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Use of Finite Element Computer Programs in Fracture Mechanics

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ABSTRACT

Since the theoretical stresses and strains at the tip of a V-notched crack in an elastic continuum are infinite, the question arises as to the accuracy of strain energy as calculated from finite element computer programs for systems containing such a crack. Two geometries for which analytical solutions are available were analyzed using a plane stress finite element computer program. Results show that accuracy in both cases depended upon proper selection of a grid network. Several methods of calculating stress intensity factors are discussed. Application of the finite element computer program in the analysis of fracture in solid propellant rocket motor cartridge or grain is included.

Introduction

Determination of the structural integrity of rocket motor grains recently has included considerable effort in predicting the useful life of motors with damaged or cracked grains. Recent applications have employed the energy balance theory of fracture mechanics for linearly viscoelastic materials. The energy approach developed by Griffith [1], and extended to linear viscoelastic materials by Williams [2], has been used to predict depth of crack propagation and crack trajectory in rocket motor grains. This approach requires a knowledge of the rate of change of the strain energy with respect to new crack surface area as the crack extends $(\partial U/\partial A)$, which can be related to the stress intensity factors. Since direct relationships exist between stress, strain and energy, the stress intensity factors are proportional to the square root of the strain energy release rate, as was shown by Irwin [3]. This information, combined with the experimentally determined cohesive fracture energy function, γ_c, permits a calculation of maximum crack depth [4, 5].

A rigorous determination of stress intensity factors or energy rates in rocket motor grains requires the exact solution of viscoelasticity problems involving complex geometries. Since these solutions are not generally available, numerical techniques must be used. The method which is currently most used in the analysis of stresses and strains in complex geometries is application of finite element computer formulations. The application of these computer programs to fracture mechanics for several different geometries is discussed in papers by Swanson [6] and Chan [7].

A method for determining the cohesive fracture energy γ_c for viscoelastic materials is outlined in Reference [8]. However, all of the fracture mechanics analyses reported herein apply to a particular instant of time after load. For this time γ_c is a constant and a quasiviscoelastic strain energy analysis is used.

Comparison of Numerical and Analytical Results

Although the general three dimensional finite element computer programs are in operation, this discussion is limited to the special case of two dimensional plane strain and plane stress for which analytical solutions are available. For these problems, nonzero stress components are

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mathematically infinite at the tip of a V-notched crack. For example, the local stresses and displacements in the vicinity of the tip of a finite crack in a plane strain continuum subjected to a uniform stress, σ, at infinity, are [9]:

$$\sigma_x = \left(\frac{a}{2r}\right)^{\frac{1}{2}} \sigma \cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_y = \left(\frac{a}{2r}\right)^{\frac{1}{2}} \sigma \cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{xy} = \left(\frac{a}{2r}\right)^{\frac{1}{2}} \sigma \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_z = \nu(\sigma_x+\sigma_y), \qquad \tau_{xz} = \tau_{yz} = 0$$

$$u = \left(\frac{ar}{2}\right)^{\frac{1}{2}} \frac{\sigma}{G}\cos\frac{\theta}{2}\left(1-2\nu+\sin^2\frac{\theta}{2}\right)$$

$$v = \left(\frac{ar}{2}\right)^{\frac{1}{2}} \frac{\sigma}{G}\sin\frac{\theta}{2}\left(2-2\nu-\cos^2\frac{\theta}{2}\right)$$

where (r, θ) = polar coordinates from crack tip
$a = \frac{1}{2}$ crack length
σ = applied stress
G = shear modulus
u = displacement in direction of crack
v = displacement perpendicular to crack

The question, therefore, arises as to the accuracy of the strain energy as calculated by finite element computer programs for a system containing such a crack. Two geometries for which closed form solutions are available were analyzed using a plane stress finite element computer program.

The first geometry consists of a finite length crack ($2a$) in an infinite sheet subjected to tension at infinity. The difference in energy in a sheet with a crack of length $2a$ and a sheet without a crack as obtained by Griffith [1] is $\Delta U = \pi a^2\sigma^2/E$.

TABLE 1

Strain energy for Griffith crack

Half-Crack Length (in.)	Analytical	300 Node Grid				720 Node Grid		
		A	B	C	D	E	F	G
0	13.2332	13.2248	—	—	—	—	—	—
0.5	13.6096	13.5740	—	—	—	—	—	—
0.8	14.8632	14.1572	—	—	—	—	—	—
0.9	14.4716	14.4224	—	—	—	—	—	—
1.0	14.7640	14.7692	14.7596	14.7312	14.7404	14.8312	14.832	—
1.2	15.4408	15.5360	15.5212	—	—	—	—	15.6356
Minimum Grid Dimension (in.)	—	0.01	0.02	0.04	0.08	0.005	0.02	0.02
Percent from Finest Grid	—	0.418	0.483	0.674	0.612	0	0.005	—

A finite sheet (6 in. high by 9 in. long) was analyzed with longitudinally oriented cracks to simulate the infinite sheet for which Griffith's energy equation applies. A modulus of 10,800,000 psi and Poisson's ratio of 0.3333 were used. The analytical results are compared in Table 1 with the strain energy as calculated using a finite element computer program. The calculations

in column A of Table 1 show that approximately three-figure accuracy may be obtained using a relatively coarse grid network.

The total energy (U) was calculated for several grid networks to determine the effect of total mesh size and node distribution in the vicinity of the crack tip. The grid mesh in the vicinity of the crack tip consisted of square elements, the first element from the crack tip having the dimensions shown in Table 1. In any direction from the crack tip, the elements increased in geometric progression with a ratio of two for most grids used. A difference of only 0.26 percent was found between the analytical and numerical results for a 1 in. crack. This small difference accounts for both the finite dimension approximation (6 in. by 9 in.) and the numerical inaccuracies in the computer results. If the finest grid (E) is used as a reference, the results indicated in Table 1 show that an error of 0.67 percent may be introduced by using a relatively coarse grid. Consequently, it was concluded that, at least for this geometry, good engineering accuracy can be obtained in calculating strain energy from finite element computer programs. For the example shown in Table 1, relatively coarse grid networks are satisfactory for computing total strain energy accurately to the second or third significant digit.

