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CHANGE IN MATHEMATICS EDUCATION SINCE THE LATE 1950's – IDEAS AND REALISATION HUNGARY

To János Surányi's sixtieth birthday
with a wish of many more
fruitful divergences of opinion

0. INTRODUCTION: THE SITUATION AROUND THE LATE FIFTIES

A general discontent, but no coherent plan of improvement: this is what can be said, in short, of those years. News of the sputnik shock and the subsequent boom of the new math with its fresh wind and dust storms did not reach us before some years later. At that time we were still involved, since 1946, in a fundamental change of our school system: the creation of an eight-year general school (six to fourteen). The goal was to provide equal opportunities for every pupil of whatever descent and habitation. It goes without saying that the last four years of this general school could offer less to *every* pupil, than what the first four years of the earlier eight grade secondary school could give to a highly *selected* population of the same age. What finally remained was seven years of arithmetic and one year of so called algebra: linear equations with one unkown, negative numbers as top level notion, not even powers. In addition, some geometry in grades five to eight. This was and still is the content on which to build the curriculum of the four secondary years.

Thin content does not necessarily entail poor teaching. Another variable may matter more: the teachers. Their number has been increased from 30 thousand in 1937/38 to 66 thousand in 1960/1 (primary plus secondary) to keep pace with the growing number of pupils, especially between twelve and eighteen, and to decrease the class effectives. But this could not be achieved without concessions in the requirements about teacher qualification and competence.

These changes did not impair the education of a mathematical élite. We had a long tradition: competitions and periodicals for mathematically gifted pupils since 1894; clubs in an outside the school; literature for the clubs, for individual enrichment; organized and occasional assistance to the brightest by research mathematicians.

Paradoxically, while mathematical excellence was sufficiently supported

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during these years of extending continued education to many more pupils than ever, we were much less successful in reaching our main goal, an efficient mass education in mathematics. Since the end of the nineteen fifties we have been working toward this goal.

In what follows we give an account of

1. these efforts: the ripening of the reform,
2. how they were and are received: reaction to change,
3. change in the objectives,
4. change in the curriculum: its structure,
5. change in the curriculum: its content,
6. how all that is being realized: classroom examples, with an attempt at highlighting some aspects of the methodology.

1. THE RIPENING OF THE REFORM

Our good traditions in mathematics education were not restricted to the mathematically gifted. Many of our research mathematicians have been keenly interested in educational problems. They voiced their views in front of students, teachers, authorities, and the wide public, in books and papers, talks and courses, and had a considerable effect on producing a climate favourable to a radical reform later on. Among those who though living abroad influenced mathematical education in Hungary, we mention G. Pólya and Z.P. Dienes. Of those at home, unfortunately, five brilliant mathematicians – Alfréd Rényi, György Hajós, László Kalmár, Rózsa Péter and Pál Turán – died recently, each of whom had a considerable impact on school mathematics.

In 1962 an international symposium was organized in collaboration with UNESCO on school mathematics. This was a major attempt to confronting different ideas and formulate recommendations acceptable for all participants. In the same year a project was initiated in the general school (OPI Mathematics Project). A few years later the Ministry of Education created a commission for studying the ways and means of updating mathematics teaching (as seen in Hungary) and prepared a new programme for the general schools based on the OPI Mathematics Project which it found alone suitable to this end. On the base of this proposition a new programme has been prepared which is being implemented during the seventies and eighties.

Interest for the problems of education grew during the last few years. On the base of a national study and decisions taken in 1972, new programmes are being introduced on all levels in each subject. During the four years of the non-vocational secondary schools 2, 4, 9, 11 weekly hours are, respectively, kept for facultative courses (2 weekly hours in each year are optional, the rest are for everybody). Mathematics will be one of the available courses in each year.

No massive experimentation has taken place so far on secondary level – similar to the OPI Mathematics Project – but in more than a dozen schools pilot work goes in much the same spirit. In this Secondary Project the content is not much affected, the final and entrance examinations being too close and menacing. Changes have to be modest before the new general school curriculum becomes effective everywhere, during the late eighties at the best.

A new feature of the Secondary Project is that during all the four years pupils are working on what we call *miscellaneous problems.* These are different kinds of not too difficult, yet non-routine problems. The topics they learn do not offer clues to their solution. They usually do not require much more than logical thinking or some unusual combination of simple knowledge. These problems often throw light to some topics of elementary mathematics not treated systematically, they may also be simple special cases of advanced problems usually ranged into higher mathematics. Other miscellaneous problems are destined for preparing topics to be treated in details later on.

