ON A POSSIBLE MODEL OF CRACK PROPAGATION IN A SOLID SUBJECTED TO A CYCLIC LOADING

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ABSTRACT

By the use of the Griffith model, the propagation of a crack in a solid subjected to a cyclic loading is studied. It is assumed that all the energy lost per cycle in the Bauschinger process is transferred to the crack to increase its size.

The paper is divided in two parts, the first dealing with materials which do not undergo work-hardening and the second with materials subjected to said phenomenon during cyclic loading. Notwithstanding the crude approximation of this model, it is shown that the main features of fatigue phenomena are accounted for.

INTRODUCTION

The purpose of this paper is to show that by the use of a very simple model it is possible (at least in a semi-quantitative way) to account for the main features of a crack propagation in a solid subjected to a cyclic loading.

In the present treatment the crack is considered in the Griffith's approximation and it is assumed that every cycle results in an energy loss due to the hysteresis phenomena. As is well known, in the Griffith's model a crack has an energy depending upon its length: it is assumed that all the energy gained by the material in a single hysteresis loop is transferred to the crack which, in this way, increases in size. This process lasts until when the crack reaches the critical radius consistent with the maximum applied stress. Naturally this is an oversimplification of the problem, since only a fraction of the energy gained per cycle actually contributes to the propagation of the crack, while the remaining part produces plastic flow and heat. Hence the theory is not expected to be in a good quantitative agreement with experiments, as the number of cycles to produce fracture is underestimated. However, the general features of this simple model allow one to explain some peculiarities of the fatigue phenomena.

In the present treatment a fundamental rôle is played by work-hardening, and it will be shown that a fatigue limit is closely connected with it.

The first section of this paper will be devoted to the case in which no work-hardening occurs, its introduction is considered in the second part of the present treatment.

CRACK PROPAGATION IN A SAMPLE WITHOUT WORK-HARDENING

For the sake of simplicity, and after Griffith [1], a disc-shaped crack of radius a is assumed to exist in an isotropic elastic medium of shear modulus μ and surface energy γ.

Under an applied stress σ the energy of the crack is easily obtained supposing that the defect relieves the stresses in a sphere of radius a. This produces the following change in the total energy

$$E = - \frac{2\pi}{3} \frac{\sigma^2}{\mu} a^3 + 2\pi\gamma a^2 \tag{1}$$

where $2\pi\gamma a^2$ is the surface energy of the solid.

The critical length for the stability of the crack, obtained as a solution of equation $\frac{\partial E}{\partial a} = 0$, is given by:

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$$a_c = \frac{2\gamma\mu}{\sigma^2} \tag{2}$$

In a recent paper[(2)], Marcus et al. have studied the Bauschinger effect in a cracked sample subjected to a cyclic loading between $-\sigma$ and $+\sigma$. They concluded that the energy gained by the crack during a hysteresis loop is very well approximated by

$$E_{hys.} \simeq \alpha \frac{a^3\pi^3\sigma^4}{3\mu\sigma_1^2} \tag{3}$$

when $\sigma \ll \sigma_1$. Here $\alpha = 1 - \nu$ (ν = Poisson ratio) and σ_1 is a frictional stress opposing the flow of the material. More precisely they assume that the stress distribution around an unloaded crack is that shown in fig. 1.

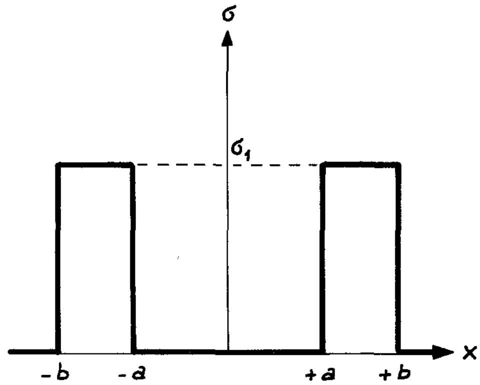

Fig. 1. Schematic behavior of the assumed stress field around an unloaded crack of radius a.

