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Delayed Failure — The Griffith Problem for Linearly Viscoelastic Materials*

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ABSTRACT

The unstable growth of a crack in a large viscoelastic plate is considered within the framework of continuum mechanics. Starting from the local stress and deformation fields at the tip of the crack, a non-linear, first order differential equation is found to describe the time history of the crack size if the stress applied far from the crack is constant. The differential equation contains the creep compliance and the intrinsic surface energy of the material. The surface energy concept for viscoelastic materials is clarified. Inertial effects are not considered, but the influence of temperature is included for thermorheologically simple materials.

Initial crack velocities are given as a function of applied load in closed form, as well as a comparison of calculated crack growth history with experiments. Above a certain high stress, crack propagation ensues at high speeds controlled by material inertia while at a lower limit infinite time is required to produce crack growth. Thus an upper and lower limit criterion of the Griffith type exists. For rate insensitive (elastic) materials the two limits coalesce and only the brittle fracture criterion of Griffith exists. The implications of these results for creep fracture in metals and inorganic glasses are examined.

1. Introduction

There are few circumstances more disconcerting to the user of structural materials than a lack of knowledge about the load bearing ability of a material which would be otherwise desirable for a particular application. There are many examples where hard and soft polymers could perform structurally in an effective manner, yet they are excluded from consideration. The reason may not be so much that they are weaker than the structural metals, but that their failure behavior appears erratic and seemingly eludes quantitative prediction.

For hard and soft polymers it is known that the lifetime under load depends to varying degrees on the size and history of the applied load, high loads leading to short life times and conversely. Although laboratory experiments can go a long way in clarifying the failure behavior, the extrapolation of laboratory tests to less restricted conditions must come from theoretical treatments.

Since fracture of structural members is the result of crack growth, it is necessary to understand the growth of cracks under arbitrary load histories. The phenomenon of delayed crack growth is not unique to polymeric materials but is observed also for the much less rate sensitive materials such as inorganic glasses [1] and metals. With regard to the latter, Johnson and Paris [2] write:

> "Of the various known modes of subcritical flaw growth, perhaps the most surprising and least understood is the subcritical flaw growth under constant load in a chemically inert environment".

While we shall deal here primarily with crack growth in strongly viscoelastic materials, it is hoped that the results may shed some new light on the delayed fracture in less rate sensitive materials such as glass and metals.

In the following we restrict our considerations to linearly viscoelastic materials. Nevertheless, we expect that the results carry implications for non-linearly viscoelastic materials to

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the same extent that Griffith's linear analysis for brittle materials has enlivened the understanding of fracture in non-linear solids.

In an earlier paper [3], the author proposed a criterion for the unstable crack growth in a viscoelastic material which was based on a critical crack speed transition. While the criterion gave very good agreement with experimental results, it left some basic questions unanswered. In particular, that earlier paper treated in detail the initiation of fracture in polymeric solids, and, for lack of a law governing the growth of a macroscopic crack, substituted the crack initiation law. The current paper supplements the earlier work by deriving the law governing the growth of a macroscopic crack.

Specifically, we shall here be concerned with the time history of crack growth in a large sheet under in-plane loading and discuss in detail the effect of a constant tensile stress applied far from the crack surfaces. In order to keep the analysis reasonably simple and to expose clearly the effect of viscoelastic properties only, material inertia is excluded from consideration.

We are thus dealing with the analog to the classical Griffith problem [4], except that our interest centers now on the detailed crack growth history rather than only on a criterion as to whether the crack will grow or not.

Although there is a large bibliography on fracture of viscoelastic materials, theories on crack propagation are few. In 1961 Williams and Schapery [5, 6] calculated, by using a maximum strain criterion and a Voigt model for the viscoelastic properties representation, an exponential crack growth rate, the same being shown to hold for a shear crack by McClintock [7]. Bueche and Halpin [8] adopted part of this model to predict failure properties of soft polymers. However, the crack propagation concept was rudimentary, if not incorrect [9]. An empirical approach for crack propagation in rubbery materials was suggested by Lake and Lindley [10]. In a recent paper [11] the author suggested a similar approach, which unlike that work [10] takes into account the detailed stress distribution around the tip of a crack and which formulates the problem of crack propagation in terms of a non-linear differential equation.

A prime difficulty in the calculation of viscoelastic crack growth is the fact that the prerequisite viscoelastic stress analysis is complicated by the prescription of time varying discontinuous boundary conditions. Following the author's observation [12] that the Griffith stability criterion for cavities of various shapes changes only by a multiplicative constant and not in form, Williams has modeled crack propagation by the more easily analyzed geometries of spherical and cylindrical voids in two- and three-dimensional stress fields [13, 14]. This was done in the hope of including dissipative processes during time-varying deformations.

In this paper we examine the local phenomena at the crack tip by considering the rate of work released by the forces at the crack tip as they unload from high values to zero as the crack passes by. This work follows that of Irwin [15] who demonstrated that this local unloading was equivalent to the global Griffith criterion. Having been successful in corroborating experiments with cracks travelling with constant speed in a long strip [16, 17] over a very wide range of crack speeds we apply this concept now to the non-steady growth of a crack in a large sheet.*

2. Local Fracture Model and Power Equation

In attempting to model the local fracture process at the tip of an advancing crack after actual observations, one may be led to insurmountable analytic difficulties because of the exceedingly complex microscopic geometries encountered at the crack tip [11]. However, if one retains the simplest notion of the continuous fracture process, one arrives at the following picture.