Our tasks loom larger now, than ten or twenty years ago. Yet our successes – in the education of the gifted, or of the average with teachers above the average, or even with average pupils and teachers, on a small scale, or on a somewhat larger scale, in the lower grades – offer us many clues to solving the bulk of the problems, still before us.

In the Introduction we mentioned efficient mass education in mathematics as our main goal. Speaking of a mathematical illiteracy and a programme of terminating it is not out of place, either. In our days a citizen cannot be in the full possession of his rights unless he acquires some literacy, and one which extends beyond mere technical skills. This statement increasingly applies to mathematical literacy as well. Reading formulae, graphs, statistical tables, understanding what is behind them is as important as reading between the lines, not just the lines; and this is just one small section of mathematical literacy. (Cf. the examples in Sections 6.1 to 6.5.)

2. REACTION TO CHANGE

Pilot work started in a few classes during the early sixties. Some years later it began to catch the attention of other teachers, near or distant. Visitors discussed what they saw, asked for more information, and many of them started similar work in their classes that often became centres again, visited by others. With very little organization something like an organic growth took place: the increment was proportional to the number of existing pilot classes, and this, of course, resulted in an exponential function, or rather a geometrical progression. (The quotient was around 1.7 per year.) During this time – more than a decade – a good reputation took shape and spread among teachers, parents, a wider public, authorities. The golden age was over after the decision of implementing the curriculum based on this pilot work in every school of the country, simultaneously with new curricula for other subjects. The implementation was gradual, reaching every first grader four years after it began in 7% of the classes in 1974, – a simulated extension of the previous organic growth – this pace happened to be quicker of the optimal. Teachers who had joined in voluntarily became outnumbered by those who were less willing or prepared, and who, on their turn, became centres spreading a less favourable reputation.

This was reinforced by news about "the failure of the new math" abroad. If Johnny cannot add, Hänschen is made ill by Mengenlehre and Jeannet, too, goes back to the good old arithmetic, then Jancsi would do better to keep learning it before he becomes the next victim! The suspicion is there, and we are exposed to be *mitgefangen, mitgehangen.*

The situation is not so bad. The reputation won previously did not evaporate. The number of those who see more than the surface, even grew. They include many or most mathematicians, educationlists, psychologists, linguists, physicists, experts of various other fields. We co-operate in many ways. The international understanding and co-operation is also substantial.

The general public is divided. They are mostly influenced by the positive or negative instances they meet. Among the teachers, who produce these instances, there is a precious nucleus, and various strata from the benevolent to the indifferent to those who jump on the band wagon and those who know, "common sense will eventually win over lunacy." To make things more complicated, the dividing lines are within persons, not between them, and common sense certainly weigh more in our case, than expert opinions. Change only can go on through the teachers who do their jobs daily and who care much less about arguments or evaluations than about their own experiences.

3. CHANGE IN THE OBJECTIVES

When a mother says: "Kitty is nine and she still cannot do the sums I could and my sons could do when we were eight", and concludes that the school does not do a good job, then one of her hidden premises is this: "The objectives of teaching should not change. Schools should continue teaching the same sums at the same age as they used to."

Whatever are our reasons for doing less sums or doing them later or doing them embedded into other types of activities, we have to state these reasons explicitly as a part of a coherent set of objectives. Whether or not the realization of the objectives can be objectively measured, they serve as checking points for us, for the parents, for anybody.

In the beginning we had a policy of popularizing those objectives which we felt were undervalued by the school or by the society. (Think of *forming an opinion of his own* as opposed to *answering forthwith.*) Then we came to the understanding, that a well balanced set of objectives is preferable to a tendency of counterbalancing, which is likely to provoke a contrary tendency. In fact, slogans such as "Developing cognitive abilities is more important than teaching addition facts" has made the "new math" a target of jokes in America and Western Europe.

We set a high value on the objectives of mathematical education in the affective domain, but we have not elaborated them, and the sensory-motor objectives either. We confine ourselves to stressing the outstanding importance of the intrinsic motivation in mathematics education. Mathematics has the qualities of appealing to pupils – especially if it appeals to their teachers – and this is, we feel, an immense reserve not sufficiently exploited.

For the cognitive objectives we found a two by two grouping and labelled them this way:

	synthesis	analysis	
Receptive-reproductive	KNOWING, DOING, USING	DISTINGUISHING, UNDERSTANDING	. . . what is given, shown or told
Productive	CONSTRUCTING, FINDING, INVENTING	FORMING AN OPINION, JUDGING	. . . adding their own contribution

The list below is exemplifying these terms. Topics are not specified in the list, items can be related to any content. They are also almost independent of age level: the four quadrants, if not each item in them, are expected to interact and support each other from the outset, both in the activity of the individuals, and in the form of a social interaction with divided and changing roles. We regard this scheme as an extension of the more usual sets of objectives which in actual practice rarely exceed the first quadrant – knowledge and skills – this being more accessible to testing than the other three.