If the crack is thought of as given by a dislocation distribution between $x = -a$ and $x = a$, σ_1 is a sort of repulsive stress by which the dislocations themselves try to keep other defects of the same kind very far apart. By this interpretation it is easily concluded that σ_1 must be of the order of magnitude of shear modulus μ (because this is the magnitude of the stress at some spacings from the core of a dislocation). This conclusion is further supported by the fact that generally the hysteresis loop, corresponding, say, to a maximum load of the order of one half the yield stress, is still negligible, while in the above theory a value of $\sigma = \sigma_1$ gives an infinite loop.*

In our approximation it is assumed that during the n^{th} cycle the crack undergoes a change in its length such that the resulting energy variation is equal, eq. (3) to

$$E(a_n) - E(a_{n-1}) = E_{hys.}(a_{n-1}) \tag{4}$$

a_n being the crack radius at the end of the n^{th} cycle. As the number of cycles to failure is generally extremely high, the change in radius per cycle is expected to be very small, so that relation (4) may be substituted by the following differential equation:

* The asymptotic formula replacing (3) for σ near σ_1 is:

$$E_{hys.} = \frac{16\ \alpha\ a^3\pi\sigma}{\mu} \operatorname{tg} \frac{\pi}{2} \frac{\sigma}{\sigma_1}$$

which gives $E_{hys.} = \infty$ for $\sigma = \sigma_1$.

$$\frac{\partial E}{\partial a}\frac{da}{dn} = E_{hys.}(a) \tag{4b}$$

By introducing eqs.(1) and (3) into (4b) it is easy to obtain:

$$\frac{da}{dn}\left(-2\pi\frac{\sigma^2}{\mu}a^2 + 4\pi\gamma a\right) = \alpha\frac{a^3\pi^3\sigma^4}{3\mu\sigma_1^2} \tag{5}$$

Integrating the differential equation with the condition that $a = a_0$ for $n = 0$ one gets

$$\log\frac{a_0}{a} = \frac{2\mu\gamma}{a_0\sigma^2}\left(\frac{a_0}{a} - 1\right) + \frac{\alpha\pi^2}{6}\frac{\sigma^2}{\sigma_1^2}n \tag{6}$$

This relation gives the dependence of radius a on the number of cycles. The general features of this dependence can be easily obtained graphically. In fig. 2 both sides of eq.(6) are plotted as a function of $\frac{a_0}{a}$. Taking into account that the slope $\frac{2\mu\gamma}{a_0\sigma^2} = \frac{a_c}{a_0}$ of the straight line appearing in the right side is greater than unity, one can see solutions result only when $a > a_0$. Moreover, with reference to fig. 2, it is seen that the y - value of point

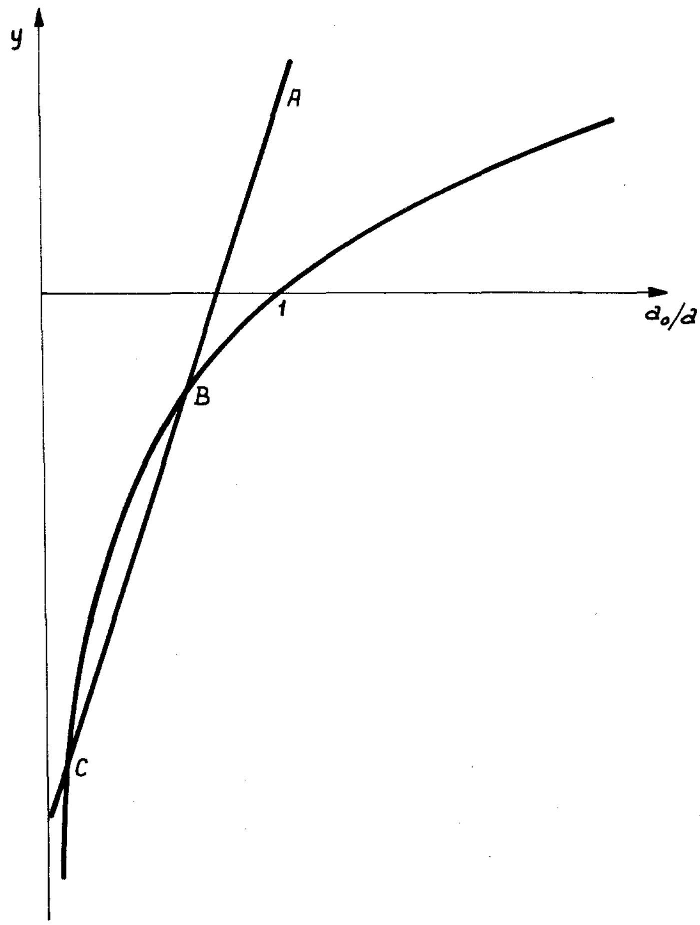

Fig. 2. Schematic representation of the functions $y = \log\frac{a_0}{a}$ and $y = \frac{a_c}{a_0}\left(\frac{a_0}{a} - 1\right) + C(n)$ necessary for the discussion of eqs.(6) and (9).