Consider a point on the line of crack propagation and ahead of the advancing crack-tip. As the crack approaches that point, the stress normal to the line of crack propagation rises to a maximum and then falls off to zero if the crack surface is traction free. Figure 1 illustrates a

* During the editing phase of this paper it was brought to the author's attention that Cherepanov [18] has treated some special cases of crack propagation in simple viscoelastic solids by an energy criterion of the type treated in this country by Rice [19]. Cherapanov's general conclusions agree with the results to be presented here in more detail.

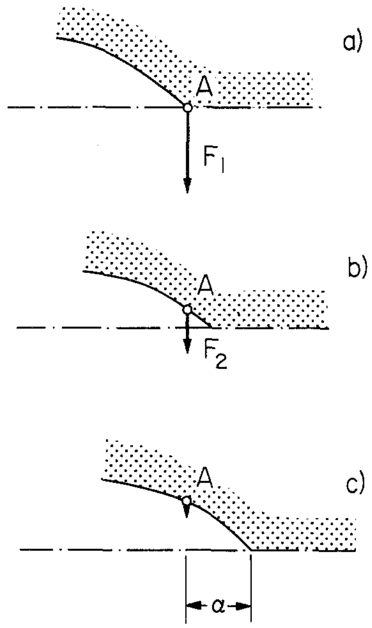

Figure 1. Crack unloading sequence (see text for explanation).

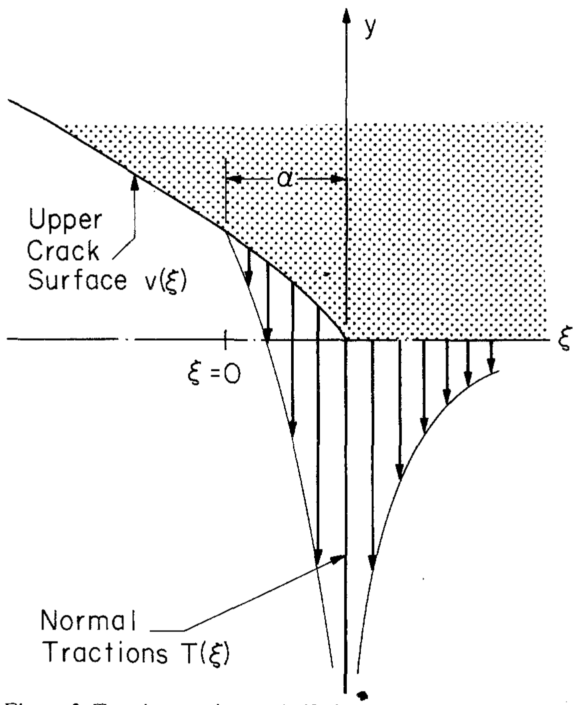

Figure 2. Tractions acting on half-plane generated by advancing crack.

sequence of the unloading process. In Figure 1a at time t, the maximum force F_1 acts at point A, a small time increment later the cohesive force at A has decreased while the crack opens up a little (Figure 1b) and at a time Δt later, during which the crack has travelled a small distance α, the force at A has dropped to zero (Figure 1c). The instantaneous stress distribution in the vicinity of the crack is then as shown qualitatively in Figure 2; it resembles the Barenblatt model rather than the classical elasticity distribution, a fact which is of relatively little consequence to our further treatment.

It is a straightforward matter to show [15, 16, 17] that the rate at which surface energy is consumed is equal to the rate at which the tractions $T(x)$ do work on the half space $y \geqq 0$. If the rate of surface energy increase is $\Gamma \dot{c}$, c being the crack half length, the dot denoting differentiation with respect to time and Γ the energy to create one unit of new surface, then one has

$$\int_{-\alpha}^{\infty} T(\xi)\dot{v}(\xi)\mathrm{d}\xi = \int_{-\alpha}^{0} T(\xi)\dot{v}(\xi)\mathrm{d}\xi = \Gamma \dot{c} \tag{1}$$

v being the displacement of the line $y=0$. The upper limit on the second integral results from the fact that $v=\dot{v}=0$ for $\xi \geqq 0$.

The immediate difficulty is that the tractions $T(\xi)$ in $-\alpha \leqq \xi \leqq 0$ are not known and because the displacement $v(\xi)$ depends on these tractions, they are not known, either.

Let us assume that the stress and displacement distribution is given by the linearly viscoelastic solution. If $\sigma_y(\xi, 0, t)$ is the traction on the upper half plane for $\xi > 0$ and $v(\xi, 0, t)$ is the displacement of the crack surface for $\xi \leqq \alpha$ then this assumption may be expressed more explicitly in terms of (1) as [17]

$$\tfrac{1}{2}\int_{0}^{\alpha} \sigma_y(\xi, 0, t)\dot{v}(\xi-\alpha, 0, t)\mathrm{d}\xi = \Gamma \dot{c} \tag{2}$$

The factor $\frac{1}{2}$ is introduced because at a given point—say point A in Figure 1—the force decays with time and the full force does not act through the whole displacements v. In effect only the average of the force at point A acts through the displacement v.

There are several compelling reasons for making this assumption:

First, the specific traction and displacement functions allow us to proceed towards explicit results which can be compared quantitatively with experiments.

Second, it is only the integrated effect of these distributions which is of final importance and the details of the distributions are, to a large degree, lost in the integration process. This observation is clearly reflected in the fact that for the time-independent material response the fracture criteria according to the Griffith–Irwin, Barenblatt and Dugdale models—the latter for very limited ductility—agree, in spite of the different stress distributions around the tip of the crack [20, 21].

Third, the desired limit cases for brittle fracture result.

Some simple checks on the effect of different stress distributions have been examined and did not appear to lead to markedly different results. Leaving the resolution of this detail to the future, we proceed to derive the necessary crack boundary displacement for a crack moving in a viscoelastic sheet in order to evaluate the fracture growth law (2).

