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An Energy Balance Criterion for Crack Growth under Fatigue Loading from Considerations of Energy of Plastic Deformation

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ABSTRACT
An approximate analysis using a bilinear representation of the stress-strain behaviour has been made for the energy of plastic deformation at the tip of a crack growing under sinusoidal loading of constant amplitude. The energy of plastic deformation results from hysteretic and non-hysteretic plastic deformation. It is shown that the energy due to hysteresis is independent of the rate of growth of the crack whereas energy due to non-hysteretic plastic deformation is dependent on growth rate. Work hardening due to hysteretic plastic deformation is not considered in the analysis.

The energy balance criterion which is basic to fracture mechanics has been applied to the problem of crack growth under cyclic loading, considering the energy due to hysteretic plastic deformation in the plastic enclave as obtained in the analysis. The equation of energy balance results in an expression for crack growth rate, consistent with the general trends observed in experiments.

Some of the merits and limitations of the energy formulation of fatigue crack growth have been discussed.

Notations

$2a$		Crack length
t		Thickness of sheet
τ_{oct}		Octahedral shear stress at any point near the crack tip
$\tilde{\tau}_{oct}$		Octahedral shear stress range at any point near the crack tip under constant amplitude cyclic loading conditions
γ_{oct}		Octahedral shear strain at any point near the crack tip
$\tilde{\gamma}_{oct}$		Octahedral shear strain range at any point near the crack tip under constant amplitude cyclic loading condition
τ^{el}_{oct}		Octahedral shear stress at any point near the crack tip given by an elastic analysis
$\tilde{\tau}^{el}_{oct}$		Octahedral shear stress range at any point near the crack tip under constant amplitude cyclic loading given by an elastic analysis
τ_{oli}		Octahedral shear stress at yield under monotonic loading (Fig. 2)
$\tilde{\tau}_{oli}$		Octahedral shear stress at yield under cyclic loading (Fig. 2)
γ	=	$\tilde{\tau}_{oli}/\tau_{oli}$
E		Young's modulus
E_s		Secant modulus
G		Shear modulus
G_s		Secant modulus of the octahedral shear stress-strain curve
G_t		As defined in Fig. 2
λ	=	G/G_t; $\xi=\tau_{oct}/\tau_{oli}$; $\tilde{\xi}=\tilde{\tau}_{oct}/\tilde{\tau}_{oli}$; $\alpha=\tau^{el}_{oct}/\tau_{oli}$; $\tilde{\alpha}=\tilde{\tau}^{el}_{oct}/\tilde{\tau}_{oli}$
$\tilde{\alpha}_0$, $\tilde{\xi}_0$	=	Critical values of $\tilde{\alpha}$ and $\tilde{\xi}$ beyond which fracture processes are active
K_{max}		Stress intensity factor corresponding to maximum stress in the cycle
K_R		Stress intensity factor corresponding to stress range
w_p		Plastic enclave width corresponding to maximum stress in the cycle and is given by $K^2_{max}/9\tau^2_{oli}$ (Refer Fig. 1)
$\tilde{w}_p$		Plastic enclave width corresponding to stress range and is given by $K^2_R/9\tilde{\tau}^2_{oli}$ (Refer Fig. 1)
u_p		Non-hysteretic energy density at any point in the plastic enclave

$\tilde{u}_p$ Hysteretic energy density at any point in the plastic enclave
$\tilde{U}_P$ Hysteresis energy absorbed in the cyclic plastic enclave per unit thickness
$\tilde{U}'_P = 2\tilde{U}_P G/3\tilde{\tau}^2_{oli}\tilde{w}^2_p$
U_P Non-hysteretic energy absorbed in the plastic enclave per unit thickness
$U'_P = 2U_P G/3\tilde{\tau}^2_{oli}\tilde{w}^2_p$
$U_{P(tot)} = \tilde{U}_P + U_P$; $U'_{P(tot)} = \tilde{U}'_P + U'_P$
da/dN Crack growth rate;
$\dfrac{d\bar{a}}{dN} = \dfrac{1}{\tilde{w}_p} \cdot \dfrac{da}{dN}$
R Minimum stress/Maximum stress
K_C Fracture toughness

Introduction

The application of fracture mechanics to the problem of fatigue crack propagation has been limited to use of the concept of the so called stress intensity factor, K, to correlate crack propagation rates. Very good correlations often achieved with stress intensity factor for different geometries of specimens and test configuration have proved valuable in predicting crack growth behaviour of actual real life structures subjected to fatigue loading. Some empirical [1, 2, 3, 4] laws relating crack propagation rates to stress intensity factor have been putforth in the past.

Application of the principle of energy balance, which forms the basis of fracture mechanics, to the fatigue crack propagation problem does not appear to have received serious attention. Some attempts have been made by V. Gallina, G. P. Galotto and M. Omini [5] in this regard.

In this paper, an attempt has been made to analyse, in some detail, the energy of plastic deformation in the plastic enclave at the tip of a crack growing under cyclic loading of constant amplitude applied normal to the crack plane, at infinity. The analysis is based on a bilinear representation of the stress-strain curve and Von Mises yield criteria, coupled with the assumption of kinematic hardening behaviour of the material. It is also assumed in the analysis, that crack extension occurs instantaneously in each cycle of loading, when the maximum stress in the cycle is reached. Work hardening due to cyclic plastic deformation is not considered in the analysis.

Using the expressions obtained for the hysteretic energy and non-hysteretic energy of plastic deformation an energy balance equation governing the extension of the crack in each cycle of loading has been formulated. The energy balance equation results in an expression for crack growth rate, which is similar in structure to the empirical expression obtained by Forman [4] *et al.* Hudson and Scardina [6] have shown that Forman's [4] expression fits the experimental data well.