Figure 1 shows the rate of change of strain energy with respect to crack area ($\partial U/\partial A$) for

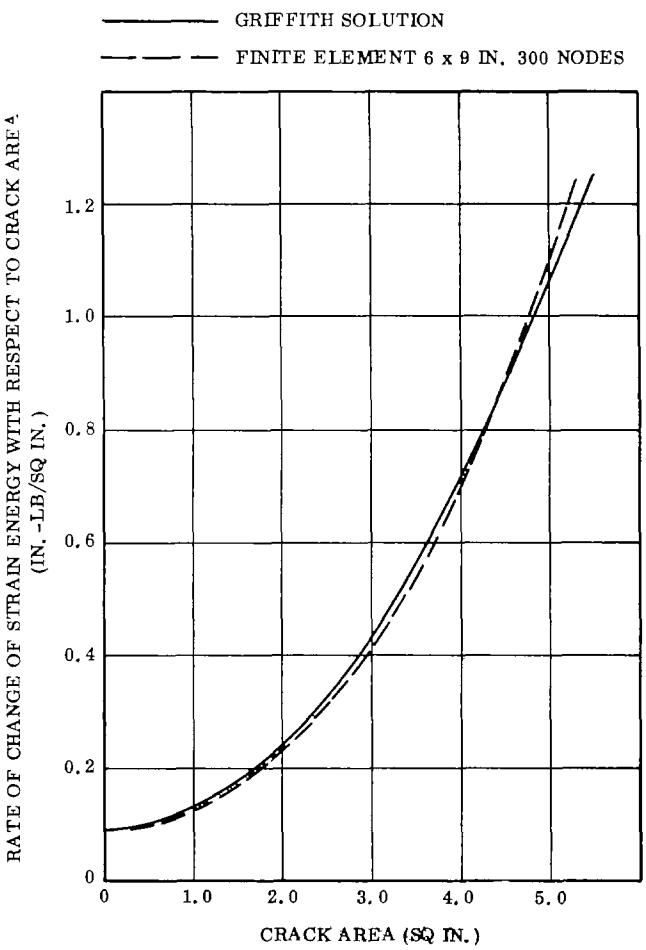

Figure 1. Rate of Change of Energy for Griffith Crack.

analytical versus the ($\Delta U/\Delta A$) numerical results. The inaccuracy for large crack areas is attributed to the finite geometry analyzed. In this analysis and all other reported herein, the $\partial U/\partial A$ was calculated by computing energy for various crack depths and using a central difference technique. The increment size, ΔA, depends upon the rate of change in the slope of the energy

versus crack area plot. Since the $\partial U/\partial A$ curve is normally a smooth function of crack area, a fairly large increment in crack area, ΔA, is used.

A second analysis was performed to determine if stress concentrations due to curved boundaries adversely affected the accuracy of finite element computer programs in analyzing V-notch cracks. The geometry, as analyzed by O. L. Bowie [10], consists of a circular hole in an infinite body with any number of radial cracks extending from the circular hole. A tensile loading is applied at infinity.

Computer determinations of strain energy for various values of crack depth were calculated using several different grid densities. These values are shown in Table 2.

TABLE 2

Total strain energy(U) for Bowie analysis

Crack Depth (in.)	Numerical Solution*			Analytical Solution
	Mesh Size			
	11 × 11	18 × 19	27 × 38	
0	3.5814	3.55525	3.547980	3.546189
0.303	—	3.71637	3.724561	3.731300
0.497	—	3.95403	3.960683	3.983400

*Internal radius = 1.0. External radius = 10.0.

Accuracies of 99.6 to 99.9 percent were obtained (depending upon grid mesh size) for a crack of 0.303 in. depth. Since the error increases with crack length, part of the error is attributed to using a finite cylinder to approximate the infinite body as analyzed by Bowie.

Results led to the conclusion that finite element programs can be used to calculate accurately the strain energy in rocket motor grains, at least when quasiviscoelastic analysis is applicable.

Calculation of Stress Intensity Factors

Several methods of calculating elastic stress intensity factors using finite element computer programs were examined. The first method consists of calculating the strain energy for several different values of crack depth. The rate of change of strain energy with respect to crack area ($\Delta U/\Delta A$) is then calculated for the crack depths of interest. These data may be converted to stress intensity factors using the equations of Reference [3] or [9].

The second method consists of selecting a grid with a sufficiently fine mesh around the crack tip that the stresses or displacements may be fit directly to the theoretical equations. Chan [7] shows that if displacements are used, the stress intensity factors may be obtained most accurately using the displacements perpendicular to the crack surface. For plane strain with no shear stress applied, the stress intensity factor (K_I) is $\sigma\sqrt{\pi a}$. These stress intensity factor in terms of crack tip displacement v becomes

$$K_I = \frac{(2\pi/r)^{\frac{1}{2}} Gv}{\sin\frac{\theta}{2}\left[2-2v-\cos^2\left(\frac{\theta}{2}\right)\right]}$$

For extremely small or very large values of r, the computed values of K_I are not very accurate. The most accurate results are obtained by plotting K_I *vs.* r and extrapolating the intermediate data to the point $r=0$.

Rice [11] has shown that for an elastic material the value of the line integral

$$J=\int_\Gamma \left(W\,dy - \boldsymbol{T}\cdot\frac{\partial \boldsymbol{u}}{\partial x}\,ds\right)$$

is proportional to the square of the crack tip stress intensity factor, where Γ is an arbitrary

contour surrounding the crack tip, W is the strain energy density, and $\boldsymbol{T}$ and $\boldsymbol{u}$ are traction and displacement vectors.

