J Popul Econ (1990) 3:73–87

Journal of
Population Economics

Capital accumulation, endogenous population growth, and Easterlin cycles *

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Received November 29, 1989 / Accepted May 7, 1990

Abstract. In this paper we attempt to explain the occurrence of population cycles in industrialised economies where the birth rate depends on the difference between the actual and the expected consumption rate. This model of an endogenously growing population brings together Easterlin's idea of an adapting aspiration level with the neoclassical optimal growth paradigm. It is shown that in this highly aggregated demo-economic system (i.e., without inclusion of the age structure of a population) swings both in the economic and demographic variables may exist. The reason behind this "strange" *optimal* behaviour is identified to be an intertemporal substitution effect between current and future levels of consumption.

1. Introduction

In this paper we formulate an economic model of endogenous population growth where fertility fluctuates with changes in the economic environment. Such an approach is in the tradition of Solow 1956 (see also Davis 1969; Sato and Davies 1971; Becker and Barro 1988; and the surveys provided by Pitchford 1974; Steinmann 1974; Krelle 1985).

One of the most discussed hypotheses of population growth originates from Easterlin who tries to explain fluctuations in birth numbers. He (Easterlin 1962, 1968, 1973, 1980) assumes that age-group sizes influence relative cohort well-being and that well-being influences fertility levels. Hence, he concludes that due to economic competition a larger population will result in lower incomes. This may cause fertility to decline as parents try to maintain a certain standard of living for themselves. Thus, past birth numbers influence fertility and thereby present and future birth numbers.

* We wish to thank A. Novak for helpful assistance and an anonymous referee for useful comments. Financial support by the Austrian Science Foundation under contract No. P6601 is acknowledged.

Although the work of Easterlin is extensively discussed in demographic and economic circles, surprisingly few economic (mathematical) models have been formulated to analyze the implications of his ideas (see, however, Lee 1974, 1987a, b; Becker and Barro 1988; Barro and Becker 1989; Benhabib and Nishimura 1989).

As already mentioned, age-group size is a crucial variable in the Easterlin framework relating current and future fertility rates. This relationship and its consequences can best be studied in terms of a "feedback" model. In fact, several authors follow this approach to explain demo-economic interactions (Lee 1974; Samuelson 1976; Swick 1981; Frauenthal and Swick 1983). Lee studies the feedback relation from prior birth numbers to present fertility. Samuelson considers a discrete time model of the Leslie type including only two age-groups to model demographic interactions (see Feichtinger and Sorger 1989 for a continuous time version). Swick and Frauenthal extend the continuous renewal model of population dynamics to study self-generated fertility waves (further references are Keyfitz 1972, 1977; Feichtinger 1979). Recently, using mathematically sophisticated results on renewal functional equations, the question has turned to the study of conditions that imply cyclical swings in age-structured population models and whether the oscillations observed in U.S. birth numbers can be explained by them (Tuljapurkar 1987; Wachter 1987; Wachter and Lee 1989). All these models, however, are purely demographic ones, i.e., they do not include economic variables. The purpose of the present paper is to formulate a demo-economic model that is able to explain population cycles as expressed in Easterlin's hypothesis. We use a framework where the choice of consumption, intergenerational transfers and the accumulation of capital is linked to the determination of fertility. Thus, we extend the literature on optimal economic growth to allow for endogenous population growth. Endogenous population growth in our model is determined indirectly through the choice of consumption (savings). In the Barro-Becker theory (Barro and Becker 1989) agents choose directly the rate of fertility according to an altruistic utility function.

Our model consists of two parts. The first one is a standard neoclassical optimal growth model where we assume that social welfare at each period of time depends on both the actual and the expected level of consumption (Ryder and Heal 1973); the expected level of consumption is calculated as an exponentially weighted average of past consumption. This makes it possible to relate current economic well-being to changes in preferences. Intertemporally dependent preferences are of importance in our model of endogenous fertility. The second part of the model links the economic variables to the demographic ones and hence relates economic decisions to generative behavior. We assume that the crude birth rate depends positively on the excess of actual over expected per capita consumption. This corresponds to Easterlin's idea of the formation of material aspiration (see, e.g., Easterlin 1980, pp 40–43) where in our model aspiration is identified with the level of expected consumption.

