, 1996.

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On Ricci Eigenvalues of Locally Homogeneous Riemannian 3-Manifolds

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(Received: 16 January 1995)

Abstract. We find the necessary and sufficient conditions for three constants $\rho_1, \rho_2, \rho_3 \in \mathbb{R}^3$ to be the principal Ricci curvatures of some 3-dimensional locally homogeneous Riemannian space.

Key words: Riemannian manifolds, homogeneous manifolds.

1. Introduction

According to Singer [Sin], a Riemannian manifold (M, g) is said to be *curvature homogeneous* if for every two points $p, q \in M$ there is a linear isometry $F : T_pM \to T_qM$ between the corresponding tangent spaces such that $F^*R_q = R_p$. Here R denotes the corresponding curvature tensor. It is obvious that every connected and locally homogeneous Riemannian manifold is automatically curvature homogeneous. Sekigawa [Sek 1] discovered the first examples of curvature homogeneous spaces which are not locally homogeneous. In recent years various classes of new examples have been found and also some specific classification problems have been solved (see [KTV 1–3], [Kow 2–3], [Yam], [ST], [KP]).

For three-dimensional Riemannian manifolds (M, g) we have the following simple criterion: (M, g) is curvature homogeneous if and only if all principal Ricci curvatures of (M, g) are constant. Thus the problem of classifying all three-dimensional Riemannian manifolds with prescribed constant Ricci eigenvalues is of considerable interest. From [Kow 2], [KP] and [ST] one can see that, for every prescribed constant $\rho_1, \rho_2, \rho_3 \in \mathbb{R}^3$, there exists a three-dimensional Riemannian manifold with constant Ricci eigenvalues ρ_1, ρ_2, ρ_3. If the numbers ρ_i are not all equal, then there always exists a corresponding Riemannian manifold (M, g) which is not locally homogeneous. On the other hand, it is *not* always possible to find a locally homogeneous Riemannian 3-manifold with arbitrary prescribed Ricci eigenvalues ρ_1, ρ_2, ρ_3. It is easy to see that for a locally homogeneous Rie-

* The first author was supported by the grant GAČR 201/93/0469; the second author was supported by the grant SFS, Project #0401.

mannian 3-manifold, the signature of the Ricci tensor is never equal to $(+,+,-)$ or $(+,0,-)$. Indeed, these signatures are never acquired by three-dimensional Lie groups with left invariant metrics, as was proved by Milnor [Mil]. On the other hand, from the paper by Sekigawa [Sek 2] it follows that every connected locally homogeneous Riemannian 3-manifold is either locally symmetric or locally isometric to a group space. Hence the result follows easily.

The purpose of the present paper is to determine precisely the range of all triplets from $\mathbb{R}^3$ which can occur as corresponding Ricci eigenvalues of locally homogeneous Riemannian 3-manifolds.

2. Ricci Eigenvalues

Let G be a connected n-dimensional Lie group, and let be $\mathcal{G}$ the associated Lie algebra. Choosing some fixed left invariant metric on G, then the resulting Riemannian manifold is *homogeneous*. That is, there exist an isometry carrying any point to any other point.

The curvature of a Riemannian manifold at a point can be described by the bi-quadratic *curvature function*

$$\mathcal{K}(x,y) = \langle R_{xy}(x), y\rangle,$$

$\langle , \rangle$ is the Riemannian inner product. Here, x, y range over all tangent vectors at the given point.

Recall that the adjoint L^* of a linear transformation L between metric vector spaces is defined by the formula

$$\langle Lx, y\rangle = \langle x, L^* y\rangle.$$

Further, we need the notation of the *Ricci quadratic form* $r(x)$, as a real-valued quadratic function of the tangent vector x, defined by the formula

$$r(x) = \sum_i \mathcal{K}(x, e_i) = \sum_i \langle R_{xe_i}(x), e_i\rangle.$$

It may be more convenient to work with the self-adjoint Ricci transformation $\hat{r}$, defined by

$$\hat{r}(x) = \sum_i R_{xe_i}(e_i).$$

This is related to the quadratic form r by the identity

$$r(x) = \langle \hat{r}(x), x\rangle.$$

The eigenvalues of $\hat{r}$ are called the *principal Ricci curvatures*. If we choose an orthonormal basis $e_1, \dots, e_n$ consisting of eigenvectors, then the numbers

$r(e_i)$ can be identified with the principal curvatures. The collection of signs $\{\text{sgn } r(e_1), \dots, \text{ sgn } r(e_n)\}$ can be called the *signature* of the quadratic form (see [Mil]).