Some of the potentialities and limitations of the energy formulation of fatigue crack propagation problem are discussed.

1. Plastic Deformation History at the Tip of a Crack Growing under Constant Amplitude Loading ($R \geqq 0$)

Let us focus our attention to the region in the immediate vicinity of the crack tip and assume this region to be initially free from stresses. Let the body containing the crack be subjected to the first cycle of loading, applied at infinity, normal to the crack plane. We shall consider the plastic deformation history at the crack tip during the loading and unloading phases of the first cycle, assuming that the crack does not grow or, in other words, that the crack tip is stationary. During the loading phase of the first cycle, i.e. loading from zero to a maximum tensile load, the region in the vicinity of the crack tip is plastically deformed and a plastic enclave or zone can be identified. During the unloading phase, i.e. unloading to a minimum tensile load, two distinct regions in the plastic enclave can be identified, namely an inner region surrounding the crack tip where reverse plastic deformation takes place and an outer region where the deformation is elastic. This inner region of reverse plastic deformation will be identified, in this paper, as the cyclic plastic enclave or zone. It can be easily seen that in the loading and unloading phases of the subsequent cycles plastic deformation takes place in a cyclic manner in the cyclic plastic enclave and the deformations in the outer zone remain elastic.

This picture of deformation history changes when the extension of the crack or the advance of the crack tip in each cycle of loading is considered. The changes in the deformation history in the plastic enclave due to crack extension can best be explained through Fig. 1.

Basically the effect of crack extension is to increase the stresses in some parts of the enclave and to decrease the stresses in the remaining parts of the plastic enclave. Hence one can identify a so called Neutral curve [7] on which the stress conditions do not vary with crack extension. Since we assume that crack extension in each cycle occurs instantaneously at the maximum stress, the neutral curve will be a straight line [7]. This implies, further, that we disregard the effects of changes in enclave size due to growth of the crack in each cycle. The neutral line [7]

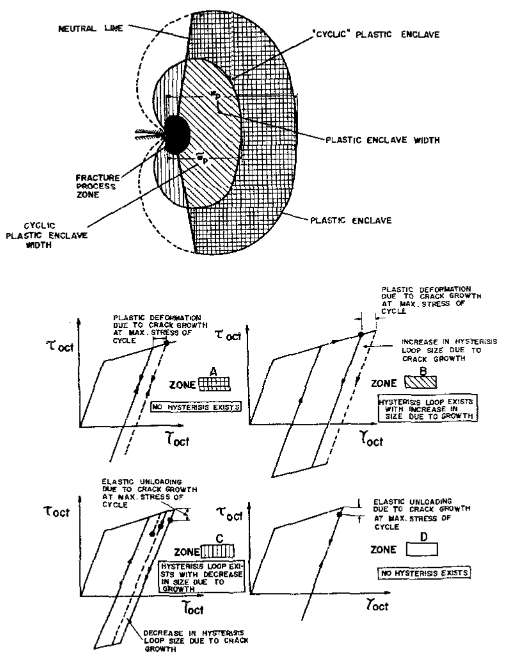

Figure 1. Plastic deformation history in the plastic enclave as affected by crack growth.

divides the plastic enclave, formed during the initial loading phase, into four distinct regions in which the changes in deformation history due to crack extension, are distinctly different. Figure 1 shows the different zones to which the plastic enclave is divided by the neutral line along with the octahedral shear stress-strain curves associated with these zones, indicating the differences in the deformation history suffered by these zones. Referring to Fig. 1, it is to be observed that

(i) In zone A, plastic deformation occurs in the initial loading phase. In the subsequent cycles plastic deformation occurs only when the crack extends at the maximum stress of each cycle.

(ii) In zone D, plastic deformation occurs only in the initial loading phase. In the subsequent cycles the deformations are completely elastic since elastic unloading occurs due to crack extension.

(iii) In zone B of the cyclic plastic enclave, crack extension, occurring at the maximum stress in each cycle, gives rise to an increase in the hysteresis loop width.

(iv) In zone C of the cyclic plastic enclave, crack extension, occurring at the maximum stress in each cycle, gives rise to a decrease in the hysteresis loop width.

Hence we see that under cyclic loading and crack extension plastic deformation occurs in a part of the plastic enclave (zones A, B and C).

2. Analysis for Energy of Plastic Deformation

The energy of plastic deformation at the tip of a crack growing under cyclic loading consists of a hysteretic and a non-hysteretic part. The hysteretic part of the energy refers to cyclic plastic deformation energy or hysteresis energy in the cyclic plastic enclave. The non-hysteretic part of the energy refers to plastic deformation energy in the outer region of the plastic enclave.

Assuming that there is no growth of the crack an expression for the hysteresis energy in the cyclic plastic enclave is obtained. Expressions to account for variation in the hysteresis energy and for the non-hysteretic energy resulting from crack extension in each cycle, are then obtained separately. The sum of the hysteresis energy, its variation due to crack extension and the non-hysteretic energy resulting from crack extension, gives the total energy of plastic deformation for each cycle of loading. Referring to the bilinear representation of the octahedral shear stress

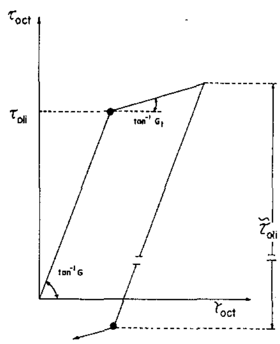

Figure 2. Idealised octahedral shear stress shear strain curve assumed in analysis.

