NEAR-FIELD DISPERSION FROM INSTANTANEOUS SOURCES IN THE SURFACE LAYER

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Abstract. This paper considers the near-field dispersion of an ensemble of tracer particles released instantaneously from an elevated source into an adiabatic surface layer. By modelling the Lagrangian vertical velocity as a Markov process which obeys the Langevin equation, we show analytically that the mean vertical drift velocity $\overline{w}(t)$ is

$$\overline{w}(\tau) = bu_*(1 - e^{-\tau}(1 + \tau)),$$

where τ is time since release (nondimensionalized with the Lagrangian time scale at the source), b Batchelor's constant, and u_* the friction velocity. Hence, the mean height and mean depth of the ensemble are calculated. Although the derivation is formally valid only when $\tau \ll 1$, the predictions for $\overline{w}$, mean height and mean depth are consistent in the downstream limit ($\tau \gg 1$) with surface-layer Lagrangian similarity theory and with the diffusion equation. By comparing the analytical predictions with numerical, random-flight solutions of the Langevin equation, the analytical predictions are shown to be good approximations at all times, both near-field and far-field.

1. Introduction

To describe the dispersal of an ensemble of tracer particles released instantaneously from a point source into a turbulent flow, we may define two regions. The *near field* is the region $t \lesssim T_L$ (where t is the time since release and T_L the Lagrangian time scale of the turbulence), in which the tracer distribution is controlled mainly by the velocities of the tracer particles at the source through the persistence of the turbulent fluid motion (Taylor, 1921). The *far field* is the region $t \gg T_L$, where the randomness of the turbulence causes the tracer distribution to be independent of the precise velocity histories of the tracer particles. In this region the dispersion tends towards a classical diffusion process, described by a diffusion, or Fokker-Planck, equation (Monin and Yaglom, 1971).

The near field is important in many turbulent dispersion problems, but it cannot be treated with the diffusion equation (still the mainstay of much turbulent dispersion research). An alternative approach is to represent the Lagrangian velocities of the tracer particles as sample functions of a Markov process $w(t)$ that obeys the Langevin equation

$$\mathrm{d}w/\mathrm{d}t = -\alpha w + \lambda \xi(t), \tag{1}$$

where $\xi(t)$ is Gaussian white noise (Arnold, 1974, p. 50), which has the properties

$$\overline{\xi}(t) = 0, \qquad \overline{\xi(s)\xi(t)} = \delta(t - s). \tag{2}$$

When $w(t)$ is stationary, which is true for dispersion in stationary, homogeneous turbulence, the coefficients α and λ obey (Legg and Raupach, 1982)

$$\alpha = 1/T_L, \qquad \lambda = \sigma_w \sqrt{2/T_L}, \tag{3}$$

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where $\sigma_w^2 = \overline{w'^2}$ is the Lagrangian velocity variance, with overbars and primes denoting ensemble averages and departures therefrom, respectively. (Though the model is rewritten here for one-dimensional dispersion, extension to more than one dimension is possible.) Numerical dispersion calculations, based on the finite-difference form of Equation (1), have been carried out by several workers (e.g., Durbin, 1980; Durbin and Hunt, 1980; Legg and Raupach, 1982; Legg, 1983).

The purpose of this paper is to present some analytical results, derived from Equation (1), for the near-field vertical dispersal of an ensemble of tracer particles released from an elevated source into a boundary layer at time $t = 0$. Interest will centre on the ensemble's mean Lagrangian vertical velocity (or vertical drift velocity) $\overline{w}(t)$, mean height $\overline{Z}(t)$ and mean depth

$$\Sigma_z(t) = (\overline{(Z - \overline{Z})^2})^{1/2}, \tag{4}$$

where

$$Z(t) = h + \int_0^t w(s)\,\mathrm{d}s \tag{5}$$

is a stochastic process whose sample functions are the heights above the ground ($z = 0$) of the individual tracer particles, h being the release height. The near-field behaviour will be asymptotically matched with the far-field behaviour implied by the diffusion equation and by similarity theories.

