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High Velocity Flow in Porous Media

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Abstract. Experimental observations have established that the proportionality between pressure head gradient and fluid velocity does not hold for high rates of fluid flow in porous media. Empirical relations such as Forchheimer equation have been proposed to account for nonlinear effects. The purpose of this work is to derive such nonlinear relationships based on fundamental laws of continuum mechanics and to identify the source of nonlinearity in equations.

Adopting the continuum approach to the description of thermodynamic processes in porous media, a general equation of motion of fluid at the macroscopic level is proposed. Using a standard order-of-magnitude argument, it is shown that at the onset of nonlinearities (which happens at Reynolds numbers around 10), macroscopic viscous and inertial forces are negligible compared to microscopic viscous forces. Therefore, it is concluded that growth of microscopic viscous forces (drag forces) at high flow velocities give rise to nonlinear effects. Then, employing the constitutive theory, a nonlinear relationship is developed for drag forces and finally a generalized form of Forchheimer equation is derived.

Key words. Porous media, high velocity flow, non-Darcy flow, Forchheimer equation, inertial effects, constitutive equations.

1. Nomenclature

a	coefficient in Equations (1) to (3) and (22); also a_1 and a_2 in (23) and (24)
b	coefficient in Equations (1) to (3) and (22); also b_1 and b_2 in (23) and (24)
c	coefficient in (2); also c_1 and c_2 in (23)
d	coefficient in (3)
da	microscopic infinitesimal element of area
d_{kl}	deformation rate tensor, $\frac{1}{2}(v_{k,l} + v_{l,k})$
e_1	coefficient in (24)
E_{KL}	solid-phase deformation tensor
g_k	gravity vector
l	microscopic (pore) characteristic length of the porous medium
L	macroscopic characteristic length of the porous medium
M	specific surface of the solid phase

n_l^{fs}	unit vector normal to the fluid-solid interfaces
p	(macroscopic) thermodynamic pressure
p'	microscopic (pore) pressure
q	order of magnitude of flow velocity
R	coefficient in Equation (20), also R_{kl}, R_{klm}, and R_{klmn} in (14)
Re	Reynolds number defined as $\rho q l/\mu$
t'_{kl}	microscopic fluid stress tensor
T	characteristic time, assumed to be equal to L/q
$\hat{T}_k$	solid-fluid interfacial drag force
v	magnitude of velocity in Equations (1) to (3)
v_k	macroscopic fluid velocity vector
$\tilde{v}_k$	deviation of pore velocity from average velocity, $\tilde{v}_k = v'_k - v_k$
v_k^d	macroscopic velocity of fluid relative to the solid
z_k	an arbitrary objective vector in Equations (15) to (20)
Z	a scalar product defined in (15)

Greek

δ_{kl}	Kronecker delta
δV	volume of the representative element of volume (REV)
δA^{fs}	solid-fluid interfacial surfaces within an REV
ϵ	porosity
θ	temperature
μ	(microscopic) fluid viscosity
ξ_a	a set of invariants defined in (18)
ρ	macroscopic fluid density
ρ'	microscopic fluid density
$\hat{\tau}_k$	dissipative part of $\hat{T}_k$
τ_{kl}	dissipative part of macroscopic fluid stress tensor

Special Notation

$\langle\ \rangle$	averaging sign
$\mathcal{O}[\ ]$	order of magnitude

2. Introduction

The equation most widely used for describing the flow of a fluid through a saturated porous medium is the well-known Darcy's law. This is a reduced form of the equation of fluid motion which predicts a linear relationship between the fluid velocity relative to the solid and the pressure head gradient.

Over the years, Darcy's law has been extended to more generalized forms in order to describe more complex flow situations. In addition, it has been observed that the proportionality between head gradient and fluid velocity does not hold for high rates of fluid flow. This phenomenon has been the subject of many experimental and theoretical investigations. These studies have centered upon two important issues: (i) establishing an upper bound for the range of validity of

Darcy's equation and providing (generalized) relationships which predict the nonlinear flow behavior properly, (ii) providing a physical basis for the generalized equation of motion and identifying mechanisms which are responsible for the nonlinear flow behavior. The latter issue is addressed in this paper.