The main focus of the analysis in this paper relates to the study of the stability behaviour of a steady state solution. In particular, we prove that smooth convergence to a unique steady state is not guaranteed. Stable limit cycles might exist and hence persistent oscillations in all variables, demographic and economic ones. The reason for these persistent demo-economic fluctuations is an intertemporal substitution effect between present and future per capita consumption that relates to the formation of the aspiration level.

The formulation of our model is in the tradition of neoclassical growth theory with endogenous population and therefore we derive *normative* statements. On the other hand this approach allows us to calculate an efficient allocation of consumption over time and its related change in population that would result in a society consisting of a large number of identical consumers each maximizing his (her) discounted stream of utility. Clearly, this way of discussing demoeconomic interactions is very different from the class of "feedback" models mentioned above. The latter simply describe relations from prior birth numbers to present fertility without postulating any optimizing behavior.

The paper is organized as follows. In Sect. 2 we present the model and its assumptions. Section 3 discusses sufficient conditions for the existence of persistent demo-economic cycles. There we provide also an economic interpretation of the mechanism that generates these cycles. Section 4 contains a numerical example. Section 5 concludes the paper and gives some remarks for possible extensions. Finally, the appendix reviews some of the mathematical background (Hopf bifurcation theorem) used to derive the existence of stable persistent oscillations.

2. The model

We assume a closed economy that consists of a large number of identical consumers and is controlled by a benevolent government that maximizes social welfare (i.e., utility of a representative consumer). Let $Y(t)$ be the output of a single good at time t that is used for consumption and as a capital good. The good is produced with a constant returns to scale technology described by the production function $Y(t) = F[K(t), L(t)]$. $K(t)$ is the capital stock at time t and $L(t)$ is the labour force.[1] F is assumed to be twice continuously differentiable and satisfies[2]

$$\begin{aligned} &F_K[K(t),L(t)]>0, \quad F_{KK}[K(t),L(t)]<0 \ , \\ &F_L[K(t),L(t)]>0, \quad F_{LL}[K(t),L(t)]<0 \quad \text{for all} \quad K(t),L(t)>0 \ . \end{aligned} \tag{1}$$

The inequalities in (1) state that both capital and labour are subject to diminishing returns with positive marginal products. Since F exhibits constant returns to scale it can be rewritten in per capita units. Denoting with $k(t) = K(t)/L(t)$ the capital per worker and with $y(t) = Y(t)/L(t)$ the output per worker we get

$$y(t) = F[k(t), 1] \equiv f[k(t)] \ , \tag{2}$$

where $f[k(t)]$ satisfies

$$f[k(t)]>0, \quad f'[k(t)]>0 \ , \quad f''[k(t)]<0 \qquad \text{for all } k(t)>0 \ . \tag{3}$$

Additionally we assume that the Inada conditions hold, i.e.,

1 For simplicity we identify the labour force with the entire population of the economy.
2 Subscripts denote partial derivatives of the function with respect to the corresponding argument.

$$\lim_{k(t)\to 0} f[k(t)] = 0, \quad \lim_{k(t)\to\infty} f[k(t)] = \infty ,$$
$$\lim_{k(t)\to 0} f'[k(t)] = \infty, \quad \lim_{k(t)\to\infty} f'[k(t)] = 0 . \tag{4}$$

The single good can either be consumed or invested. Consumption at time t is denoted by $C(t)$. With a constant rate of depreciation, δ, and equilibrium in the goods market we get the following accumulation equation

$$\dot{K}(t) = F[K(t), L(t)] - C(t) - \delta K(t) . \tag{5}$$

A dot over a variable denotes the time derivative.

Labour force is assumed to grow exponentially according to

$$\dot{L}(t) = [b(t) - d] L(t) , \tag{6}$$

where $b(t)$ is the time varying crude birth rate and d is the constant crude death rate of the population. Combining (2), (5) and (6) we get the reduced form equation of our model with all variables expressed in units per worker, i.e.,

$$\dot{k}(t) = f[k(t)] - c(t) - \delta k(t) - [b(t) - d] k(t), \quad k(0) = k_0 \tag{7}$$

where $c(t) = C(t)/L(t)$ is consumption per worker.

