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Quasi-orderings and population ethics

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Abstract. Population ethics contains several principles that avoid the repugnant conclusion. These rules rank all possible alternatives, leaving no room for moral ambiguity. Building on a suggestion of Parfit, this paper characterizes principles that provide incomplete but ethically attractive rankings of alternatives with different population sizes. All of them rank same-number alternatives with generalized utilitarianism.

1. Introduction

Population ethics contains several theories that are consistent with fixed-population utilitarianism or its generalized counterpart. Examples are classical (or complete) utilitarianism, average utilitarianism, and critical-level utilitarianism[1].

We adopt the standard convention that a utility level of zero represents *neutrality* so that life is worth living if and only if utility is positive[2]. Given this, Classical Utilitarianism (CU) evaluates states of affairs by means of the sum of the utilities of these alive. It suffers from a serious flaw – the *repugnant conclusion*[3].

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[1] All of these principles and a suggestion of Narveson (1967) are discussed in Broome (1992a). Critical-level utilitarianism was introduced in Blackorby and Donaldson (1984)

[2] If a multi-period model is employed (see Blackorby et al. 1994, 1995), a life at neutrality is represented by a utility level of zero. A life, taken as a whole, is worth living if and only if lifetime utility – an aggregate of utilities in each period – is positive

[3] See Parfit (1976, 1982, 1984)

Classical utilitarianism declares any state with positive average utility inferior to state in which everyone has an arbitrarily small but positive utility and the population is suitably large.

Average Utilitarianism (AU) evaluates states by comparing the average utility of those alive. It favours the addition of another person to an existing population – when the well-being of everyone alive in both states is unaffected – if and only if the utility of the added person is above the existing average. Because the desirability of such additions depends on the utilities of the existing population and because rankings of states depend on the utility levels of people who are long deceased, average utilitarianism is often rejected as ethically unsatisfactory.

Critical-Level Utilitarianism (CLU) is a family of ethical rules, one for each critical level of well-being. The value function for CLU is constructed by subtracting the critical level from the lifetime utility of each person alive in the state and summing. We recommended a positive critical level; it represents a lifetime utility above neutrality. If CLU is applied to lifetime utilities, killing people whose utilities are below the critical level reduces their lifetime utilities and is not, other things equal, a good thing.

CLU with a positive critical level avoids Parfit's criticism of classical utilitarianism – the repugnant conclusion[4]. Broome (1991, 1992a,b,c)[5] has criticized CLU because of its fixed critical level. According to CLU, any state with any average utility level above the critical level is inferior to another state with a suitably large population and an average level above the critical level but arbitrarily close to it. Therefore, Broome argues, we should reject CLU for low critical levels. On the other hand, if the critical level is high, CLU tells us to prevent the existence of people whose lifetime utilities would be just below it when the existing population's levels of well-being are unaffected. Because of this, Broome rejects fixed critical levels.

Sen has addressed the first of these objections, noting that the critical utility level should be high enough so that a "scenario in which more people enjoy a utility level ... [above the critical level] ... must be seen as a better outcome" (Sen 1991, p. 19). This supports Broome's view that the critical level should not be too low but suggests that tradeoffs above it between average utility and population size are appropriate. If the social ranking is complete, rejection of a fixed critical level necessarily makes critical utility levels depend on the utilities or size of the existing population.

The sharp cutoff of CLU can be softened somewhat by choosing an interval of critical levels and declaring one state better than another if and only if it is better according to CLU for all critical levels in the interval. This results in a social quasi-ordering of states: a ranking that is reflexive and transitive but not necessarily complete[6]. The addition of a person to an unaffected population should be prevented when his or her well-being is below all of the utility levels in the interval and should be welcomed when his or her utility is above all of them. This rule is silent when the utility level of the added person is in the interval. It is this family of principles that we characterize in the present paper.

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[4] See Blackorby and Donaldson (1984, Corollary 4.1)

[5] Broome (1992b) is a response to Blackorby and Donaldson (1992a)

[6] Orderings are quasi-orderings that are complete. An example of a quasi-ordering is the ranking of states of affairs produced by the standard Pareto principle. Quasi-orderings are fully rational but allow a degree of ambiguity or incommensurability in social comparisons. Quasi-orderings are discussed in Sen (1970)

All of these principles may be generalized to allow for social inequality aversion in utilities. Fixed-population utilitarianism is replaced with *Generalized Utilitarianism* (GU). It retains the additive structure of utilitarianism but replaces utility levels with transformed levels. The transformations used must be concave to ensure inequality aversion[7]. *Critical-Level Generalized Utilitarianism* (CLGU) subtracts the transformed critical level from transformed individual utility levels.