For plane strain conditions, the following relation is given.

$$K_I^2 = \frac{JE}{1-\nu^2}$$

This equation may be used as a third method of calculating stress intensity factors.

Several check cases have been investigated by Chan using the line integral method and the stresses and displacements in the vicinity of the crack tip. He concludes that approximately the same accuracy may be obtained using either method. The accuracy of these methods for the grids used by Chan is generally 95 to 97 percent.

One disadvantage of both methods is that they require an extremely fine grid mesh in the vicinity of the crack tip. Chan uses a system of automatic refinement and renumbering of the nodes at the crack tip. The refinement of this portion of the grid increases the stiffness matrix bandwidth to such an extent that direct solution schemes are no longer feasible because of storage limitations. This problem may be resolved by using an iterative scheme. Although iterative techniques work well for materials with low Poisson's ratios they converge very slowly for nearly incompressible materials ($\nu \to \frac{1}{2}$) such as solid propellant fuel.

A possible program improvement to make these methods more feasible for propellant grain analyses and to decrease the number of elements required for a given accuracy is replacement of the linear displacement triangular elements in the neighborhood of the crack tip with annular elements. The displacements in these elements would be chosen proportional to the square root of the distance from the crack tip, which would satisfy the theoretical displacement equations in this area. Wilson is presently studying this feature at Westinghouse.

Several computer runs were completed to determine the relative accuracy of calculation methods for stress intensity factors using a Griffith sheet geometry. In the first method, the strain energy was calculated for two cracks of length 0.998 and 1.0 inch (6 in. × 9 in. sheet). The $\Delta U/\Delta A$ was calculated and converted to stress intensity using the formula

$$K_I = (2E\Delta U/\Delta A)^{\frac{1}{2}}$$

In the second method, the displacements perpendicular to the crack surface were converted to stress intensity factors for the 1.0 in. crack using the equation

$$K_I = \frac{Ev}{4}\left(\frac{2\pi}{r}\right)^{\frac{1}{2}}$$

These results are compared with the analytical solution

$$K_I = \sigma(\pi a)^{\frac{1}{2}}$$

in Table 3. With the same material properties and geometry as used previously in this report,

TABLE 3

Stress intensity factor (K_I) comparison for Griffith sheet

Grid	Grid Dimensions	Number of Nodes Distributed with $\sqrt{r}$	Distance from Crack Tip with Fine Grid	K_I from Energy	Percent Error	K_I from Displacement	Percent Error
1	15 × 22	0	0	2,887	0.143	2,580	−10.5
2	15 × 22	5	0.25	2,912	1.01	2,720	− 5.7
3	15 × 22	5	0.05	2,862	−0.72	2,820	− 2.2
4	15 × 22	5	0.005	2,729	−5.31	2,965	2.8
5	30 × 49	13	0.0034	2,881	−0.07	2,920	1.3
Analytical				2,883			