A macroscopic balance of forces containing all mechanisms relevant to the fluid flow is presented as the starting point. Then an order-of-magnitude argument is employed to illustrate that at the onset of nonlinear flow behavior, the solid-fluid drag force is much larger than inertial and viscous stress forces. Finally, based on principles of continuum mechanics and the general equation of the balance of forces, a physical basis for the high-velocity flow equations found in the literature is provided. The generality of the approach allows one to avoid making restrictive assumptions about the pore geometry, fluid compressibility, microscopic velocity distribution, rigidity of the medium, and so forth.

3. Previous Works on High Velocity Flow

Many laboratory and numerical studies have been devoted to the determination of the upper range of validity of Darcy's law. Customarily, this limit has been signified by means of a critical value for the Reynolds number (defined by $\mathrm{Re} = \rho q l/\mu$) beyond which the head gradient is no longer proportional to the flow velocity. Critical values of Re at the onset of nonlinear flow, according to most experiments, range between 1 and 15. Typical values given by various investigators are 1 by Tek (1957), 2 by Wright (1968), 5 by deVries (1979), and 1–10 by Dybbs and Edwards (1984). Also, numerical experiments consisting of the solution of Navier–Stokes equations in the pore space of an idealized porous medium give values of Reynolds number between 5 and 13 (e.g., Stark, 1972; Couland *et al.*, 1986).

Many investigators have developed nonlinear relationships between the head gradient and the flow velocity following different approaches. Various classes of approaches may be identified and are discussed here. More extensive reviews of the subject exist in the works of Scheidegger (1974), Bear (1972), and Hannoura and Barends (1981).

Some of the theories developed to account for nonlinear effects in porous media flow have relied on descriptive models of a more or less intuitive or empirical nature. Empiricism, fortified with dimensional analysis and other theoretical considerations, forms the basis of these theories. Early theories of high velocity flow could be classified within this group of models. The first equation of motion to account for nonlinear effects was proposed by Forchheimer (Bear, 1972) who suggested the following one-dimensional forms:

$$-\Delta p/\Delta x = av + bv^2 \tag{1}$$

$$-\Delta p/\Delta x = av + bv^2 + cv^3 \tag{2}$$

where p is the fluid pressure, v is the magnitude of the flow velocity, and a, b, and c are constants. Forchheimer postulated Equation (1) by analogy with pipe flow. A generalization of (1) to include unsteady-state effects was proposed by Polubarinova-Kochina (1959) in the following form:

$$-\Delta p/\Delta x = av + bv^2 + d\frac{\partial v}{\partial t} \tag{3}$$

Much work has been done to obtain correlations for a, b, c and d in terms of microscale properties such as grain size and shape, porosity, viscosity, etc. (see, e.g., Tek, 1957; Geertsma, 1974).

A number of empirical correlations have been given between the Reynolds number and a friction factor accounting for the resistance of a porous medium to the flow of fluids. For example, using this approach Sunada (1965) obtained a one-dimensional nonlinear flow equation similar to (1).

Some theoretical derivations of (1) are based on averaging the Navier–Stokes equation. For example, Irmay (1958) averages the Navier–Stokes equation for a model of spheres of equal diameters representing a homogeneous isotropic medium. The flow is assumed to be macroscopically one-dimensional. Employing certain assumptions and heuristic arguments, he arrives at Equation (3) with a, b, and d given in terms of the fluid and medium properties such as viscosity, density, grain size, and two shape factors. A similar approach has been adopted by Ahmed and Sunada (1969) and Dullien and Azzam (1973). Also, in a recent article Cvetković (1986) averages the general form of the linear momentum balance to obtain a macroscopic equation of motion. He treats the average of the microscopic convective term in a special way such that in his averaged equations, he obtains a term which is of second order in the flow velocity. Furthermore, he employs a linear constitutive relation for the interfacial drag force and arrives at a generalized momentum equation for unsteady flow including nonlinear velocity terms.