3. Stress and Displacement Analysis

The stress distribution in a cracked, large viscoelastic sheet under in-plane tractions far away from the crack is independent of the material properties (generalized plane stress) and hence is the same as in an elastic material. The same is, of course, not true with respect to the displacements which depend on the material properties through the stress-strain relations.

Consider the thin sheet geometry in Figure 3 under uniaxial loading $\sigma_0(t)$. If $\sigma_0(t)$ is a step function in time, $\sigma_0 1(t)$, then the displacement along the crack axis for a non growing crack in a viscoelastic sheet is

$$v(x, c, t) = \begin{cases} 2cD_{\mathrm{cr}}(t)\sigma_0\left\{1-\left(\dfrac{x}{c}\right)^2\right\}^{\frac{1}{2}} & x \leqq c \\ 0 & x \geqq c \end{cases} \tag{3}$$

The displacement for a growing crack in a viscoelastic sheet can be constructed from (3) by a sequence of loading and unloading steps [16].

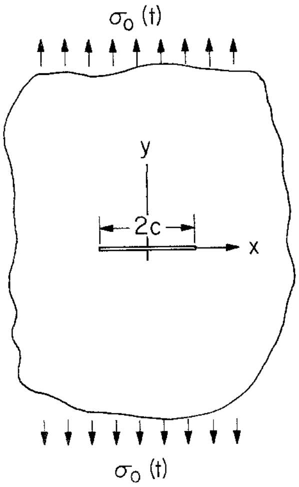

Figure 3. Sheet geometry and load definition.

$$v(x, c, t) = \begin{cases} 2\int_0^t \{c(\tau)^2 - x^2\}^{\frac{1}{2}} \sigma_0(\tau) \dot{D}_{cr}(t-\tau)\mathrm{d}\tau & x \leqq c(t) \\ 0 & x \geqq c(t) \end{cases} \tag{4}$$

where $\dot{D}_{cr}$ is the rate of change of the creep compliance with respect to its total argument. The formality of the integral consists in the meaning of the lower limit of the integral; for points $x \leqq c(0)$ it implies the time at which the crack tip reaches the point x.

The displacement rate $\dot{v}$ is now given by

$$\dot{v}(x, c, t) = 2\int_0^t \frac{\partial}{\partial t}\{[c(\tau)^2 - x^2]^{\frac{1}{2}}\} \sigma_0(\tau) \dot{D}_{cr}(t-\tau)\mathrm{d}\tau + \begin{cases} \left[\dfrac{c\dot{c}\sigma_0(t)}{[c^2(t)-x^2]^{\frac{1}{2}}} + [c^2(t)-x^2]^{\frac{1}{2}}\dot{\sigma}_0(t)\right] D(0)\, ; & x \leqq c(t) \\ 0\, ; & x \geqq c(t) \end{cases} \tag{5}$$

if the initial and later jumps in the applied stress or crack length are absent. For step loading $\sigma(t) = \sigma_0 1(t)$ one obtains

$$\dot{v}(x, c, t) = 2\sigma_0 D_{cr}(0) \frac{c(t)\dot{c}(t)}{(c(t)^2 - x^2)^{\frac{1}{2}}} + 2\sigma_0 [\{c(0)^2 - x^2\}^{\frac{1}{2}}]_{x \leqq c(0)} \dot{D}_{cr}(t) + \begin{cases} 2\sigma_0 \displaystyle\int_0^t \frac{c(\tau)\dot{c}(\tau)\dot{D}_{cr}(t-\tau)\mathrm{d}\tau}{[c(\tau)^2 - x^2]^{\frac{1}{2}}}\, ; & x \leqq c(t) \\ 0\, ; & x \geqq c(t) \end{cases} \tag{6}$$

When dealing with crack growth the term containing $\dot{D}_{cr}(0)$ is of little consequence since it is valid only for the initial geometry. It is therefore only important in calculating that time after load application at which crack growth begins. This time t^* (incubation time), is on the order of the relaxation times of the material and related to the applied load σ_0 [22]

$$\sigma_0^2 D_{cr}(t^*) = \sigma_g^2 D_{cr}(0) \tag{7}$$

where σ_g is the minimum stress required to cause instantaneous crack propagation. A similar result for the incubation time has been given by Williams [13]. In contrast, subsequent times to reach high crack speeds, indicative of catastrophic failures, are larger by several orders of magnitude. Invoking the assumption that crack propagation starts immediately after the step load has been applied allows us to simplify (6). Inclusion of the incubation time in the time scale merely requires the addition of t^* from equation (7). Alternately, one may write equation (6) for $1-x/c(t)\ll 1$ as

$$\dot{v}(x,c,t)=\begin{cases}\sqrt{2}\,\sigma_0 D_{cr}(0)\dfrac{[c(t)]^{\frac{1}{2}}\dot{c}(t)}{[c(t)-x]^{\frac{1}{2}}}+\sqrt{2}\,\sigma_0\displaystyle\int_{t^*\doteq 0}^{t}\dfrac{[c(\tau)]^{\frac{1}{2}}\dot{c}(\tau)}{[c(\tau)-x]^{\frac{1}{2}}}\dot{D}_{cr}(t-\tau)\,d\tau & x\leqq c(t)\\ 0 & ;\ x\geqq c(t);\end{cases} \tag{8}$$

We must now resolve the formality of the lower limit in the integral, taking into account that the absolute lower limit $t^*\doteq 0$ is valid only for the initial configuration while actually it depends on time through the crack propagation because of the discontinuous displacement rates around the advancing crack tip. Let us consider the integral

$$J(x,t)=\int_0^t\frac{[c(\tau)]^{\frac{1}{2}}\dot{c}(\tau)}{[c(\tau)-x]^{\frac{1}{2}}}\dot{D}_{cr}(t-\tau)d\tau\,. \tag{9}$$

