Journal of Risk and Uncertainty, 2: 219-234 (1989)

The Impact of Risk Sharing on Efficient Decision

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Key words: risk aversion, group decision, syndicate, risk sharing, utility, lotteries

Abstract

A group of risk-averse members must choose among monetary risks and payoff-sharing rules. Departure from the status quo requires unanimous consent. Such groups drill for oil, bail out nations, and make hostile takeover bids. Assume agreement on probabilities. As is well known, if all members have identically shaped HARA utility functions, efficient group act-choices follow another such function independently of payoff sharing. We show that *all* other groups inevitably have complex efficient behavior, accepting gambles among individually unacceptable lotteries in almost every status quo position. We also develop *proper* risk aversion for groups, and treat disagreement on probabilities.

A consortium of three medium-sized oil companies has an opportunity to purchase a major lease. Need they agree on how to divide up drilling expenses and revenues before they determine whether to buy the lease? More generally, our central question is this: When will a group's choice among lotteries not depend on the weightings applied to its members' welfares? This article describes the answer in relation to the nature of syndicate members' utility functions.

Because squabbling is virtually inevitable among members of an investment group—whether its purpose is to bail out debtor nations, operate a privately held firm, succeed in a hostile takeover, or build a nuclear plant—and because many financial opportunities have a short fuse, this question is of considerable practical importance. It is also important in the many areas of economic theory where risk sharing plays a role, especially since its answer shows how readily surprising behavior can have normative underpinnings even in the simplest of general decision and risk-sharing problems. This is the problem of a *syndicate* (Christenson,

Support of the Associates of the Harvard Business School (Pratt) and the Business and Government and Energy and Environmental Policy Centers (Zeckhauser) is gratefully acknowledged. We received helpful comments from Scott Johnson, Mark Machina, and a referee. Jay Patel provided assistance beyond the call of collegiality.

1965; Wilson, 1968), but its implications, especially as it turns out, are far broader than the term *syndicate* might suggest.

Consider a group of n agents who must agree on a decision among monetery risks and on the division of the group payoff among members. To undertake action—that is, to depart from the status quo—requires unanimous consent. Suppose that each agent i has a strictly risk-averse utility function depending only on his total monetary position and, for the moment, that all agents agree about all relevant probabilities. Mutually advantageous bets are thus impossible, as is a strategy of making the group effectively risk-neutral by shifting all risk to risk-neutral members. Nonmonetary considerations play no role. Moreover, agents cannot influence the outcome by their own efforts. In this context, how does the nature of the agents' utility functions affect efficient group choice among risks? We make no assertions or assumptions beyond ex ante efficiency, as indicated by Pareto comparisons, about how groups do or should bargain internally or make external choices. Because of the indeterminacies and complexities of bargaining strength, fairness, and so on, group choices can be expected to display properties that would be unreasonable for individuals. Indeed, in many group decision problems, the need for unanimous consent by itself makes randomization desirable, violating the sure-thing principle or substitution axiom for the group, though the reason is more subtle in pure risk sharing than in a couple's choice of where to vacation (see note 3).

An important class of principal–agent relationships has the agent play a major role as decision maker. To establish such a relationship, say between the limited and general partners in a real estate investment venture, requires unanimous assent. Risk sharing is often a significant factor. Thus many principal–agent problems have the features of concern here. They usually have the additional complication of ensuring that the agent adheres to the contingent decisions he would promise on behalf of the principals, even though he has relevant private information. Often principals or agents have multiattribute utility functions. Though we do not deal with the incentive-compatibility or multiattribute aspects of principal–agent problems, our results are relevant to principal–agent and group decision problems broadly, not merely to tightly defined syndicates.

After laying out the group decision problem (section 1), we contrast the well-known, simple, and "reasonable" external behavior of HARA groups—groups whose members all have HARA utility functions of the same shape (section 2)—with the complex patterns of choices of all other groups (section 3). Our major theorem proves that *every* non-HARA group in essentially *every* possible status quo will inevitably be randomization-preferring in some choice problems, i.e., would accept gambles all of whose payoffs are unacceptable lotteries. Thus the *only* groups for which no two gambles' Pareto frontiers cross are HARA, and for all others, crossings are possible almost everywhere. We also explore the group analogue of the recently introduced concept of proper risk aversion, namely that an undesirable lottery will not be made attractive because another undesirable gamble is pressed upon the group (section 4). Our final analytic discussion addresses situations in which the group members disagree on probabilities (section

5). Section 6 presents conclusions. As will be explained, sections 1–3 and 5 complete in a strong sense all parts of a program initiated by Wilson (1968). One part was completed in a weaker sense by Rosing (1970) and the rest in an unpublished thesis by Wallace (1974), but our treatment is independent, self-contained, technically stronger, and, we believe, substantially simpler than all earlier work.

1. The group decision problem

Suppose that the risk $\tilde{z}$ is to be shared among agents $i = 1, 2, \ldots, n$, and agent i has risk-averse utility function u_i and initial wealth w_i. As is well known (e.g., Borch, 1962; Wilson, 1968), efficiency requires that $\tilde{z}$ be divided into shares $\tilde{z}_i$ so as to maximize $\Sigma\lambda_i Eu_i(w_i + \tilde{z}_i)$ for some $\lambda_i \geqq 0$, subject to $\Sigma\tilde{z}_i = \tilde{z}$. At the solution, for each $\tilde{z}$, the values of $\lambda_i u_i'(w + \tilde{z}_i)$ are the same for all i. Hence the $u_i'(w_i + \tilde{z}_i)$ values maintain the same proportions for all outcomes $\tilde{z}$; that is, $u_i'/u_j' = \lambda_j/\lambda_i$ independent of the outcome. Varying the proportions, the λ's, generates the Pareto frontier. Continuity of the possible shares and risk aversion make the frontier continuous and the feasible set convex in $\underline{u}$ space; hence extraneous randomization is inefficient. Since all u_i' are strictly decreasing functions, all $\tilde{z}_i$ are strictly increasing functions of $\tilde{z}$, but the $\tilde{z}_i$ need not all have the same sign; in particular, they incorporate all initial payments and side payments among the members of the group.