An important concept in our model is the expected level of per capita consumption. It is denoted by $z(t)$ and measured as an exponentially weighted average of past consumption levels. Thus, we have

$$z(t) = \gamma \int_{-\infty}^{t} \exp[-\gamma(t-s)] c(s) \mathrm{d}s . \tag{8}$$

From the weighting scheme in (8) it is clear that past consumption levels are less heavily counted then present ones. Differentiation of (8) with respect to time t yields the standard continuous time first order adjustment process between actual and expected consumption, i.e.,

$$\dot{z}(t) = \gamma [c(t) - z(t)], \quad z(0) = z_0 , \tag{9}$$

where γ indicates the speed of adjustment.

With the concept of expected level of consumption consumers are able to compare their present well being – expressed in terms of $c(t)$ – with that of the past and use this information when making their choices of fertility. This suggests that we use the expected level of consumption, $z(t)$, as the aspiration level influencing the crude birth rate. In particular, we assume that $b(t)$ is a function of the difference between the actual and the expected per capita consumption,

$$b(t) = b[c(t) - z(t)] , \tag{10}$$

with

$$b(\cdot)>0, \quad b'(\cdot)>0, \quad b''(\cdot)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0 \quad \text{as} \quad c(t)-z(t)\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}0 \tag{11}$$

This specification links the accumulation equation and hence the choice of consumption and savings to fertility. Assumption (11) implies that the higher (lower) the actual level of consumption is relative to the expected one, the (lower) higher is the crude birth rate (see the paper by Feichtinger and Sorger 1990 for a similar approach).

The decision how much to consume and how much to save at each period of time is made by the social planner (government) based on intertemporal welfare maximization. We assume that a consumption path is chosen that maximizes the discounted stream of utility (welfare) over an infinite planning horizon. Instantaneous welfare is dependent on both, current and expected (past) levels of consumption (cf. Ryder and Heal 1973)

$$U = U[c(t), z(t)] \ . \tag{12}$$

With a utility function of the type (12) instantaneous levels of satisfaction are dependent on current as well as past levels of consumption. The justification for such an assumption is obvious and takes into account that there is strong complementarity between consumption at successive dates.

The utility function $U[c(t), z(t)]$ is assumed to satisfy (cf. Ryder and Heal 1973)

$$U_c[c(t), z(t)]>0, \quad U_z[c(t), z(t)]\leq 0, \quad U_c(c,c)+U_z(c,c)>0 \ , \tag{13}$$

$$U_{cc}[c(t), z(t)]<0, \quad U_{zz}[c(t), z(t)]<0 \ , \tag{14}$$

$$U_{cc}[c(t), z(t)]\, U_{zz}[c(t), z(t)] - U_{cz}^2[c(t), z(t)]>0 \tag{15}$$

$$\lim_{c(t)\to 0} U_c[c(t), z(t)] = \infty, \quad \lim_{c\to 0} [U_c(c,c)+U_z(c,c)] = \infty \ . \tag{16}$$

Economically, assumptions (13)–(16) mean the following. An increase in current consumption increases utility, ceteris paribus. An increase in past consumption with no change in present consumption decreases utility or keeps it constant. An increase in a uniformly maintained consumption level will increase utility. Utility is concave jointly in $c(t)$ and $z(t)$.

The intertemporal decision problem then becomes

$$\max_{c(t)} \int_0^\infty e^{-rt} U[c(t), z(t)]\, \mathrm{d}t \tag{17}$$

subject to

$$\dot{k}(t) = f[k(t)] - c(t) - k(t)\, b[c(t)-z(t)] - \alpha k(t), \qquad k(0) = k_0 \ , \tag{18}$$

$$\dot{z}(t) = \gamma[c(t)-z(t)], \quad z(0) = z_0 \ , \tag{19}$$

where $\alpha \equiv \delta - d$.

In the next section we use Pontryagin's maximum principle (see Feichtinger and Hartl 1986) to solve problem (17)–(19) by studying existence and stability of a steady state.

3. Long-run population growth and Easterlin cycles

Section 2 specifies the demo-economic model that is analyzed in this section by means of a stability analysis. This amounts to showing existence and possible uniqueness of a steady state and linearizing the canonical system around the steady state. From the stability analysis we can infer the behaviour of the optimal consumption decisions and hence the growth of income and population over time. Before doing that it is necessary, however, to discuss the existence of an optimal solution to the infinite horizon control problem (17)–(19). The first theorem gives a definite answer to this question.