1. Receptive-reproductive synthesis
   KNOWING, DOING, USING
   - terminology, symbols, devices,
   - mathematical statements,
   - routine problems, algorithms
   - manual skills

2. Receptive-reproductive analysis
   DISTINGUISHING, UNDERSTANDING
   - Identifying objects, properties, concepts
   - Seeing relationships
   - Appreciating patterns
   - Understanding a statement, rewording, re-coding, translating it
   - Following a line of thought e.g. a reasoning

3. Productive synthesis
   CONSTRUCTING, FINDING, INVENTING
   - Formulating a problem
   - Stating hypotheses, educated guesses
   - Planning a solution
   - Finding tools for the solution
   - Finding objects (concrete or not) which satisfy given conditions
   - Finding all such objects
   - Finding a definition for a concept
   - Developing a proof
   - Generalizing, extending by analogy . . .

4. Productive analysis
   FORMING AN OPINION, JUDGING if
   - a statement makes sense,
   - a statement is true
   - a problem is clearly defined:
     - if it contains enough data,
     - if there are in it superfluous or contradictory data or conditions,
   - a symbol, a definition, a suggested way of solving a problem is suitable or appropriate or promising,
   - a reasoning is correct,
   - a given object satisfies the conditions (some or all),
   - a solution is reasonable, if it satisfies practical requirements, standards etc.

4. THE CURRICULUM: ITS STRUCTURE

It is safe to say that the structure of the curriculum changes more than its content. For example, combinatorics and probability were repeatedly included into former curricula, but never at an age below 17. In our recent curriculum they appear eleven years earlier, in grade one, and they are spread over the whole of it – an essential difference in the structure. Instead of the question, *when* to teach probability, we ask a different one: *what of it* to teach – and how – in grade one, grade two etc. Similarly with geometry and other disciplines.

The philosophy behind this structure is well known. It is related to the new priorities: developing ideas more highly appreciated than imparting knowledge. It is possible – though difficult – to expound a body of knowledge sequentially, but development is global and harmonic or it is not development. Developmental psychology has shown how childrens' ideas about many aspects of mathematics grow even before they enter school – ideas about number, space, chance etc. – and it is difficult to see why should we renounce to creating favourable conditions to their continued growth thereafter. It is sad to see children regressing in their mental development after they enter school. The first lesson schools are to learn is how to avoid this decline, how to harness children's natural learning abilities. Then comes their second lesson: how to induce children to more mature learning. By the altered structure of the curriculum we intend to face both tasks.

A number of other features of this structure will be expounded. Not only do topics appear soon and run through the whole curriculum, they are also *interwoven* in it, with the intention of being integrated into mathematics as a whole. Like threads in a cloth, they disappear and appear again in various combinations.

Though integration is intended through the whole curriculum (6 to 18 years of age), its meaning slightly changes from the lower to the higher grades. Whole lessons assigned to one part or aspect of mathematics are rare in the lower, more frequent in the higher grades, where even weeks of work may be concentrated on organizing the knowledge from one aspect.

Suggested activities linking mathematics with other fields of knowledge – language, sciences, arts – are more frequent in lower than in higher grades, but an alimentation and motivation of mathematics by its applications to many fields is invariably intended later on.

The main ideas of most mathematical disciplines appear rather early, most of them in the first few grades, so far as, and in the way as, they can be motivated to and assimilated by average children of the given age. Thus the approach to numbers through counting and *measuring* and the early use of the number line suggests some vague idea of the real numbers as early as in grade two, though the existence of irrational ratios only becomes clear six to eight years later.

The education being centrally organized, with syllabi for each grade valid over the country, they are to be flexible enough to fit the needs of every child in every classroom. For this reason:

the syllabus for each grade consists of *compulsory* topics to be covered by the whole class, and *suggested* topics for enrichment to be assigned to, or tackled by, only some of them;

a part of the compulsory topics is summed up at the end of each year in the form of *requirements,* some of which are distinguished from the rest as *minimal requirements;* many concepts and skills not appearing as requirements in the schoolyear where they are first mentioned in the syllabus, get enrolled to them in subsequent years when they are supposed to become ripened.

There is some flexibility even in the interpretation of the topics to be covered and the requirements to be fulfilled so that the teacher can adapt them to the actual situation.