This paper will consider single-particle dispersion only; that is, the particles making up the ensemble will be assumed to move independently. Thus, the results will describe the average behaviour of a succession of independent instantaneous puffs, in contrast to a single puff, for which a two-particle analysis is required. For simplicity, attention is restricted at this stage to the adiabatic surface layer (in which $T_\mathrm{L} \propto z$ and σ_w = constant).

2. Review of Lagrangian Properties of the Adiabatic Surface Layer

In the adiabatic surface layer, the only velocity scale for vertical velocity statistics (either Lagrangian or Eulerian) is the friction velocity u_*, and the only length scale is the height z. Dimensional analysis then fixes σ_w and T_L:

$$\sigma_w = au_* \tag{6}$$

and

$$T_\mathrm{L} = cz/u_*, \tag{7}$$

where a and c are constants of proportionality. Furthermore, the vertical drift velocity of tracer particles in the far field is given by the Lagrangian similarity theory of Batchelor (1964)

$$\overline{w} = bu_* \quad (t \gg T_h), \tag{8}$$

where b is another proportionality constant (Batchelor's constant) and T_h is the value of T_L at the source:

$$T_h = ch/u_* . \tag{9}$$

Values are required for a, b, and c. By adopting the common assumption (Hanna, 1982) that $\sigma_w^2 = \sigma_{w\mathrm{E}}^2$ (σ_w^2 and $\sigma_{w\mathrm{E}}^2$ being the Lagrangian and Eulerian vertical velocity variances, respectively) we obtain $a = \sigma_{wE}/u_* \approx 1.25$, a typical value for both laboratory and atmospheric adiabatic surface layers. To fix b and c, we follow Chatwin (1968) and Hunt and Weber (1979). Consider a ground source. Since $h = 0$ and $T_h = 0$, the far field extends over all time, the near field vanishes, and the diffusion equation can be applied at all times. Therefore, values of $\overline{Z}(t)$ and $\Sigma_z^2(t)$ deduced from the diffusion equation must agree with those from the extension to the surface layer of the statistical dispersion theory of Taylor (1921). The diffusion equation predicts (Chatwin, 1968) that

$$\overline{Z} = \Sigma_z = ku_*t/\mathrm{Pr} , \tag{10}$$

where k is the von Karman constant and Pr the turbulent Prandtl number for the dispersing tracer. The statistical dispersion theory predicts (Hunt and Weber, 1979) that

$$\overline{Z} = bu_*t , \qquad \Sigma_z = a(bc)^{1/2}\, u_*t . \tag{11}$$

Hence, equating these predictions, it follows that $b = k/\mathrm{Pr} \approx 0.4$ (assuming $k \approx 0.4$ and $\mathrm{Pr} \approx 1$, both experimentally well verified) and that

$$a^2c/b = 1 , \tag{12}$$

giving $c \approx 0.26$.

3. The Vertical Drift Velocity

To find $\overline{w}(t)$ in the near field ($t \lesssim T_h$), we consider each tracer particle to be governed by Equation (1) with values of T_L and σ_w appropriate to its height $Z(t)$. An ensemble average of Equation (1) gives (since $\overline{\lambda\xi} = 0$):

$$\mathrm{d}\overline{w}/\mathrm{d}t = -\,\overline{w(t)/T_L(t)} \tag{13}$$

$$= -(u_*/c)\overline{w(t)/Z(t)} . \tag{14}$$

By defining the dimensionless height

$$Y(t) = (Z(t) - h)/h , \tag{15}$$

for which $Y(0) = 0$, a binomial expansion can be performed on Equation (14) when t, and hence $Y(t)$, is small:

$$\frac{\mathrm{d}\overline{w}}{\mathrm{d}t} = -\left(\frac{u_*}{ch}\right)(\overline{w} - \overline{wY} + \overline{wY^2} - \cdots) . \tag{16}$$

