NOTES ON THE PATHS AND STABILITY OF CRACKS

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ABSTRACT

The stress distribution at the tip of a crack can be expanded as a power series. The first term, usually called the stress intensity factor, determines the initiation of fracture in a brittle material. In this paper it is shown that the second, third and fourth terms have the following effects:

(a) the second term controls the stability of the crack's direction,
(b) the third term controls the stability of the crack's propagation,
(c) the fourth term determines whether the maximum shear stress on the prolongation of the crack increases or decreases with distance from the crack tip.

THE INFLUENCE OF THE STRESS DISTRIBUTION AT THE TIP OF A CRACK

The stress distribution near a tip of a slowly propagating crack will be symmetrical with respect to the prolongation of the crack [1,2] and, provided that the curvature is not great, can be expressed by the power series. [3]

$$\begin{aligned}\sigma_r &= \frac{a_1}{4}\left(\frac{\ell}{r}\right)^{\frac{1}{2}}\left[5\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] + a_2\cos^2\theta \\ &+ \frac{a_3}{4}\left(\frac{r}{\ell}\right)^{\frac{1}{2}}\left[3\cos\frac{\theta}{2} + 5\cos\frac{5\theta}{2}\right] + \frac{a_4}{2}\left(\frac{r}{\ell}\right)\left[\cos\theta + \cos 3\theta\right] \\ &+ 0\left(\frac{r}{\ell}\right)^{3/2}\end{aligned} \tag{1}$$

$$\begin{aligned}\sigma_\theta &= \frac{a_1}{4}\left(\frac{\ell}{r}\right)^{\frac{1}{2}}\left[3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2}\right] + a_2\sin^2\theta \\ &+ \frac{a_3}{4}\left(\frac{r}{\ell}\right)^{\frac{1}{2}}\left[5\cos\theta - \cos\frac{5\theta}{2}\right] + 3a_4\left(\frac{r}{\ell}\right)\left[\cos\theta - \cos 3\theta\right] \\ &+ 0\left(\frac{r}{\ell}\right)^{3/2}\end{aligned} \tag{2}$$

$$\begin{aligned}\sigma_{r\theta} &= \frac{a_1}{4}\left(\frac{\ell}{r}\right)^{\frac{1}{2}}\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] - \frac{a_2}{2}\sin 2\theta \\ &+ \frac{a_3}{4}\left(\frac{r}{\ell}\right)^{\frac{1}{2}}\left[\sin\frac{\theta}{2} - \sin\frac{5\theta}{2}\right] + a_4\left(\frac{r}{\ell}\right)\left[\sin\theta - 3\sin 3\theta\right] \\ &+ 0\left(\frac{r}{\ell}\right)^{3/2}\end{aligned} \tag{3}$$

where r and θ are polar coordinates (r being measured from the crack tip and θ being measured from the prolongation of the crack) and ℓ is the distance between the tips of the crack. The first term in these expressions has, quite rightly, dominated the study of fracture mechanics over the last ten years. However, the higher order terms also have an influence on fracture.

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(a) *Effect of coefficient* a_2

The second term in the symmetrical stress distribution represents a direct stress σ_x in the local direction of the crack.

$$\sigma_x = a_2 \tag{4}$$

In a perfect isotropic elastic solid a crack will grow in the direction of the principal stress trajectory which passes through the crack tip. [1,2] However, in a real material there may be deviation from the perfect path caused by slight irregularities. The ideal direction for crack growth when the stress distribution is symmetrical is obviously along the line $\theta=0$. Let us assume that local irregularities cause the growth to occur along the line $\theta=d\theta$. At any stage in this growth, the ideal (or most probable path) can either be directed back towards the original ideal path or not. If the path returns towards the original ideal path, the crack will propagate in a zig-zag manner (see fig. 1a). This mode of crack growth was called

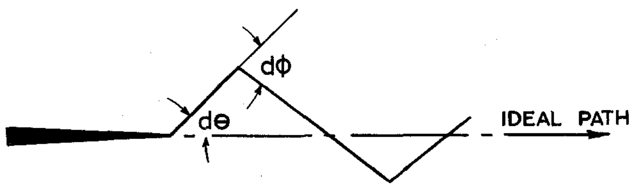

Fig. 1(a). Class I fracture.