shear strain curve in Fig. 2, and to Appendix I (a), it can be seen that the hysteretic energy density $\tilde{u}_p$ in an element of volume dv located at (r, θ) in the cyclic plastic enclave is given by

$$\tilde{u}_p = \frac{3}{2}\frac{\tilde{\tau}_{oli}^2}{G}(\tilde{\xi}-1)(\lambda-1)\,. \tag{1}$$

Using total laws of plasticity one can write

$$\tau_{oct} = \eta\,\tau_{oct}^{el}$$

where η is a scalar factor depending on the stress-strain characteristics of the material. Using Dixon's [8] approximate relation between the total strains and the corresponding strains obtained by elastic analysis and assuming that $\nu = 0.5$, we have

$$\eta = \left(\frac{E_s}{E}\right)^{\frac{1}{2}} \quad \text{or} \quad \left(\frac{G_s}{G}\right)^{\frac{1}{2}}.$$

Also we have from Appendix I(a)

$$\tilde{\alpha} = \{\tilde{\xi}[1+\lambda(\tilde{\xi}-1)]\}^{\frac{1}{2}} = \left(\frac{\tilde{W}_p f_e}{r}\right)^{\frac{1}{2}} \tag{2}$$

where

$$f_e = f_1^2 - f_1 f_2 + f_2^2 + 3 f_3^2$$

for plane stress conditions.

Here f_1, f_2, f_3, are functions of θ defining the elastic stress distribution in the immediate vicinity of the crack tip. The elastic stress distribution is given by

$$\sigma_x = \frac{K}{(2r)^{\frac{1}{2}}} f_1 \; ; \quad \sigma_y = \frac{K}{(2r)^{\frac{1}{2}}} f_2 \; ; \quad \tau_{xy} = \frac{K}{(2r)^{\frac{1}{2}}} \cdot f_3$$

where K is the stress intensity factor referring to either

(i) the applied stress range, in which case the stresses refer to stress ranges at the crack tip. Or

(ii) the maximum stress, in which case the stresses refer to the maximum stresses at the crack tip.

Now the hysteresis energy $\tilde{U}_P$ per unit thickness absorbed in the cyclic plastic enclave is given by

$$\tilde{U}_P = \int_0^{\pi} \int_{r=\tilde{w}_p f_e/\tilde{\alpha}_0}^{r=\tilde{w}_p f_e} \frac{3\tilde{\tau}_{\text{oli}}^2}{G} (\xi - 1)(\lambda - 1) r \, dr \, d\theta \,. \tag{3}$$

It is to be observed here that the integration is not carried down to the crack tip. This is because it is presumed that a "Fracture Zone" [7] defined by $\xi = \xi_0$ participating in the actual process of crack extension at the maximum stress in each cycle can be identified as shown in Fig. (1). Changing the variables to ξ, θ, we obtain

$$\tilde{U}_P = \frac{3\tilde{\tau}_{\text{oli}}^2 (\lambda - 1) \tilde{w}_p^2}{G} \int_1^{\xi_0} \int_0^{\pi} \frac{(\xi - 1)(2\lambda\xi - \lambda + 1)}{\xi^3 (\lambda\xi - \lambda + 1)^3} f_e^2 \, d\xi \, d\theta \,. \tag{4}$$

Integrating and putting

$$\zeta_0 = \frac{1 + \lambda(\xi_0 - 1)}{\xi_0} \,.$$

we get for plane stress conditions

$$\tilde{U}_P = \frac{4.526 \tilde{\tau}_{\text{oli}}^2 \tilde{w}_p^2}{G(\lambda - 1)^3 \zeta_0^2} [\zeta_0^4 - 2\zeta_0^3 (1 + \lambda) + \zeta_0^2 (1 + \lambda + \lambda^2)(1 + \lambda)$$
$$- 2\zeta_0 \lambda^2 (1 + \lambda) + \lambda^3 - 2\lambda(\lambda - 1)\zeta_0^2 \log_e \zeta_0] \,. \tag{5}$$

We shall now consider the variation in the hysteresis energy in the plastic enclave and the non-hysteretic energy resulting from crack extension in each cycle.

The variation in the hysteresis energy density at (r, θ) in the cyclic plastic enclave is given by

$$\frac{d\tilde{u}_p}{dN} = \frac{3}{2} \frac{\tilde{\tau}_{\text{oli}}^2}{G} (\lambda - 1) \frac{d\xi}{dN} \,. \tag{6}$$

The change in the hysteresis energy in the cyclic plastic enclave is given by

$$\frac{d\tilde{U}_P}{dN} = \frac{3\tilde{\tau}_{\text{oli}}^2 (\lambda - 1)}{G} \int_0^{\pi} \int_{r=w_p f_e/\tilde{\alpha}_0}^{r=w_p f_e} \frac{d\xi}{d\tilde{\alpha}} \cdot \frac{d\tilde{\alpha}}{dN} r \, dr \, d\theta \,.$$

Changing the variables to ξ and θ, we have

$$\frac{d\tilde{U}_P}{dN} = \frac{3\tilde{\tau}_{\text{oli}}^2 (\lambda - 1)}{G} \int_0^{\pi} \int_1^{\xi_0} \frac{d\tilde{\alpha}}{dN} \cdot \frac{2\tilde{w}_p^2 f_e^2}{[\xi(1 + \lambda\xi - \lambda)]^{\frac{5}{2}}} d\xi \, d\theta \,. \tag{7}$$

