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Notes on Craven's conjecture

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Abstract. Craven's conjecture says that 2^{n-1} is the maximum number of linear orders on $\{1,2,\ldots,n\}$ that simultaneously satisfy certain restrictions on three-element subsets of $\{1,2,\ldots n\}$. This is true for $n=3$, but the maximum exceeds 2^{n-1} for $n \geqq 4$. There is a set of nine linear orders on $\{1, 2, 3, 4\}$ that satisfy the restrictions. As n gets large, the ratio of the size of the maximum satisfying set to 2^{n-1} approaches infinity.

In a recent column, Kelly [3] observed that for every $n \geqq 3$, there is a set of 2^{n-1} linear (strict, strong) orders on $\{1, 2,\ldots, n\}$ which simultaneously satisfy the conditions of Sen and Pattanaik [4] (see also [1, 2]) on triples in $\{1, 2,\ldots, n\}$ which guarantee that in every profile of linear orders thus restricted every nonempty subset of $\{1, 2,\ldots, n\}$ has an element which beats or ties every other element in the subset under simple majority comparisons. Kelly attributes to John Craven the conjecture that 2^{n-1} is best possible, i.e., if S is any set of more than 2^{n-1} linear orders on $\{1, 2,\ldots, n\}$ then the Sen-Pattanaik restrictions must be violated for some three-element subset of $\{1, 2,\ldots, n\}$ with respect to S.

We note here that Craven's conjecture is false for every $n \geqq 4$. In fact, an ostensibly weaker conjecture is false for all $n \geqq 4$. The weaker conjecture is obtained from the original by invoking only the constraint of value restriction on triples from the set of conditions (value restriction, limited agreement, extremal restriction) specified by Sen and Pattanaik.

For each $n \geqq 3$ let $f(n)$ denote the maximum number of linear orders in a set S of linear orders on $\{1, 2,\ldots, n\}$ that simultaneously satisfy value restriction: that is, for every ordered triple of integers (i, j, k), $1 \leqq i < j < k \leqq n$, there is an $x \in \{i, j, k\}$ such that either x is never first, or never second, or never third, among $\{i, j, k\}$ in the orders in S. The constructions of Craven and Kelly show that $f(n) \geqq 2^{n-1}$, and it is easily checked that $f(3) = 2^{3-1} = 4$. If Craven's conjecture were true, it would imply $f(n) = 2^{n-1}$.

Theorem 1. $f(4) \geqq 9$.

Proof. Let S_1 consist of the nine orders

1234 1432 4123

1243 2134 4132

1423 2143 4312 .

Then 1 is never third among {1, 2, 3} and never third among {1, 2, 4}, and 3 is never first among {1, 3, 4} or among {2, 3, 4}. Hence S_1 satisfies value restriction. □

If a fifth element is inserted into the orders of S_1 in next-to-last place and then in last place, we get 18 orders for $n = 5$ that satisfy value restriction. Similar insertions for a sixth element, then a seventh element,..., show that

$$f(n) \geqq (9/8)\, 2^{n-1} \quad \text{for all} \quad n \geqq 4 \ .$$

However, we can say more.

Theorem 2. *As n gets large,* $f(n)/2^{n-1} \to \infty$.

Proof. We note shortly that

$$f(2^m) \geqq 2^{2^{m-2}-1}\, 3^{2^{m-1}} \quad \text{for all} \quad m \geqq 2 \ .$$

It follows from this that

$$f(2^m)/2^{2^m-1} \geqq (9/8)^{2^{m-2}} \ .$$

The approach of the preceding paragraph then gives

$$f(n)/2^{n-1} \geqq (9/8)^{2^{m-2}} \quad \text{when} \quad 2^m \leqq n < 2^{m+1} \ ,$$

so $f(n)/2^{n-1} \to \infty$.

The initial lower bound on $f(2^m)$ is derived by an iterative procedure. For the first step, let S_1 be given as in the proof of Theorem 1, let $\{1^*, 2^*, 3^*, 4^*\}$ be a disjoint copy of {1, 2, 3, 4}, and let S_1^* consist of the nine orders in S_1 with asterisks superscribed. Let S_2 consist of the $2 \times 9 \times 9$ linear orders on

$$A = \{1, 2, 3, 4\} \cup \{1^*, 2^*, 3^*, 4^*\}$$

that either begin with an S_1 order followed by an S_1^* order (e.g. 12342*1*4*3*) or begin with an S_1^* order followed by an S_1 order (e.g. 4*3*1*2*2143).