If several risks are accepted, the agents' shares should depend only on the total $\tilde{z}$, ordinarily nonlinearly. If it is necessary to share the risks linearly or separately, efficiency is typically reduced. For example, if a group of individuals jointly owns a portfolio of stocks, efficiency may require that individual j's share of the appreciation in stock A depend also on the level of appreciation in all other stocks, and that his share of the tenth dollar of appreciation differ from his share of the first dollar.[1]

If agents' initial positions $\tilde{w}_i$ are uncertain, we assume that the sharing of any risks accepted by the group depends only on the outcomes thereof, not on the $\tilde{w}_i$'s, which are beyond the purview of the group. Although it would be more efficient to share the grand total including $\Sigma\tilde{w}_i$, there are three common reasons why the $\tilde{w}_i$ would usually have to be divorced in this manner: agents may disagree about the distributions of the $\tilde{w}_i$'s, they may be able to influence the distribution of their own $\tilde{w}_i$'s, or the values of the $\tilde{w}_i$'s may not be monitorable. For risks that agent i considers independent of $\tilde{w}_i$, this constraint amounts to replacing the utility u_i by the derived utility $U_i(x) = Eu_i(\tilde{w}_i + x)$ and the initial position by 0. The unconstrained case will be dealt with briefly later.

We shall call a risk $\tilde{x}$ *unacceptable* at initial wealth position $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$ if no division of $\tilde{x}$ exists that all agents would accept; that is, for every division $\tilde{x} = \sum \tilde{x}_i$ with each $\tilde{x}_i$ a function of $\tilde{x}$, some agent i strictly prefers $\tilde{w}_i$ to $\tilde{w}_i + \tilde{x}_i$. This says that the initial position lies beyond the Pareto frontier obtainable by accepting and dividing up $\tilde{x}$. Barring pathologies, which we do, it makes no interesting difference whether strict preference is required for one or all agents or for acceptability or unacceptability. Additional randomization would serve no purpose.

2. HARA groups

Everything becomes particularly simple if initial positions are certain and all agents have HARA utility functions of the same shape, that is, either all agents have constant absolute risk aversion (exponential utility), or else all have the same constant proportional risk aversion (power or logarithmic utility) when their wealths are measured from suitable base levels, possibly different.[2] Equivalently, all agents have constant or hyperbolic absolute risk-aversion functions whose reciprocals (risk tolerances) are linear with the same slope $b \geqq 0$, say $-u'_i(w)/u''_i(w) = bw + a_i$, with $w > -a_i/b$ if $b > 0$. We call such a group a HARA group and b its shape parameter. If $b = 0$, the agents have constant absolute risk aversions $1/a_i$. If $b > 0$, all have constant proportional risk aversion $1/b$ when agent i's wealth is measured from the origin $-a_i/b$, and their utilities are power functions with the same exponent $1 - 1/b$ or, if $b = 1$, logarithms. Thus b determines the shape of the u_i. As noted by Wilson (1968), in this situation straightforward calculation shows that (1) efficiency requires all risks to be shared in the same constant proportions after fixed side payments (linear sharing); (2) the Pareto frontiers generated by different lotteries never cross, so that all risks are Pareto-comparable and choices among them should not depend on how they will be shared; and (3) the group's decisions should agree with a utility function having risk tolerance $bw + \Sigma a_i$. From conclusion (1) it also follows that (4) separate sharing of multiple risks, independent or not, entails no loss of efficiency.

If some or all of the initial positions $\tilde{w}_i$ are uncertain and excluded from risk sharing, then the agents' attitudes toward new, independent risks are governed by the derived utilities $U_i(x) = Eu_i(\tilde{w}_i + x)$ in place of the u_i. If the U_i are HARA of the same shape, then conclusions (1) to (4) apply to risks independent of the $\tilde{w}_i$. If the u_i are exponential, then so are all the U_i, since $E\{-e^{-(\tilde{w}_i+x)/a_i}\} = -k_i e^{-x/a_i}$ where $k_i = E\{e^{-\tilde{w}_i/a_i}\}$, which is a positive constant, and the conclusions above therefore apply to all uncertain initial positions. Otherwise, however, if u_i is HARA, then U_i generally is not, and the conclusions above do not apply to uncertain initial positions. Similarly, if U_i is HARA for one distribution of $\tilde{w}_i$, then it is HARA for translations $\tilde{w}_i + a_i$ but generally not for other distributions of $\tilde{w}_i$.

If the initial positions $\tilde{w}_i$ are uncertain but there is agreement about their distribution, and if, contrary to our earlier assumption, they are included in the risk sharing, then the u_i are again relevant, not the U_i. If the u_i are HARA of the same shape, then conclusions (1) to (4) remain valid.

3. Groups that are not HARA

The main developments of this section will show that a group's behavior is inevitably complex and quite possibly troubling if the members' utilities u_i or derived utilities U_i, whichever are relevant, are not all HARA of the same shape.

One striking anomaly is that a lottery on unacceptable lotteries may be desirable. We shall call this a *Horror*, since the group Happily Okays Randomized Risks Over Rejects, but it may be less unreasonable than it appears. It makes sense to divide each risk a group might accept so as to favor the members whose utility functions enable them to benefit most from its risk characteristics, and to balance matters by randomizing among risks favorable to different members.[3]

The result to which this section leads (theorem 4) implies that for every non-HARA group, there exist gambles whose Pareto frontiers cross each other at essentially all points. Thus which of the two gambles is better for the group depends on how the gambles would be divided, randomization between them permits Pareto improvement, and the other simplifications of section 2 do not occur. To our knowledge, the existence of frontier crossings for all non-HARA groups has never been stated, much less proved, in the published literature, and their ubiquitousness has never appeared anywhere. The pioneering paper by Wilson (1968) initiated the problem and made considerable progress, but his assumptions and conclusions are subtle, and his results are interesting mainly as historical landmarks in view of theorem 4 and its counterpart for groups that disagree on probabilities (theorem 6, discussed in section 5). Rosing (1970) resolved the central existence question in the case of disagreement but reached essentially the same point as Wilson in the case of agreement. We know of no further published progress, but Wallace (1974), in an unpublished dissertation, obtained the central result in both cases by a very long and difficult argument. Our route to somewhat stronger results has little obvious overlap with any of the paths carved out by earlier investigators, and appears to be much simpler. Three theorems provide milestones on our way.