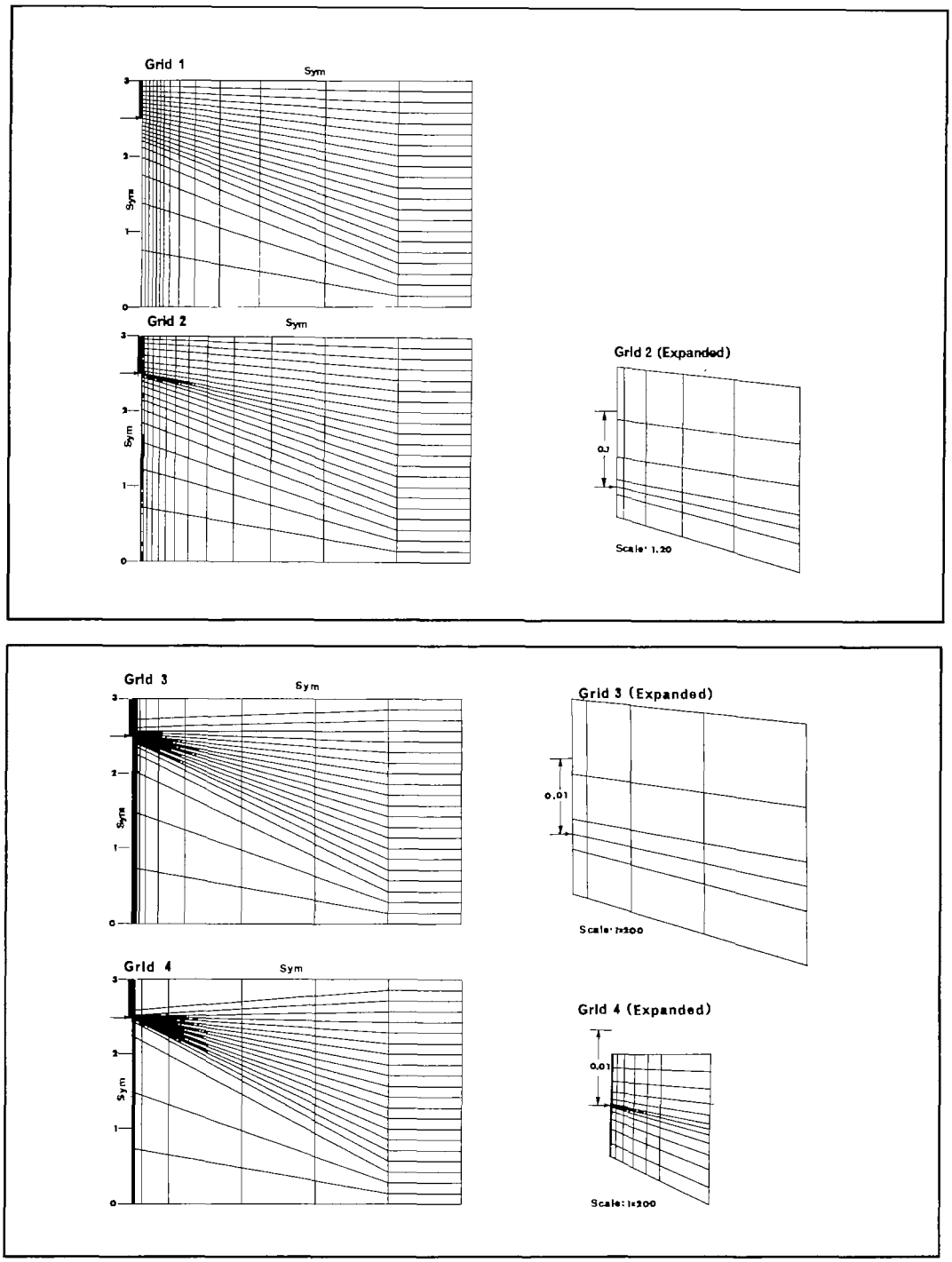

Figure 2. Grid Network for Griffith Sheet.

the stress intensity factor from the above equation has the numerical value of 2,883.

The number of nodes used in the first four grids of the study (Figure 2) was 330, which is considerably less than the program capability. With the exception of the first grid, the nodes in the vicinity of the crack tip were selected so that equal changes in the square root of the distance from the crack tip were obtained over each element.

The data in Table 3 show that for a given number of nodes a grid mesh can be selected so

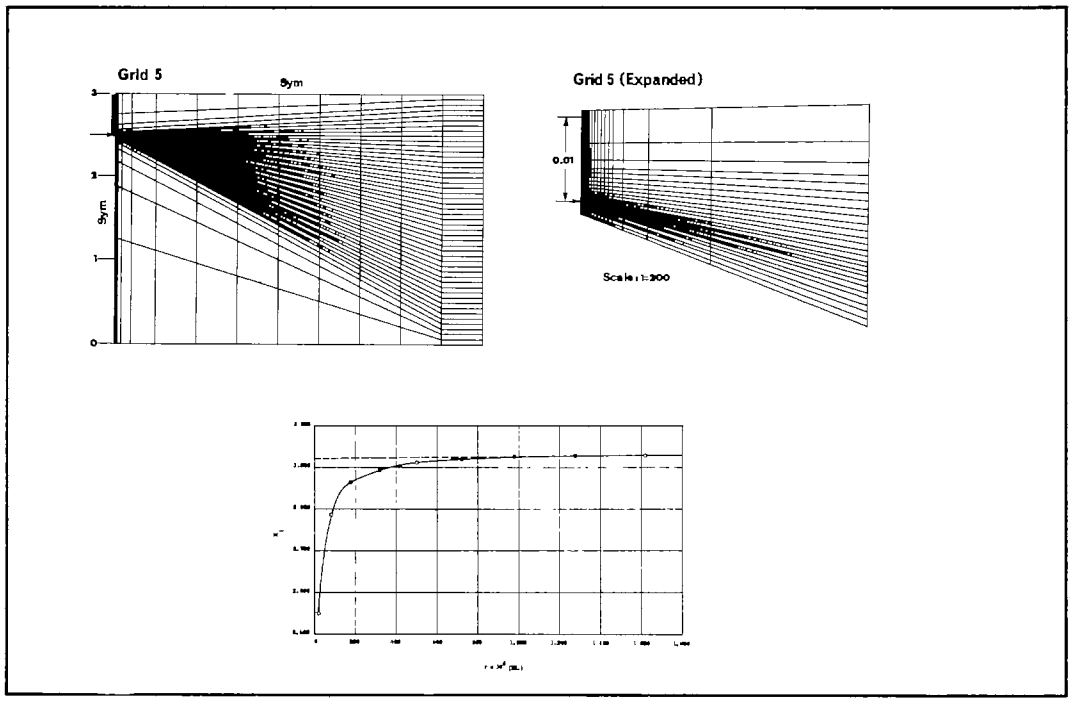

Figure 3 Stress Intensity Factor for Griffith Sheet.

that the stress intensity factor can be more accurately obtained from strain energy than from displacements. Then by redistributing the nodes (fines in vicinity of the crack tip), the displacements can produce the more accurate results.

The values of K_I listed in Table 3 for grids 1 thru 4, as calculated from displacements, are the maximum values for nodes in the vicinity of the crack tip. For the small mesh used, too few nodes were sufficiently close to the crack tip to obtain a straight line extrapolation to $r=0$. A fifth run was conducted to demonstrate the grid requirements and accuracy obtained from displacements when sufficient nodes are available. Figure 3 shows K_I versus r for grid 5. When this curve is extrapolated to the crack tip, a stress intensity factor of 2,920 is obtained, which differs from the analytical results by approximately 1 percent. The error in stress intensity factor as calculated from strain energy for this grid is less than 0.1 percent.

Using the finite element programs as presently formulated for a given number of nodes and the proper selection of a grid, the stress intensity factors may be calculated more accurately using strain energy than from either displacements or the integral method, as discussed above. When methods of refining grid networks or modifying the elements in the vicinity of the crack tip are incorporated into the program, the displacements may produce the more accurate results.

In the calculation of elastic strain energy change in rocket motor grains analyzed to date, the energy in the case changes radically for even fairly small crack depths; consequently, this change in energy must be combined with the change in energy in the propellant grain to predict crack behavior.

Selection of Grid Mesh

The basic assumption inherent in the development of our computer programs is that the displacements within each triangular element are linear functions of the coordinates. The optimum placement of nodes in any region where the functional form of the displacement field is known is such that the displacements may be fit most accurately by straight lines. Selection of the optimum node distribution often requires estimation of the displacement gradients.