Suppose $\{a, b, c\}$ is a triple in A. If $\{a, b, c\} \subset \{1, 2, 3, 4\}$ or $\{a, b, c\} \subset \{1^*, 2^*, 3^*, 4^*\}$, then it obviously satisfies value restriction in S_2. Suppose $\{a, b, c\}$ contains elements from both $\{1, 2, 3, 4\}$ and $\{1^*, 2^*, 3^*, 4^*\}$. Assume with no loss of generality that $a \in \{1, 2, 3, 4\}$ and $b, c \in \{1^*, 2^*, 3^*, 4^*\}$. Then, with respect to $\{a, b, c\}$, a is first or third in every order in S_2, so it is never second. We conclude that every triple in A satisfies value restriction, so

$$f(2^3) \geqq |S_2| = 2 \times 3^4 \ .$$

For the second iteration let A^* be a disjoint copy of A, so $|A \cup A^*| = 2^4$, and let S_2^* consist of the $|S_2|$ orders on A^* that duplicate those in S_2 using the elements in A^* in place of those in A. Let

$$S_3=\{XY\colon \text{either } X\in S_2 \text{ and } Y\in S_2^*, \text{ or } X\in S_2^* \text{ and } Y\in S_2\} .$$

The arguments of the preceding paragraph applied to A in place of $\{1,2,3,4\}$ and A^* in place of $\{1^*,2^*,3^*,4^*\}$ show that every triple in $A\cup A^*$ satisfies value restriction in S_3. Therefore

$$f(2^4)\geqq |S_3|=2|S_2|^2=2^3 3^8 .$$

Further iterations yield $f(2^5)\geqq 2|S_3|^2$, $f(2^6)\geqq 2|S_4|^2,\ldots$, which are tantamount to the initial inequality of this proof for $m\geqq 5$. □

The problem of determining $f(n)$ for all n seems daunting. It is true but tedious to show that $f(4)=9$, and $f(n)$ could probably be determined for the next few n with computer assistance. I cannot suggest a good upper bound on f, and the lower bound in the preceding proof is much too small. For example, at $m=8$ that proof gives

$$f(256)/2^{255}\geqq(9/8)^{64}=1878.28\ldots ,$$

but the iterative procedure of the next few paragraphs yields

$$f(256)/2^{255}\geqq 3^{170}/2^{255}=22282.53\ldots .$$

We begin again with S_1. For each $i\in\{1,2,3,4\}$ let B_i be a four-element set with $B_i\cap B_j=\emptyset$ whenever $i\neq j$, and let T_i be a set of nine linear orders on B_i such that $T_i\approx S_1$, i.e., T_i is isomorphic to S_1 under relabeling. Let $B=\cup B_i$, so $|B|=16$. Define S_2' as the set of 9^5 linear orders on B formed by replacing each i in an order in S_1 by any one of the 9 orders in T_i. For example, order $2143\in S_1$ generates the 9^4 orders in

$$\{X_2X_1X_4X_3\colon X_i\in T_i \quad\text{for}\quad i=1,2,3,4\} .$$

Each of the other 8 orders in S_1 generates a different 9^4 orders for S_2'.

It is easily seen that each triple $\{a,b,c\}$ in B satisfies value restriction within S_2'. If $\{a,b,c\}\subseteq B_i$, this follows from $T_i\approx S_1$. If each of a, b and c is from a different B_i, value restriction holds for the same reason it held for S_1. And if two of a, b and c are in B_i and the third is in B_j, $j\neq i$, then the third is never second among $\{a,b,c\}$ in an order of S_2'. Hence

$$f(2^4)=f(16)\geqq|S_2'|=9^5 .$$

For convenience, relabel the 16 elments in B as $1,2,\ldots,16$. To repeat the above process, for each $i\in\{1,2,\ldots,16\}$ let C_i be a 16-element set with $C_i\cap C_j=\emptyset$ whenever $i\neq j$, and let U_i be a set of 9^5 linear orders on C_i with $U_i\approx S_2'$. Let $C=\cup C_i$, so $|C|=2^8=256$. Define S_3' as the set of $(9^5)^{17}$ linear orders on C formed by replacing each $i\in\{1,2,\ldots,16\}$ in an order in S_2' by any one of the 9^5 orders in U_i. As before, value restriction holds within S_3', so

$$f(2^8)\geqq|S_3'|=(9^5)^{17}=3^{170} ,$$

as claimed earlier.

The process continues in the manner indicated: the next step gives

$$f(2^{16})\geqq(3^{170})^{257}=3^{43690} .$$

It can be checked that the ratio of the lower bound thus generated to the corresponding bound from the proof of Theorem 2 for $n = 2^{2^m}$ approaches ∞ as n gets large.

The challenge remains to specify tight bounds on $f(n)$ for all n.

References

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