From the general to the particular is a feature which distinguishes, we feel, our curriculum from earlier ones. The direction is not from triangles to quadrilaterals to polygons to curvilinear figures of the plane to solid geometry, but from sets of various objects to sets of points – subsets of the space – to its particular subsets, planar or not. Not from the linear function to the quadratic to other polynomials, to exponential, trigonometric functions etc., all of one variable before realizing that something else than one real number can also serve as input or output, but functions as mappings in a very general sense, even before children can read or write numbers, without using scholarly words like *set* or *function.* Not from linear equations (with the idea that x means "the unknown number") to polynomial and other equations to inequalities, but from the general idea of an open sentence with one or more variable and its truth set to particular open sentences. No strict order from natural numbers to positive rationals to all rationals to real numbers, but from a vague idea of positive real numbers as quantitites measured by some unit (using but immediately exceeding natural numbers) quickly to directed numbers (e.g. turns in two possible directions), using and gradually filling the number line, generalizing to vectors before its completion. Not from classical problems of probability based on equally likely events to more general assignments, but from a general setting where probabilities may be taken from experience alone, and "more likely" comes before "equally likely".

From the particular to the general is another characteristic feature of our curriculum. This is complementary rather than contradictory to the statement expounded above. For example, we begin fractions with the halves in the following way: counting numbers have their names already: one, two, . . . ; measuring leads children to "more than one, less than two", "more than two, less than three" etc.; then we introduce numbers "half", "one and a half", "two and a half" before any systematic or semi-systematic introduction of other fractions. Children may even count convenient objects that way: half, one, one and half, two, . . . and visualize on the number line what they say. Though geometry is about sets of points in general, exploring particular sets of points on square grids (geoboards, squared papers) etc. lend themselves better to integrating geometry with arithmetic, algebra and functions than does an exclusive use of blank sheets. Babylon sets – balls with 26 holes in points with both of their spherical co-ordinates multiples of 45° and joining sticks – do the same service in developing spherical and solid geometry. Simple finite sample spaces too are excellent fields for exploring ideas about probability.

5. THE CURRICULUM: ITS CONTENT

Our general attitude toward the content has been rather cautious, both in reducing traditional topics and in introducing new ones. Rather than excluding topics and replacing them with other ones, we prefer thinking in terms of shift in the stress. We look at the reform of mathematical education as a long process. The next ten to twenty years are a period of transition and it will be followed by another one. These relatively long transitions seem to be necessary for the teachers to become familiar with new topics and ideas, introduced smoothly. As for the pupils, the pace could be quicker, if only *we* can be sure that more will be gained than lost by the change. Teachers in pilot classes test such doubts, they furnish evidence for further decisions and help to make them popular.

The *woven* structure of the curriculum creates favourable conditions to introducing new topics within the frames of the old. Let us give three examples.

1. The idea of *multiplying* natural numbers becomes more deeply rooted in children's minds, and fields for developing skills also become wider, with an approach through combinatorial problems leading to the Cartesian product of two – or more – sets. (To the idea, through problem situations, not yet to names and symbols!) This approach complements rather than replaces the traditional one through "repeated addition" or – in set language – the union of equipotent disjunct sets.

2. For *powers* with natural numbers again some combinatorial problem

situations lend themselves as a good starting point: permutations with repetitions in the old wording. In the new: "mapping into" between two finite sets. Problem situations arise around the age of nine, systematization is left to a few years later. Teachers at that level hardly notice what they teach is about cominatorics and functions, not just arithmetic. We gradually enlighten them on this fact – it is better if they, too, see the broader context –, but we are not particularly keen on shocking or dazzling them either with advanced terminology.

A third example, within the same range of ideas, is about weaving probability with fractions. Ten to twelve is the age we find suitable to developing skills with *fractions,* after some preliminaries on an intuitive level. This is also the age when probability reaches the level of calculations: if from five beads, three red and two white, one is drawn at random, then another one without replacement, and this experiment is repeated many times, then about two fifth of the trials results in a white bead first and about one fourth of such cases in another white bead. The expected relative frequencies – the respective theoretical values, called probabilities – are exactly 2/5 és 1/4. Such situations serve as more natural starting points toward the multiplication of fractions – and transforming them to decimals or percentages – than most of those used traditionally. Also to addition (find the probability of drawing two beads of equal colour!) etc. Fractions and probability motivate and elucidate each other.

These examples enable us to state a general aspect of our curriculum as related to the former one:

Most of the traditional topics occur in it in some form, usually in connection with other topics.

In many instances traditional topics appear earlier than they used to, on a less formal level. This happens with the number systems, geometric transformations, much of co-ordinate geometry, functions and their graphs, equations and inequations.

Another aspect of our cautiousness in introducing new topics is that much of what is new belongs – for the time being – to the non-compulsory part of curriculum, to the enrichment topics. This is the case, for example, with most of the combinatorial topology.