We assume that t is small enough that $\overline{Y}(t) \ll 1$ and that the higher-order terms, $\overline{wY^2}, \ldots,$

can be neglected. The term $\overline{wY}$ can be written as

$$\overline{wY} = \overline{w}\overline{Y} + \frac{1}{h}\int_0^t \overline{w'(s)w'(t)}\,\mathrm{d}s \tag{17}$$

$$\approx \overline{w}\overline{Y} + \left(\frac{\sigma_w^2 T_h}{h}\right)(1 - e^{-t/T_h}), \tag{18}$$

where we have assumed an exponential form for the autocorrelation function of $w(t)$:

$$\overline{w'(s)w'(t)} = \sigma_w^2\, e^{(s-t)/T_\mathrm{L}} \quad (s \leq t)\,. \tag{19}$$

Here T_L should be evaluated at a height somewhere between $\overline{Z}(s)$ and $\overline{Z}(t)$, but it may be evaluated at h when $\overline{Y}(t) \ll 1$. Combining Equations (18) and (16) gives:

$$\frac{\mathrm{d}\overline{w}}{\mathrm{d}t} = -\frac{\overline{w}(t)}{T_h} + \frac{\sigma_w^2}{h}(1 - e^{-t/T_h}), \tag{20}$$

where $\overline{w}\overline{Y}$ has been neglected in comparison with $\overline{w}$. This is a simple linear ordinary differential equation in $\overline{w}(t)$, which may be solved with the initial condition

$$\overline{w}(0) = 0\,, \tag{21}$$

which arises from the requirement that Lagrangian and Eulerian velocities be statistically identical at $t = 0$, the 'labelling time' for tracer particles. The solution is

$$\overline{w}(t) = bu_*[1 - e^{-t/T_h}(1 + t/T_h)]\,, \tag{22}$$

in which the equality $\sigma_w^2 T_h/h = bu_*$, established by Equation (12), has been used.

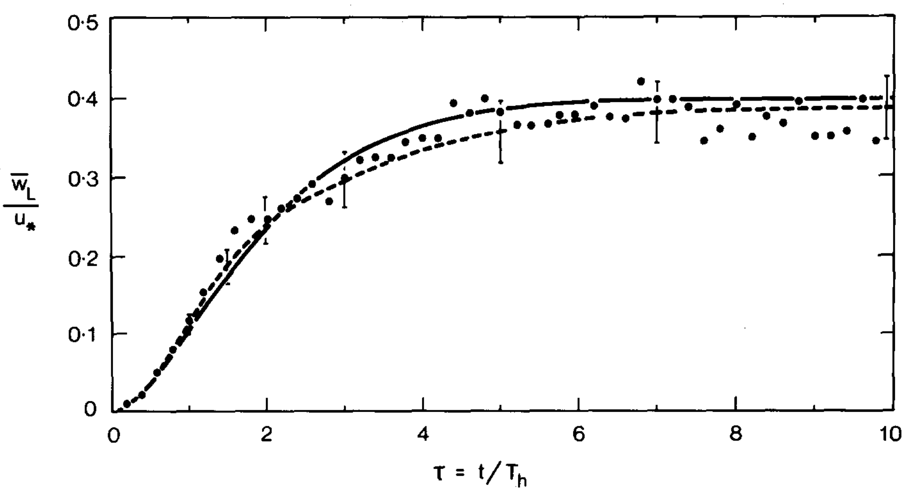

Fig. 1. Comparison of analytical prediction (22) and numerical prediction of vertical drift velocity $\overline{w}(t)$. Solid line: Equation (22); points: one run of numerical model (1000 particles, time step $0.02T_\mathrm{L}$); dashed line with error bars: average of 10 independent runs of numerical model.