We shall see that whether a Pareto-efficient group behaves like a rational individual or in a more complicated fashion depends on the linearity or nonlinearity of what we shall call the *expansion paths*, the curves in u-space on which the efficiency-determining tradeoffs λ_j/λ_i are constant. (These ratios may be viewed as either the group's marginal tradeoffs u_i'/u_j' at an efficient point, or the relative scalings in the implicit criterion or objective function $\Sigma\lambda_i u_i$ maximized.) The following example may motivate and clarify our approach.

Example. Two agents have utility functions $u_1(w) = \log(w)$ and $u_2(w) = -e^{-w}$. If total wealth w is fixed, any nonrandom sharing w_1, w_2 with $w_1 + w_2 = w$ and $w_1 > 0$ is Pareto-efficient, and the frontier for sharing w is given by $u_1 = u_1(w_1) = \log(w_1)$, $u_2 = u_2(w_2) = u_2(w - w_1) = u_2(w - e^{u_1}) = -\exp(-w + e^{u_1})$, or equivalently $u_1 = \log(w + \log(-u_2))$. Pareto frontiers F in u-space for four values of w are shown in figure 1.

If total wealth is uncertain, Pareto efficiency requires that the shares w_1 and w_2 give always the same value of

$$\frac{u_2'(w_2)}{u_1'(w_1)} = w_1 e^{-w_2} = -u_2 e^{u_1}$$

and hence lie on the same expansion path $-u_2 e^{u_1} = \lambda_1/\lambda_2$. Where the pareto

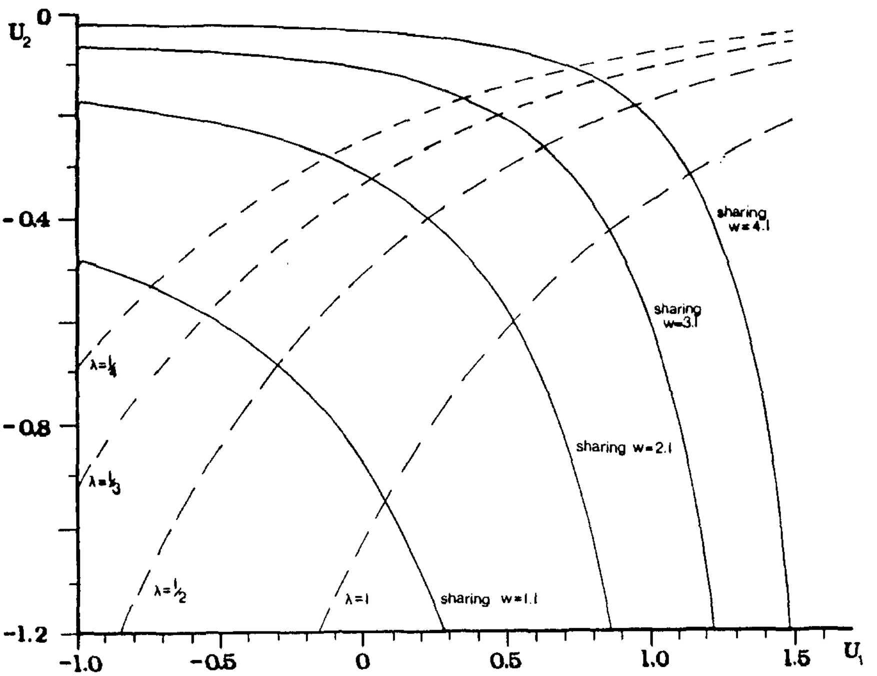

Fig. 1. Pareto frontiers (———) and expansion paths (- - -) for a group of two members with utility functions log w_1 and $-e^{-w_2}$.

frontiers for sharing different amounts w cross this path, they all have the same slope $-\lambda_1/\lambda_2$. We call $\lambda = \lambda_1/\lambda_2$ the tradeoff and $-\lambda$ the slope corresponding to the path. Expansion paths P for four values of λ are also shown in figure 1.

If a risk $\tilde{w}$ is shared efficiently as $\tilde{w}_1$, $\tilde{w}_2$, the outcomes must lie on a single expansion path, corresponding to some tradeoff λ. At the expected utility point $\bar{u}_1 = Eu_1(\tilde{w}_1)$, $\bar{u}_2 = Eu_2(\tilde{w}_2)$, the slope of the frontier for sharing $\tilde{w}$ is $-\lambda$. Because the expansion paths are strictly concave in this example, the point $(\bar{u}_1, \bar{u}_2)$ lies off the path corresponding to λ and hence on some other path, corresponding to some other tradeoff λ'. The slope of the frontier for sharing a certainty $\bar{w}$ at the point $(\bar{u}_1, \bar{u}_2)$ is $-\lambda'$. (The certainty $\bar{w}$ yielding $(\bar{u}_1, \bar{u}_2)$ is $\bar{w}_1 + \bar{w}_2$, where $\bar{w}_i$ is the certainty equivalent of $\tilde{w}_i$ given by $u_i(\bar{w}_i) = \bar{u}_i$.) Thus the frontiers for sharing $\tilde{w}$ and $\bar{w}$ pass through $(\bar{u}_1, \bar{u}_2)$ with different slopes and hence must cross there. Figure 2 gives an example for a 50–50 lottery $\tilde{w}$.