Another general remark: we find that our curriculum is less *structural* on the one hand, and more *numerical* on the other, than most of the recent curricula. It is very nice to show to pupils, possibly to all of them, the beautiful architecture of mathematics. Yet our ambition, in the present phase of the reform, is more down-to-earth: to show mathematics in its interaction with the environment, through its applications. Without denying the usefulness of the great structures, we find that at present most applications are strictly numerical.

The stress on the numerical side can be exemplified by the role probability and statistics play in it.

As to the structures, our policy is that of making pupils ready to their appreciation during several years. No doubt, they can learn to manipulate their rules – e.g. that of a group – very early, like the rules of a game, but we prefer waiting until they can also see their mathematical importance. After a long preparatory period the majority of the pupils may reach this maturity. This way we do not force on them a ware they do not want to buy or answer questions they would never ask.

This latter remark throws some light on still another decision of ours regarding the content: we do not teach any kind of axiomatic geometry in school, not even in secondary. The level we adopt is that of local organization, in geometry just as in any other branch of mathematics. This means a smooth introduction of the idea of deduction: defining and proving as activities. (Not limited to mathematical subject matter.) Within this smooth introduction the idea of what primitive concepts or axioms mean are also exemplified, without the claim of a complete system.

In order to tell more of the content we present below the way we have grouped the topics – or threads – into five bunches with two or three sub-bunches in each. We were led to this subdivision by practical considerations. Both the grouping, and the order is rather arbitrary. The percentages after the titles suggest the estimated weights of the bunches. The estimations cannot be but vague, due to the woven structure of the curriculum. The remarks after the titles give a rough interpretation of the titles. Further highlighting is left to the examples in sections 6.1 to 6.5.

Sets, logic (5 to 10 %)

Sets of concrete objects, of mathematical and other abstract entities including numbers, points, sets. Operations with sets. Their relationships with logical operations. Quantification. Topics related to computer science such as flow diagrams are included into this bunch.

Relations, functions, sequences (20 to 30 %)

Relations: mostly binary at the beginning, both within a set and with sets for each variable. Open sentences. Their truth sets within given sets. Functions: in the general sense of *mapping into,* but with numerical functions in the centre. Sequences: at first independently, later revealed as functions of natural numbers.

Arithmetic, algebra (25 to 40%)

Numbers: natural to real, in various normal and other forms (terms without variables). Operations with numbers as functions. Rudiments of number theory (divisibility, modular arithmetic). Manipulations of terms (without or with variables). Numerical open sentences (equations, inequations and other) with one or more variables. An outlook to algebraic structures, to linear algebra.

Geometry, measurements (20 to 30%)

Geometrical objects as sets of points (subsets of R^3). Transformations as mappings within R^3. Measurements, both in the physical space and within a mathematical model. Some combinatorial topology (mainly graphs, also as models of relations).

Combinatorics, probability, statistics (10 to 15%)

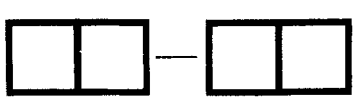
At first probability is based on frequencies and estimations, not on combinatorics, which appears independently, and statistics is on the descriptive level, not based on probability. Later their interdependences come to light.

6. CLASSROOM EXAMPLES

6.1. Arithmetic, probability, strategy planning (Age: 7 to 20 or more)

Children who can subtract two digit numbers can play the game below. They will become involved in thinking of probability without knowing what probability means. They will develop strategies without knowing what a strategy is. Students who know a lot of probability may find it difficult to assess the merits and de-merits of these strategies.

The game is about filling the four places below with random digits,



appearing one by one. The goal is to make the difference of the two two-digit numbers as large as possible. Some of the children produce and announce the random digits appearing on ordinary dice (or dice with faces marked by 0, 2, 4, 5, 7, 9 or some device generating decimal random digits etc.), others working individually or in teams, write the digits to one of the empty places before the next digit is announced. After the fourth they perform the subtraction to decide who won (possibly more of them). An easier variant: addition rather than subtraction; a harder one: numbers with more than two digits.

From the point of view of arithmetic the game serves, apart from developing skills in arithmetic, to calling attention to the place value and to the respective roles of minuend and subtrahend (etc.) – ideas belonging also to the realm of functions.

This game – together with many other games – are expected to contribute to some more sophisticated ideas, such as the following:

1. *Strategy* as a set of instructions telling you what to do (how to move) in any given situation. Here: where to write any digit, possibly depending on how some places are filled.

2. Distinguishing between moves you *cannot regret* (using a 6 as a tens digit in the minuend if that place is still empty) and one you *can possibly regret* (you write a 5 to the tens digit of the minuend and then a 6 appears).