Neglecting changes in $\tilde{w}_p$ due to crack extension we obtain from equation (2)

$$\frac{d\tilde{\alpha}}{dN} = \frac{\tilde{\alpha}^3}{2\tilde{w}_p f_e^2} \cdot \frac{da}{dN} \cdot \frac{d}{d\theta} (f_e \cdot \sin\theta) \tag{8}$$

substituting equation (8) in (7), we obtain

$$\frac{d\tilde{U}_P}{dN} = \frac{3\tilde{\tau}_{\text{oli}}^2 (\lambda - 1)}{G} \tilde{w}_p^2 \cdot \frac{da}{dN} \left[\int_1^{\xi_0} \frac{d\xi}{\xi(1 + \lambda\xi - \lambda)} \times \int_0^{\pi} \frac{d}{d\theta} (f_e \sin\theta) \, d\theta \right] .$$

It is observed that

$$\int_0^{\pi} \frac{d}{d\theta}(f_e \sin\theta)\,d\theta = 0\,.$$

Hence we have

$$\frac{d\tilde{U}_P}{dN} = 0\,. \tag{9}$$

This is an interesting result. This physically means that the increase in hysteresis energy in zone B, (Fig. 1) due to crack extension is exactly balanced by the decrease in the hysteresis energy in zone C. Therefore the hysteretic part of the energy of plastic deformation is given by equation (5).

The non-hysteretic part of the energy of plastic deformation in the outer region (Zone A, Fig. 1) resulting from crack extension will now be considered.

The non-hysteretic plastic deformation energy density at (r, θ) is given by [see Appendix I(a)]

$$u_p = \frac{3}{2}\frac{\tau_{oli}^2}{G}\xi(\lambda-1)\frac{d\xi}{da}\cdot\frac{da}{dN} + \frac{3}{4}\frac{\tau_{oli}^2}{G}(\lambda-1)\left(\frac{d\xi}{da}\cdot\frac{da}{dN}\right)^2. \tag{10}$$

Neglecting the second term involving $(da/dN)^2$ as being relatively small, we have

$$u_p = \frac{3}{2}\frac{\tau_{oli}^2}{G}\xi(\lambda-1)\cdot\frac{d\xi}{d\alpha}\cdot\frac{d\alpha}{da}\cdot\frac{da}{dN}\,. \tag{11}$$

The non-hysteretic plastic deformation energy per unit thickness, in the outer zone is given by

$$U_P = \int_0^{\theta_0}\int_{r=w_p f_e(1-R)^2/\gamma^2}^{r=w_p f_e} \frac{3\tau_{oli}^2}{G}\frac{\xi(\lambda-1)\alpha^3}{2w_p f_e^2}\cdot\frac{d}{d\theta}(f_e\cdot\sin\theta)\cdot\frac{d\xi}{d\alpha}\cdot\frac{da}{dN}\cdot r\,dr\,d\theta \tag{12}$$

where θ_0, the inclination of the neutral line [7], is equal to 79.9°. Changing the variables to ξ, θ and simplifying, we get

$$U_P = \frac{3(\lambda-1)}{G}\tau_{oli}^2 w_p\cdot\frac{da}{dN}\int_0^{79.9^\circ}\int_1^{\xi_1}\frac{\frac{d}{d\theta}(f_e\cdot\sin\theta)}{1+\lambda(\xi-1)}\cdot d\xi\,d\theta\,,$$

where

$$\xi_1 = \frac{\lambda-1}{2\lambda}\left[1+\left\{1+\frac{4\gamma^2\lambda}{(1-R)^2(\lambda-1)^2}\right\}^{\frac{1}{2}}\right].$$

Integrating and using the relation between $\tilde{w}_p$ to w_p [see Appendix I(b)], we obtain

$$U_P = \frac{3.885(\lambda-1)}{G\lambda}\cdot\frac{\tau_{oli}^2}{(1-R)^2}\tilde{w}_p\cdot\frac{da}{dN}\log_e\left\{\frac{\lambda-1}{2}\left[\left(1+\frac{4\gamma^2\lambda}{(1-R)^2(\lambda-1)^2}\right)^{\frac{1}{2}}-1\right]\right\}. \tag{13}$$

The total plastic deformation energy going into the plastic enclave, in each cycle is given by

$$U_{P(tot)} = \tilde{U}_P + U_P\,. \tag{14}$$

From equations (5) and (13) we have

$$\begin{aligned} U_{P(tot)} = {} & \frac{4.526\tilde{\tau}_{oli}^2\tilde{w}_p^2}{G(\lambda-1)^3\tilde{\zeta}_0^2}\left[\tilde{\zeta}_0^4 - 2\tilde{\zeta}_0^3(1+\lambda)+\tilde{\zeta}_0^2(1+\lambda)(1+\lambda+\lambda^2)\right. \\ & \left. -2\tilde{\zeta}_0\lambda^2(1+\lambda)+\lambda^3-2\lambda(\lambda-1)\tilde{\zeta}_0^2\log_e\tilde{\zeta}_0\right] \\ & + \frac{3.885(\lambda-1)\tilde{\tau}_{oli}^2\tilde{w}_p^2}{G\lambda(1-R)^2}\cdot\frac{d\bar{a}}{dN}\log_e\left\{\frac{\lambda-1}{2}\left[\left(1+\frac{4\gamma^2\lambda}{(1-R)^2(\lambda-1)^2}\right)^{\frac{1}{2}}-1\right]\right\} \end{aligned} \tag{15}$$

where $d\bar{a}/dN$ is the non-dimensionalised growth rate given by

$$\frac{\mathrm{d}\bar{a}}{\mathrm{d}N} = \frac{1}{\bar{w}_\mathrm{p}} \cdot \frac{\mathrm{d}a}{\mathrm{d}N}. \tag{16}$$