Class I in a previous paper and was shown to be highly probable on the macroscopic scale, although the microscopic crack growth at any instant was not very certain. In Class II fracture (see fig. 1b) the most probable

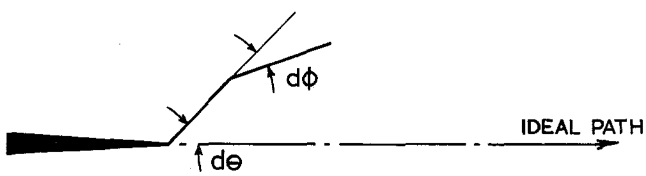

Fig. 1(b). Class II fracture.

crack path never returns to the original direction.

The most probable path for crack growth after a small deviation $d\theta$ will be again in the direction of the principal stress trajectory which passes through the new crack tip. The dominant term in the stress distribution at the new crack tip will be that due to the growing of the crack along the line $\theta=d\theta$ by the stresses.

$$(\sigma_\theta)_{\theta=d\theta} = \frac{a_1}{4}\left(\frac{\ell}{r}\right)^{\frac{1}{2}}\left[3\cos\frac{d\theta}{2} + \cos\frac{3d\theta}{2}\right] + a_2\sin^2 d\theta + 0(r^{\frac{1}{2}}) \tag{5}$$

$$(\sigma_{r\theta})_{\theta=d\theta} = \frac{a_1}{4}\left(\frac{\ell}{r}\right)^{\frac{1}{2}}\left[\sin\frac{d\theta}{2} + \sin\frac{3d\theta}{2}\right] - \frac{a_2}{2}\sin^2 d\theta + 0(r^{\frac{1}{2}}) \tag{6}$$

(see fig. 2), which for small $d\theta$ can be written

$$(\sigma_\theta)_{\theta=d\theta} = a_1\left(\frac{\ell}{r}\right)^{\frac{1}{2}} + 0(d\theta^2, r^{\frac{1}{2}}) \tag{7}$$

$$(\sigma_{r\theta})_{\theta=d\theta} = \left[\frac{a_1}{2}\left(\frac{\ell}{r}\right)^{\frac{1}{2}} - a_2\right] d\theta + 0(d\theta^2, r^{\frac{1}{2}}) \tag{8}$$

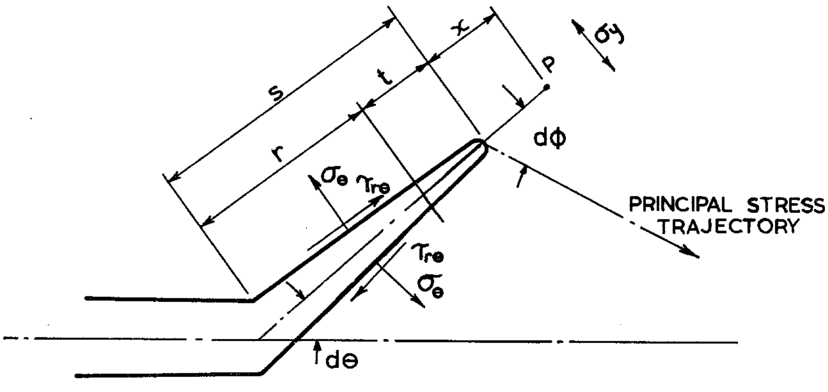

Fig. 2. Crack tip after deviation through angle $d\theta$.

If $d\theta$ is small, the bend in the crack can be ignored and the dominant stress distribution obtained by Muskhelishvili's method. The two analytic functions $\Phi(z)$ and $\Omega(z)$ that describe the stress distribution are[4]

$$\Phi(z) = \Omega(z) = -\frac{1}{2\pi i z^{\frac{1}{2}}(z-c)^{\frac{1}{2}}}\int_0^s \frac{t^{\frac{1}{2}}(t-c)^{\frac{1}{2}}}{(t-z)}\, p(t)\, dt \tag{9}$$

where the tips of the crack are located at $z=0$ and $z=c$ and $p(t) = [(\sigma_\theta)_{\theta=d\theta} + i(\sigma_{r\theta})_{\theta=d\theta}]$. The symmetric ($k_1$) and skew symmetric ($k_2$) stress intensity factors[2] are from equation (9).

$$k_1 + i\,k_2 = \frac{2^{\frac{1}{2}}}{\pi}\int_0^s \frac{\left[1 - \frac{t}{c}\right]^{\frac{1}{2}}}{t^{\frac{1}{2}}}\, p(t)\, dt \tag{10}$$