A increases indefinitely with n: hence, of the two possible solutions corresponding to points B and C, in fig. 2, only the former is acceptable on physical grounds: in fact it gives a as a monotonic increasing function of n, starting from $a = a_o$ for $n = 0$, while the latter solution corresponds to a non-physical crack with initial radius greater than a_c and decreasing in size with time. As n increases, the straight line AB moves to the left keeping a constant slope, and for a given value n^* of n it becomes tangent to the logarithmic curve. For $n > n^*$ no real solution of eq. (6) exists. This critical value n^* is easily interpreted as the number of cycles to failure. The value of radius a corresponding to it is obtained by taking the derivative of eq. (6) with respect to $\frac{a_o}{a}$ i.e.

$$a = \frac{2\mu\gamma}{\sigma^2} = a_c$$

As expected, the crack propagates until when it reaches the maximum size consistent with the applied stress σ, that is, the Griffith critical radius. Putting $a = a_c$ in eq. (6), and defining the static fracture load $\sigma_f^2 = \frac{2\mu\gamma}{a_o}$ one gets

$$\log\left(\frac{\sigma}{\sigma_f}\right)^2 + \left(\frac{\sigma_f}{\sigma}\right)^2 - 1 = \frac{\alpha\pi^2}{6}\left(\frac{\sigma_f}{\sigma_1}\right)^2\left(\frac{\sigma}{\sigma_f}\right)^2 n^* \tag{7}$$

This is the analytical equation of the fatigue curve. As the y-value of point A in fig. 2 increases indefinitely with n, whatever σ is, it is seen that the instability condition is always reached even when σ is very small. No fatigue limit is obtained in the present model. For very small stresses, we can neglect, in the left side of (7), unity and the logarithm term with respect to $(\sigma_f/\sigma)^2$ so that the asymptotic equation of the fatigue curve is given by

$$\sigma \simeq \left(\frac{12\mu\gamma}{a_o}\frac{\sigma_1^2}{\alpha\pi^2}\right)^{\frac{1}{4}} n^{*-\frac{1}{4}} = \left(\frac{6}{\alpha\pi^2}\sigma_1^2\sigma_f^2\right)^{\frac{1}{4}} n^{*-\frac{1}{4}} \tag{8}$$

In order to estimate the magnitude of the parameter σ_1 appearing in the present model, one can use as typical values in eq. (7), $\sigma = 30$ Kg/mm^2, $n^* \simeq 10^5$, $\sigma_{f/\sigma} \simeq 3$. In this way a value of $\sigma_1 \simeq 40\,\sigma_f = 3,600$ Kg/mm^2 is obtained. Thus σ_1 is of the same order of magnitude as the shear modulus μ, as expected.

3. EFFECT OF WORK-HARDENING

According to Marcus et al. (2), the simplest way to take work-hardening into account is to assume that σ_1 undergoes a variation $\Delta\sigma_1$ per cycle. For the sake of simplicity, this variation is supposed independent of n. In this model eq. (5) is replaced by

$$\frac{da}{dn}\left(-2\pi\frac{\sigma^2}{\mu}a^2 + 4\pi\gamma a\right) = \frac{\alpha a^3\pi^3\sigma^4}{3\mu(\sigma_1 + n\Delta\sigma_1)^2}$$

which can be easily integrated to give:

$$\log\frac{a_o}{a} = \frac{2\mu\gamma}{a_o\sigma^2}\left(\frac{a_o}{a} - 1\right) + \frac{\alpha\pi^2}{6}\frac{\sigma^2}{\sigma_1\Delta\sigma_1}\left(1 - \frac{1}{1 + n\frac{\Delta\sigma_1}{\sigma_1}}\right) \tag{9}$$

It is easy to see that this equation implies the existence of a fatigue limit. In fact, eq. (9) can be solved by the same graphical method as eq.(5). The only difference is that, with reference to fig. 2, the y-coordinate of point A approaches now a finite limit $\left(= \frac{\alpha\pi^2}{6} \frac{\sigma^2}{\sigma_1 \Delta\sigma_1} \right)$ when n goes to infinity. Since this limit increases with σ, it turns out that if σ is too small, the straight line cannot become tangent to the logarithmic curve, and no fracture is possible. In order to obtain the minimum value of σ allowing fracture, one has to put into eq. (9) $a = a_c$, $n = \infty$. In this way the equation defining the fatigue limit σ_L becomes

$$\log \left(\frac{\sigma_L}{\sigma_f} \right)^2 = 1 - \left(\frac{\sigma_f}{\sigma_L} \right)^2 + \frac{\alpha\pi^2}{6} \left(\frac{\sigma_f}{\sigma_1} \right)^2 \left(\frac{\sigma_1}{\Delta\sigma_1} \right) \left(\frac{\sigma_L}{\sigma_f} \right)^2 \tag{10}$$