The transformation

$$t=\int_{c(0)}^{c}\frac{dc}{\dot{c}}\geqq\tau=\int_{c(0)}^{\beta}\frac{dc}{\dot{c}} \tag{10}$$

permits the integral (9) to be written as

$$J(x,t)=\int_x^{c(t)}\left(\frac{\beta}{\beta-x}\right)^{\frac{1}{2}}\dot{D}_{cr}\left[\int_\beta^c\frac{dc}{\dot{c}}\right]d\beta \tag{11}$$

In order to evaluate (11) further*, we observe that the function $J(x,t)$ is required only for $1-x/c(t)\ll 1$. Indeed, the maximum value of $c(t)-x=a$ was found in [17] to be extremely small. Therefore, if the rate of velocity, i.e., the acceleration is small in any small interval $x\leqq\beta\leqq c(t)$ and $1-x/c(t)\ll 1$, for the purpose of the integration (11) the integral in the argument of $\dot{D}_{cr}$ may be replaced by $(c-\beta)/\dot{c}$.

$$J(x,t)=\int_x^{c(t)}\left(\frac{\beta}{\beta-x}\right)^{\frac{1}{2}}\dot{D}_{cr}\left[\frac{c(t)-\beta}{\dot{c}}\right]d\beta \tag{12}$$

Let the creep compliance be represented by the spectral representation

$$D_{cr}(t)=D_\infty-\int_0^\infty L(\theta)\exp(-t/\theta)d\theta\,. \tag{13}$$

Using (13) the integral (12) may then be written as**

$$J(x,t)=\int_0^\infty\frac{L(\theta)}{\theta}\int_x^{c(t)}\left(\frac{\beta}{\beta-x}\right)^{\frac{1}{2}}\exp\left[-\frac{c(t)-\beta}{\dot{c}\theta}\right]d\beta\,d\theta\,. \tag{14}$$

The major variation of the internal integrand arises from the singular factor $(\beta-x)^{-\frac{1}{2}}$, modulated by the exponential factor, the variation of $\sqrt{\beta}$ being insignificant. This fact allows one to evaluate the integral, which evaluation results, after some algebraic manipulation, in

$$\dot{v}[x,c(t)]=\sigma_0[2c(t)]^{\frac{1}{2}}\left\{\frac{D_c(0)\dot{c}}{[c(t)-x]^{\frac{1}{2}}}+[c(t)-x]^{\frac{1}{2}}\int_0^\infty 2\frac{L\theta}{\theta}\sum_{n=0}^{\infty}\frac{\left[-2\dfrac{c(t)-x}{\dot{c}\theta}\right]^n}{(1;2;n+1)}d\theta\right\}. \tag{15}$$

where $(1;2n+1)\equiv 1\cdot 3\cdot 5\ldots(1+2n)$.

* It is of interest to point out that if (11) along with (8) were used in the power equation (2) there would result a non-linear integro-differential equation for the crack size c, similar to the one derived by Chrepanov [18]. However, we seek here a more explicit and detailed solution.

** Here as later we assume that interchange of orders of integration is permissible.

4. Evaluation of the Power Equation and Griffith Limits

As pointed out during the development of the model we shall use the linearly (visco)elastic stress distribution

$$\sigma_y(\xi, 0, t) = \frac{\sigma_0[c(t)]^{\frac{1}{2}}}{(2\xi)^{\frac{1}{2}}} \tag{16}$$

Substitution of (15) and (16) into the power equation (2) leads, after division by $\dot{c}$ to the following expression

$$\int_0^\alpha \frac{\sigma_0^2 c(t)}{\sqrt{\xi}} \left\{ \frac{D_{cr}(0)}{(\alpha-\xi)^{\frac{1}{2}}} + (\xi-\alpha)^{\frac{1}{2}} \int_0^\infty 2\frac{L(\theta)}{\theta \dot{c}} \sum_{n=0}^\infty \frac{\left[-2\dfrac{\alpha-\xi}{\dot{c}\theta}\right]^n}{(1;2;n+1)} \mathrm{d}\theta \right\} \mathrm{d}\xi = 2\Gamma \tag{17}$$

By use of elementary integrals with respect to ξ and subsequent summation of the resulting infinite series this expression may be reduced without approximation to

$$\frac{2\Gamma}{\pi\sigma_0^2 c(t)} = D_{cr}\left(\frac{\alpha}{\dot{c}}\right) \tag{18}$$

Equation (18) is a non-linear, first order differential equation for the crack length $c(t)$, with the initial condition $c(0)=c_0$. For the subsequent discussion it is advantageous to transform this

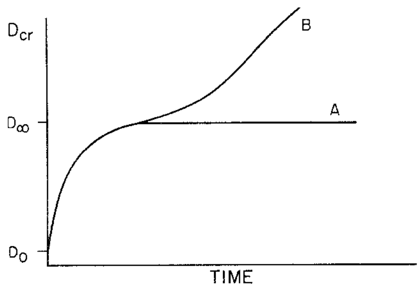

Figure 4. Creep compliance for a) cross linked polymer, b) uncross linked polymer.

equation slightly, taking into account also the time-temperature reduction if the material is thermo-rheologically simple [23].