Theorem 1. *Given assumptions (3), (11), (14) and (15) there exists a unique consumption path that solves the optimal control problem (17)–(19).*

Proof: A proof of Theorem 1 is identical to the one given in Ryder and Heal (1973) and will not be repeated here.

The optimal path can be characterized by Pontryagin's Maximum Principle.[3] We formulate the current-value Hamiltonian.[4]

$$H = U(c,z) + \lambda_1 [f(k) - c - kb(c-z) - \alpha k] + \lambda_2 \gamma (c-z) \ , \tag{20}$$

where λ_i are the current-value shadow prices associated with the state Eqs. (7) and (9). The Maximum Principle consists of the following conditions[5]

$$H_c = 0 \Rightarrow U_c - \lambda_1 (1 + b'k) + \gamma \lambda_2 = 0 \ , \tag{21}$$

$$\dot{\lambda}_1 = (r + \alpha + b - f')\lambda_1 \ , \tag{22}$$

$$\dot{\lambda}_2 = (r + \gamma)\lambda_2 - \lambda_1 k b' - U_z \ . \tag{23}$$

Given concavity of the model, conditions (21)–(23) are also sufficient provided the limiting transversality condition

$$\lim_{t \to \infty} e^{-rt} \{\lambda_1(t)[k'(t) - k(t)] + \lambda_2(t)[z'(t) - z(t)]\} = 0 \tag{24}$$

for any feasible states $z'(t), k'(t)$ holds.

A steady state solution to problem (17)–(19) is a solution of the system of equations.

3 The interested reader not familiar with the Maximum Principle is referred to Kamien and Schwartz 1981 or Feichtinger and Hartl 1986 for an introduction to optimal control theory.

4 From now on unless otherwise stated we suppress the time argument t.

5 With the Inada-type assumptions (4) and (16) the optimal path will be an interior one.

$$\begin{aligned} c^{\infty} &= z^{\infty} , \\ c^{\infty} &= f(k^{\infty}) - [\alpha + b(0)]k^{\infty} , \\ f'(k^{\infty}) &= r + \alpha + b(0) , \\ \lambda_1^{\infty} &= [U_c + \gamma\lambda_2^{\infty}]\frac{1}{(1+b'(0)k^{\infty})} , \\ \lambda_2^{\infty} &= \frac{1}{r+\gamma}[\lambda_1^{\infty}k^{\infty}b'(0) + U_z] . \end{aligned} \tag{25}$$

It is easily checked that given the regularity conditions of our model system (25) admits a unique steady state. It is of some interest to interpret the steady state conditions (25) economically. First, we note that the steady state corresponds to the conventional modified golden rule solution adjusted for endogenous population growth. Second, the steady state capital labour ratio is determined by the society's discount rate, the crude birth and death rate. Hence, implicitly it depends on the utility function and therefore on the wishes of society through the choice of c and z. With exogenous population growth this dependence disappears (see also Sato and Davis 1971 on this point). This remark will be of importance when we discuss optimal population cycles.

From the maximizing condition (21) it is clear that c can be expressed as a function of $(k, z, \lambda_1, \lambda_2)$. This enables us to summarize the necessary and sufficient optimality conditions in terms of the canonical system:

$$\begin{aligned} \dot{k} &= f(k) - c - [\alpha + b(c-z)]k , \\ \dot{z} &= \gamma[c-z] , \\ \dot{\lambda}_1 &= [r + \alpha + b(c-z) - f'(k)]\lambda_1 , \\ \dot{\lambda}_2 &= (r+\gamma)\lambda_2 - \lambda_1 k b'(c-z) - U_z(c,z) . \end{aligned} \tag{26}$$

In the neighbourhood of the steady state we can approximate system (26) by a linear system of the following type

$$\dot{x} = J(x^{\infty})(x - x^{\infty}) , \tag{27}$$

where $x \equiv (k, z, \lambda_1, \lambda_2)^T, x^{\infty} \equiv (k^{\infty}, z^{\infty}, \lambda_1^{\infty}, \lambda_2^{\infty})^T$ and $J(x^{\infty})$ is the Jacobian of (26) evaluated at the steady state. In the appendix we present the explicit form of the elements of the Jacobian.