For $\sigma > \sigma_L$, the fatigue curve corresponds to the equation

$$\log \left(\frac{\sigma}{\sigma_f} \right)^2 = 1 - \left(\frac{\sigma_f}{\sigma} \right)^2 + \frac{\alpha\pi^2}{6} \left(\frac{\sigma_f}{\sigma_1} \right)^2 \frac{\sigma_1}{\Delta\sigma_1} \left(\frac{\sigma}{\sigma_f} \right)^2 \left(1 - \frac{1}{1 + n^* \frac{\Delta\sigma_1}{\sigma_1}} \right) \tag{11}$$

In the asymptotic region where σ is near σ_L, one has approximately

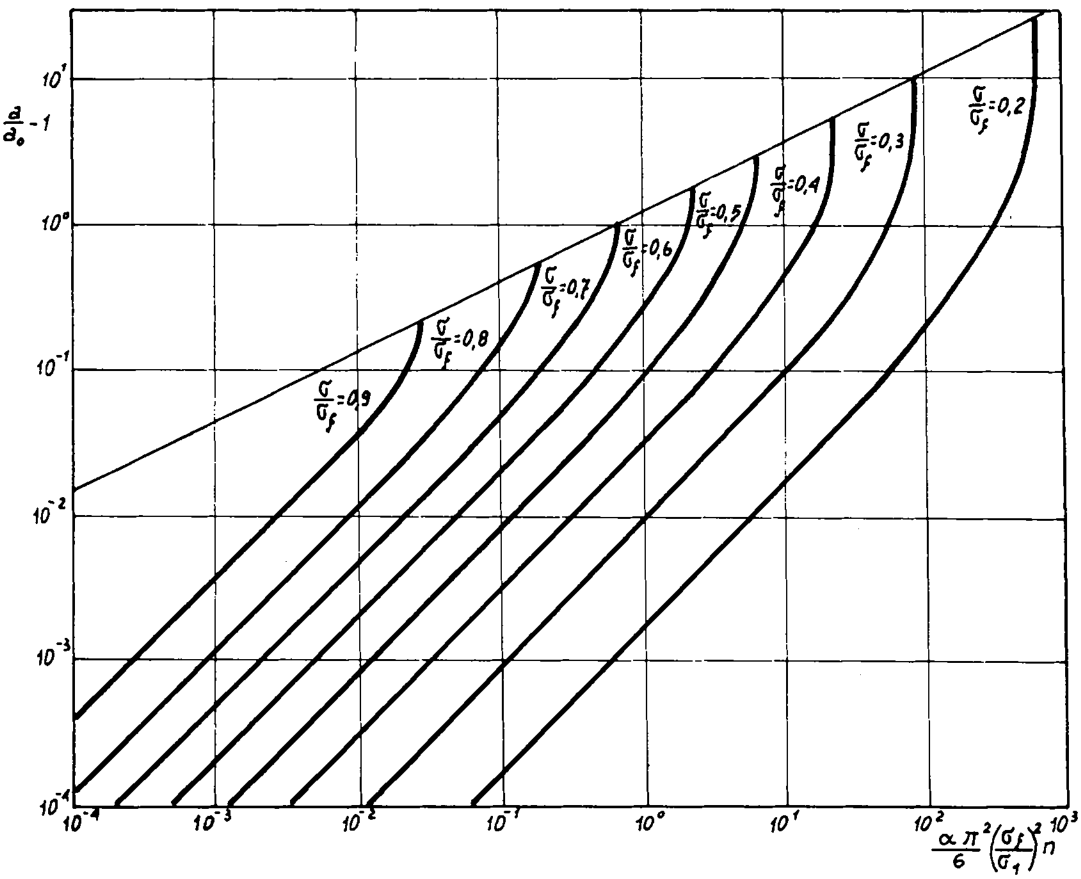

Fig. 3. Values of the relative increase of the crack radius as a function of the cycle number n for some values of the maximum applied stress σ.