The total non-dimensionalised energy of plastic deformation is given by

$$U'_{\mathrm{p(tot)}} = \frac{3.0173}{(\lambda-1)^3 \tilde{\zeta}_0^2} [\tilde{\zeta}_0^4 - 2\tilde{\zeta}_0^3(1+\lambda) + \tilde{\zeta}_0^2(1+\lambda)(1+\lambda+\lambda^2) - 2\tilde{\zeta}_0\lambda^2(1+\lambda) + \lambda^3 - 2\lambda(\lambda-1)\tilde{\zeta}_0^2 \log_e \tilde{\zeta}_0] + \frac{2.589}{\lambda(1-R)^2} \cdot (\lambda-1) \cdot \frac{\mathrm{d}\bar{a}}{\mathrm{d}N} \cdot \log_e \left\{ \frac{\lambda-1}{2} \left[\left(1 + \frac{4\gamma^2 \lambda}{(1-R)^2(\lambda-1)^2} \right)^{\frac{1}{2}} - 1 \right] \right\}. \tag{17}$$

3. Energy Balance During Fatigue Crack Growth

The changes in external work done, elastic strain energy and energy of plastic deformation in the plastic enclave, at the instant of crack extension are considered in constructing an equation of energy balance. It is to be observed that material "elements" at the crack tip which fracture resulting in crack extension, will require lower energy for fracture than the material elements which are located further away. This is because the elements at the crack tip, will have suffered damage due to the plastic deformation history they have undergone. To make this point clear, let us consider a material element located outside the plastic enclave. As the crack tip advances in each cycle, the material element gets nearer to the crack tip. In this process at some instant, the material element enters the outer region of the plastic enclave, where it suffers plastic deformation, only due to crack extension in each cycle. After a certain number of cycles, the element enters the cyclic plastic enclave where it undergoes hysteretic plastic deformation. With this history of plastic deformation the material element, when it approaches the crack tip, will be in a damaged state and hence would need lesser energy for fracture than it would have required if it had not suffered any such prior plastic deformation history. With this observation we proceed to construct the equation of energy balance at the instant of crack extension in each cycle.

The energy balance equation can be written as

$$\frac{\mathrm{d}W_{\mathrm{ext}}}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}N} - \frac{\mathrm{d}U_\mathrm{e}}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}N} - \frac{\mathrm{d}\overline{U}_\mathrm{P}}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}N} = u'_f \cdot \frac{\mathrm{d}a}{\mathrm{d}N} - \beta\tilde{U}_\mathrm{P}$$

where

$\frac{\mathrm{d}a}{\mathrm{d}N}$ is crack growth rate,

$\frac{\mathrm{d}W_{\mathrm{ext}}}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}N}$ is the change in external work done in each cycle due to crack extension,

$\frac{\mathrm{d}U_\mathrm{e}}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}N}$ is the change in the elastic strain energy in each cycle due to crack extension, and

$\frac{\mathrm{d}\overline{U}_\mathrm{P}}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}N}$ is the change in the energy of plastic deformation in each cycle due to crack extension [7].

The left-hand side of the equation of energy balance, represents the net energy available in each cycle for the fracture process at the crack tip, resulting in crack extension. The right-hand side of the equation represents the energy required for fracture, taking into account the reduction in fracture energy due to prior hysteretic plastic deformation. The term $\beta\tilde{U}_\mathrm{P}$ is the reduction in the fracture energy $u'_f(da/dN)$ per cycle due to prior hysteretic plastic deformation. It is to be observed that this reduction should also depend on the extent of the non-hysteretic plastic deformation that accumulates before the material elements enter into the cyclic plastic enclave. This dependence on non-hysteretic plastic deformation can be accounted for by taking β as

$$\beta = A\left(\frac{w_p}{\tilde{w}_p}\right)^n . \tag{19}$$

Using the relation between w_p and $\tilde{w}_p$ derived in Appendix I(b), we obtain

$$\beta = A\gamma^{2n}\left(\frac{1}{1-R}\right)^{2n}$$

putting

$$A\gamma^{2n} = A'$$
$$2n \quad = m$$

we get

$$\beta = A'\left(\frac{1}{1-R}\right)^m . \tag{20}$$

This relation indicates that for a given hysteretic energy input $\tilde{U}_P$ into the cyclic plastic enclave, greater damage and hence larger reduction in fracture energy results at higher stress ratios which is physically consistent. In equation (20) A' and m can be considered as constants depending only on material characteristics.

Rewriting equation (18) we obtain the following expression for crack growth rate

$$\frac{da}{dN} = \frac{A'\left(\frac{1}{1-R}\right)^m B\tilde{\tau}_{oli}^2 \cdot \tilde{w}_p^2}{\left[u'_f - \frac{dW_{ext}}{da} - \frac{dU_e}{da} - \frac{d\bar{U}_P}{da}\right]} . \tag{21}$$

where B is a function of ζ_0, G and λ as in equation (5). It is to be observed that

$$u'_f - \left[\frac{dW_{ext}}{da} - \frac{dU_e}{da} - \frac{d\bar{U}_P}{da}\right] = 0$$

represents the energy balance equation governing fracture instability [7]. If one assumes linear elastic fracture mechanics as applicable to the residual strength problem, equation (21) can be written as

$$\frac{da}{dN} = \frac{\bar{A}(1-R)^{4-m} \cdot \sigma_y^2 w_p^2}{w_{p_{cr}} - w_p} . \tag{22}$$

since u'_f and

$$\left(\frac{dW_{ext}}{da} - \frac{dU_e}{da} - \frac{d\bar{U}_P}{da}\right)$$

are proportional to $w_{p_{cr}}$ and w_p respectively. Here $w_{p_{cr}}$ is the plastic zone width at final fracture. In the above equation $\bar{A}$ is a constant depending only on material characteristics, and σ_y is the yield stress in tension.