Examples of such crossings can always be constructed unless the expansion paths are straight lines. Thus the basic question is what combinations of utility functions make all expansion paths linear. We first detail analytical conditions equivalent to linearity, next describe its relationship to HARA, and then turn to its behavioral implications.[4]

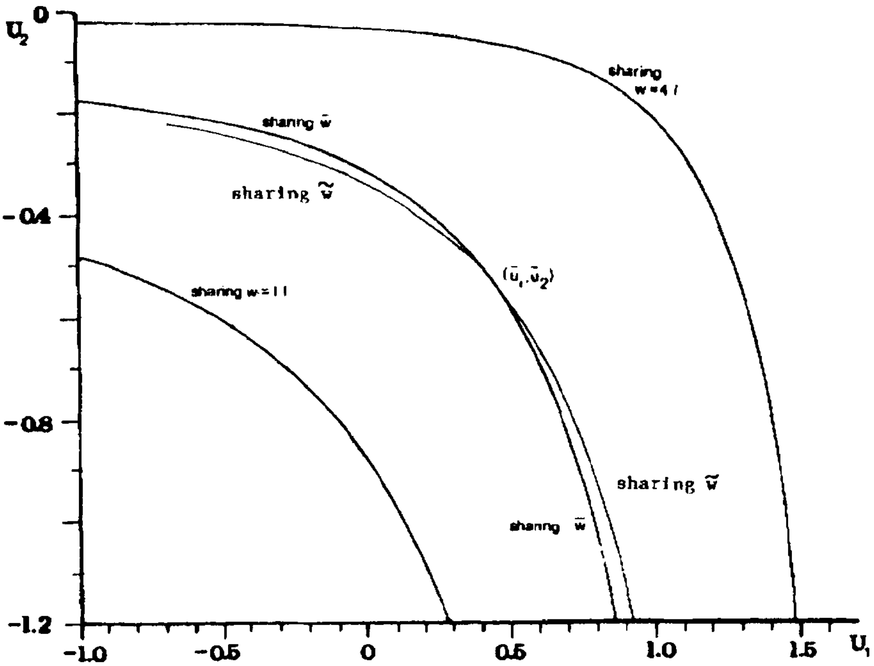

Fig. 2. Example of crossing frontiers.

Theorem 1. Given any tradeoffs $\lambda_i > 0$, the following conditions are equivalent on any intervals of the agents' wealths, with a_i, b_i, and c_i denoting the same constants throughout and $b_i > 0$, $i = 1, \ldots, n$.
(a) The Pareto-efficient shares w_i given by $\underline{\lambda}$ are linearly related, say $w_i = b_i w + a_i$ as w varies.
(b) The agents' utilities u_i are equivalent after linear transformations of their wealth, say $u_i(b_i w + a_i) = b_i t(w)/\lambda_i + c_i$.
(c) The expansion path for $\underline{\lambda}$ is linear in $\underline{u}$ space, say $u_i = b_i t/\lambda_i + c_i$ as t varies.

Proof. Condition (a) is eqivalent to

$$\lambda_i u_i'(b_i w + a_i) \text{ depends only on } w. \tag{1}$$

This is obviously equivalent to (b) by integration and differentiation. Conditions (a) and (b) imply (c) directly. Condition (c) is equivalent to

$$\lambda_i u_i'(u_i^{-1}(b_i t/\lambda_i + c_i)) \text{ depends only on } t. \tag{2}$$

Recalling that $u_i^{-1\prime}(s) = 1/u_i'(u_i^{-1}(s))$ and integrating the reciprocal of (2) gives

$$(u_i^{-1}(b_i t/\lambda_i + c_i) - a_i)/b_i \text{ depends only on } t. \tag{3}$$

Denoting this by w and solving for t gives (b). Q.E.D.

Theorem 2. If in given intervals of its members' wealths a group is HARA with shape parameter b (all members have HARA utility functions with parameter b), then for all $\underline{\lambda}$ the linearity conditions (a) to (c) of theorem 1 hold in these intervals with $b_i = b$. Otherwise, none of the linearity conditions of theorem 1 hold, except possibly at isolated values of $\underline{\lambda}$.

Proof. The first sentence is easily verified directly; see also section 2. If (a) holds on an open set, differentiating (1) with respect to λ_i gives

$$u_i'(b_i w + a_i) + \lambda_i \frac{\partial b_i}{\partial \lambda_i}\left(w + \frac{\partial a_i}{\partial \lambda_i}\right)u_i''(b_i w + a_i) = 0. \tag{4}$$

For $\underline{\lambda}$ fixed and $W = b_i w + a_i$, we see that $u_i'(W)/u_i''(W)$ is linear in W, and hence u_i is HARA. Since linear transformation of wealth carries a HARA utility into another of the same shape, it follows from (b) that all u_j are HARA of the same shape. A similar proof is possible, provided some value of $\underline{\lambda}$ at which (a) holds is a limit of others. Q.E.D.

We shall say that Horrors exist at given wealths of the agents or at the corresponding point $\underline{u}$ if there exist randomizations between two risks that permit strict Pareto improvement over $\underline{u}$ even though $\underline{u}$ lies strictly beyond the Pareto frontier of each risk individually. Any one Horror clearly has a continuum of others nearby, since small variations in either the risks or the randomization preserve the Horror. The same is true if one of the risks is replaced by a certainty. The proof below demonstrates the ubiquity of Horrors with one risk and one certainty; small variation then demonstrates it for two risks.

Theorem 3. If the linearity conditions of theorem 1 fail anywhere on the expansion path through a given interior position $\underline{u}$, then Horrors exist at $\underline{u}$.

The main result of this section follows immediately from theorems 2 and 3.

Theorem 4. If a group is not HARA throughout given intervals of its members' wealths, then Horrors exist at all interior values of wealth except possibly on isolated expansion paths.