With the value of $p(t)$ given by equations (7) and (8) we have

$$k_1 + i\,k_2 = (2\ell)^{\frac{1}{2}}\left\{a_1 + i\left[\frac{a_1}{2} - \frac{2a_2}{\pi}\left(\frac{s}{\ell}\right)^{\frac{1}{2}}\right] d\theta\right\} + 0(s, d\theta^2) \tag{11}$$

Erodogan and Sih[2] show that the direction of the principal stress trajectory is given by

$$\cos\frac{\phi}{2}\left\{k_1 \sin\phi - k_2\,(3\cos\phi - 1)\right\} = 0. \tag{12}$$

Disregarding the trivial solution $\phi = \pi$ we have if $d\phi$ is small

$$k_1\, d\phi - 2k_2 = 0. \tag{13}$$

Substitution of the values of k_1 and k_2 (equation 11) into equation (13) yields, for small crack growths

$$d\phi = \left[1 - \frac{4}{\pi}\left(\frac{a_2}{a_1}\right)\left(\frac{s}{\ell}\right)^{\frac{1}{2}}\right] d\theta \tag{14}$$

Thus if a_2 is negative $d\phi > d\theta$ and the fracture path has the tendency to return to its original path, i. e. it is Class I fracture. If a_2 is positive $d\phi < d\theta$ and the path does not return to the original path, i. e. it is Class II fracture.

The sign of the coefficient a_2 can readily be determined from the isochromatic pattern obtained from a photoelastic model. In fig. 3 the iso-

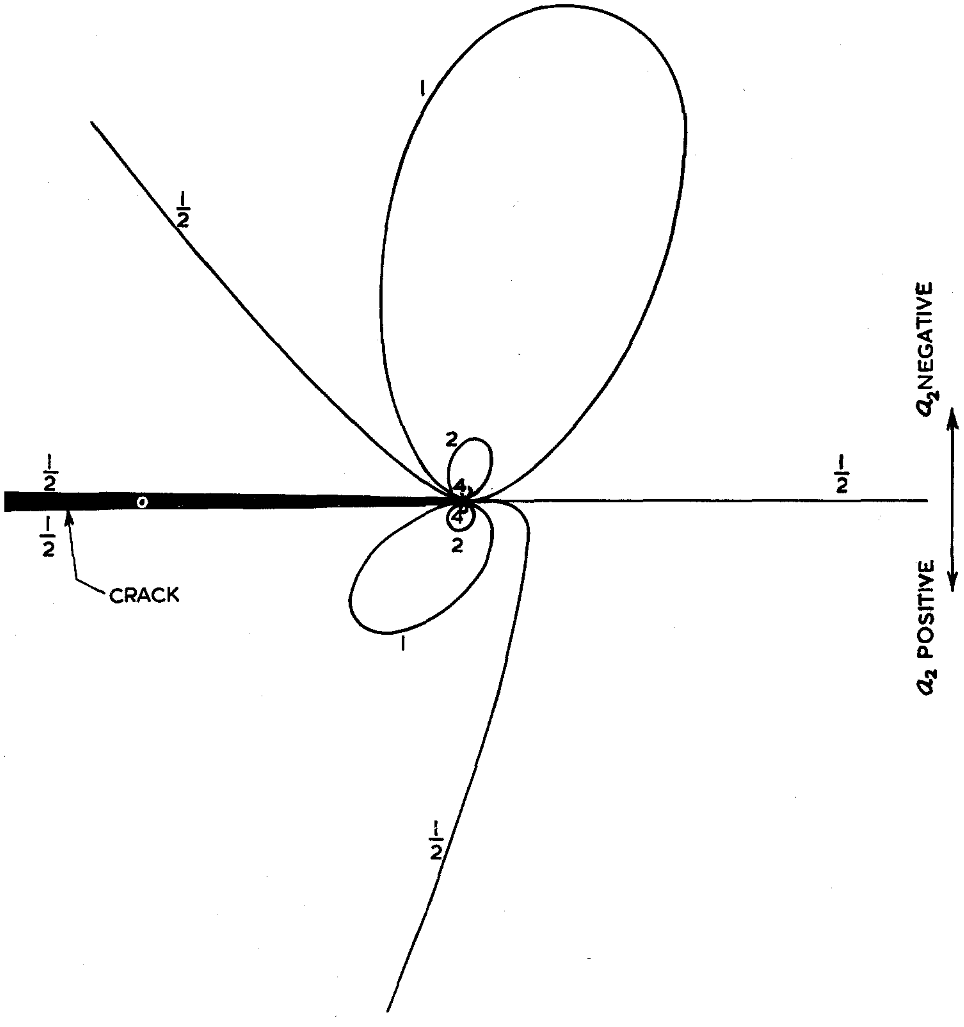

Fig. 3. Theoretical isochromatic pattern near tip of crack.