3. Distinguishing between a move you will *probably regret* and one you will *probably not regret.* (More precisely: one you are more likely to regret than not and one you are less likely to regret, than not to regret. There are actually three such cases, not two.) Which is the case with using a 5 as the tens digit of the minuend? Does it depend on how many throws are over?

4. *Better and worse strategies.* What does it mean? How can you distinguish them? Does the user of a better strategy a) always win, b) more often win than not? (Not even that. But he is more likely to win at any time – that is the definition. And he is *a fortiori* more likely to win more often than not.)

Games like the above are powerful means of giving pupils a first intuitive grasp of deep ideas. Much evidence is still needed, however, about the way and efficiency of building upon such basing a solid conceptual structure in mass education.

6.2. Arithmetic and algebra through functions (Age: 8 to 10)

Children worked so far with natural numbers mostly below one thousand. They also can handle some other rational numbers. They have some practice in the use of variables (boxes or letters). They performed earlier such activities as putting together and dividing between them collections of objects each of which can be cut into two (example: horse-chestnuts) and locating numbers on the number line. The teacher exhibits some couples of numbers as inputs, each of them with its output:

x	y	z (output)
8	4	6
20	10	15
6	3	4 and a half

Depending on the number of children who are working with him – not necessarily the whole class – he may suggest several things to do:

a. Add further triples – either on their own papers or for the whole group – which they think follow the same rule.

b. Ask for input numbers to find their outputs.

c. Mark each triple on a separate number line, distinguishing them (e.g. first input blue, second input orange, output red).

d. Suggest some activity where such inputs give such outputs.

e. Devise open sentences valid for each of these triples.

f. Choose among given open sentences those which are valid for each triple. Suggested open sentences:

A. $x + x - y = z + z$ E. $x - y + 2 = z$

B. $x - \frac{y}{2} = z$ F. $\frac{x}{4} + y = z$

C. $\frac{x}{2} + \frac{y}{2} = z$ G. $\frac{x + y}{2} = z$

D. $y + \frac{x - y}{2} = z$ H. $x - \frac{x - y}{2} = z$

All but one of the open sentences is satisfied by the above triples. As soon as somebody suggests input numbers with different ratio, this will not be the case any more. (The teacher can do this in task b.)

These and similar activities lend themselves to developing quite a number of ideas and skills:

1. The meaning of *input* (we can choose them) and *output* (determined).

2. *Determined* means *unique* (unique number, unique object, possibly a unique set etc.).

3. We might have chosen 12 and 6 first, or 6 and 3 first, the *order* in this sense *does not matter.*

4. Choosing 4 and 8 is, generally speaking, different from choosing 8 and 4. (The idea of an ordered pair – or couple –, ordered triple etc.; numerical and non-numerical examples taken from the environment.) The *order* in this sense *does matter.*

5. The role of the *first set* (the *domain*). Choosing means choosing among. Should we include negative numbers? (With a different source set the function is different.) In our example choosing x always as the double of y was only an artifice.

6. The role of the *second set.* Does the machine produce numbers such as four and a half or it begins blinking or buzzing at the input of the couple (6, 3)? It may behave this way or that way depending on our programme. (Both sets are essential parts of the function.)

7. *Rules* as prescriptions enabling us to find outputs for further inputs. Rules expressed by addition, multiplication and their inverses are used frequently in view of the importance we attribute to basic skills, but we come back again and again to non-numerical functions or numerical functions with rules such as *the larger of* (two numbers), *the integer part of* (implicitly appearing with the Euclidean division), later *the least common multiple,* arctan (appearing informally as *steepness of a slope "one forward, two up" in degrees* etc.)

8. *Functions in the broad sense,* possibly without apparent rules. (Sets of observations, arbitrary graphs, arbitrary n-tuples with the last element as the output.)

9. The idea that *a rule should work with all listed items* (here: with each triple of numbers). Among the above rules "$x - y + 2 = z$" works with the first item only, "$x + y - 15 = z$" with the second, "z equals 2 in $x + y$" (in the sense of the integer part of $(x + y)/2$ with both, but not the third, "$x - y/2 = z$" with all three; further items may only satisfy one but not the other.

10. Apparently contradicting 8: for any number of items *there is always some rule.* If inputs are single numbers, then any two inputs with their outputs can be characterized by a linear rule – more generally any n by exactly one polynomial rule of at most $(n - 1)$th degree and infinitely many of nth degree – and the choice is broader, if there is more than one variable. These advanced ideas, are beyond the reach of the 7 to 10 year olds, but something important can be conveyed by always suggesting them to *find a rule* rather than *find the rule.* Starting with a few items they experience the manifold of possible rules. (See also Section 6.3: Steps toward calculus.)