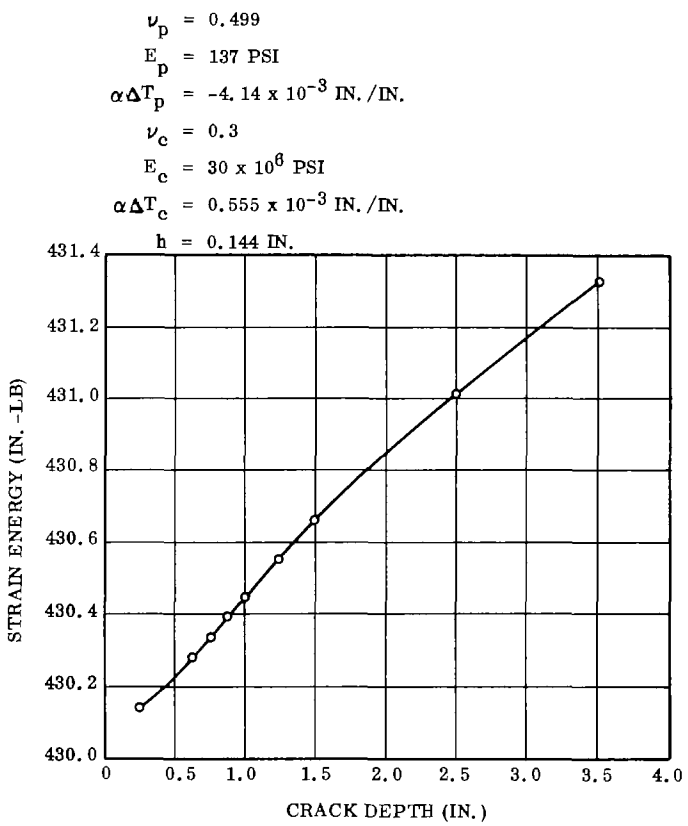

Figure 4. Strain Energy *vs.* Crack Depth – Thermal Load.

However, since the displacements are known to be proportional to the square root of r in the vicinity of crack tips, the optimum node distribution is such that each element has the same change in the function $\sqrt{r}$. This distribution is obtained automatically if the nodes are distributed in an arithmetical progression in all directions from the crack tip.

In most of our analyses, the grids have been selected in the above manner. Normally the first five nodes in every direction from the crack tip are kept in exactly the same relation to the crack as crack angle or crack depth is changed. Use of this selection technique has been necessary to prevent grid bias from distorting the energy calculations in the fourth or fifth decimal place.

A second method has recently been employed which appears to provide approximately the same accuracy. In this method the grid network is kept constant and crack depth is changed by changing boundary constraints or introducing zero modulus in continuum elements. A fine mesh cannot be obtained in the vicinity of the crack tip if many depths are required. However, as shown in the test cases, the energy can be calculated to 3 or 4 place accuracy using a fairly coarse grid. Since there is no grid bias with this method, the changes in energy provide sufficient accuracy to calculate a $\Delta U/\Delta A$ curve accurately. This constant grid method was used on a complex motor grain which had previously been analyzed using a fine grid for each crack depth. The greatest difference between the two resulting $\Delta U/\Delta A$ curves was less than two percent.

Figures 4 and 5 show typical strain energy curves for thermal and pressure loads from a motor grain in a steel case. Note that changes in the fourth significant digit will contribute to relatively large changes in slope of these curves. Since this slope ($\Delta U/\Delta A$) is used in calculating stress intensity factors; and because, as shown previously, the check cases are only accurate to approximately three significant figures, some care must be used in performing the calculations.

In the initial crack propagation analyses, severe oscillations were noted in the fourth digit for small cracks under pressure load. The oscillations were decreased by redistributing the

nodes so that a finer mesh distribution, proportional to $\sqrt{r}$, was used in the vicinity of the crack tip. However, a program change to calculate energy directly from the displacements and the

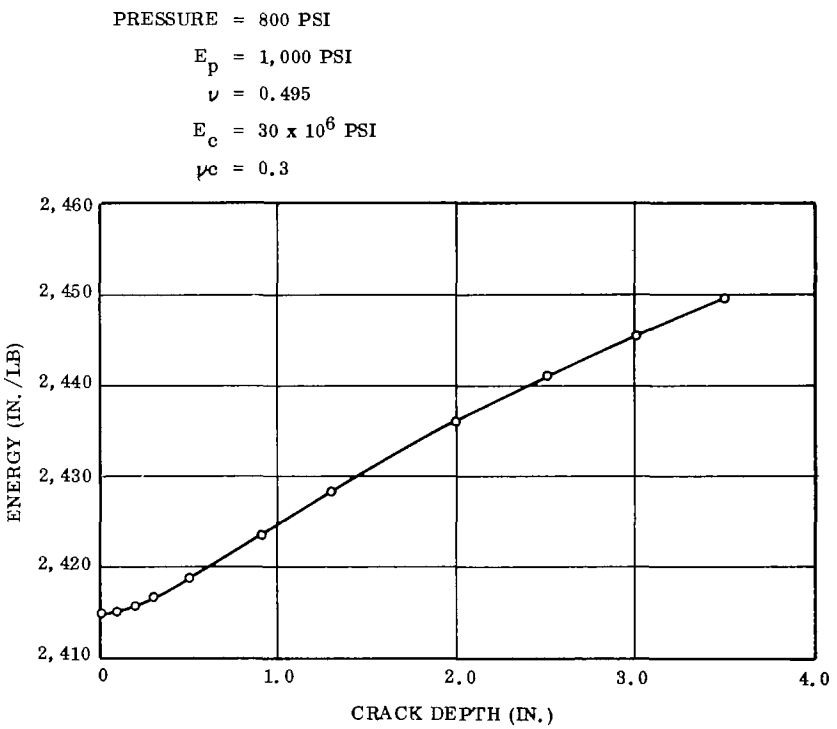

Figure 5. Strain Energy *vs.* Crack Depth – Pressure Load.

stiffness matrix, rather than from stresses and strains, was required to remove the oscillations. The analytical solution for a hollow cylinder was used to check the accuracy of the new strain energy calculations. The error was reduced from 0.063 percent to 0.0046 percent, thus increasing the accuracy by approximately one digit.