3. Unimodular Lie Groups

Let G be a three-dimensional unimodular Lie group which is oriented and endowed with a left invariant Riemannian metric and let $\mathcal{G}$ denote its Lie algebra. Then the usual vector cross-product is defined on $\mathcal{G}$, besides the Lie bracket [,]. The Lie group is unimodular if and only if the linear transformation L, defined by $L(u \times v) = [u, v]$ is self-adjoint (see Lemma 4.1 [Mil]). Then there exists an orthonormal basis e_1, e_2, e_3 consisting of eigenvectors, $Le_i = \lambda_i e_i$, $i = 1, 2, 3$. Replacing e_1 by $-e_1$ if necessary, we may assume that the basis e_1, e_2, e_3 is positively oriented. For the Lie bracket we obtain the following normal form

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2, \tag{3.1}$$

(for further details see [Mil]).

THEOREM 3.1. *Let ρ_1, ρ_2, ρ_3 be real numbers. Then a unimodular Lie group with a left invariant metric and with the principal Ricci curvatures ρ_1, ρ_2, ρ_3 exists if and only if $\rho_1\rho_2\rho_3 > 0$ or if at least two of ρ_i, $i = 1, 2, 3$, are zero.*

Proof. For the orthonormal basis e_1, e_2, e_3, chosen as above, the corresponding principal Ricci curvatures are given by

$$r(e_1) = 2\mu_2\mu_3, \qquad r(e_2) = 2\mu_3\mu_1, \qquad r(e_3) = 2\mu_1\mu_2, \tag{3.2}$$

where

$$2\mu_i = (\lambda_1 + \lambda_2 + \lambda_3) - 2\lambda_i, \tag{3.3}$$

$i = 1, 2, 3$ (see [Mil], Th. 4.3).

The product of the Ricci principal curvatures will be

$$\rho_1\rho_2\rho_3 = r(e_1)r(e_2)r(e_3) = 8\mu_1^2\mu_2^2\mu_3^2. \tag{3.4}$$

By (3.4) the determinant of the Ricci quadratic form is always non-negative. If this determinant is zero, then at least one of μ_i, $i = 1, 2, 3$ will be zero and this implies that at least two of ρ_i, $i = 1, 2, 3$, will be zero.

Conversely, let ρ_1, ρ_2, ρ_3 be real numbers and suppose that either $\rho_1\rho_2\rho_3 > 0$ or at least two of ρ_i, $i = 1, 2, 3$, are zero. It is enough to express the structure constants λ_i, $i = 1, 2, 3$, from (3.1) as functions of the numbers ρ_i, $i = 1, 2, 3$. First, we will consider the case when $\rho_1\rho_2\rho_3 > 0$. Due to (3.2) we must have

$$\rho_1 = 2\mu_2\mu_3, \qquad \rho_2 = 2\mu_3\mu_1, \qquad \rho_3 = 2\mu_1\mu_2, \tag{3.5}$$

where μ_i are the numbers from (3.3).

Solving (3.5) with respect to $\mu_i, i = 1, 2, 3$, we get

$$\mu_1 = \pm\left(\frac{\rho_2\rho_3}{2\rho_1}\right)^{1/2}, \quad \mu_2 = \pm\left(\frac{\rho_3\rho_1}{2\rho_2}\right)^{1/2}, \quad \mu_3 = \pm\left(\frac{\rho_1\rho_2}{2\rho_3}\right)^{1/2}. \tag{3.6}$$

Now, by using (3.3) in the form $\lambda_i = (\mu_1 + \mu_2 + \mu_3) - \mu_i, i = 1, 2, 3$, we can get the structure constants $\lambda_i, i = 1, 2, 3$, as functions of numbers $\rho_i, i = 1, 2, 3$ (uniquely up to a sign).

This gives the desired unimodular Lie group with left invariant metric.

Further, if, for example, $\rho_1 \neq 0, \rho_2 = \rho_3 = 0$, we have $2\lambda_2\lambda_3 = \rho_1$ and $\lambda_1 = \lambda_2 + \lambda_3$; or if $\rho_1 = \rho_2 = \rho_3 = 0$ then, for example, $\lambda_1 = 0$ and $\lambda_2 = \lambda_3$ and again the corresponding Lie algebra can be defined (depending on one arbitrary parameter). □

Remark Some examples connected with this topic can be found in [Las].