It is well understood (Kurz 1968) that the canonical system (26) constitutes a modified Hamiltonian system, hence the eigenvalues of $J(x^{\infty})$ are symmetric around $r/2$. In the present case an even stronger result holds. Given the low dimensionality of the system (26) the eigenvalues of the Jacobian can be calculated (see Dockner and Feichtinger 1989 for details). They are given by

$$\xi_{1,2,3,4} = \frac{r}{2} \pm \sqrt{\left(\frac{r}{2}\right)^2 - \frac{K}{2} \pm \sqrt{\left(\frac{K}{2}\right)^2 - \det J}} \tag{28}$$

where K is defined as the following sum of determinants

$$K \equiv \begin{vmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial \lambda_1} \\ \frac{\partial \dot{\lambda}_1}{\partial k} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_2}{\partial z} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} \end{vmatrix} + 2 \begin{vmatrix} \frac{\partial \dot{k}}{\partial z} & \frac{\partial \dot{k}}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_1}{\partial z} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} \end{vmatrix} . \tag{29}$$

For our specific model K turns out to be

$$K = -\gamma(r+\gamma) - \frac{1}{H_{cc}} \{-(1+kb')H_{ck}[r+(\alpha-f'+b)] + (1+kb')^2 H_{kk}\} + \frac{2\gamma\lambda_1 b'}{H_{cc}} - \frac{\gamma}{H_{cc}} [(r+2\gamma)H_{cz} + \gamma H_{zz}] , \tag{30}$$

where all elements are evaluated at the steady state. Making use of the assumptions (11), (29) can be simplified to

$$K = -\gamma(r+\gamma) - \frac{(1+kb')}{H_{cc}} [-r\lambda_1 b' + (1+kb')H_{kk}] + \frac{2\gamma\lambda_1 b'}{H_{cc}} - \frac{\gamma}{U_{cc}} [(r+2\gamma)U_{cz} + \gamma U_{zz}] . \tag{31}$$

Our purpose here is to find conditions under which system (26) exhibits a stable limit cycle as optimal policy. To establish a stable limit cycle we make use of Hopf's theorem (Guckenheimer and Holmes 1983). Among other things this requires two pure imaginary roots of the Jacobian $J(z^\infty)$. In Dockner and Feichtinger (1989) we derive conditions that guarantee the existence of two pure imaginary roots. They are given by

$$\det J > \left(\frac{K}{2}\right)^2 \quad \text{and} \quad \det J = \left(\frac{K}{2}\right)^2 + r^2 \frac{K}{2} . \tag{32}$$

From (32) it is clear that $K>0$ is a necessary condition for two pure imaginary roots. But what is the economic mechanism that guarantees K to be strictly positive? From (31) we know that the first three terms are negative. Hence, K can become positive only if

$$A \equiv -\frac{\gamma}{U_{cc}} [(r+2\gamma)U_{cz} + \gamma U_{zz}] > 0 \tag{33}$$

holds. This last condition, however, has a nice economic interpretation if we refer to the concepts of *adjacent* and *distant complementarity* as used in the paper by Ryder and Heal (1973). We say that consumer preferences exhibit *adjacent complementarity* if increasing consumption c at some date t raises marginal utility at nearby dates (t_1) relative to distant ones (t_2). In other words, the marginal rate of substitution between consumption at dates t_1 and t_2 increases with a small increment of c at date t. Evaluating this condition gives the following expression for adjacent complementarity

$$A \equiv -\frac{\gamma}{U_{cc}}[(r+2\gamma)U_{cz}+\gamma U_{zz}]>0 \ , \tag{34}$$

which is identical to (33). We say that *distant complementarity* holds if $A<0$. From this discussion it is clear that adjacent complementarity is a necessary condition for the existence of limit cycles. If on the contrary distant complementarity holds, i.e., $A<0, K$ is negative which together with $\det J>0$ implies saddle-point stability (see Dockner and Feichtinger 1989) and convergence to the unique steady state.

According to (32) $K<0$, however, is not the only necessary condition for two characteristic roots to be pure imaginary. As a second condition $\det J-(K/2)^2>0$ has to hold. In order to give an economic interpretation for this last inequality let us calculate the determinant of the Jacobian for the case that the impact of relative income (consumption) on fertility is very small, i.e., $b'(\cdot)\approx 0$. It is given by

$$\det J \approx \frac{f''\lambda_1}{U_{cc}}[(r+\gamma)\gamma]>0 \ .$$

This last inequality demonstrates that in the case of $b'\approx 0$ a limit cycle can only exist if $f''\lambda_1$ is sufficiently large. Put differently, we can argue that a limit cycle can be ruled out if the impact of relative income on fertility is small and the production technology is close being linear. Hence, the existence of a limit cycle requires at least one of the effects to be numerically important.[6]

So far we have discussed the logical possibility of a limit cycle. In the next section we present a numerical example that establishes a *stable limit cycle* as optimal policy.