Let

$$\sigma_{g0} \equiv \left(\frac{2\Gamma E_0}{\pi c_0}\right)^{\frac{1}{2}}$$

be the Griffith instability stress based on the short time modulus E_0. Furthermore, let $c(t)/c_0 = u(t)$ and

$$\Psi(t) \equiv D_{cr}(t)/D_0 \; ; \quad \Psi(0) = 1$$

be the normalized creep function, two typical functions being given in Figure 4. Then one has from (18)

$$\left(\frac{\sigma_{g0}}{\sigma_0}\right)^2 = \frac{T_0}{T} u\Psi\left(\frac{\alpha}{c_0} \frac{1}{\dot{u}a_T}\right) \tag{19}$$ ★

and upon defining the inverse of $s = \Psi(t)$ as $t = \Psi^{-1}(s)$

★ The temperature ratio T_0/T has been shown valid only for the long time behavior, although it is often used throughout the whole time range. Its effect on the time scale is small compared to that of a_T. We carry the temperature ratio along as a remainder of a more complete temperature reduction, but will not use it in later calculations.

$$\dot{u} = \frac{\alpha}{c_0 a_T} \left\{ \Psi^{-1} \left[\frac{T}{T_0} \left(\frac{\sigma_{g0}}{\sigma_0} \right)^2 \frac{1}{u} \right] \right\}^{-1} \tag{20}$$

At time $t=0$ when $u=1$ one has for the initial speed of crack growth

$$c_0 \dot{u} = \frac{\alpha}{a_T} \left\{ \Psi^{-1} \left[\frac{T}{T_0} \left(\frac{\sigma_{g0}}{\sigma_0} \right)^2 \right] \right\}^{-1} \tag{21}$$

Because the right hand side of Eq. (20) is a monotonically increasing function of u it suffices to study the stability* of a crack through the initial relation (21).

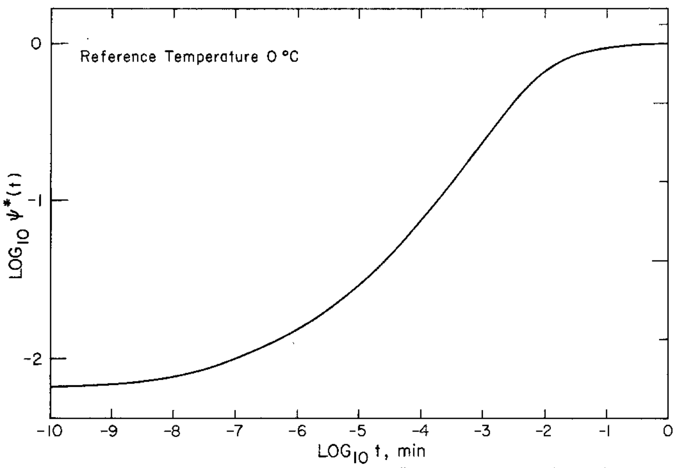

Figure 5. Normalized creep compliance, Ψ^*, solithane 113 (50/50) from ref. 25.

Consider first curve A in Figure 5. As the argument of Ψ^{-1} approaches $\Psi(\infty)$, Ψ^{-1} approaches infinity rapidly and thus $u(0) \to 0$. Therefore,

$$u = 0 \quad \text{for} \quad \frac{T}{T_0} \left(\frac{\sigma_{g0}}{\sigma_0} \right)^2 = \Psi(\infty) \tag{22}$$

Since $\Psi(\infty) = D_{cr}(\infty)/D_{cr}(0) = E_0/E_\infty$ for a material of the type A in Figure 5, E_∞ being the long time Young's modulus, (22) may be written as

$$\frac{T}{T_0} \left(\frac{\sigma_{g\infty}}{\sigma_0} \right)^2 = 1 \tag{23}$$

where

$$\sigma_{g\infty} = \left(\frac{2\Gamma E_\infty}{\pi c_0} \right)^{\frac{1}{2}}$$

is the Griffith stress based on the long time equilibrium modulus. If the argument of Ψ^{-1} exceeds Ψ_∞, i.e., if

$$T_0 \sigma_0^2 < T \sigma_{g\infty}^2 \tag{24}$$

Eq. (21) has no solution and thus crack propagation is not possible.

Now consider the other extreme; when the argument of Ψ^{-1} tends toward zero the velocity $\dot{v}$ tends toward infinity.

* For our purposes we define stability as did Shield and Green [24]: "An equilibrium state is stable whenever in the motion following any sufficiently small changes in the body and surface forces, these changes being made for a finite interval of time only, the displacement $\boldsymbol{u}(x, t)$ from the equilibrium state and the velocity $\partial \boldsymbol{u}/\partial t$ are everywhere arbitrarily small in magnitude". This interpretation denotes a crack as unstable if it grows at all and does not admit the motion of a growing subcritical crack. The common definition of a crack instability by virtue of a slow-to-fast crack growth transition is fraught with arbitrariness since it depends on time resolution of the observer's experimental ability.

Upon examining the limit cases for the material represented by curve B in Figure 5 we can establish the following limit theorems.

1) If the creep compliance (relaxation modulus) of a material possesses an upper (lower) limit, then there exists a limit stress $\sigma_{g\infty}$ below which a crack will not propagate. Such a limit is not assured if the compliance (modulus) has no (zero) limit.

2) In the absence of inertial constraints on crack growth the crack will propagate immediately at infinite (high) velocity if the applied stress σ_0 exceeds the Griffith stress based on the zero time elastic modulus, $\sigma_{g\infty}$.

3) If the applied stress is intermediate to these two limits the crack will start, in the absence of inertial constraints, with a finite velocity (equation 21) and accelerate monotonically to infinite (high) velocity.

Having established physically reasonable and expected limit behavior of the derived differential equation we proceed now to examine the growth history of a crack in more detail.