Proof of theorem 3. Using the notation $\underline{u}(\underline{x}) = [u_1(x_1), \ldots, u_n(x_n)]$, suppose that $\underline{u}(\underline{x})$ and $\underline{u}(\underline{z})$ lie on the expansion path given by $\underline{\lambda}$, while a convex combination $\underline{u}(\underline{w}) =$

$p\underline{u}(\underline{x}) + (1-p)\underline{u}(\underline{z})$ does not. Then the Pareto frontier for the lottery with probability p at $x = \Sigma x_i$ and probability $1-p$ at $z = \Sigma z_i$ passes through $\underline{u}(\underline{w})$ perpendicular to $\underline{\lambda}$, while the frontier for sharing the certainty $w = \Sigma w_i$ passes through $\underline{u}(\underline{w})$ with a different normal. Hence some convex combination of the lottery and the certainty is strictly Pareto-superior to $\underline{u}(\underline{w})$. The conclusion follows for $\underline{u} = \underline{u}(\underline{w})$ since the lottery and the certainty can be made slightly worse and the certainty slightly uncertain without losing strict Pareto superiority. It remains only to show that every interior point on a nonlinear expansion path is a convex combination of two points on some other expansion path. This is fairly easy to see for two agents but harder for more than two. It is an immediate consequence of the following lemma. Q.E.D.

We title the lemma to reflect its interpretation. It demonstrates, in effect, that if a filament in a rope is not straight, then every point that is on the filament but not on the surface of the rope is on a chord between two points on some other filament, assuming that all filaments run the length of the rope.

Rope Lemma. Let $\underline{u}$ be a smooth, one-to-one function of $(w, \underline{v}) = (w, v_1, \ldots, v_{n-1})$ on an open set S, with values in R^n. If $(w_0, \underline{v}_0)\varepsilon S$, and if $\underline{u}(w, \underline{v}_0)$ does not lie wholly on one line as w varies, then there exist w_1, w_2, and $\underline{v}_* \neq \underline{v}_0$ such that $\underline{u}(w_0, \underline{v}_0)$ is a convex combination of $\underline{u}(w_1, \underline{v}_*)$ and $\underline{u}(w_2, \underline{v}_*)$.

Proof. Let $\underline{u}_0 = \underline{u}(w_0, \underline{v}_0)$. Choose w_1 so that $\underline{u}(w_1, \underline{v}_0)$ is not on the tangent to $\underline{u}$ at $\underline{u}_0$. Let V be a compact set with $\underline{v}_0$ in its interior such that $(w_1, \underline{v}) \varepsilon S$ for $\underline{v} \varepsilon V$. Choose $\varepsilon > 0$ so small that $(1+\varepsilon)\underline{u}_0 - \varepsilon\underline{u}(w_1, \underline{v})$ is in the image of $\underline{u}$ for every $\underline{v} \varepsilon V$ and that, when g and $\underline{f}$ are defined on V by $\underline{u}(g(\underline{v}), \underline{f}(\underline{v})) = (1+\varepsilon)\underline{u}_0 - \varepsilon\underline{u}(w_1, \underline{v})$, $\underline{f}$ is a contraction and $\underline{f}(\underline{v}_0) \neq \underline{v}_0$. By the contraction mapping theorem, $\underline{f}$ has a fixed point, say $\underline{v}_* = \underline{f}(\underline{v}_*) \neq \underline{v}_0$. Then $\underline{u}_0$ is a convex combination of $\underline{u}(w_1, \underline{v}_*)$ and $u(g(\underline{v}_*), \underline{v}_*)$. Q.E.D.

The proof requires smooth enough filaments to make $\underline{f}$ a contraction. One-to-one transformations of $\underline{u}$, w, and $\underline{v}$ of course have no effect on the essential question since they do not change the filaments, but they might create (or destroy) smoothness.[5]

4. Proper risk aversion for groups

If the members of a group are risk-averse, then the group will be risk-averse as well, in the sense that the Pareto frontier for sharing any strict lottery lies inside the Pareto frontier derived by sharing its expected value. If the members are decreasingly risk-averse, then so is the group—that is, an acceptable lottery remains acceptable if any agent's initial wealth is increased. Thus far the situation is as we would have hoped. But when we step further in virtually any direction, the similarity of group behavior to the behavioral characteristics of its individual members vanishes. We have already seen, for example, that only in very special groups are all randomizations between unacceptable lotteries unacceptable. We can, however, move beyond decreasing risk aversion at least as far as one natural group analogue of proper risk aversion (Pratt and Zeckhauser, 1987).

An individual finds each of two independent lotteries undesirable. If he is required to take one, should he not continue to find the other undesirable? If in all such cases he would, he displays a property we call *proper risk aversion.* We believe that proper risk aversion is a desirable property for an individual's utility function, and suspect that if it were well understood it would be required by real-world decision makers of the functions they employ to guide their decisions. The concept has important implications. For instance, insuring against one risk (i.e., rejecting one lottery) will make a proper individual readier to accept other independent risks. Thus, additional options or futures markets could stimulate, not drain, activity on competitor markets.

To begin our discussion of proper risk aversion, recall that an individual whose preferences are determined by the utility function u is called *risk-averse* if $\tilde{w} \preccurlyeq E\tilde{w}$ for all $\tilde{w}$, which is equivalent to concavity of u. Also, one of several trivially equivalent, intuitively natural definitions of *decreasingly risk-averse* is that a certain decrease in certain wealth never makes an undesirable gamble desirable, that is,

$$w + \tilde{x} + y \preccurlyeq w + y \text{ whenever } w + \tilde{x} \preccurlyeq w \text{ and } y < 0. \tag{5}$$

Several analytical conditions on u have been shown equivalent to (5) and useful (Pratt, 1964, 1988; Arrow, 1971), including the condition that the *local risk aversion* $r(w) = -u''(w)/u'(w)$ be decreasing in w.