This new method of calculating strain energy was applied to an operational missile motor for a pressure load, the results are shown in Table 4.

TABLE 4

Results of pressure load calculation in operational motor

	Energy from Stress and Strain	Energy from Displacements
Original Grid	2500.983	2505 7654
Refined Grid	2501.943	2505.7594
Percent Change	0.038	0.00024

Not only has the strain energy changed in the fourth digit, but it appears that the more accurate calculation method is much less sensitive to grid changes.

Critical Parameters in Fracture Propagation

A strong dependence of $\Delta U/\Delta A$ on Poisson's ratio exists for the propellant grains under ignition load (nearly incompressible). For example, a plane strain analysis of a rocket motor grain showed that the $\Delta U/\Delta A$ ranged from 27.0 to 5.0 in.-lb/sq in. for values of ν of 0.495 to 0.499, respectively. Such a large variation was not expected, so a check case consisting of a smooth

bore grain with a thin shell in plane strain was investigated. The bore was treated as a cylindrical flaw. A closed form solution for the energy balance of this configuration was obtained together with a finite element computer analysis for internal pressure loading. The results are shown in Table 5. The configuration, material properties and results of a second check case are presented in Reference [4]. Both check cases verified the sensitivity of the energy balance to Poisson's ratio. It is, therefore, concluded that Poisson's ratio is an extremely critical parameter in fracture propagation analysis of pressurized grains enclosed by steel cases. Poisson's ratio must be known to at least three places to obtain reasonably accurate estimates of crack growth.

TABLE 5

Closed form solution comparison

Poisson's Ratio	$\partial U/\partial A$ (Closed Solution)	$\Delta U/\Delta A$ (Computer)
0.495	120.6	120.5
0.499	22.8	22.8

The sensitivity of $\partial U/\partial A$ to Poisson's ratio for internal pressure load can be attributed to the apparent effect on grain-case interface pressure. This effect leads to another interesting phenomenon when applying the energy balance method to fracture in propellant grains. The crack driving parameter $\Delta U/\Delta A$ was compared for a typical smooth bore grain with a steel case *vs.* a fiberglass case for the ignition load condition. The change in pressure (chamber pressure-interface pressure), which is much greater in the fiberglass case, results in a $\Delta U/\Delta A$ approximately 10 times that with the steel case. These limited results indicate a potential catastrophic failure if the grain contained in a fiberglass case has surface flaws such as scratches and cracks.

For thermally loaded grains, the value of Poisson's ratio is not extremely critical. However, a second parameter becomes very important, as can be seen from the thermal analysis of a cylindrical grain in a rigid case [12]. If the temperature in the grain is uniform, the strain energy and energy rate may be written

$$U = C_1 E(\alpha \Delta T)^2, \quad \frac{\partial U}{\partial A} = C_2 E(\alpha \Delta T)^2$$

Since the term $\alpha \Delta T$ in the above equations is squared, the measurement of the thermal coefficient of linear expansion (α) becomes extremely important. Consequently, it is felt that emphasis should be directed to these measurements and to determining the effect of strain on α when fracture mechanics programs are initiated.

Crack Trajectory Analysis

The first step in determining crack trajectory is to locate the most probable point of crack initiation. For complex geometries, the crack initiation point is normally located by determining the stress and strain distribution in the system using a finite element computer program and applying an appropriate failure criterion: *e.g.*, the point of maximum principal strain could be selected.

The direction of propagation of a crack from this point can be determined using the principle that a crack will travel in the direction of maximum strain energy release. For a crack of a given length (unit thickness), this will be the direction of greatest change in stored energy. Therefore, the procedure is to input a small length crack, initiating at the point of maximum principal strain, to a finite element computer program for several crack angles. The calculated strain energy is then plotted *vs.* crack angle. For a thermal load where stored strain energy is a maximum for the uncracked condition, the angle of crack for which strain energy is minimum is the predicted initial angle of propagation (maximum strain energy release from initial uncracked configuration to cracked configuration).

To determine the remaining trajectory, a short crack extension is input to the initial crack selected. Energies are then calculated for a system with cracks of different orientation initiating at the terminus of the first crack. The orientation of the second crack for which the system has minimum stored energy (or maximum change in strain energy) is the crack direction predicted. This procedure can be repeated to trace the entire trajectory.

A trajectory analysis was performed on a star grain motor (case O.D. ~65 in.) and the results were compared to the actual crack trajectory in a motor which was cracked by cold soaking. A 1 in. crack was assumed to have initiated at the point of maximum strain (approximately 0.4 in. from the starpoint centerline). The finite element computer program indicated a minimum energy with a crack angle of 20 deg from the starpoint centerline, as shown in Figure 6.