5. Crack Growth in a Realistic Material

Equation (20) can be integrated explicitly for the time t to find

$$\frac{\alpha t}{c_0 a_T} = \int_1^u \Psi^{*-1}\left[\frac{T}{T_0}\left(\frac{\sigma_{g\infty}}{\sigma_0}\right)^2 \frac{1}{u}\right] \mathrm{d}u \tag{24}$$

where Ψ^{*-1} is the inverse of $\Psi^*(t) = D_{cr}(t)/D_{cr}(\infty)$. Let us define the function $\mathfrak{F}^*(\mu)$ by the integral

$$\mathfrak{F}^*(\mu) = \int_\varepsilon^\mu \Psi^{*-1}(v)\frac{\mathrm{d}v}{v^2} \tag{25}$$

where $\varepsilon = D_{cr}(0)/D_{cr}(\infty)$. Through the transformation

$$\mu = \frac{T}{T_0}\left(\frac{\sigma_{g\infty}}{\sigma_0}\right)^2 \frac{1}{u}$$

equation (25) can be written as

$$\frac{\alpha t}{c_0 a_T} = \frac{T}{T_0}\left(\frac{\sigma_{g\infty}}{\sigma_0}\right)^2 \left\{\mathfrak{F}^*\left[\frac{T}{T_0}\left(\frac{\sigma_{g\infty}}{\sigma_0}\right)^2\right] - \mathfrak{F}^*\left[\frac{T}{T_0}\left(\frac{\sigma_{g\infty}}{\sigma_0}\right)^2 \frac{1}{u}\right]\right\} \tag{26}$$

The function $\mathfrak{F}^*(\mu)$ is a material property and can be calculated without reference to the

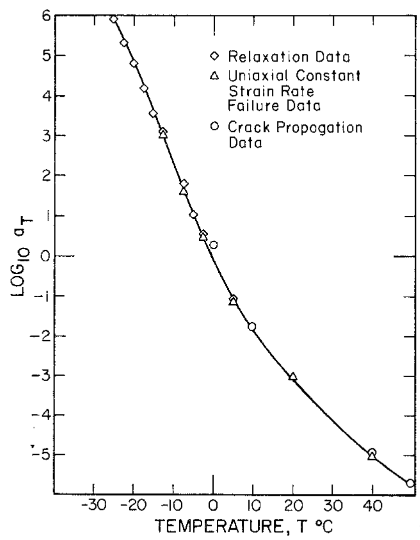

Figure 6. Time–temperature shift factor for Solithane 113 (50/50) ref. 25.

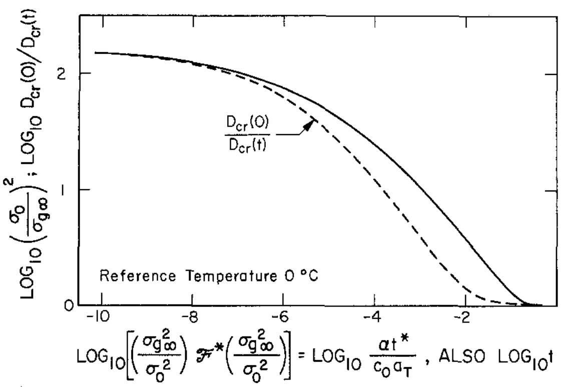

Figure 7. Function $\nu\mathfrak{F}^*(\nu)$ *vs* ν compared with reciprocal creep compliances $D_{cr}(0)/D_{cr}(t)$.

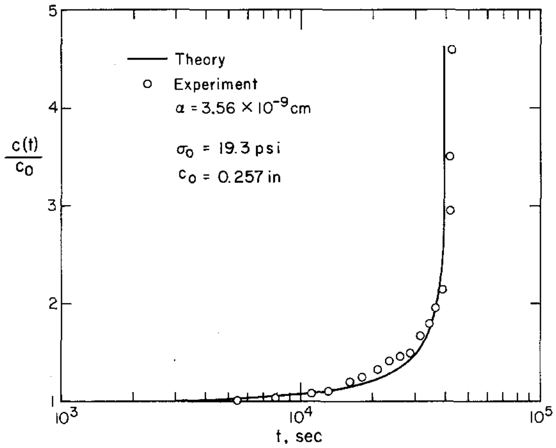

Figure 8. Theoretical and experimental crack growth history.

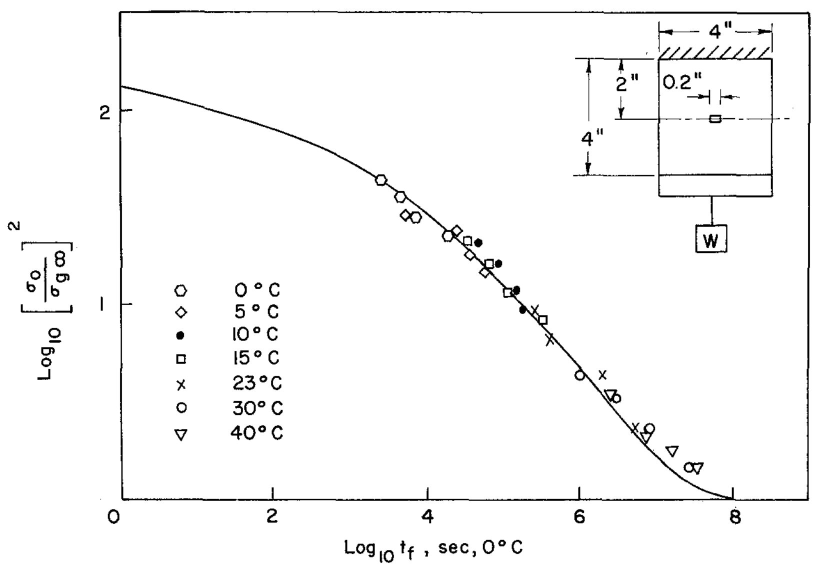

Figure 9. Failure times t_f as a function of the applied gross stress σ_0. Comparison of experimental points and theoretical relation. (Eq. 27, solid line.)

cracked sheet geometry. Figure 5 shows the normalized creep compliance Ψ^* of Solithane 113 (50/50), a polyurethane rubber [25] and Figure 6 the corresponding shift factor a_T. The function $\mu\mathfrak{F}^*(\mu)$ is shown in Figure 7.