4. Non-unimodular Lie Groups

Let G be a three-dimensional non-unimodular Lie group endowed with a Riemannian metric and let $\mathcal{G}$ denote its Lie algebra. Then one has a positive scalar product on $\mathcal{G}$ and the Lie bracket can be expressed in the following way.

LEMMA 4.1 *(see [Mil, p. 321]). If the connected three-dimensional Lie group G is not unimodular, then there is an orthogonal basis e_1, e_2, e_3 in $\mathcal{G}$ such that*

$$[e_1, e_2] = \alpha e_2 + \beta e_3,$$
$$[e_1, e_3] = \gamma e_2 + \delta e_3,$$

and $[e_2, e_3] = 0$, with $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$. □

LEMMA 4.2 (A modification of [Mil, Lemma 6.5]). *Any basis as in Lemma 4.1 diagonalizes the Ricci quadratic form, the principal Ricci curvatures being given by*

$$\begin{aligned}
\lambda^2\rho_1 &= r(e_1) = -\alpha^2 - \delta^2 - \tfrac{1}{2}(\beta + \gamma)^2, \\
\lambda^2\rho_2 &= r(e_2) = -\alpha(\alpha + \delta) + \tfrac{1}{2}(\gamma^2 - \beta^2), \\
\lambda^2\rho_3 &= r(e_3) = -\delta(\alpha + \delta) - \tfrac{1}{2}(\gamma^2 - \beta^2),
\end{aligned}$$

where $\lambda = ||e_i||, i = 1, 2, 3$. □

THEOREM 4.1. *A non-unimodular Lie group with left invariant metric and with the principal Ricci curvatures ρ_1, ρ_2, ρ_3 exists if and only if, either $\rho_1 = \rho_2 = \rho_3 < 0$ or, up to a possible re-numeration,*

$$\begin{aligned} &2\rho_1 < \rho_2 + \rho_3 < 0, \\ &\rho_1(\rho_2 + \rho_3) \leq \rho_2^2 + \rho_3^2, \\ &\rho_1 - \rho_2 = k(\rho_2 - \rho_3) \quad \text{holds for some constant } k. \end{aligned} \tag{4.1}$$

Proof. (A) The 'only if' part. Let $\{e_1, e_2, e_3\}$ be an orthogonal basis from Lemma 4.1, which is normalized in such a way that $\alpha + \delta = 2$. Thus the formulas

$$\begin{aligned} \alpha &= 1 + \xi, & \beta &= (1 + \xi)\eta, \\ \gamma &= -(1 - \xi)\eta, & \delta &= 1 - \xi, \end{aligned} \tag{4.2}$$

hold where ξ, η are arbitrary parameters. Then from Lemma 4.2 we get

$$\begin{aligned} \lambda^2\rho_1 &= -2(1 + \xi^2(1 + \eta^2)), \\ \lambda^2\rho_2 &= -2(1 + \xi(1 + \eta^2)), \\ \lambda^2\rho_3 &= -2(1 - \xi(1 + \eta^2)). \end{aligned} \tag{4.3}$$

Suppose first that the ρ_i are not equal, i.e. $\xi \neq 0$. Because $\lambda > 0$, the second inequality $\rho_2 + \rho_3 < 0$ of (4.1) follows immediately. To prove the remaining inequalities, we derive first from (4.3)

$$\lambda^2 = \frac{-4}{\rho_2 + \rho_3}, \tag{4.4}$$

$$\xi(1 + \eta^2) = \frac{\rho_2 - \rho_3}{\rho_2 + \rho_3} \neq 0. \tag{4.5}$$

(4.5) can be rewritten in the form

$$\xi = \frac{1}{(1 + \eta^2)} \left(\frac{\rho_2 - \rho_3}{\rho_2 + \rho_3} \right). \tag{4.6}$$

Substituting into the first equation of (4.3) from (4.4) and (4.6) we get

$$\frac{-4\rho_1}{\rho_2 + \rho_3} = -2\left[1 + \left(\frac{\rho_2 - \rho_3}{\rho_2 + \rho_3}\right)^2 \frac{1}{(1 + \eta^2)}\right]. \tag{4.7}$$