Still some other theories of nonlinear flow are based on an idealized geometrical description of a porous medium simple enough to allow the governing continuum equations to be solved. An example of this type of model is that of Blick (1966) which represents a porous medium as a bundle of parallel capillary tubes filled with fluid and with orifice plates spaced along each tube. A static balance of forces is applied to obtain an equation of the Forchheimer's type when the medium is assumed rigid and the fluid is considered to be homogeneous and Newtonian. Another interesting example is the work of Barak and Bear (1981) in which they consider five different 'physical models' with various degrees of complexity. Using these models, they obtain nonlinear relationships between the pressure gradient and the flow velocity and compare these equations with generic mathematical models and experimental results.

Opinions on the mechanism(s) responsible for the onset of nonlinearity at high flow velocities are also diverse. Early descriptions of high velocity flow have attributed nonlinearity to the occurrence of turbulence. The definition and employment of the Reynolds number to signify the onset of 'non-Darcian flow' stems from this perception. Although, deviations from Darcy's law have been observed at Reynolds numbers of order 10, experiments have indicated that the onset of turbulence occurs at much higher velocities. For example, values up to 300 have been reported (Dybbs and Edwards, 1984). Thus, one can conclude that deviations from Darcy's law are not initiated by turbulence. Indeed, such deviations do not necessarily correspond to a different regime of flow. Thus, the term 'non-Darcian flow' should not be used to imply a particular type of flow. Firoozabadi and Katz (1979) have previously expanded upon this point and suggest the term 'high-velocity flow' be used for describing the situation where nonlinear effects become significant.

Another point of view attributes deviations from Darcy's law to the effect of microscopic inertial forces. This point of view has been widely accepted among authors in porous media flow (e.g., Schneebeli, 1955; Bear, 1972; Hubbert, 1956; Scheidegger, 1974; Geertsma, 1974; Happel and Brenner, 1965; MacDonald *et al.*, 1979; Cvetković, 1986).

Finally, a third point of view attributes the rise of nonlinear terms to the effects of increased microscopic drag forces on the pore walls (e.g., Firoozabadi and Katz, 1979; Slattery, 1972). The derivation of Forchheimer-type equations presented herein supports this point of view.

From the brief discussion of models on high velocity flow, it may be deduced that present approaches to this problem are diverse. They are often restricted, through assumptions and constitutive relations, to certain materials and/or special cases (e.g., incompressible, steady-state, one-dimensional, etc). Thus, their general validity is questionable and, usually, the steps required to extend the results to more general situations are rather obscure. All in all, it seems that a rational and systematic framework within which the mechanics of flow through porous media could be advanced might prove useful.

4. General Equation of Motion in Porous Media

Hassanizadeh and Gray (1979a,b, 1980) have developed a continuum approach to the study of thermodynamic processes in porous media. Using a systematic averaging technique, they obtain a set of macroscopic equations of balance by averaging microscopic balance laws of continuum mechanics. Constitutive equations are introduced at the macroscopic level of observation where the behaviors of materials are normally studied. The combination of balance equations and constitutive relations furnishes general field equations of mass, momenta, and energy which may account for the various phenomena occurring in porous media. The resulting equations may further be linearized and/or simplified by restrictive assumptions in order to apply to special cases of practical interest. Here, this approach is employed to illustrate the theoretical basis of Forchheimer's equation.

Consider the macroscopic equation of momentum conservation for a fluid flowing in a saturated deformable porous medium with no phase change (Hassanizadeh and Gray, 1979b):

$$\epsilon\rho\frac{\partial v_k}{\partial t}+\epsilon\rho v_l v_{k,l}=(\langle t'_{kl}\rangle-\langle\rho'\tilde{v}_k\tilde{v}_l\rangle)_{,l}+\epsilon\rho g_k+\hat{T}_k \tag{4}$$

where the primed variables are microscopic quantities, $\langle\quad\rangle$ denotes the phase average, $\tilde{v}_k=v'_k-v_k$ is the deviation of pore velocity from average velocity, and $\hat{T}_k$ is the solid-fluid interfacial drag force defined by

$$\hat{T}_k=\frac{1}{\delta V}\int_{\delta A^{\mathrm{fs}}}t'_{kl}n_l^{\mathrm{fs}}\,\mathrm{d}a. \tag{5}$$