The formal derivation of this solution is valid only in the limit $t \to 0$, but the solution has the correct behaviour $\overline{w} = bu_*$ in the limit $t \to \infty$. Therefore, it is reasonable to enquire if Equation (22) is a good approximation at all times. This has been tested by solving Equation (1) numerically, in finite-difference form, using a random-flight model similar to that of Durbin (1980) and Legg (1983). The comparison between the analytical and numerical solutions, given in Figure 1, shows that Equation (22) adequately specifies $\overline{w}(t)$ at all values of t, both near- and far-field. In essence, the reason for the success of Equation (22) is that the assumptions in its derivation (especially the assumption $\overline{Y} \ll 1$) are valid throughout the near field.

4. The Mean Height and Mean Depth of the Ensemble

It is desirable to evaluate $\overline{Z}(t)$ and $\Sigma_z(t)$ in dimensionless form. Accordingly, we rewrite Equation (22) as

$$\overline{w}/u_* = b(1 - e^{-\tau}(1 + \tau)), \tag{23}$$

where $\tau = t/T_h$. Since $\overline{w} = \mathrm{d}\overline{Z}/\mathrm{d}t$, $\overline{Z}$ follows directly by integration:

$$\overline{Z}/h = 1 + bc(\tau - 2(1 - e^{-\tau}) + \tau e^{-\tau}). \tag{24}$$

For Σ_z, we return temporarily to dimensional variables and use the dispersion theorem of Taylor (1921), which gives (on taking mean and fluctuating parts of Z and w):

$$\begin{aligned} \mathrm{d}\Sigma_z^2/\mathrm{d}t &= 2 \int_0^t \overline{w'(t)w'(s)}\, \mathrm{d}s \\ &= \frac{2\sigma_w^2 c\overline{Z}(t)}{u_*}\left[1 - \exp\left(\frac{-u_* t}{c\overline{Z}(t)}\right)\right], \end{aligned} \tag{25}$$

where Equation (19) has been used for the autocorrelation function, evaluating T_{L} at $\overline{Z}(t)$. Since the exponential is negligible when $t \gg T_h$, it is a good approximation to replace $\overline{Z}(t)$ by h within it, giving

$$\mathrm{d}\Sigma_z^2/\mathrm{d}t \approx 2bu_*\overline{Z}(t)\,[1 - e^{-t/T_h}], \tag{26}$$

which can be integrated thus:

$$\begin{aligned} \Sigma_z^2 &= 2bu_* \int_0^t \overline{Z}(s)\left(\frac{\overline{w}(s)}{bu_*} + \frac{T_h}{bu_*}\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}\right)\mathrm{d}s \\ &= 2\int_0^t \overline{Z}(s)\frac{\mathrm{d}\overline{Z}}{\mathrm{d}s}\,\mathrm{d}s + 2T_h\int_0^t \overline{Z}(s)\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}\,\mathrm{d}s \\ &\approx \overline{Z}(t)^2 - h^2 + 2hT_h\overline{w}(t). \end{aligned} \tag{27}$$

In the first line, Equation (20) has been used; in the second line, the term $\overline{Z}(s)$ in the second integral is replaced by h since $d\overline{w}/ds$ (the other term in the integral) approaches zero when $t \gg T_h$. Numerical integration of Equation (25) has shown that the approximation (27) is good to within 0.2%. In dimensionless variables, the result is

$$\Sigma_z^2/h^2 = (\overline{Z}/h)^2 - 1 + 2bc(1 - e^{-\tau}(1 + \tau)). \tag{28}$$

In Figures 2 and 3, the analytical results for $\overline{Z}$ and Σ_z are compared with the predictions of the numerical model used in the computation of $\overline{w}$. For $\overline{Z}$, the two results are in good agreement. For Σ_z, the analytical prediction slightly exceeds the numerical prediction (by about 10%) except when $t < T_h$ or $\tau < 1$. This happens because a single time scale $T_L(\overline{Z})$ has been used for the entire cloud in Equation (25) instead of giving each fluid particle a different time-scale according to its height, as in the numerical model.