Because u increases with time the argument of the last term in (26) decreases and thus that term decreases; when it is equal to zero, time does not increase anymore and the crack travels at infinite velocity. The remaining equation

$$\frac{\alpha t^*}{c_0 a_T} = \frac{T}{T_0}\left(\frac{\sigma_{g\infty}}{\sigma_0}\right)^2 \mathfrak{F}^*\left[\frac{T}{T_0}\left(\frac{\sigma_{g\infty}}{\sigma_0}\right)^2\right] \tag{27}$$

relates, therefore, the time t^*, after which crack propagation occurs with unbounded or at least kinetically controlled speed to the applied stress σ_0. If one interprets this time t^* which differs only very little from the time at which the crack speed increases rapidly (cf. Figure 9) as the failure time of a cracked sheet, then Figure 7 predicts with possibly good approximation the failure of precracked sheets.

6. Comparison with Experiments

A number of crack propagation tests have been performed on sheets of the aforementioned material 10 inches × 10 inches and $\frac{1}{32}$ of an inch thick containing crack lengths varying between 0.25 and 1.5 inches. While inevitable data scatter exists, a representative example of a crack growth history is shown in Figure 8. The theoretical curve is calculated for $\alpha = 3.56 \times 10^{-9}$ cm. We shall comment on this value later.

An observation on the experimental results is of interest here. At times the film records showed that the crack slowed down temporarily as if it were overcoming an obstacle. This was true particularly for the initial stage of slow crack growth. In a few cases, when this occurred several times, each portion of the crack length-time trace was well reproduced by the theoretical curve while the overall agreement was worse. It appears, therefore, that discrepancies between theory and experiment may always arise from material inhomogeneities not considered in the theory but present in any real material.

An additional test series was run at various temperatures on 4 inch × 4 inch sheets containing 0.25 inch large cracks by subjecting them to dead-weight loading. The resulting failure time, reduced by the temperature shift factor in Figure 6, are resolved in Figure 9. The theoretical failure time t^* as given by equations (27) is shown as the solid line for the same value of α as before.

It should be mentioned parenthetically that to achieve the relatively small scatter in these data the long time modulus of each specimen was determined separately to eliminate the usual 10 to 15 per cent variations due to sample to sample variability.

7. Additional Comments and Observations

The foregoing results contain two quantities, α and the surface energy Γ, which deserve clarification and interpretation. Several other items related to thermal effects, generalization of the present calculations as well as creep failure in inorganic glasses and metals will now be discussed.

The factor α. We have seen that the quantity α is extremely small, so small in fact that a continuum interpretation is questionable. The best that one can say safely is that its small size reflects the high strain rates that occur at the tip of the advancing crack. On the other hand a more realistic stress distribution would have produced a different value. An indication of the truth of this statement is afforded in the work recorded in reference [17]. In that paper the authors considered the stress to unload as a linear function of time at the crack tip rather than assuming that the average stress acts, as was done in writing equation (2). The net results of that calculation was to change the value of α, but nothing else was much changed.

Indeed the small size parameter α may be constructed as an artifact of the standard continuum theory of fracture mechanics. If one were to formulate this problem as one of non-linear mechanics [11] wherein the material disintegrates continuously ahead of the crack by hole growth or any similar process a distinct value of α may be difficult to define. Nevertheless, in that realistic case the disintegration domain could be related to such a small length parameter as α.

It is of further interest to note that for fracture in the absence of dissipation, i.e., at the limits of short time, brittle, or long time failure as well as for rate insensitive materials, the factor α vanishes entirely from consideration which is in agreement with [4] and [15].

The surface energy Γ. After Rivlin and Thomas [26] suggested the energy concept for the fracture of rubber, Thomas [27] demonstrated that the energy required to propagate a crack depended on the speed of propagation. As a consequence, surface energy of rate sensitive materials has become looked upon as being also rate sensitive.

This viewpoint does not differentiate between the energy required to break molecular bonds and the energy dissipated into heat in the strain field around the advancing crack tip. In contrast the present approach [16, 17] accounts for the bond energy Γ directly and for the energy dissipated against internal friction forces in the material only indirectly through the velocity dependence of the crack-tip displacement. In order to differentiate clearly between the energy dissipation in the material (rate sensitive fracture energy) and the surface energy Γ required to break interatomic forces, one might call the latter "intrinsic surface energy". This energy was determined from crack propagation tests in swollen Solithane 113 as $\Gamma = 3.21 \times 10^{-2}$ lb/in [16].

For rubbery materials which exhibit very little stress relaxation behavior but strong time or rate dependence in their failure properties under similar environmental and loading conditions, it lies near at hand to postulate that the fracture energy is a rate or time-dependent quantity. For tests involving constant crack speeds this has been done by Thomas [27]. For constant load tests Figure 9 would represent the time dependent fracture energy since σ_0^2 is proportional to the fracture energy. However, the relation between the time dependence of the latter and the rate dependence of the constant crack rate energy would not be clear without the calculations put forth in this paper.

Time-Temperature Effects. It has been known for some time [28, 29] that many thermorheologically simple materials follow the WLF [23] time-temperature reduction both with regard to small strain deformation data and failure response. Although an explanation could have been derived from the results obtained by Williams and Schapery [5, 6], and Bueche and Halpin [8] did make use of such an explanation in postulating their simplified crack propagation model, it was Mueller [16] who exposed in a rational way for realistic materials how the temperature dependence enters the crack propagation and thus the failure process.