11. *Two rules are told to be different* if they give different outputs for at least one input within the domain. Otherwise it is the same rule in another

form. This becomes interesting when the domains are sufficiently broad (children may use any number they know). Open sentences expressing the same rule serve then as stepping stones toward *algebraic manipulations:* rules of transforming a term or an equation to an equivalent one (one which takes the same values, respectively truth values, at the same inputs).

Examples to this point: A and B are equivalent, though neither their left hand, nor their right hand terms (members) are. The left hand terms of C, D, G, H are all equivalent. These are true far beyond the set of the listed input values. On that set all but E are equivalent, but the set is too restricted to make this fact interesting.

D and E are awkward, but the idea behind them is more natural for children working with number lines, than the idea behind C and G. This latter is more natural for those working with objects which they put together and divide. What is the number precisely between 288 and 284? You add half of their difference to the second number or you subtract it from the first. The fact that adding their halves or halving their sum give the same result is something to discover and analyse. If the first number is smaller than the second, then the rules

$$\text{I. } x + \frac{y-x}{2} = z \qquad\qquad \text{J. } y - \frac{y-x}{2} = z$$

are the more natural expressions. But are those or these really restricted to some couples? Or do all the six give the same output for any couple? (The use of calculators is advisable here, to furnish heuristic arguments, "empirical evidence".)

6.3. Steps toward calculus (Age: 8 to 12)

The idea hehind the suggestion "look for *a rule,* not for *the rule*" (see in Section 6.2.) becomes clearer when pupils understand that for any number of input and corresponding output values there are an infinity of rules satisfying all of them. One of our approaches to convey this idea is via sequences. *Suggest arbitrary numbers to begin a sequence!* We try to find a rule by looking at the differences:

$$\begin{array}{ccccccccc} 3 & & 1 & & 5 & & 12 & & \\ & -2 & & 4 & & 7 & & & \\ & & 6 & & 3 & & 0 & -3 & -6 \end{array}$$

Is there a rule here? Look at *their* differences:

6 and 3 fit well into a decreasing sequence with a constant difference -3.

Taken this as granted we can go back to the first differences and then the original sequence as well:

3		1		5		12		19		23			21...
	−2		4		7		7		4		−2		...
		6		3		0		−3		−6			...

If more elements are given, then we may have to go on to further differences. And any time we go one step further, we find an infinity of rules of the given type.

This easy and natural technique foreshadows what later will be known as polinomial approximation. The main idea can be grasped without any formula: two arbitrary numbers determine an arithmetic sequence (linear function), three of them a second order arithmetic sequence (quadratic function), n of them an arithmetic sequence of order $n - 1$ (a polynomial of degree $n - 1$). If the numbers are not consecutive elements of a series, then the technique becomes clumsy:

1		4		..		15
	3		$3 + x$		$3 + 2x$	

and even more so, if instead of elements of a sequence – i.e. values of a function at integer inputs – the inputs are arbitrary numbers. These difficulties can serve as incentives to looking for better techniques. Yet the main idea once grasped serves as a thread to be followed. Taylor series are a natural follow-up of polinomial approximations. The analogy between integers written in some base b as the polynomials of b and real numbers in the form of infinite b-cimals as their extensions to infinite series is revealing.

The early study of sequences foreruns many more ideas pertaining to calculus. The n-th differences of the sequences of order n are constant just as the n-th derivatives of the n-th degree polynomials. But children also meet sequences following a given rule (with a constant quotient) where the differences are constant times the numbers in the original sequences:

1 3 9 27....   1 0.5 0.25 0.125...

2 6 18...(c = 2)   −0.5 −0.25 −0.125...(c = −0.5)

and a certain value of the quotient, incidentally 2, makes c = 1. Again a thread, leading to exponential functions and their derivatives.

6.4. Statistics: an estimated distribution (Age: 15 to 18)

How much time before starting classes have you to leave your home in order to arrive at school in time?

The wording is deliberately vague. Pupils are expected to contribute to formulating the problem.

1. "In time" should mean: "not to be late". They may arrive earlier.

2. They use the vehicles they usually do. (If they make it depend on the time of leaving, then the problem becomes more complex.)

3. "To be certain to arrive" is a too strong assumption. Even if you leave one hour earlier something very special may happen that prevents you from arriving.

100% certainty is unrealistic but a probability like 99% is not. What do you mean by this probability? Write your estimation of the time you need for that. Also for 90%, 67%, 50%, 33%, 10%, 1%. Plot a graph based on your estimations. Connect the points smoothly. The curve will represent an extension of your estimations to further probabilities.