4. A numerical example

The general analysis from above has revealed that different configurations for optimal policies are possible depending on the complementarity in consumption. In this section we present a numerical example that yields a stable limit cycle as optimal policy.

The technology is specified as Cobb-Douglas

$$f(k) = nk^{\sigma} \ , \tag{35}$$

with $n>0$ and $0<\sigma<1$. The birth rate is given as

$$b(c-z) = \mu \tanh[\beta(c-z)]+d \ , \tag{36}$$

6 We thank the referee for drawing our attention to this point.

with $\mu, \beta > 0$. The preferences are specified as the strictly concave utility function

$$U(c,z) = -a\mathrm{e}^{-c+z} + \frac{b}{2}c^2 + mcz + \frac{f}{2}z^2 + gc + hz \tag{37}$$

where $a, m, g > 0, b, f, h < 0$ and $bf - m^2 > 0$ holds. As the inequalities (13) are not globally valid with the utility function (37) we restrict our numerical calculations onto that region of the (c, z)-plane where (13) is satisfied.

We make use of the following set of parameter values.

$$\begin{aligned} &\mu = 0.2, \quad \beta = 0.5, \quad \gamma = 0.5, \quad \alpha = 9.0, \quad r = 1.0, \quad n = 10.187914, \\ &a = 1.0, \quad b = -1.\dot{3}, \quad m = 1.98, \quad f = -3, \quad g = 0.5633491\ , \\ &h = -3.7339424\ . \end{aligned} \tag{38}$$

The parameter values (38) were chosen such that $f'(k^\infty) = 10.0$, $U_c(c^\infty, z^\infty) = 3.0$ and $U_z(c^\infty, z^\infty) = -7.0$.

To prove the existence and stability of the limit cycle we make use of the *Hopf Bifurcation Theorem* that is stated in the appendix. Choosing σ as the bifurcation parameter, the Jacobian evaluated at the unique steady state possesses two pure imaginary roots for the critical value $\sigma_{\text{crit}} = 0.98904$. The unique steady state is given by

$$k^\infty = 2.0, \quad c^\infty = z^\infty = 2.221625, \quad \lambda_1^\infty = 0.588235, \quad \lambda_2^\infty = -4.588235\ . \tag{39}$$

If we evaluate A as given by (34) along the steady state we get

$$A = 0.848571428\ . \tag{40}$$

The pair of pure imaginary roots gives rise to the existence of limit cycles for values of σ near the critical value σ_{crit}. The stability of the cycles and the direction of the bifurcation are determined along with the sign of the coefficients $\tilde{A}$ and $\tilde{A}/D$ in the so called normal form (see the appendix for details). Using the codes "BIFOR2" and "BIFDD" (Hassard et al. 1981) we calculate

$$\tilde{A} = -0.00288723, \quad D = -2.2306353\ . \tag{41}$$

Thus the limit cycles are stable and occur for values of σ smaller than σ_{crit}. In order to illustrate the cyclical solution for $\sigma = 0.98903$ the boundary value problem solver *COLSYS* is applied (see Ascher et al. 1978; Steindl 1981).[7] The optimal time paths are depicted in Figs. 1–3. Looking at them provides us with additional insights into the demo-economic dynamics.

[7] With $\sigma = 0.98903$ instead of σ_{crit} the steady state is shifted to

$$k^\infty = 1.99, \quad c^\infty = z^\infty = 2.22156, \quad \lambda_1^\infty = 0.58822, \quad \lambda_2^\infty = -4.588197.$$