chromatic pattern has been calculated using only the first two terms in equations (1)-(3). The fringes either lean towards the direction of propagation (a_2 negative, Class I fracture) or they lean backwards (a_2 positive, Class II fracture). For the particular case when $a_2 = a_4 = \ldots a_{2n} = 0$ (straight crack in a biaxial stress field) the fringes will be symmetrical with respect to $\theta = \pi/2$ for small values of r.[3] The deviation from the symmetrical arrangement for larger values of r will depend on the term a_3.

When a brittle sheet with a center or edge crack is fractured by a load applied normally to the crack, the fracture is straight and symmetrical, provided that the crack velocity is not too great[1]. The isochromatic pattern at the tip of an edge crack loaded in such a manner is shown in fig. 4, The fringes lean forward, thus a_2 is negative and the fracture behaves as

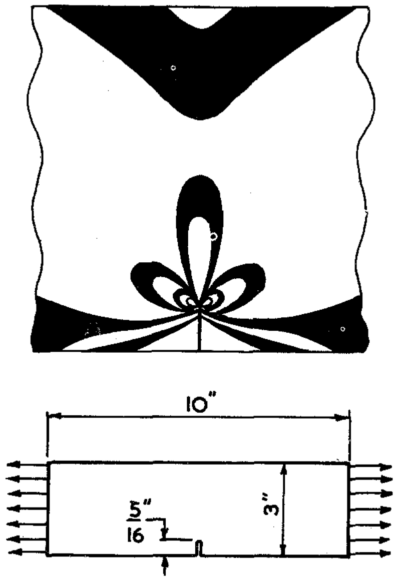

Fig.4. Isochromatic pattern of a tensile specimen with an edge crack.

predicted by the theory.

A cleavage type specimen has been used by several workers to measure the fracture toughness of plastics. Although the loading and the initial crack are perfectly symmetrical, the fracture will not of its own accord run along the line of symmetry. Berry[5] used a groove to control the fracture path. Benbow and Roesler[6] found it possible to control the fracture by introducing a compressive stress in the direction of the crack. A photoelastic study of the cleavage specimen by Guernsey and Gilman[7] showed that without a compressive stress the isochromatic fringes leaned backwards, (the results from a similar model are shown in fig. 5). Thus in

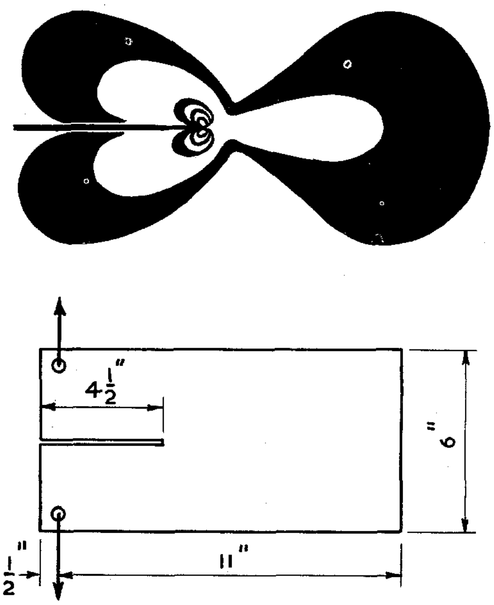

Fig.5. Isochromatic pattern of a cleavage specimen with a long crack.

this case a_2 is positive and, as predicted by the theory, the fracture is Class II and will not run along the line of symmetry. The compressive stress used by Benbow and Roesler to make the fracture run along the line of symmetry reverses the sign of a_2 and hence produces a Class I type of fracture.