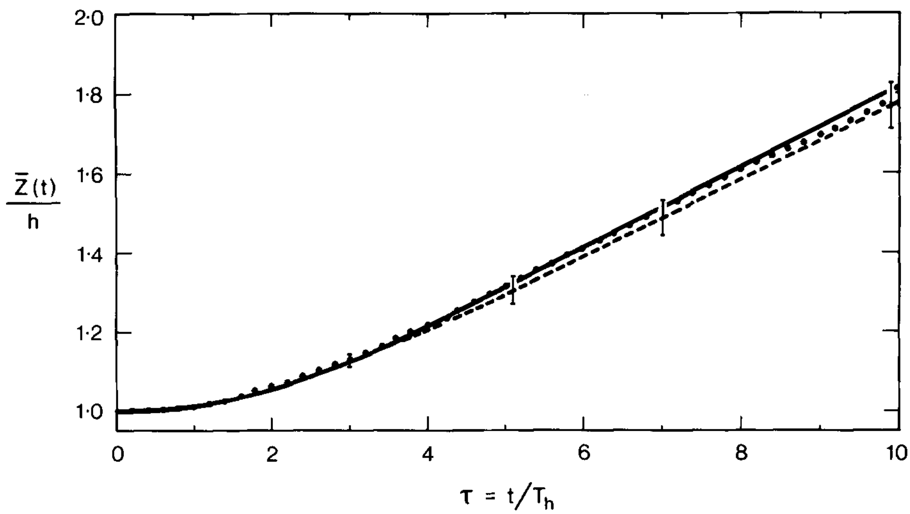

Fig. 2. Comparison of analytical prediction (24) and numerical prediction of $\overline{Z}(t)$. Symbols as for Figure 1.

5. Discussion

The far-field solution for $\overline{Z}$, obtained by letting $t \to \infty$ in Equation (24), is

$$\overline{Z}(t) = h + bu_*(t - 2T_h) \quad (t \gg T_h), \tag{29}$$

which is equivalent to the prediction of Lagrangian similarity theory

$$\overline{Z}(t) = bu_*(t + t_v), \tag{30}$$

where t_v is a displacement for the time origin which accounts for the elevation of the

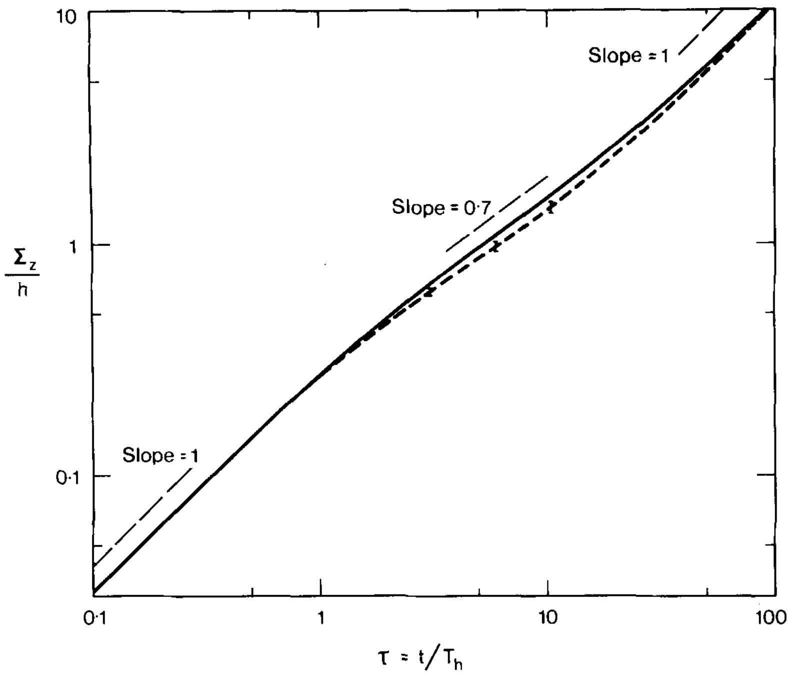

Fig. 3. Comparison of analytical prediction (27) and numerical prediction $\Sigma_z(t)$. Symbols as for Figure 1 (single run of numerical model not shown).