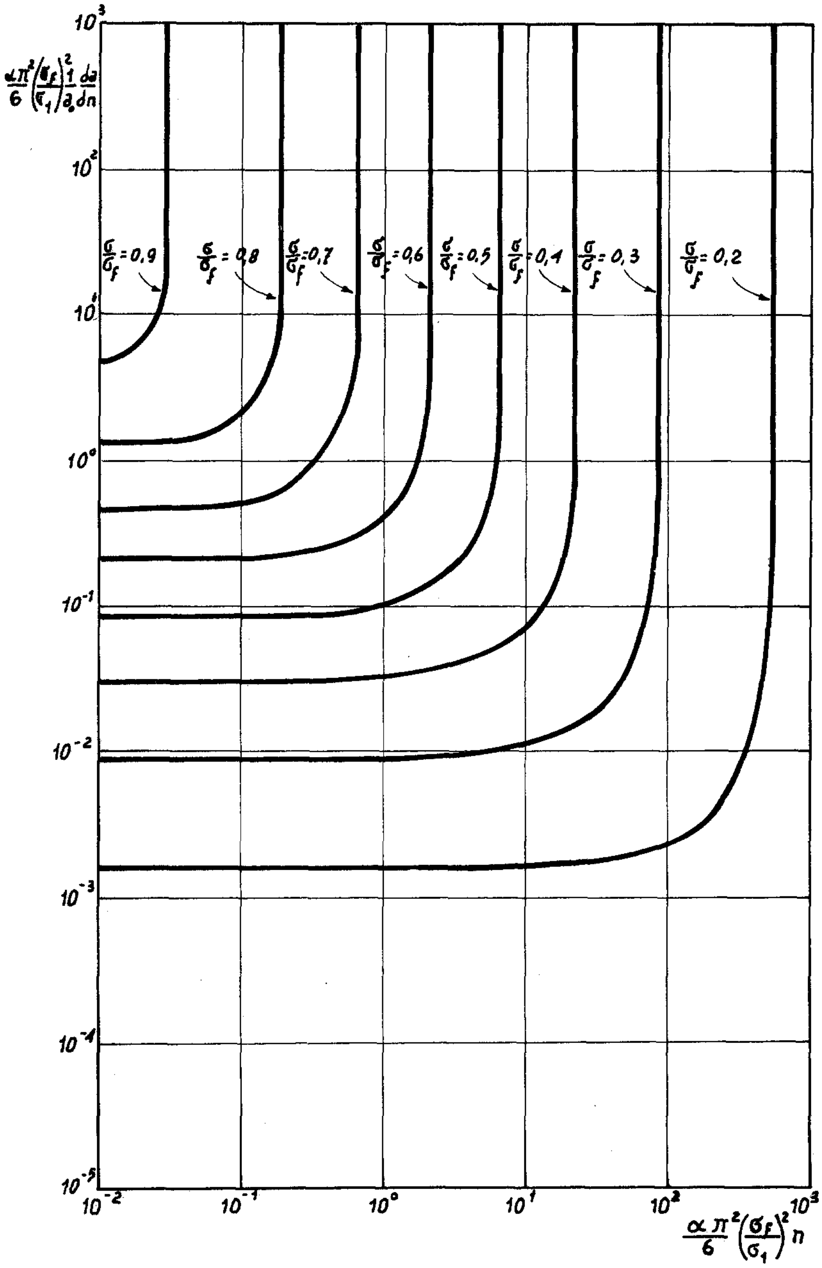

Fig. 4. Plot of da/dn as a function of $\underline{n}$ for some values of the maximum applied stress σ.

$$\frac{\sigma - \sigma_L}{\sigma_L} \simeq \frac{\alpha\pi^2}{12}\,\frac{\sigma_L^2}{\sigma_1 \Delta\sigma_1}\left[\left(\frac{\sigma_f}{\sigma_L}\right)^2 - 1 + n\left(\frac{\alpha\pi^2}{6}\left(\frac{\sigma_L}{\sigma_1}\right)^2 + \frac{\Delta\sigma_1}{\sigma_1}\sqrt{\frac{\sigma_f^2}{\sigma_L^2} - 1}\right)\right]^{-1}$$

The order of magnitude of $\Delta\sigma_1$ may be estimated by neglecting the term $\log\left(\frac{\sigma_L}{\sigma_f}\right)^2 - 1$ with respect to $\left(\frac{\sigma_f}{\sigma_L}\right)^2$. In this way

$$\frac{\Delta\sigma_1}{\sigma_1} \simeq \frac{\alpha\pi^2}{6}\,\frac{\sigma_L^4}{\sigma_1^2\sigma_f^2}$$

Since in most cases $\left(\frac{\sigma_L}{\sigma_f}\right)^2 \simeq \frac{1}{10}$ and $\left(\frac{\sigma_L}{\sigma_1}\right)^2 \simeq 10^{-4} \div 10^{-3}$, one has $\frac{\Delta\sigma_1}{\sigma_1} \simeq 10^{-5} \div 10^{-4}$.