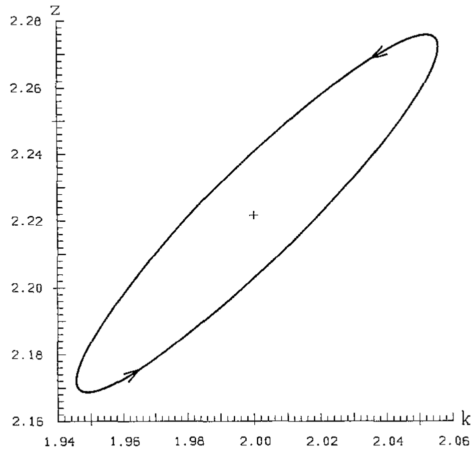

Fig. 1. Limit cycle in the state space

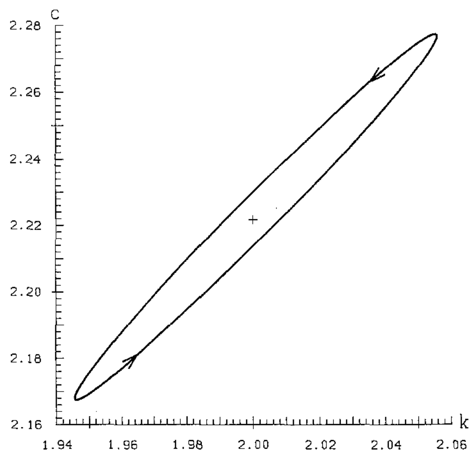

Fig. 2. Limit cycle in the capital stock – consumption plane

Figure 1 plots the phase diagram of the capital stock and expected consumption, while Fig. 2 shows the capital stock – consumption phase plane. For both diagrams the stable limit cycle moves counter-clockwise. Figure 2 allows us to examine the time profiles of optimal per capita consumption and the resulting per capita capital stock. At an initial condition with a low capital stock but a high level of current consumption, consumption (and hence population growth) should be decreased gradually while capital continues to fall. After some time the reduced levels of consumption and population growth permit capital to rise again. As capital accumulates consumption (population growth) can be increased again eventually reaching a high enough level where capital stops to grow and starts to fall. As this process continues we arrive at the initial conditions of high consumption and a low capital stock and the cyclical motion starts all over again. Figure 3 presents these cyclical time paths of all three variables. In addition it shows that the length of the period is about 59 years which corresponds to empirically observable Easterlin cycles.

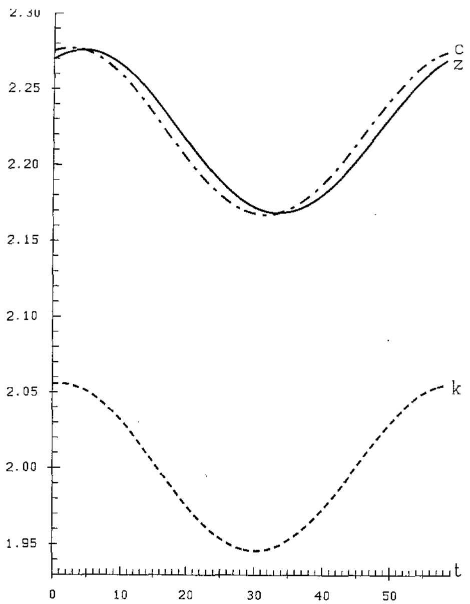

Fig. 3. Optimal time paths of k, z and c

5. Concluding remarks

In this paper we presented a theoretical explanation of cycles of the Easterlin type. Our model makes use of the neoclassical growth framework with endogenous population. The economic and demographic variables are linked through the crude birth rate depending on the excess of actual consumption over the aspiration level measured as the level of expected consumption. In this setting we identify an intertemporal substitution effect as source of demo-economic fluctuations.

The model proposed in this paper has two features that distinguishes it from traditional approaches to demographic oscillations. First, it is highly aggregated and omits any age structure. Second, the fluctuations are the result of optimal choices of consumption over time. Hence, we conclude that, in a simple framework as ours, population cycles can arise from intertemporal rational decision making and be efficient (Pareto-optimal).

Appendix

Hopf Bifurcation Theorem

In the numerical example we made use of the computer codes "BIFOR2" and "BIFDD" that are based on the Hopf Bifurcation Theorem (see Hassard et al. 1981). It reads as follows (see, e.g., Guckenheimer and Holmes 1983).