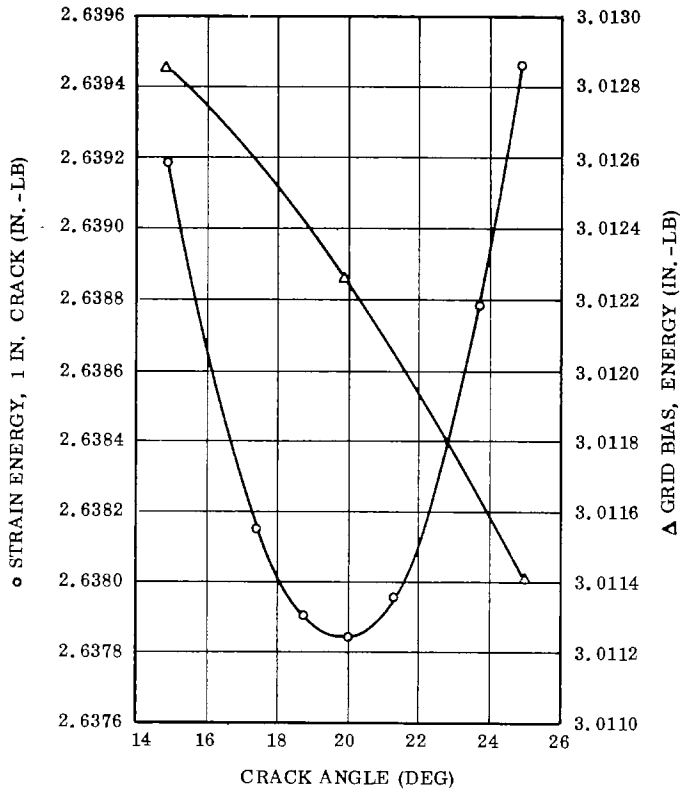

Figure 6. Strain Energy *vs.* Crack Angle, 1 in. Crack.

Particular attention should be paid to the scale on Figure 6. To obtain a smooth curve, energy changes in the fifth significant digit become important. These changes were anticipated, since the orientation of a 1 in. crack in a motor with a minimum web of 17 in. should not affect the total stored energy greatly; however, it indicates that extreme care must be used in setting up the grid. In the vicinity of the crack tip, the grid for this particular problem was kept constant for all crack angles. The small energy change with respect to crack angle also indicates that the crack trajectory may be very sensitive to local nonhomogeneities in the propellant.

In the computer analysis, the crack was simulated with a row of zero modulus elements (0.001 in. wide). To verify that energy changes were not due to grid network, the modulus in the crack was changed from zero to the propellant modulus and three of the grids were rerun (no crack). The grid network changes were of approximately the same magnitude as the energy changes, However, as can be seen from the grid bias curve in Figure 6, if the grid bias is subtracted from the energy curve, the minimum energy is changed by only approximately 1 deg.

A second series of runs was made to determine if the crack trajectory changes direction after propagating into the propellant. This study was conducted for an initial crack of 0.35 in. at 20

deg with a 0.2 in. extension. The energy versus angle curve for this configuration is shown in Figure 7, indicating that the trajectory changes from 20 deg to approximately 5 deg. The grid bias for this configuration is almost zero and would not change the predicted minimum.

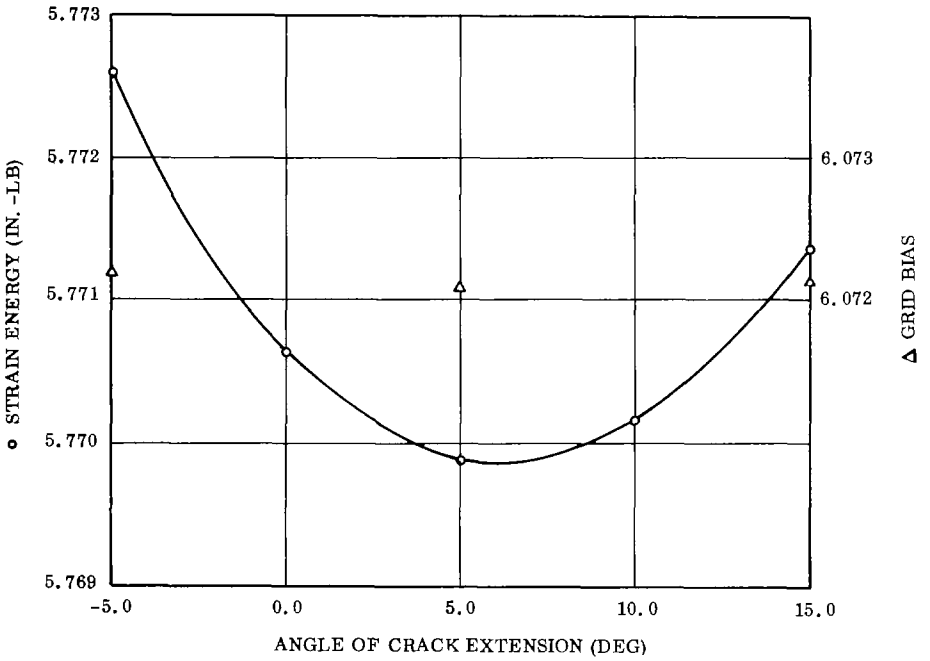

Figure 7. Energy *vs.* Angle of Crack Extension.

The motor for which this analysis was performed was cooled to failure. Dye was placed in the cracks and the cracked segments were removed. The average initial crack angle was 20 deg, as predicted; however, the data ranged from 0 to 45 deg. The data scatter demonstrates that the trajectory is very sensitive to local nonhomogeneity of the propellant. This effect is also noted on a smaller scale by the jagged crack edges shown in Figure 8.

Conclusions

The finite element computer program is a useful means of performing fracture analysis of continua with complex geometries for the special cases of two dimensional plane stress or plane strain. Results of this study were fruitful to warrant extension to general three dimensional finite element computer programs.

Using the finite element computer program, stress intensity factors were computed with acceptable engineering accuracy by three methods: strain energy, displacements, and integral method. With present program formulation, the strain energy change ($\Delta U/\Delta A$) method was the most accurate.

Some interesting aspects of fracture analysis of solid propellant rocket motors can be concluded as a result of this study.

For the thermal load, the significant parameter with energy balance is the ($\alpha \Delta T$) term where α is the propellant coefficient of linear expansion and ΔT is the bulk change in temperature from the zero stress temperature.

For the ignition load with nearly incompressible propellant and a steel case, the energy change ($\Delta U/\Delta A$) is very sensitive to the propellant Poisson's ratio, varying by a factor of five when ν varies from 0.495 to 0.499.

For the ignition load, the case or chamber stiffness has a significant influence on the energy change ($\Delta U/\Delta A$).