It is not necessary for a fracture to belong to one class or the other for its entire history. If the initial crack in the cleavage specimen shown in fig. 5 is very short, the fringes lean forwards indicating Class I fracture. As the crack length is increased the fringes become more upright and finally, for cracks longer than about $\frac{3}{4}$ in., lean backwards as shown in fig. 5. The deviation from the line of symmetry for cracks of various initial lengths is shown in fig. 6. Only those fractures initiated from initial

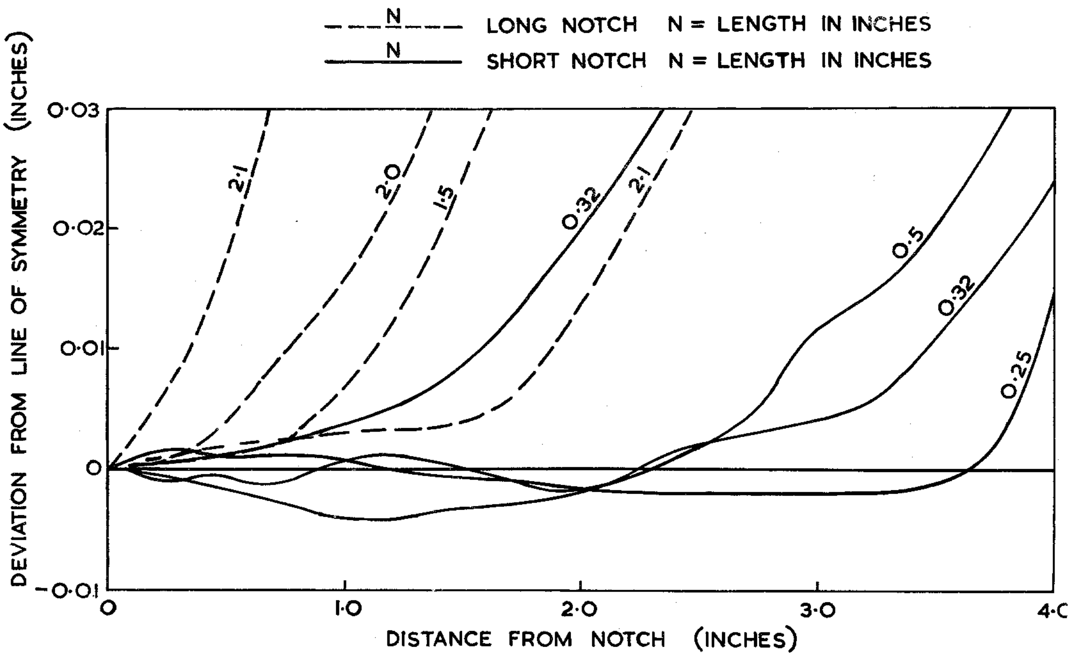

Fig. 6. Crack paths for cleavage specimens of various original crack lengths.

cracks short enough to cause the isochromatic fringes to lean forwards run straight for any appreciable distance before deviating to one side or another.

(b) *Effect of coefficient* a_3

A fracture is unstable if the crack extension force, G, increases with crack length, ℓ, without any change in the external conditions. Since $G \propto k^2$ the condition for instability can be written

$$\frac{dk}{d\ell} > 0 \tag{15}$$

Assume that the surface tractions T_i are specified on part of the surface Σ_1 and the displacements specified on the surface Σ_2. If the surface tractions on the surface Σ_2 can be expressed as $T_i = \lambda(\ell) f(\gamma)$ (where $\lambda(\ell)$ is a function of the surface coordinates γ) and for small crack extensions the functional form of λ does not change, then the gradient of the stress intensity factor can be written as

$$\frac{dk}{d\ell} = \frac{\partial\lambda}{\partial\ell}\frac{\partial k}{\partial\lambda}\Bigg|_{\substack{T_i \text{ constant on} \\ \Sigma_1,\ \ell \text{ constant}}} + \frac{\partial k}{\partial\ell}\Bigg|_{\substack{T_i \text{ constant on} \\ \Sigma_1 \ \&\ \Sigma_2}} \tag{16}$$

The first term in equation (16) cannot be deduced from knowledge of the stress distribution at any one particular crack length. However, the second term is a function of the coefficients a_1 and a_3. Only in the case of the body with the surface tractions specified over the entire surface can the stability of the fracture be deduced from the stress distribution at any one particular crack length. Since this condition is fairly common in practice it is thought useful to calculate $\partial k/\partial\ell$, the variation of stress intensity factor with crack length for constant surface tractions.