Replacing y in (5) by an undesirable gamble $\tilde{y}$ independent of $\tilde{x}$, still holding w certain, leads to the following definition: u is *fixed-wealth proper* if

$$w + \tilde{x} + \tilde{y} \preccurlyeq w + \tilde{y} \text{ whenever } w + \tilde{x} \preccurlyeq w \text{ and } w + \tilde{y} \preccurlyeq w. \tag{6}$$

In other words, if independent lotteries $\tilde{x}$ and $\tilde{y}$ are individually unattractive, the package lottery offering both together is less attractive than either alone.

We call u *proper* if (6) holds for uncertain and independent $\tilde{w}$ also, that is, if

$$\tilde{w} + \tilde{x} + \tilde{y} \preccurlyeq \tilde{w} + \tilde{y} \text{ whenever } \tilde{w} + \tilde{x} \preccurlyeq \tilde{w} \text{ and } \tilde{w} + \tilde{y} \preccurlyeq \tilde{w}. \tag{7}$$

Obviously, proper implies fixed-wealth proper, and fixed-wealth proper implies decreasing risk aversion.

Here we examine proper risk aversion in the group context. We call a group *totally proper at* $\underline{\tilde{w}}$ if $\tilde{x} + \tilde{y}$ is unacceptable whenever $\tilde{x}$ and $\tilde{y}$ are unacceptable and $\tilde{x}$, $\tilde{y}$, and $\tilde{w}_i$ are independent for all i; some or all $\tilde{w}_i$ may be nonrandom. *Simply proper at* $\underline{\tilde{w}}$ means no separate divisions of independent unacceptable gambles $\tilde{x} = \Sigma\tilde{x}_i$ and $\tilde{y} = \Sigma\tilde{y}_i$ exist such that every agent i weakly prefers $\tilde{w}_i + \tilde{x}_i + \tilde{y}_i$ to $\tilde{w}_i$. If a group is totally proper at all $\underline{\tilde{w}}$ (or all nonrandom $\underline{w}$), we call it *totally proper* (or *totally fixed-wealth proper*), and similarly with *simply* in place of *totally*. Since separate division of $\tilde{x} + \tilde{y}$ can never increase and may reduce efficiency, totally implies simply in each case. Other relations are equally immediate.

Of course if any member is not proper as an individual, then, except in special circumstances that we have not investigated, we would not expect other members' preferences to offset his improper choices in such a way that the group is even simply proper. We shall see that, if all members are proper, the group is simply proper, even though in non-HARA groups, by theorem 4, Pareto noncomparability is ubiquitous.

HARA groups. The simplest case arises when all members of the group have HARA utilities of the same shape. Then (i) the group is totally fixed-wealth proper, since utilities with linear risk tolerance are proper; and (ii) if $\tilde{x}$ and $\tilde{y}$ are unacceptable and independent, then $\tilde{x} + \tilde{y}$ is not only unacceptable but also Pareto-inferior to either alone.

If some or all the initial positions $\tilde{w}_i$ are uncertain and excluded from risk sharing, replace the u_i with the derived utilities U_i. Then (i) and (ii) apply to risks independent of the $\tilde{w}_i$ with fixed-wealth proper replaced by proper at $\underline{\tilde{w}}$ in (i). If the u_i are exponential, then, as remarked earlier, so are all U_i, the conclusions above apply to all uncertain initial positions, and the group is totally proper.

If the initial positions $\tilde{w}_i$ are uncertain but agreed upon and are included in the risk sharing, then the u_i are again relevant, not the U_i. Conditions (i) and (ii) need reinterpretation, but take a very strong form. Reinterpretation is needed because the initial position will ordinarily be Pareto-inferior to a portion of the starting frontier, the Pareto frontier achievable through risk sharing on the $\tilde{w}_i$. Since all risks are Pareto-comparable, however, for all $\tilde{w}_i$ we can state the following: if neither $\tilde{x}$ nor $\tilde{y}$ permits Pareto improvement over the starting frontier, then accepting either alone is Pareto-inferior to the starting frontier, and accepting both is Pareto-inferior to accepting either alone. This says that the group is totally proper, even when *unacceptable* is interpreted as *not Pareto-superior to the starting frontier* in the hypothesis and as *Pareto-inferior to the starting frontier* in the conclusion.

Non-HARA groups. In section 3 we found that non-HARA groups encounter numerous complications. Nevertheless, if all of the members of a non-HARA group are proper, then it will be simply proper, as we demonstrate in theorem 5. Before turning to that result, let us mention two of the complications.

First, if the $\tilde{w}_i$ are included in risk sharing or if the group is forced to accept an unacceptable risk $\tilde{y}$, the ambiguity about how the $\tilde{w}_i$ or $\tilde{y}$ will be shared creates difficulty. Suppose, for example, that agent 1's share of a small unacceptable lottery $\tilde{y}$ is $\tilde{y}/2 + z$ and agent 2's share is $\tilde{y}/2 - z$, where z is large. Then accepting $\tilde{y}$ may well make agent 1 sufficiently less risk-averse to desire a previously unacceptable risk $\tilde{x}$, even if agent 2 shares none of this risk. Similar examples apply to the $\tilde{w}_i$. Such difficulties do not arise for individuals, and Pareto comparability bypasses them for HARA groups.

Second, as the example above illustrates, for unacceptable lotteries $\tilde{x}$ and $\tilde{y}$, even if $\tilde{x} + \tilde{y}$ is unacceptable, it need not be Pareto-comparable to $\tilde{x}$ or $\tilde{y}$ alone. One might restrict the sharing so that ex ante no agent may gain if another loses. Then whether $\tilde{x} + \tilde{y}$ is always Pareto-inferior to $\tilde{x}$ or $\tilde{y}$, and even whether $\tilde{x} + \tilde{y}$ is always unacceptable for total sharing, are unresolved questions. However, theorem 5 proves unacceptability of $\tilde{x} + \tilde{y}$ for separate sharing.

Theorem 5. A group is simply (fixed-wealth) proper if all its members are (fixed-wealth) proper. It is simply proper at $\tilde{w}$ if every member i is proper at $\tilde{w}_i$.