Theorem 2. *Suppose that a system* $\dot{x} = f_\sigma(x), x \in R^n, \sigma \in R$ *has an equilibrium* $(x_{crit}, \sigma_{crit})$ *at which the following property is satisfied:*

The Jacobian $D_x f\sigma_{crit}(x_{crit})$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts . (42)

Then (42) implies that there is a smooth curve of equilibria $[\hat{x}(\sigma), \sigma]$ with $\hat{x}(\sigma_{crit}) = x_{crit}$. The eigenvalues $\psi(\sigma), \bar{\psi}(\sigma)$ of $D_x f_{\sigma_{crit}}$ $[\hat{x}(\sigma)]$ which are imaginary at $\sigma = \sigma_{crit}$, i.e., $\pm i\omega$, vary smoothly with σ. If, moreover,

$$\frac{d}{d\sigma}[Re\,\psi(\sigma)] = D \neq 0 \;, \quad if \quad \sigma = \sigma_{crit} \;, \tag{43}$$

then there is a unique three-dimensional center manifold passing through $(x_{crit}, \sigma_{crit})$ in $R^n \times R$ and a smooth system of coordinates for which the Taylor expansion of degree 3 on the center manifold is given by the following normal form

$$\begin{aligned} \dot{x}_1 &= [D\sigma + \tilde{A}(x_1^2+x_2^2)]x_1 - [\omega + C\sigma + B(x_1^2+x_2^2)]x_2 \;, \\ \dot{x}_2 &= [\omega + C\sigma B(x_1^2+x_2^2)]x_1 + [D\sigma + \tilde{A}(x_1^2+x_2^2)]x_2 \;. \end{aligned} \tag{44}$$

If $\tilde{A} \neq 0$, there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\psi(\sigma_{crit}), \bar{\psi}(\sigma_{crit})$ agreeing to second order with the paraboloid $\sigma = -(\tilde{A}/D)(x_1^2+x_2^2)$. Moreover, if $\tilde{A} < 0$, then the periodic solutions are stable limit cycles, while in the case of $\tilde{A} > 0$, the periodic solutions are repelling.

Elements of Jacobian

The elements of the Jacobian $J(x^\infty)$ are given by the following expressions. It is to be understood that they are all evaluated at the steady state.

$$\frac{\partial \dot{k}}{\partial k} = f' - a - b - \frac{\partial c}{\partial k}[1 + b'k] \;,$$

$$\frac{\partial \dot{k}}{\partial z} = b'k - \frac{\partial c}{\partial z}[1 + b'k] \;,$$

$$\frac{\partial \dot{k}}{\partial \lambda_1} = -\frac{\partial c}{\partial \lambda_1}[1 + b'k] \;,$$

$$\frac{\partial \dot{k}}{\partial \lambda_2} = -\frac{\partial c}{\partial \lambda_2}[1 + b'k] \;,$$

$$\frac{\partial \dot{z}}{\partial k} = \gamma \frac{\partial c}{\partial k} \;,$$

$$\frac{\partial \dot{z}}{\partial z} = -\gamma + \gamma \frac{\partial c}{\partial z} \;,$$

$$\frac{\partial \dot{z}}{\partial \lambda_1} = \gamma \frac{\partial c}{\partial \lambda_1} \, ,$$

$$\frac{\partial \dot{z}}{\partial \lambda_2} = \gamma \frac{\partial c}{\partial \lambda_2} \, ,$$

$$\frac{\partial \dot{\lambda}_1}{\partial k} = -f'' \lambda_1 + b' \frac{\partial c}{\partial k} \lambda_1 \, ,$$

$$\frac{\partial \dot{\lambda}_1}{\partial z} = b' \lambda_1 \frac{\partial c}{\partial z} - b' \lambda_1 \, ,$$

$$\frac{\partial \dot{\lambda}_1}{\partial \lambda_1} r + \alpha + b - f' + \lambda_1 b' \frac{\partial c}{\partial \lambda_1} \, ,$$

$$\frac{\partial \dot{\lambda}_1}{\partial \lambda_2} = \lambda_1 b' \frac{\partial c}{\partial \lambda_2} \, ,$$

$$\frac{\partial \dot{\lambda}_2}{\partial k} = -\lambda_1 b' - U_{cz} \frac{\partial c}{\partial k} \, ,$$

$$\frac{\partial \dot{\lambda}_2}{\partial z} = -U_{zz} - U_{cz} \frac{\partial c}{\partial z} \, ,$$

$$\frac{\partial \dot{\lambda}_2}{\partial \lambda_1} = -k b' - U_{cz} \frac{\partial c}{\partial \lambda_1} \, ,$$

$$\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} = r + \gamma - U_{cz} \frac{\partial c}{\partial \lambda_2} \, .$$

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