Growth of Subcritical Cracks in Weakly Rate Sensitive Materials. In the introduction we alluded to the time effects in the fracture of inorganic glasses and metals. While it is not intended here to insist on transposing the problem of linearly viscoelastic fracture onto materials commonly not regarded as viscoelastic, certain parallel phenomena seem striking and appear worth mentioning. In doing so, it should be fully recognized that metals are crystalline composites and the current considerations hardly apply without reservation.

The point to be made is that if failure is observed within an experimental time scale of, say, seconds or minutes, then the time scale of importance at the tip of the crack is many orders of magnitude smaller [6, 8]. Consequently, a material may appear as purely elastic in the time scale of the experiment while measurable time dependence occurs in the fracture process due to creep phenomena occurring at higher rates at the crack tip, rates normally not explored. Although microstructural differences and material non-linearity play an important role in a more complete picture, the effect of time dependence on fracture in the absence of a detrimental atmosphere should be traceable to the creep or relaxation behavior of the material at short time scales.

Crack Growth Under Arbitrary Load Histories. Having dealt in detail with the growth of a crack under constant load, it is of considerable interest to investigate the effect of more general loading histories. Although the details of a more general treatment are left for a future paper, several observations can be made here which are crucial in such a development.

Regardless of the physical interpretation of the quantity α and its exact size, it is probably always a very small number. Such a fact would allow one to make always the assumption made here with regard to equations (15) and (20). These assumptions amount to the statement that the shape of the immediate crack tip vicinity may be determined only by the current crack tip velocity and the current stress intensity factor. This observation would permit the derivation of a more general equation analogous to the crack growth equations (26)

$$K_1^2[\sigma_0(t), c(t)]\, D_{cr}\left(\frac{\alpha}{\dot{c}}\right) = \frac{2\Gamma}{\pi} = \text{const} \tag{28}$$

where K_1 is the stress intensity factor which depends on the time varying stress $\sigma_0(t)$ and the crack size $c(t)$ for arbitrary geometries in two dimensions. For three-dimensional cracks the differential equation (28) may be applied locally to determine subsequent crack boundaries, provided the requisite stress analysis can be performed to calculate the local stress intensity factor along the perimeter of the crack.

Current work along these lines of investigation have shown that equation (28) holds approximately if the load history increases monotonically without rapid or sudden changes. For general loading histories the rates of change of the stress intensity factor become important and equation (28) must be modified accordingly. The details of this treatment are the subject of a future paper.

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RÉSUMÉ

L'accroissement instable d'une fissure dans une tôle viscoélastique de grande dimension est examiné sous l'angle de la mécanique des milieux continus.

En partant des distributions locales des contraintes et des déformations à l'extrémité d'une fissure, on a trouvé une équation différentielle non linéaire et du premier ordre, qui décrit l'évolution de la dimension d'une fissure, dans le cas où une contrainte constante est appliquée à une distance suffisante de cette dernière. Dans l'équation différentielle interviennent le fluage et l'énergie intrinsèque de surface du matériau. Le concept d'énergie de surface est éclairci dans le cas des matériaux viscoélastiques. Les effets d'inertie ne sont pas pris en considération, mais l'influence de la température est étudiée pour des matériaux à rhéologie thermique simple. On exprime les vitesses initiales de fissuration en fonction de la charge appliquée, et on établit une comparaison entre la progression de la propagation de la fissure, déduite de calcul, et les résultats fournis par l'expérience.

Au delà d'un certain seuil de contrainte, la propagation de la fissure se fait à une grande vitesse qui dépend de l'inertie du matériau; sous une certaine limite inférieure, l'accroissement de la fissure ne se produit qu'après un temps infini. Il existe dès lors un critère, du type de celui de Griffith, à limites supérieure et inférieure.

Dans le cas de matériaux insensibles à l'effet de la vitesse (matériaux élastiques) ces deux limites convergent et seul demeure le critère de rupture fragile de Griffith.

On examine ce qu'impliquent ces résultats dans les ruptures par fluage des métaux et des verres inorganiques.

ZUSAMMENFASSUNG

Es wird das ungehinderte Wachstum eines Risses in einer grossen viskoelastischen Platte vom Gesichtspunkt der Kontinuumsmechanik betrachtet. Unter Gebrauch der lokalen Spannungen und Verformungen an den Spitzen des Risses wird eine Differentialgleichung erster Ordnung abgeleitet, welche die Rissgrösse in Abhängigkeit von der Zeit für eine vom Riss weit entfernte, konstante Spannung gibt. Die Differentialgleichung enthält die viskoelastische Kriechdehnungsfunktion und die dem Material eigene Oberflächenenergie. Die Auffassung der Oberflächenenergie für viskoelastische Materialien ist erläutert. Massenträgkeit ist nicht in Betracht genommen, aber der Einfluss der Temperatur ist für thermorheologisch einfache Materialien eingeschlossen.

Die Anfangsgeschwindigkeiten der Rissbildung werden als Funktion der angelegten Spannung ausgedrückt, wie auch Vergleiche zwischen Berechnung und Versuchsresultaten. Über einer gewissen Spannung breitet sich der Riss so schnell aus, dass die Geschwindigkeit von der Wellenmechanik kontrolliert wird, während an einer unteren Spannungsgrenze unendliche Zeit für Rissausbreitung benötigt ist. So bestehen zwei Kriterien des Griffith Types mit einer oberen und einer unteren Grenze. Für Materialien die keine Dämpfung aufweisen (elastische Materialien) fallen die zwei Grenzen zusammen und ergeben die Griffith Formel für den Sprödbruch. Folgerungen für den zeitbedingten Bruch in Metallen und inorganischen Gläsern werden angestellt.

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