As a final remark, it is interesting to investigate the change with time of the crack length, as given by eqs.(6) and (9) for $a \leq a_c$. From these equations it follows

$$\frac{da}{dn} = \frac{\alpha\pi^2}{6}\left(\frac{\sigma}{\sigma_1}\right)^2 \frac{a^2}{a_c - a} \frac{1}{\left(1 + n\frac{\Delta\sigma_1}{\sigma_1}\right)^2}$$

Hence it is seen that the crack propagates with a speed strongly increasing with time and approaching infinity when the radius approaches the critical value a_c. On physical grounds, this means that fatigue fracture is something of an avalanche process, confined in practice to the last cycles of the fatigue test. The curves giving a(n) and $\frac{da}{dn}$ for $\Delta\sigma_1 = 0$ are shown in fig. 3 and 4.

4. CONCLUSIONS

Although the roughness of the model does not allow quantitative conclusions, nevertheless some interesting remarks are possible.

First of all, the fatigue phenomenon is closely connected to the existence of a hysteresis loop, which, in turn, is a measure of the degree of plasticity of the sample. In a perfectly elastic medium no fatigue is allowed. This conclusion agrees with that of Mason (3),(4),(5) who showed that in germanium (a material without plastic deformation) there are no fatigue failures. In the same manner Coffin (6),(7) showed that the lower the plastic deformation the greater the duration before rupture.

In the present model it has been found that work hardening, which decreases the degree of plasticity in the material, is responsible for the existence of a fatigue limit, i.e. for a greater duration before failure. This is also consistent with the fact that lowering the temperature of the sample brings about a lower degree of plasticity and a greater fatigue limit.

In the same manner it is expected that aging, consisting in our model of a decreasing parameter σ_1, lowers the fatigue limit. All salient facts are in a good agreement with experience (8). It is well known, for example, that some alloys (9) which prevent aging, have no fatigue limit. Moreover, as the presence of impurities reduces the dislocation mobility, parameter ν_1 in this model is increased by said defects and a sharp fatigue limit is induced. This has been found in some magnesium alloys and in steels (10).

As a final remark, a consequence of the present theory, is the possibility of the so called "coaxing" of a material having a fatigue limit. If such a material undergoes a cyclic loading with maximum stress lower than σ_L, it has been observed that in some cases its fatigue limit is increased. This phenomenon is easily explained in the present model. In fact, besides increasing the radius of its cracks, a solid which has undergone n cycles at a load $\sigma < \sigma_L$ does generally show a work-hardening effect as well. When the latter effect is larger than the former one, the fatigue resistance of the solid is enhaced.

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RESUMÉ - L'étude de la propagation d'une fissure dans un solide soumis à des charges cycliques a été entreprise en partant du modèle de Griffith, et d'une hypothèse selon laquelle l'énergie dissipée au cours de chaque cycle d'hystéresis sert intégralement à accroitre la dimension de la fissure.

On considère séparément le cas des matériaux qui ne subissent pas un durcissement en cours de sollicitation cyclique, et le cas des matériaux qui sont sujets à un tel durcissement.

Malgré la faiblesse des hypothèses de base, due à leur caractère de grossière approximation, les résultats de l'analyse concordent qualitativement avec les caractéristiques principales du phénomène de fatigue.

ZUSAMMENFASSUNG - Durch Gebrauch des Griffithschen Models, wird die Fortpflanzung eines Risses in einem zyklisch beladenen festen Körper studiert. Es wird angenommen, dass alle in einem Zyklus verlorene Energie während des Bauschingerverfahrens zum Riss übergehen wird, um dessen Grösse zu erweiters.

Das Werk umfasst zwei Teile. Der erste Teil behandelt Stoffe, die der Bearbeitungsverhärtung nicht ausgesetzt sind; und der zweite Teil Stoffe, die dem Verfahren während zyklischer Belastung unterworfen werden. Trotz der grossen Annäherung dieses Models, wird gezeigt, dass die Hauptzüge der Zermürphässomene erklärt werden.