Do you think that the curve should have some sort of symmetry? Behave somehow the same way at 99% as at 1% etc.?

Could we simulate these estimated distributions by some chance experiment – throwing dice, generating random digits on a calculator?

Suggestion 1: If the estimated difference between the minimum and the maximum travelling time is n minutes, then choose a random digit from O to n and add it to the minimum.

Suggestion 2: To avoid certainty (at minimum $+ n$) let n be the surplus-time at some high probability, and if the digit is n, then you produce a third, a fourth etc., until you get a digit less than n.

Suggestion 3: Digits from O to n or $n - 1$ with equal probability do not fit to reality. Numbers in the middle should be more frequent. Adding two random digits? Or three?

Threads lead to discovering, generating and using normally (or otherwise) distributed random numbers.

6.5. Calculators used in solving a statistical problem (Age: 16 to 18)

One of the many uses of statistics – and a particularly important one – is to help us distinguishing between causality and chance. We want to equip pupils with some statistical tools they can reliably use to this purpose.

We illustrate this point by the following example.

Headline in a newspaper: KOMÁROM, TOWN OF TWINS. The news:

"In 1977 five pairs of twins were born among 270 babies in that town. Statistics show that in Hungary the probability of giving birth to twins is 1%. Experts could not yet explain what may have caused the higher rate in Komárom. Some suggest that it may be a side effect of oral contraceptives."

We would like school leavers – whether they become journalists or just readers, anybody – to be able and willing to verify if this sort of news is a news. Did something particular happen in Komárom in 1977? We expect them to develop some intuition and ability of estimating probabilities. Beyond that, they should be able to *choose* one of the suitable tools and to *use* it properly. In our example the Poisson distribution is a good choice. You need a table, a few pages in a book. Or you may prefer to use a calculator. Some calculators have a key for the Poisson distribution, others can be programmed to it by a card. Less is enough: a simple scientific calculator with an e^x key, and a recollection of $e^x = \Sigma\, x^k/k!$ and hence of $1 = \Sigma\, x^k/e^x\, k!$ as the formula giving the Poisson probabilities, term by term, if x is the expectation, Np. In our case $N = 265$ and $p = .01$ give $x = 2.65$. From the formula for $k = 5, 6, 7, \ldots$ the approximate probabilities of five, six, seven . . . pairs of twins are .08, .03, .01, . . . , respectively. This gives .12 as the probability of *five or more* pairs of twins, a more interesting value, than that of exactly five. Interpretations:

1. From the point of view of just one town, Komárom, this means: if N and p are supposed to be and remain 265 and .01, respectively, then five or more pairs of twins can be expected in every eight years or so. Not very rarely.

2. From the point of view of the whole country the event is much less unusual. The population of the country is 800 times that of Komárom. The country can be subdivided to 800 Komároms; five or more pairs of twins are born *yearly in one hundred of these,* roughly. Under the assumption, to be sure, of the same rate of natality and the same probability (.01) of twins.

Conclusion: there is nothing special about twins in Komárom. Even oral contraceptives can be absolved on this occasion.

We do not expect schools to develop skill in those particular keystrokes which are needed for solving such particular problems with their calculators. What we expect is less and more: to put together what they know – including the whereabouts of some books to help their memories – and enabling themselves to many more problem solving procedures than they could ever meet in school. In other words, we expect them to *produce,* rather than *stock,* problem solving programmes. (Note an analogy: calculators produce data, while tables stock them.)

We also expect them to adapt their problem solving programmes to the actual circumstances. What if they only have a calculator with four rules, not with e^x, at disposal?

If mathematical education is sufficiently oriented to *understanding,* then the trainees have a fair chance of seeing here a key idea: the only role of e^x is to make the sum of the series equal to one.

If mathematical education is, besides, sufficiently oriented to *solving real problems* – not just in principle, but actually, until the end – then they have a fair chance of exploiting this idea. The numbers $x^k/k!$ $(k = 0, 1, 2, \ldots)$ are themselves proportional to the probabilities of 0, 1, 2, . . . pairs of twins. The coefficient of proportionality can be determined by summing them, up to a certain k, depending on the desired precision. (They will get an approximation of e^x). Now they can translate the key idea into the language of keystrokes on their simple machine.

They can do it with paper and pencil alone – possibly five to ten times slower. Or even mentally. We want the various means and techniques to compete. The competition will reduce the role of written arithmetic to what it still merits. It will underline, we predict, the continued – and in some respects, increased – importance of mental arithmetic. (Especially of estimations.) It will demonstrate the due place of calculators in education. And it will help to convince those who expect too much of the technical means, that for the ideas they are no substitutes.