In terms of stress intensity factor, equation (22) can be written as

$$\frac{da}{dN} = \frac{\bar{A}(1-R)^{4-m} K_{max}^4}{2(K_C^2 - K_{max}^2)} . \tag{23}$$

It is to be observed that the relation derived above does not indicate any effect of interaction in variable amplitude loading which result from varying residual stress field at the crack tip. This is because (i) the analysis for hysteretic plastic deformation energy, does not take into account the hardening or softening that occurs under cyclic plastic deformation. (ii) The effect of hydrostatic tension, which is dependent on the residual stresses and thickness, on the damage accumulation due to hysteretic energy input into the cyclic plastic zone has not been considered in deriving equation (22) or (23).

The effect of hydrostatic tensile stress, σ_{hyd} which is dependent on residual stress field, can be accounted for by rewriting equation (23) as

$$\frac{da}{dN} = \frac{\bar{A}}{2} \frac{(1-R)^{4-m} K_{\max}^4 f(\sigma_{\text{hyd}})}{K_C^2 - K_{\max}^2}. \tag{24}$$

Equation (24) represents a somewhat generalised crack growth relation which involves effect of thickness through K_C and $f(\sigma_{\text{hyd}})$. The sequence or history effect is brought in through the function f which is dependent on the residual stress fields relating to the previous history of loading. The nature of the function f in the present state of development remains undefined.

4. Results and Discussion

(a) Analysis for energy of plastic deformation

Fig. 3 shows the non-dimensionalised plastic energy $U'_{P(\text{tot})}$ dissipated per cycle, as a function of ξ_0 with R and $d\bar{a}/dN$ as parameters. In this figure $\lambda = 20$ and $\gamma = 2.0$. It can be seen from this figure that $U'_{P(\text{tot})}$ increases with ξ_0 and appears to reach an asymptotic value.

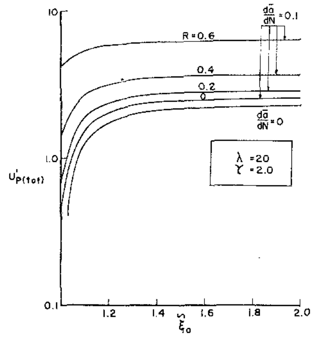

Figure 3. Non-dimensionalised plastic energy dissipated per cycle as a function of ξ_0 with R and $d\bar{a}/dN$ as parameters.

Fig. 4 shows the variation of $U'_{P(\text{tot})}$ with λ for $\xi_0 = 1.10$ and $\gamma = 2.0$. R and $d\bar{a}/dN$ are shown as parameters. It can be seen that increase in λ results in an increase in $U'_{P(\text{tot})}$.

Fig. 5 shows the variation of $U'_{P(\text{tot})}$ with $d\bar{a}/dN$, the non-dimensionalised crack growth rate, for different values of stress ratio R. It appears from this figure that at very low values of $d\bar{a}/dN$,

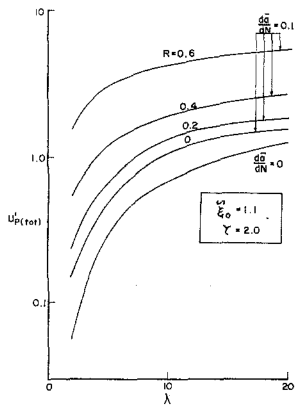

Figure 4. Non-dimensionalised plastic energy dissipated per cycle as a function of λ with R and $d\bar{a}/dN$ as parameters.

the effect of R is very small for a considerable range of R (say from 0.7 to 0). At larger values of $d\bar{a}/dN$ (higher than 0.01), R has a very significant effect on $U'_{P(\text{tot})}$.

It is to be observed from equation (18) that the hysteretic part of $U'_{P(\text{tot})}$ namely $\tilde{U}'_P$ is independent of stress ratio R and the non-dimensionalised growth rate $d\bar{a}/dN$, whereas the non-hysteretic part namely U'_P is a function of R and $d\bar{a}/dN$. Fig. 6 shows the variation of the ratio $U'_P/(d\bar{a}/dN)$ with λ for different values of stress ratio R. The curves shown in this figure refer to $\gamma = 2.0$. Fig. 7 shows the variation of the hysteretic part of $U'_{P(tot)}$ namely $\tilde{U}'_P$ as a function of $\bar{\xi}_0$ with λ as a parameter.

To summarise figures 3 to 7 show the variation of $U'_{P(tot)}$ and its hysteretic and non-hysteretic parts, $\tilde{U}'_P$ and U'_P, with the material parameters λ, γ, and the parameters, stress ratio R; crack growth rate $d\bar{a}/dN$ and the fracture zone size parameter $\bar{\xi}_0$.

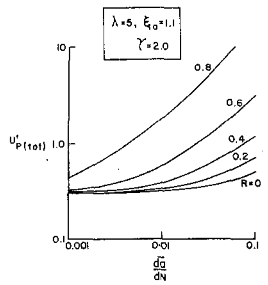

Figure 5. Variation of $U'_{P(tot)}$ with $d\bar{a}/dN$ for different values of stress ratio, R.

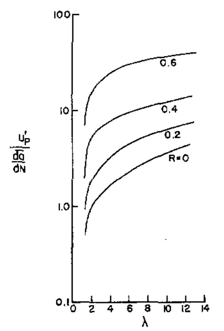

Figure 6. Variation of $U'_P/(d\bar{a}/dN)$ with λ for different values of stress ratio, R.