Proof. By considering derived utility functions, we may take $\tilde{w}_i = 0$ without loss of generality. Suppose that $\tilde{x} = \sum \tilde{x}_i$, $\tilde{y} = \sum \tilde{y}_i$, and $\tilde{x}_i + \tilde{y}_i \succcurlyeq_i 0$. Define a_i and b_i such that $\tilde{x}_i + a_i \sim_i 0 \sim_i \tilde{y}_i + b_i$. (Their negatives are selling prices, not certainly equivalents.) Then

$$i \text{ proper} \Rightarrow \tilde{x}_i + a_i + \tilde{y}_i + b_i \preccurlyeq_i 0 \preccurlyeq_i \tilde{x}_i + \tilde{y}_i \Rightarrow a_i + b_i \leqslant 0. \quad (8)$$

Therefore, either $\Sigma a_i \leqq 0$, and $\tilde{x}$ is surely as large as $\Sigma(\tilde{x}_i + a_i)$ and hence acceptable, or $\Sigma b_i \leqq 0$ and $\tilde{y}$ is unacceptable.[6] Q.E.D.

5. Heterogeneous beliefs

We will now discuss briefly the situation when the agents do not have identical beliefs (probability distributions). We utilize the same model as Wilson (1968) and Rosing (1970). Some of its implicit assumptions are worth making explicit.

Each agent's beliefs are supposed to be uninfluenced by anything the other agents do or say; perhaps the agents have shared information fully but "agreed to disagree," despite Aumann (1976). We allow an arbitrary state variable or event ξ and consider group choice among risks $\tilde{z}$ determined by ξ. It is in the spirit of theories of choice that all functions of ξ are considered, although only a few are available. Refinement in the definition of ξ can always make all available risks functions of ξ, can never shrink the Pareto frontier, and may expand it by enlarging the opportunities for implicit betting among the agents. We assume, however, that the same event ξ will become public knowledge whatever decisions the agents and the group make—that is, they cannot influence what information will be available for side betting.

If the agents' initial positions w_i are certain, efficient sharing of a risk $\tilde{z} = z(\xi)$ now maximizes $\Sigma\lambda_i E_i u_i(w_i + \tilde{z}_i)$ for some $\lambda_i \geqq 0$, where E_i denotes agent i's expectation.[7] At the solution, the $u_i'(w_i + z_i)f_i(\xi)$ maintain the same proportions for all ξ, where f_i is the mass function or density function (with respect to an arbitrary measure on ξ-space) determining E_i.

Everything is still simple if each agent has constant risk aversion, say $u_i(w) = -e^{-w/a_i}$ (Wallace, 1974; see also Wilson, 1968). Agent i should receive $\{z + \Sigma_k a_k \log [f_i(\xi)/f_k(\xi)]\}a_i/\Sigma_k a_k$ after fixed side payments. The group's payoff z is always shared in the same proportions. The optimal side bets depend only on ξ, not the risk chosen or its value. The Pareto frontiers are contours of $\Pi_i(-u_i)^{a_i}$ with all $u_i < 0$. They never cross and are generated solely by varying the side payments. The group's decisions should agree with a utility function having constant risk tolerance Σa_i and a density proportional to $\Pi_i[f_i(\xi)]^{a_i/\Sigma_k a_k}$. A sum of individually unacceptable risks is Pareto-inferior to each alone (or any subset). The expansion paths are rays through the origin in the negative orthant. If the initial positions $\tilde{w}_i$ are uncertain, whether or not they are included in the risk sharing and side betting, the analysis given above for the exponential case with agreement on probabilities can easily be extended to incorporate them. Even if the initial positions are certain, side bets permit Pareto improvement over standing pat, and the situation is similar to that described near the end of section 2.

If agents without identical beliefs do not all have constant risk aversion, then crossing Pareto frontiers, and hence randomizations permitting strict Pareto improvement, exist everywhere. Rosing (1970) shows that crossings exist in this case, and perhaps his proof can be extended to show that they exist everywhere. His analysis is already difficult, however, and adapting and extending our results above seems easier (at least to us) and accomplishes the whole task. For simplicity, we omit reference to the ranges of wealth over which the agents' risk aversions must be constant to avoid crossing frontiers, but note that optimal side bets enlarge them.

Two fundamental points are worth making before getting into details. First, if agents' beliefs agree in some respects but not others, then in general the optimal side bets on matters of disagreement depend on how agreed matters turn out, though of course no such dependence occurs when all agents have constant risk aversion, as noted above. Second, better contracts may be available ex ante than ex post; that is, the side bets to be made should be specified before any results come in, provided that the bets are enforceable.

The counterparts of theorems 1 and 3 will be evident in the proof of our final theorem, which may be stated as follows.

Theorem 6. If the members of a group do not have identical beliefs and do not all have constant risk aversion, then Horrors exist at all interior values of wealth except possibly those on paths corresponding to isolated tradeoff vectors $\underline{\lambda}$.

Proof. As we vary the group payoff function $\tilde{z}$ and hence the value z for fixed ξ, the Pareto-efficient shares w_i given $\underline{\lambda}$ equate $\lambda_i f_i(\xi) u_i'(w_i)$ across i. Hence, for $\underline{\lambda}$ and ξ fixed, the situation is essentially that of theorem 1 with $\lambda_i f_i(\xi)$ in place of λ_i, and

$$\begin{aligned} w_i = b_i w + a_i &\text{ is Pareto optimal,} \\ &\Leftrightarrow \lambda_i f_i(\xi) u_i'(b_i w + a_i) \text{ depends only on } w, \\ &\Leftrightarrow u_i(b_i w + a_i) = b_i t(w)/\lambda_i f_i(\xi) + c_i, \\ &\Leftrightarrow u_i = b_i t/\lambda_i f_i(\xi) + c_i \text{ is an expansion path.} \end{aligned}$$

If these linearities hold as $\underline{\lambda}$ varies, a_i and b_i depend on $\underline{\lambda}$ and ξ, but (4) is obtained as before with a factor $f_i(\xi)$ that can be divided out, and the conclusion follows as before unless the u_i are HARA of the same shape.