source ($t_v = 0$ for a ground source). Comparing these two results, it follows that

$$t_v = h/bu_* - 2T_h \approx 0.32h/u_* , \tag{31}$$

which quantifies the order-of-magnitude estimate $t_v \sim h/u_*$ obtained from similarity theory. If an ensemble is released from a hypothetical ground source at time $(-t_v)$, its trajectory passes beneath the real elevated source at height $0.8h$.

At very small times ($t \ll T_h$), the ensemble has the following behaviour, in dimensionless variables:

$$\overline{w}/u_* = b\tau^2/2 + 0(\tau^3) , \tag{32}$$

$$\overline{Z}/h = 1 + bc\tau^3/6 + 0(\tau^4) , \tag{33}$$

$$\Sigma_z^2/h^2 = bc\tau^2 + 0(\tau^3) . \tag{34}$$

Equation (32) shows that the ensemble has a well-defined vertical acceleration at all times, including $t = 0$ when the acceleration is zero. This happens even though the sample functions of the Markov process $w(t)$ have undefined (infinite) derivatives. From Equation (34), it follows that $\Sigma_z = \sigma_w t$ when $t \ll T_h$, as in homogeneous turbulence.

Three asymptotic ranges can be identified for Σ_z: the first is the near field where $\Sigma_z \to \sigma_w t$, and the last is a far downstream range where $Z \gg h$ and $\Sigma_z \to \overline{Z}$ (this happens so far downstream that the distinction between elevated and ground sources is lost). In between lies an intermediate range where $t \gg T_h$ but $\overline{Z} \sim h$; here $\Sigma_z^2 \to 2bu_*th$, so $\Sigma_z \propto t^{1/2}$. This is the classical result for diffusion in a homogeneous medium with diffusivity $\sigma_w^2 T_h$. Figure 3 shows that, in practice, the intermediate range is 'squeezed out' between the outer ranges, so that the exponent s in the power law $\Sigma_z \propto t^s$ passes from 1 (near field) through a minimum of about 0.7 (the squeezed-out intermediate range) and back to 1 (far downstream).

6. Conclusions

For an ensemble of tracer particles released instantaneously from an elevated source into an adiabatic surface layer, analytical expressions have been obtained for the vertical drift velocity $\overline{w}(t)$, mean height $\overline{Z}(t)$ and ensemble mean depth $\Sigma_z(t)$ (Equations (22), (24), and (28), respectively). The expressions describe the near field, merging into well-known far-field expressions (from similarity theory and the diffusion equation) at large times. The analysis is based on the assumption that the particle velocities are sample functions of a Markov process which obeys the Langevin equation with a time-scale T_L proportional to height.

There is no fundamental reason why the Langevin equation should describe fluid particle velocities in turbulence; instead, its use is justified by the fact that its solution $w(t)$ is a Gaussian process with an exponential autocorrelation function (Arnold, 1974, p. 132). Velocities in homogeneous turbulence are approximately Gaussian, with nearly exponential Lagrangian autocorrelation functions (Deardorff, 1978); the vertical velocity in an adiabatic surface layer is also nearly Gaussian, in the weak sense that it has a skewness close to zero and a kurtosis of about 3 (Raupach, 1981). Therefore, the Langevin equation appears to offer a tenable model in these circumstances. However, when velocity distributions become highly nonGaussian (in crop canopies, strong convection, or intermittent turbulence, for example), the Langevin model is far from secure.

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