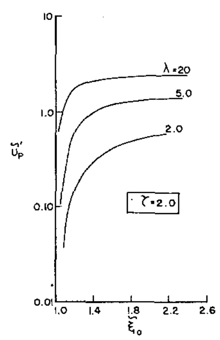

Figure 7. Non-dimensionalised hysteresis energy dissipated per cycle as a function of $\bar{\xi}_0$ with λ as parameter.

(b) *Energy balance equation for fatigue crack growth*

At the outset it is in order to comment here that the energy balance equation can be generalised to include damaging effects due to corrosion, creep, etc. In such cases, interactions will have to be considered. Generalisation for multiaxial stress system is also feasible in principle. The exact way of doing it is a question which can only be left unanswered at the present stage of development of this formulation. However it is worth noting that the important effect of thickness can be brought in through the term $w_{P_{cr}}$ in equation (22) or K_C in equation (23).

It is important to notice that equation (22) is valid at very high growth rates (nearing final fracture) only when the basic assumption that linear fracture mechanics applies, is valid. In ductile materials this is not usually the case. For such conditions, i.e., at very high growth rates, equation (21) has to be used. These comments generally indicate the limitations of the formulation.

As has been mentioned earlier, it is difficult to define the nature of the function $f(\sigma_{\text{hyd}})$ in equation (21). It is however felt that an insight into the nature of this function can probably be obtained by studying the delay effect in crack growth produced by intermittent high stress cycles. In the analysis for energy of plastic deformation, presented in this paper, no consideration has been given to the effect of cyclic strain hardening on the energy of plastic deformation at the crack tip. The changes that are produced in the stress field at the crack tip due to cyclic strain hardening are complex.

Conclusions

An approximate analysis for energy of plastic deformation occurring in the plastic enclave at the tip of a crack growing under sinusoidal loading with $R \geq 0$ has been presented. The analysis is based on a detailed consideration of the plastic deformation history in the plastic enclave.

The analysis indicates

(i) The hysteretic part of the energy of plastic deformation occurring in the cyclic plastic enclave is independent of the rate of growth of the crack.

(ii) The non-hysteretic part of the energy of plastic deformation comes about only due to the extension of the crack. It is also a function of the stress ratio R and the material parameters γ and λ.

An energy balance equation applicable to crack extension occurring in each cycle of fatigue loading has been presented. The limitations and potential for generalisation of the energy formulation of the fatigue crack growth problem have been discussed.

Acknowledgements

The author wishes to express his appreciation to Mr. M. V. V. Murthy and Mr. C. G. Shah for their help in numerical part of the work.

Appendix I

(a) *Derivation of Equations* (1), (2) *and* (10)

Equation 1

Referring to Figs. (1) and (2) let $\tilde{\tau}_{\text{oct}}$ be the octahedral shear stress range at any point (r, θ) in the cyclic plastic enclave. The hysteresis energy density $\tilde{u}_{\text{p}}$ at (r, θ) in the cyclic plastic enclave is given by

$$\tilde{u}_{\text{p}} = \tfrac{3}{2}\tilde{\tau}_{\text{oli}}\tilde{\gamma}_{\text{p}} \tag{a}$$

where $\tilde{\gamma}_{\text{p}}$ is the cyclic plastic strain

$\tilde{\gamma}_{\text{p}}$ is given by

$$\tilde{\gamma}_{\text{p}} = (\tilde{\tau}_{\text{oct}} - \tilde{\tau}_{\text{oli}})\left(\frac{1}{G_{\text{t}}} - \frac{1}{G}\right) \tag{b}$$

substituting in (a) we obtain

$$\tilde{u}_{\text{p}} = \tfrac{3}{2}\tilde{\tau}_{\text{oli}}(\tilde{\tau}_{\text{oct}} - \tilde{\tau}_{\text{oli}})\left(\frac{1}{G_{\text{t}}} - \frac{1}{G}\right).$$

Simplifying, we get

$$\tilde{u}_{\text{p}} = \frac{3}{2}\frac{\tilde{\tau}_{\text{oli}}^2}{G}(\tilde{\xi} - 1)(\lambda - 1)\,. \tag{c}$$

Equation 2

Referring to Fig. (2), we have

$$\tilde{G}_s = \frac{\tilde{\tau}_{oct}}{\tilde{\gamma}_{oct}} = \frac{\tilde{\tau}_{oct}}{\dfrac{\tilde{\tau}_{oli}}{G} + \dfrac{\tilde{\tau}_{oct} - \tilde{\tau}_{oli}}{G_t}}$$

Simplifying we have

$$\frac{\tilde{G}_s}{G} = \frac{\tilde{\xi}}{1+\lambda(\tilde{\xi}-1)} . \tag{d}$$

Using Dixon's approximate relation between total strains and the corresponding strains obtained by elastic analysis and assuming $\nu = 0.5$, we have

$$\tilde{\tau}_{oct} = \left(\frac{\tilde{G}_s}{G}\right)^{\frac{1}{2}} \cdot \tilde{\tau}^{el}_{oct} . \tag{e}$$

Substituting for $\tilde{G}_s/G$ from equation (d) and normalising with $\tilde{\tau}_{oli}$, equation (e) can be written as

$$\tilde{\alpha} = \{\tilde{\xi}[1+\lambda(\tilde{\xi}-1)]\}^{\frac{1}{2}} . \tag{f}$$

Observing that $\tilde{\tau}^{el}_{oct}$ is given by

$$\tilde{\tau}^{el}_{oct} = \frac{K_R}{(2r)^{\frac{1}{2}}} f_e$$

and $r = \tilde{w}_p$ when $\tilde{\tau}^{el}_{oct} = \tilde{\tau}_{oli}$ we obtain

$$\tilde{\alpha} = \left(\frac{w_p f_e}{r}\right)^{\frac{1}{2}} . \tag{g}$$

From equations (f) and (g) we have the relation

$$\tilde{\alpha} = \{\tilde{\xi}[1+\lambda(\tilde{\xi}-1)]\}^{\frac{1}{2}} = \left(\frac{\tilde{w}_p \cdot f_e}{r}\right)^{\frac{1}{2}} .$$