The term $\langle\rho'\tilde{v}_k\tilde{v}_l\rangle$ is the volume average of the extraneous microscopic interial terms. At the macroscopic level, this term manifests itself as an apparent stress tensor. Based on thermodynamic considerations, Hassanizadeh and Gray (1980) have obtained the following constitutive equations:

$$\langle t'_{kl}\rangle-\langle\rho'\tilde{v}_k\tilde{v}_l\rangle=-\epsilon p\,\delta_{kl}+\tau_{kl}, \tag{6}$$

$$\hat{T}_k=p\epsilon_{,k}+\hat{\tau}_k, \tag{7}$$

where τ_{kl} and $\hat{\tau}_k$ are dissipative parts of the intra-phase fluid stress tensor and the solid-fluid drag force, respectively. They are objective functions of the following set of dependent variables.

$$\rho,\ \epsilon_{,k},\ E_{KL},\ d_{kl},\ \theta,\ \theta_{,k},\ v_k^d. \tag{8}$$

At equilibrium states, where d_{kl}, $\theta_{,k}$, and v_k^d are zero, τ_{kl} and $\hat{\tau}_k$ must identically vanish. Combination of Equations (4) to (7) yields the following field equation of motion.

$$\epsilon\rho\frac{\partial v_k}{\partial t}+\epsilon\rho v_l v_{k,l}-\tau_{kl,l}=-\epsilon(p_{,k}-\rho g_k)+\hat{\tau}_k \tag{9}$$

5. High Velocity Flow Equations

In general, Equation (9) contains nonlinearities arising from inertial, viscous, compressibility, thermal, anisotropy, and inhomogenity effects. But, for most commonly-encountered porous flow situations, one may neglect the left-hand side of this equation in comparison with the right-hand side. In this section, an order-of-magnitude analysis is employed to verify this statement even at high velocities where the nonlinear flow behavior is observed.

Note that $\tilde{v}_k$, v'_k, and t'_{kl} are microscopic quantities which vary over the pore scale of the medium, whereas v_k, $\langle\rho'\tilde{v}_k\tilde{v}_l\rangle$, and $\langle t'_{kl}\rangle$ (or τ_{kl}) are macroscopic quantities which vary over the macroscopic scale of the medium. Also, recall the fundamental relationship between the microscopic (pore) and macroscopic characteristic lengths (cf. Whitaker, 1969; Hassanizadeh and Gray, 1979a):

$$l \ll L \tag{10}$$

Further, the order of magnitude of t'_{kl} is given by $-p'\,\delta_{kl} + \mu v'_{(k,l)}$. Now, if the order of magnitude of the flow velocity is designated by q, the magnitudes of the terms in Equation (9) (or equivalently, Equation (4)) may be estimated as:

$$\mathcal{O}\left[\epsilon\rho \frac{\partial v_k}{\partial t}\right] = \frac{\rho q}{T} = \frac{\rho q^2}{L},$$

$$\mathcal{O}[\langle \rho \tilde{v}_k \tilde{v}_l \rangle_{,l}] = \frac{\rho q^2}{L},$$

$$\mathcal{O}[\epsilon \rho v_l v_{k,l}] = \frac{\rho q^2}{L},$$

$$\mathcal{O}[\langle t'_{kl} \rangle_{,l}] = \frac{\mu q}{lL} \quad \text{(excluding pressure order of magnitude)},$$

$$\mathcal{O}[\hat{T}_k] = \frac{\mu M q}{l} \quad \text{(excluding pressure order of magnitude)},$$

$$\frac{\text{Inertial Forces}}{\text{Interfacial Drag Forces}} = \frac{\text{Re}}{ML}, \tag{11}$$

$$\frac{\text{Macroscopic Stress}}{\text{Interfacial Drag Forces}} = \frac{1}{ML}. \tag{12}$$