Assume that a crack is propagating in the direction of the principal stress trajectory so that the stress acting across the prolongation of the crack is

$$\sigma_y = a_1\left(\frac{\ell}{r}\right)^{\frac{1}{2}} + a_3\left(\frac{r}{\ell}\right)^{\frac{1}{2}} + 0(r) \tag{17}$$

If the crack propagates a small distance s, the dominant term in the stress distribution at the new crack tip will be due to the opening of the crack by the stresses of equation (17). The stress intensity factor at the new crack tip can be calculated from equation (10) and is

$$k = (2\ell)^{\frac{1}{2}}\left\{a_1 + \frac{1}{2}\left(\frac{s}{\ell}\right)\left[a_3 - \frac{a_1}{2}\right]\right\} \tag{18}$$

This equation is strictly true only for a straight crack, because only the first term in the equation for the stress distribution at the tip of a crack is valid for a crack of any shape. Equation (18) contains the first two terms of the Taylor expansion for the stress intensity factor when the crack is $\ell + s$ in length. Thus the gradient of the stress intensity factor is

$$\frac{\partial k}{\partial\ell} = \frac{1}{(2\ell)^{\frac{1}{2}}}\left[a_3 - \frac{a_1}{2}\right] \tag{19}$$

(c) *The effect of coefficient* a_4

The odd order terms in the expansion of the stress distribution at the tip of a crack produce a biaxial stress on the prolongation of the crack. This biaxial stress has been suggested as one of the reasons for the embrittling effect of a notch in a metal, because if there is restraint in the third direction (which will be the case if the ratio of the root radius of the notch to the plate thickness is small) the state of stress at the notch will be almost triaxial and there will be little relief of the stress by plastic flow. However, the odd order terms produce a uniaxial stress in the direction of the prolongation of the crack and form the major contribution to the octahedral shear stress on the prolongation of the crack. The maximum shear stress, in the plane of the plate, along the prolongation of the crack is

$$\tau = \frac{a_2}{2}\left\{1 + \frac{a_4}{a_2}\ \frac{r}{\ell}\right\} + 0(r^{3/2}) \tag{20}$$

Thus the sign of the ratio a_4/a_2 determines whether the maximum shear stress increases or decreases with distance from the crack tip. In the fracture of metals this may indicate a dependence on the coefficients a_2 and a_4. However, Sullivan[8] in a discussion of a paper by Manjoine[9] discounts the effect of coefficient a_2. She does not discuss the effect of coefficient a_4.

The sign of the ratio a_4/a_2 affects a photoelastic pattern. If a_4/a_2 is positive, as it is in the case of a rectangular sheet with a center or edge crack loaded normally to the crack, the isochromatic loop attached to the tip of the crack pointing in the direction of the crack growth (see fig. 4) collapses towards the crack tip as the load is increased, and another loop is formed. In the cleavage specimen, if the crack is long, the isochromatic loop in the direction of the prolongation of the crack grows as the load increases (see fig. 5) indicating that the ratio a_4/a_2 is negative.

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RÉSUMÉ - Le champ des containtes de tension à l'extrémité d'une fissure peut être développé en séries. Le premier terme, qu'on appelle ordinairement le facteur d'intensité de tension, determine l'amorçage de la fracture dans un matériel fragile. Dans cet article on montre que le deuxième, troisième et quatrième termes ont les effets suivants:

a) le second terme controle la stabilité de la direction de la fissure.
b) le troisième terme controle la stabilité de la propagation de la fissure.
c) le quatrième terme détermine si la valeur maximale de la contrainte de cisaillement sur la prolongation de la fissure augmente ou diminue par rapport à la distance avec l'extrémité de la fissure.

ZUSAMMENFASSUNG - Das Spannungsfeld einer Riss-spitze kann als eine Potenzreihe geschrieben werden. Das erste Glied ist gewöhnlich als Spannungsintensitätsfaktor bekant und bestimmt die Bildung des Bruchs in spröden Materialen. Dieser Beitrag zeigt das die nächsten drei Glieder die folgenden Wirkungen haben:

a) Das zweite Glied befehlt die Stabilität der Richtung des Risses,
b) Das dritte Glied befehlt die Stabilität der Fortpflanzung des Risses,
c) Das vierte Glied entscheidet ob die maximale Schubkraft auf der Verbängerung des Risses mit Abstand von der Riss-spitze sich vermindert oder verstärkt.