Equation 10

Referring to Fig. (1) the non-hysteretic plastic deformation energy density at (r, θ), resulting from crack extension in each cycle, is given by

$$u_p = \frac{3}{2} \tau_{oct} \cdot \frac{d\gamma_p}{dN} + \frac{3}{4} \cdot \frac{d\tau_{oct}}{dN} \cdot \frac{d\gamma_p}{dN} . \tag{h}$$

But $d\gamma_p/dN$ is given by

$$\frac{d\gamma_p}{dN} = \frac{(\lambda-1)}{G} \cdot \frac{d\tau_{oct}}{dN} . \tag{i}$$

Substituting equation (i) in equation (h), we have

$$u_p = \frac{3}{2} \frac{\tau_{oct}}{G} \cdot \frac{d\tau_{oct}}{dN} (\lambda-1) + \frac{3}{4G} \frac{d\tau_{oct}}{dN}^{2} (\lambda-1) . \tag{j}$$

Normalising, bu dividing equation (j) by τ_{oli} we have

$$u_p = \frac{3}{2} \frac{\tau^2_{oli}}{G} \xi(\lambda-1) \frac{d\xi}{dN} + \frac{3}{4} \frac{\tau^2_{oli}}{G} (\lambda-1) \left(\frac{d\xi}{dN}\right)^2 .$$

Since $d\xi/dN$ can be written as

$$\frac{d\xi}{dN} = \frac{d\xi}{da} \cdot \frac{da}{dN} .$$

We have

$$u_p = \frac{3}{2}\frac{\tau_{oli}^2}{G}\xi(\lambda-1)\frac{d\xi}{da}\cdot\frac{da}{dN} + \frac{3}{4}\frac{\tau_{oli}^2}{G}(\lambda-1)\left(\frac{d\xi}{da}\cdot\frac{da}{dN}\right)^2 .$$

(b) *Derivation of relation between* w_p *and* $\tilde{w}_p$

We have

$$\tilde{w}_p = \frac{K_R^2}{9\tilde{\tau}_{oli}^2}$$

and

$$w_p = \frac{K_{max}^2}{9\tau_{oli}^2} .$$

Let K_{min} be the stress intensity factor corresponding to the minimum stress in the cycle. Then we have

$$\tilde{w}_p = \frac{(K_{max} - K_{min})^2}{9\tilde{\tau}_{oli}^2} .$$

Simplifying we get

$$\tilde{w}_p = w_p \left(\frac{1-R}{\gamma}\right)^2 . \qquad \text{(k)}$$

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RÉSUMÉ

On a effectué une analyse approchée de l'énergie de déformation plastique à l'extrémité d'une fissure au cours de propagation sous contraintes sinusoïdales d'amplitude constante, en recourant à une représentation bilinéaire de la fonction tension-déformation. L'énergie de déformation plastique provient de déformations plastiques de caractères hystérétique et non hystérétique.

On démontre que l'énergie associée à l'hystérésis ne dépend pas de la vitesse de propagation de la fissure, tandis que l'énergie associée aux déformations plastiques non hystérétiques en dépend. On ne considère pas dans l'analyse l'écrouissage résultant des déformations plastiques hystérétiques.

Le critère d'équilibre des énergies, fondamental en mécanique de la rupture, a été appliqué au problème de la propagation des fissures sous charges cycliques. L'énergie considérée est celle qui résulte de la déformation plastique hystérétique dans l'enclave plastique à la pointe de la fissure. En exprimant l'équilibre énergétique sous sa forme analytique, on trouve une expression de la propagation des fissures qui satisfait aux tendances générales observées expérimentalement.

On discute enfin les avantages et les limitations d'une formulation de la propagation des fissures de fatigue, basée sur des considérations énergétiques.

ZUSAMMENFASSUNG

Mit Hilfe einer bilinearen Darstellung der Abhängigkeit Spannung-Verformung wird eine angenäherte Analyse der plastischen Verformungsenergie an der Spitze eines sich unter sinusoïdalen Spannungen konstanter Amplitude entstehenden Risses vorgenommen. Diese Energie ergibt sich aus hysteretischen und nicht hysteretischen Verformungen. Es wird gezeigt, daß die sich aus der Hysteresis ergebende Energie nicht von der Fortpflanzungsenergie abhängig ist, im Gegensatz zum Energieanteil der nicht hysteretischen plastischen Verformung, welche von der Rißwachstumsgeschwindigkeit abhängt. Bei der vorliegenden Analyse wird die Verformungshärtung, die sich aus der hysteretischen plastischen Verformung ergibt, nicht berücksichtigt.

Das Energiegleichsgewichtkriterium, welches für die Bruchmechanik von grundlegender Bedeutung ist, wurde auf den Fall der Rißfortpflanzung unter zyklischer Beanspruchung angewendet, wobei die sich durch die hysteretische plastische Verformung in der plastischen Enklave ergebende Energie aus der oben erwähnten Analyse erhalten wurde. Die Energiebilanzgleichung führt zu einem Ausdruck der Rißfortpflanzungsgeschwindigkeit, welcher mit dem bei den Versuchen beobachteten allgemeinen Verlauf übereinstimmt.

Es werden einige Vorteile und Begrenzungen einer auf Energieüberlegungen begründeten Formulierung der Ermüdungsrißfortpflanzung erörtert.

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