We now exploit disagreement, assuming that the group is HARA but not exponential, that is, the agents' risk tolerances have common, constant, positive slope b. Disagreement implies that, for some i and j, f_i/f_j is not constant. Given any risk $\tilde{w}$, consider adding Δ to $w(\xi)$. In the discrete case, this adds $b_i(\xi)\tau(\xi)$ to $\lambda_i u_i$, where u_i is agent i's expected utility and $\tau(\xi) = t(w(\xi) + \Delta) - t(w(\xi))$. It adds the same amount to the corresponding integrand in the continuous case. It is easy to see that $\tau(\xi) > 0$ for $\Delta > 0$, and to calculate that $b_i(\xi)/b_j(\xi) = [f_i(\xi)/f_j(\xi)]^b$. Hence $b_i(\xi) > b_j(\xi)$ if $f_i(\xi) > f_j(\xi)$. Increasing $w(\xi)$ where $f_i(\xi) > f_j(\xi)$ and decreasing it elsewhere will therefore increase $\lambda_i u_i - \lambda_j u_j$. The $\underline{\lambda}$-point for the new risk must therefore differ from that for $\tilde{w}$. However, we can adjust the increases and decreases so that the Pareto frontier for the new risk passes through the $\underline{\lambda}$-point for $\tilde{w}$. Hence the frontiers must cross there. Q.E.D.

6. Conclusion

The normative theory of individual decision making under uncertainty, although well established, remains a subject of lively debate. For group (or syndicate) decisions, by contrast, norms for behavior hardly exist; discussion is sparse. Yet if decisions are weighted by the number of dollars at stake, group choices probably dwarf individual ones.

Only HARA groups behave nicely. From all other groups, we must expect such unreasonable behavior as accepting a gamble between two lotteries each of which by itself would be unacceptable. And when the valuations of members' welfare depend on personal, disparate probability assessments, all groups falter save those whose members have exponential utility functions. Additional criteria for reasonable group behavior seem called for. The concept of properness may prove helpful; a group will be simply proper if its members are proper.

The flurry of research providing new axiomatic approaches to individual decision making was stimulated by laboratory and field observations indicating that individuals do not adhere to the prescriptions of the Savage Axioms. A disturbing reality led theory. (For a recent summary, see Machina (1987), who reviews the empirical evidence on deviations from the standard theory and reports "on how these findings have changed, are likely to change, or ought to change, the way we view and model economic behavior under uncertainty" (pages 121-122). See also the surveys by Sugden (1986) and Weber and Camerer (1987).) For group decision making, since the time of the Arrow Possibility Theorem, researchers have found that the behavior implied by reasonable conditions is often disturbing. Our results reinforce that finding. Perhaps laboratory and field results on group decisions would be more disturbing still. Then again, the structure of individual preferences or group decision procedures may be such that bizarre behaviors rarely arise.

Society has produced a rich array of mechanisms for syndicates to employ in choosing and dividing risks. Tax laws, rules of subchapter S corporations, and regulations on bidding for oil leases, for example, dramatically affect the ways individuals or organizations can cooperate through risk selection and sharing. Once we are thrust out of the elegant Eden described by the modern theory of finance, where all risks are spread and all information captured, we need tools—such as the concepts of properness or of HARA and non-HARA groups—to help us explain and evaluate decisions taken by individuals acting together.

Notes

1. Mutual funds and conglomerates, in theory, could simultaneously offer low-transaction-cost diversification and varying sharing percentages by employing multiple classes of stock. Informational asymmetries and difficulties of drawing contingent contracts would exacerbate conflicts of interest that are well known from the principal-agent literature. See Spence and Zeckhauser (1971).

2. This case can be described as constant size-of-risk aversion at initial wealth equal to the base level, in the terminology of Zeckhauser and Keeler (1970), or similarly as constant partial risk aversion in the terminology of Menezes and Hanson (1970).

3. By way of analogy, consider a couple contemplating the duration and location of a vacation, as in the *Battle of the Sexes* (Luce and Raiffa, 1957, pp. 90-94). The choice is tennis in the Bahamas or shopping in Paris. The spouse whose passion is tennis dislikes shopping, and vice versa. For each, a vacation of any length in the other's preferred location is worse than no trip at all. Thus, neither vacation has a duration that is Pareto-superior to staying home. But rather than stay home, both spouses might agree to flip a coin to determine whether to go to Paris or to the Bahamas for a one-week trip. A referee reminded us that the potentially beneficial role of randomization in social choice is discussed by, among others, Fishburn (1972) and Zeckhauser (1969). And Mark Machina reminded us that, even for an individual, "We know from Spence and Zeckhauser (1972) that mitigating circumstances such as delayed resolution of risks could make it 'rational' to violate the independence axiom over lotteries."

4. *Similarity* as defined by Ross (1974) becomes (b) in the next theorem when his fee schedule is linear but not otherwise.

5. An alternative proof is possible using the Brouwer Fixed-Point Theorem. It may require less smoothness and can be elaborated to show that the convex combination can be made a simple average (50–50 lottery) except perhaps when all values of $\underline{u}(w, \underline{v}_0)$ nearer to $\underline{u}(w_0, \underline{v}_0)$ than is the nearer endpoint lie on a line. We are grateful to Andrew M. Gleason for valuable discussion of the Rope Lemma and alternative proofs.

6. We conjecture that *simply* can be replaced by *totally* in this theorem. In a computer-aided search we found no counterexample. Moreover, our reflections on the crossing frontiers that are inevitably encountered in non-HARA groups, though superficially promising, did not yield fruit.

7. It can be argued that differences among beliefs make ex ante efficiency less compelling as a normative criterion, and that they can be more appropriately handled by a concept of interim efficiency (Holmström and Myerson, 1983), but we do not explore this direction here.

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