Typically both M, the specific surface, and L have large values. It is known that the specific surface is inversely proportional to the characteristic pore length, l. For example, for a cubal packing of identical spheres of radius R, $M = \pi/2R$, (Bear, 1972). Therefore $1/ML \approx l/L$, and according to the fundamental requirement (10), this is a very small number. This indicates that macroscopic intrafluid stress terms normally will not be of importance in porous media flow. Only perhaps in the case of very coarse soils (with low values of M) and near medium boundaries (e.g., near walls of a packed tube) might they attain values comparable to drag forces.

To evaluate Re$/ML$, first recall that at the onset of nonlinear flow, Re is of the order of 10. Then, taking typical values of $M = 1.5 \times 10^2 - 2.2 \times 10^2\ \text{cm}^{-1}$ for sand (Bear, 1972), and assuming $L = 1$ meter, Re$/ML$ is of the order of 10^{-3}. That is, at the onset of nonlinear flow, inertial forces are still three order of magnitude smaller than interfacial drag forces. Therefore, inertial forces cannot be responsible for the *onset* of nonlinear flow in porous media. This does not mean that inertial forces will never become important. Perhaps for coarse grained soils and at a Reynolds number of about 100, where the flow is still laminar, inertial forces could become as important as drag forces. Then, the nonlinearity will be due to the combined effects of both inertial and drag forces.

Note that an order-of-magnitude analysis for the *microscopic* equation may yield one result for the relative importance of various *microscopic* terms.

However, when considering macroscopic phenomena, the relative importance of terms in the macroscopic equations must be analyzed.

Returning to Equation (9), the terms on the left-hand side appear to be negligible. Also, the soil deformation, E_{KL}, and the temperature gradient, $\theta_{,k}$, as well as the macroscopic velocity gradient, d_{kl}, are assumed to have negligible direct effect on the motion of the fluid. (They might, however, have indirect effects if the medium properties are considered to be temperature-dependent and the porosity varies with time and space.) Therefore, Equation (9) may be written as

$$\epsilon(p_{,k} - \rho g_k) = \hat{\tau}_k(\epsilon, \rho, \epsilon_{,k}, \theta, v_k^d). \tag{13}$$

Hassanizadeh and Gray (1980) have developed a linear constitutive equation for $\hat{\tau}_k$ which is valid only if terms of order of $|\epsilon_{,k}|^2$ and $|v_k|^2, \ldots$, are negligible compared to terms of order of $|\epsilon_{,k}|$, $|v_k|, \ldots$, respectively. Their end result, obtained after further simplifications, is the generalized Darcy's equation. Apparently, for high velocity, one should follow a similar procedure but retain second- (and higher-) order terms of velocity. As a result one may obtain a relation similar to that suggested by Cvetković (1986):

$$-\epsilon(p_{,k} - \rho g_k) = R_{kl} v_l^d + R_{klm} v_l^d v_m^d + R_{klmn} v_l^d v_m^d v_n^d \tag{14}$$

where R-coefficients are understood to be functions of ϵ, ρ, θ, and $\epsilon_{,k}$. Relations similar to this have been proposed and tested by Barak and Bear (1981). This equation is only suitable for anisotropic media. If the medium is isotropic and has a uniform porosity (i.e., $\epsilon_{,k} = 0$), then $R_{klm} = 0$ and the important second-order term disappears from the equation of motion. Therefore, some authors have argued that "the final form of the macroscopic linear momentum balance is not adequate for a theoretical interpretation of experimental results on non-Darcy flow" (Cvetković, 1986). However, this argument is, in fact, incorrect as will be demonstrated subsequently for isotropic media. To keep the development lucid, the dependence of $\hat{\tau}_k$ on $\epsilon_{,k}$ is neglected (in fact, for isotropic media, one would expect the porosity to be uniform).