Hence, because $\rho_2 - \rho_3 \neq 0$ as $\xi \neq 0$, we easily obtain

$$\frac{1}{1+\eta^2} = \left(\frac{\rho_2+\rho_3}{\rho_2-\rho_3}\right)^2 \left(\frac{2\rho_1-\rho_2-\rho_3}{\rho_2+\rho_3}\right)$$
$$= \frac{(\rho_2+\rho_3)(2\rho_1-\rho_2-\rho_3)}{(\rho_2-\rho_3)^2},$$

and, consequently,

$$0 < \frac{(\rho_2+\rho_3)(2\rho_1-\rho_2-\rho_3)}{(\rho_2-\rho_3)^2} \leq 1,$$

which is equivalent to the first and third inequality of (4.1). The last condition in (4.1) obviously holds for $k = \frac{1}{2}(\xi - 1)$. The case $\rho_1 = \rho_2 = \rho_3$ corresponds to $\xi = 0$ and then the common value is $-2/\lambda^2 < 0$.

(B) Conversely, let the conditions of Theorem 4.1 be satisfied. If ρ_i are all equal and negative, then we obtain the necessary parameters from (4.4) and (4.2) with $\xi = 0$ and η arbitrary. If the ρ_i are not equal, then $\rho_2 - \rho_3 \neq 0$ due to the last condition of (4.1). Then we can reverse our argument and prove that the desired parameters $\lambda > 0, \xi, \eta$ always exist. Hence we can construct the corresponding non-unimodular Lie algebra equipped with a scalar product, and the corresponding Riemannian group space is determined. □

5. Locally Homogeneous Riemannian 3-Manifolds

We shall start with

PROPOSITION 5.1. *A locally homogeneous Riemannian 3-manifold (M, g) with the principal Ricci curvature ρ_1, ρ_2, ρ_3 exists if and only if ρ_i satisfy at least one of the following (partially overlapping) three sets of conditions*

(i) *All ρ_i are equal, or two of them are equal and the last one is zero.*
(ii) $\rho_1\rho_2\rho_3 > 0$, *or at least two of ρ_i are zero.*
(iii) *All ρ_i are non-positive, at most one of them is zero and, up to a re-numeration, they satisfy the inequalities*

$$2\rho_1 < \rho_2 + \rho_3, \qquad \rho_1(\rho_2+\rho_3) \leq \rho_2^2 + \rho_3^2. \tag{5.1}$$

Proof. Following Sekigawa [Sek 2], each connected locally homogeneous Riemannian manifold of dimension 3 is either locally homothetic to one of the symmetric spaces $E^3, S^3, H^3, S^2 \times E^1, H^2 \times E^1$ or locally isometric to a Lie group with a left invariant metric. The symmetric spaces correspond to case (i). The case of a unimodular Lie group corresponds to case (ii) according to Theorem 3.1.

Let now the ρ_i correspond to a non-unimodular Lie group, i.e. they satisfy (4.1) for some numeration. Then at least two of the ρ_i must be negative. If the third one is positive, we obtain case (ii). There remains the case when the last Ricci root is non-positive. Then inequalities (5.1) alone guarantee that $\rho_2 \neq \rho_3$ and the proof of the 'if' part of Theorem 4.1 works. Hence conditions (iii) always lead to a non-unimodular Lie group with a left invariant metric. □

Now, we obtain immediately our main result:

THEOREM 5.2. *A locally homogeneous Riemannian 3-manifold (M, g) with the principal Ricci curvatures $\rho_1 \geq \rho_2 \geq \rho_3$ exists if and only if*

(a) *the Ricci form does not have the signature $(+,+,-)$ or $(+,0,-)$,*

(b) *for the Ricci signature $(+,+,0)$ one has $\rho_1 = \rho_2$,*

(c) *for the Ricci signature $(-,-,-)$ or $(0,-,-)$ one has either $\rho_1 = \rho_2 = \rho_3$, or*

$$\frac{\rho_1^2 + \rho_3^2}{\rho_1 + \rho_3} \leq \rho_2 < \frac{\rho_1 + \rho_3}{2}. \tag{5.2}$$

Remark. Inequalities (5.2) will be never satisfied if we make the transposition of indices $1 \leftrightarrow 2$, and they are never satisfied if we make the transposition $2 \leftrightarrow 3$ unless $\rho_2 = \rho_3$ (and $\rho_1 = 0$). Thus we have chosen the only correct numeration of ρ_i to give sense to formula (5.2). □

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