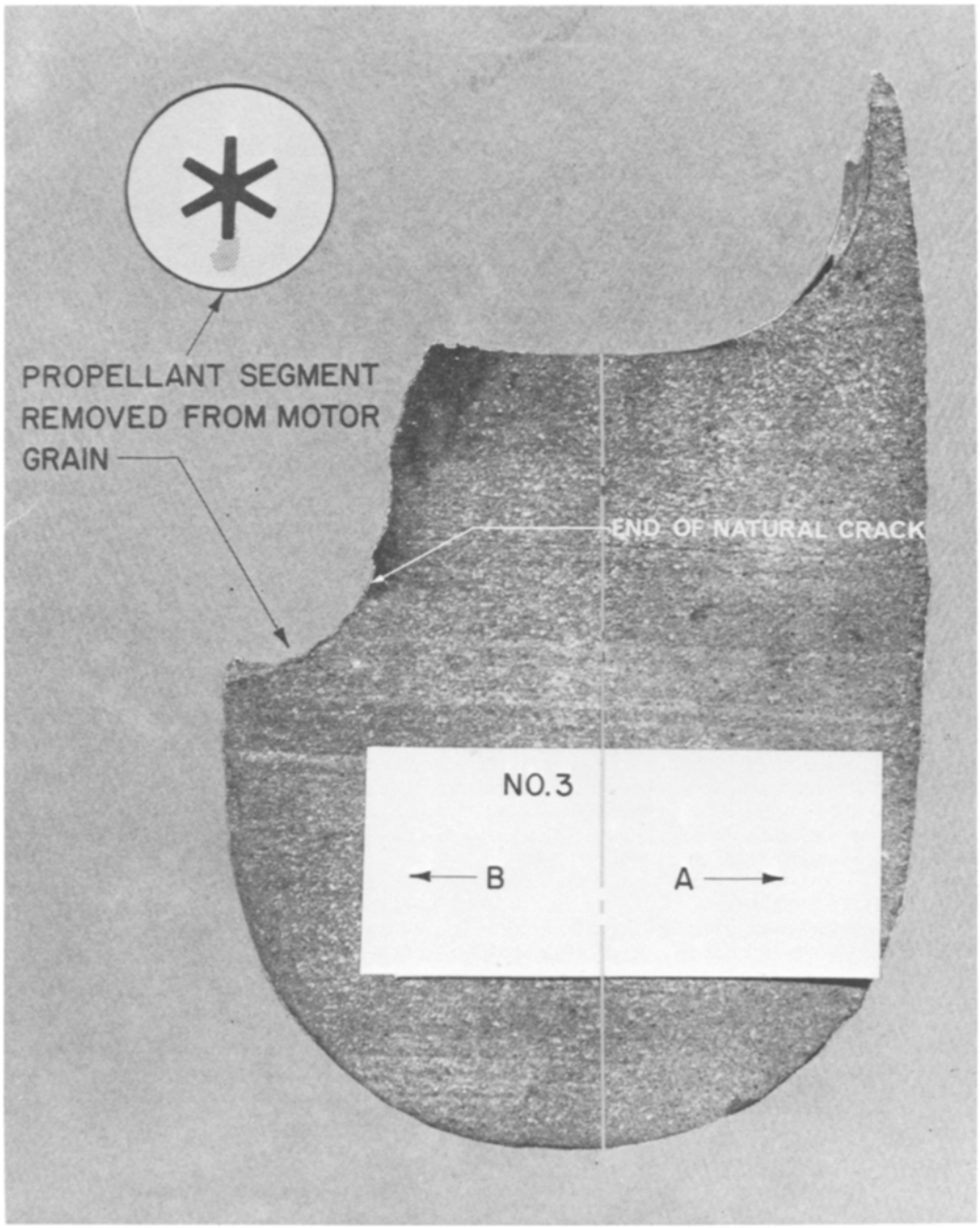

Figure 8. Propellant Sample After Crack Propagation Test.

Finally, a method was presented for which crack trajectory can be determined using the strain energies calculated from finite element computer programs.

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RÉSUMÉ

Dans un continuum élastique, les contraintes et les déformations à l'extrémité d'une fissure en Vé sont théoriquement infinies. La question se pose donc de savoir quel est le degré de précision que l'on peut attendre du calcul à l'ordinateur par éléments finis de l'énergie de déformation correspondant à des systèmes où se rencontrent de telles fissures.

On a analysé deux types de géométries différentes, pour lesquelles existent des solutions analytiques, en utilisant un programme d'ordinateur pour l'étude par éléments finis de l'état plan de tension.

Les résultats montrent que, dans les deux cas, la précision dépend du choix du réseau le plus adéquat.

Diverses méthodes de calcul des facteurs d'intensité des contraintes sont discutées. On traite de l'application des programmes de calcul par ordinateur des élements finis à l'analyse des conditions de rupture dans les cartouches ou éléments de combustible solide pour fusées.

ZUSAMMENFASSUNG

Die theoretischen Spannungen und Dehnungen an der Spitze eines V-förmigen Risses in einem elastischen Kontinuum sind unendlich groß. Es stellt sich die Frage der Genauigkeit der, für Systeme mit solchen Rissen, nach dem Verfahren der endlichen Elementen errechneten Dehnungsenergie.

Zwei geometrische Formen, für welche analytische Ergebnisse vorlagen, wurden an Hand eines Rechenprogramms für die Ermittlung des planen Spannungszustandes durch endliche Elemente untersucht.

Die Ergebnisse zeigen, daß in beiden Fällen die Genauigkeit von der Wahl eines passenden Netzes abhängt.

Es werden verschiedene Verfahren zur Bestimmung der Spannungsintensitätsfaktoren besprochen; die Anwendung des Rechenprogramms zur Ermittlung der Bruchbedingungen in Festbrennstoffelementen für Raketen wird behandelt.