Consider an arbitrary objective vector z_k. One can form the following scalar product between z_k and the vector $\hat{\tau}_k$:

$$Z = \hat{\tau}_k z_k . \tag{15}$$

From Equation (13), it follows that

$$Z = Z(\rho, \epsilon, \theta, v_k^d, z_k) \tag{16}$$

where

$$\hat{\tau}_k = \left.\frac{\partial Z}{\partial z_k}\right|_{z_k=0} \tag{17}$$

The scalar variable Z must be an objective function of its arguments. Therefore, Z can depend on v_k^d and z_k only through their scalar products which are listed below (Spencer, 1971):

$$\xi_1 = v_k^d v_k^d, \qquad \xi_2 = v_k^d z_k, \qquad \xi_3 = z_k z_k. \tag{18}$$

That is

$$Z = Z(\epsilon, \rho, \theta, \xi_a) \quad a = 1, 2, 3. \tag{19}$$

Now, substitution of (19) into (17) yields

$$\hat{\tau}_k = \sum_{a=1}^{3} \left[\frac{\partial \hat{Z}}{\partial \xi_a}\right]\left[\frac{\partial \xi_a}{\partial z_k}\right]\Bigg|_{z_k=0} = -R v_k^d \tag{20}$$

where

$$R = -\frac{\partial Z}{\partial \xi_2}\Bigg|_{z_k=0} = R(\rho, \epsilon, \theta, |v_k^d|). \tag{21}$$

The negative sign for R has been introduced to ensure that R will be a positive quantity (see Hassanizadeh and Gray, 1980). A series expansion of R in terms of v_k^d is now performed retaining terms of second order and smaller. Substitution of this result into (20) and (13) yields

$$-\epsilon(p_{,k} - \rho g_k) = (a + b|v_k^d| + c|v_k^d|^2) v_k^d \tag{22}$$

where coefficients a, b, and c are functions of ϵ, ρ and θ.

Clearly, Equation (22) is the three-dimensional analogue of Equation (2) suggested by Forchheimer. This equation is valid for high velocity isothermal flow of a macroscopically inviscid fluid through a uniform isotropic elastic porous medium.

If necessary, it is possible to go back and relax any of the assumptions made during the course of this development. For example, in the estimation of order of magnitude of terms, the characteristic time for the processes being considered, T, was assumed implicitly to be equal to L/q. If, however, smaller time scales are of interest, such that $\epsilon\rho(\partial v_k/\partial t)$ will have a higher order of magnitude, this term may be kept throughout the development and eventually a three-dimensional analogue of the Polubarinova-Kochina Equation (3) will be obtained. Also, if it is desirable to retain the temperature gradient or porosity gradient in the development, the following equations will be obtained, respectively:

$$\begin{aligned} -\epsilon(p_{,k} - \rho g_k) \\ = (a_1 + b_1|v_k^d| + c_1|\theta_{,k}|) v_k^d + (a_2 + b_2|v_k^d| + c_2|\theta_{,k}|)\theta_{,k} \end{aligned} \tag{23}$$

$$-\epsilon(p_{,k} - \rho g_k) = (a_1 + b_1|v_k^d| + c_1|\epsilon_{,k}|) v_k^d. \tag{24}$$

Note that in the latter equation, terms such as $(a_2 + b_2|v_k^d| + c_2|\epsilon_{,k}|)\epsilon_{,k}$ cannot be allowed because at equilibrium both v_k^d and $\hat{\tau}_k$ $(k = 1, 2, 3)$ must vanish identically, while $\epsilon_{,k}$ may remain nonzero.

6. Conclusion

The general continuum approach to the description of thermodynamic processes in porous media (Hassanizadeh and Gray, 1979a,b, 1980) has been employed to develop a nonlinear relationship between the pressure gradient and the flow velocity. The development is based on physical and mathematical principles and is carried out at the level of observations (the macroscopic level). Using a standard order-of-magnitude argument, microscopic inertial forces have been shown to be small at the onset of nonlinear flow ($\mathrm{Re} \approx 10$). The nonlinear dependence of interfacial drag forces on the flow velocity has been shown to give rise to the nonlinear behavior of flow at high velocities.

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