Our intention is to show that fixed-population utilitarian principles can be extended to cover population problems in a sensible, ethically attractive way. Our arguments do not depend on a particular notion of well-being (for a discussion of the different possibilities, see Griffin 1986).

Throughout this paper, we work with a world of certainty. Each state is a complete description of the world – from the distant past to the remote future[8]. In addition, we do not consider the possibility of discounting the utilities of people who are born in the future. We realize that a kind of "as if" discounting may be appropriate under conditions of uncertainty, but, in a world of certainty, the principle of *equal consideration of interests* rules out discounting[9].

In sections two, three, and four, we introduce notation, review the arguments for and against CLU in an atemporal framework, and summarize the arguments for CLU in an intertemporal framework. Following that, in section five, we introduce *Incomplete Critical-Level Generalized Utilitarianism* (ICLGU) and its special case *Incomplete Critical-Level Utilitarianism* (ICLU) and characterize them with a set of axioms. These principles produce quasi-orderings of social alternatives. We then investigate the performance of ICLGU in choosing population size and apply it to the simplest population problem – the *pure population problem.*

2. Value functions and population ethics

Parfit (1984) distinguishes three kinds of policy options: same-people choices, same-number choices, and different-number choices. The first affects neither the number of people alive nor their personal identities, the second affects the identities of those who are alive but not their number, and the third affects both. Our model allows all three kinds of options to be evaluated.

We begin with notation. $X = \{x, y, z, \ldots ,\}$ is a set of possible social states, $\mathbf{N}(x)$ is the set of people alive in state x, and $\mathbf{n}(x)$ is the number of people alive in x. If $\mathbf{N}(x) = \mathbf{N}(y)$, the same people are alive in x and y. If $\mathbf{n}(x) = \mathbf{n}(y)$, then we know only that the same number of people are alive in x and y.

We assume that social evaluations are based on individual well-being; that is, the principles we investigate are welfarist. For same-number choices we impose *anonymity*. It requires that the identities of people should not count in ranking social states. If $\mathbf{n}(x) = \mathbf{n}(y)$ and the utilities in x are the same set of numbers as the utilities in y, x and y are socially indifferent.

[7] As an example, negative utility levels could be multiplied by two while positive utility levels are given a weight of one

[8] Critical-level (generalized) utilitarianism and its incomplete counterparts can be generalized to take account of uncertainty

[9] If birth dates may be different for the same person in different states (a physiological possibility), then discounting is ruled out by the standard Pareto indifference condition. See Blackorby et al. (1994, 1995) for a discussion

Suppose that

$$\bar{\mathbf{u}} = (\bar{u}^1, \ldots, \bar{u}^{\bar{n}}) = (\{U^i(x)\}_{i \in \mathrm{N}(x)}) \tag{2.1}$$

and

$$\hat{\mathbf{u}} = (\hat{u}^1, \ldots, \hat{u}^{\hat{n}}) = (\{U^i(y)\}_{i \in \mathrm{N}(y)}) \tag{2.2}$$

where $\bar{n} = \mathbf{n}(x)$ and $\hat{n} = \mathbf{n}(y)$. $\bar{\mathbf{u}}$ is the vector of utilities in state x and $\hat{\mathbf{u}}$ is the vector of utilities in state y. There is a social preference relation R which is an ordering or quasi-ordering (not necessarily complete) of utility vectors, and x is at least as good as y if and only if the vector $\bar{\mathbf{u}}$ is ranked as no worse than the vector $\hat{\mathbf{u}}$, or

$$\bar{\mathbf{u}} R \hat{\mathbf{u}}. \tag{2.3}$$

If $\bar{\mathbf{u}} R \hat{\mathbf{u}}$ and $\hat{\mathbf{u}} R \bar{\mathbf{u}}$, $\bar{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are socially indifferent; if $\bar{\mathbf{u}} R \hat{\mathbf{u}}$ and not $\hat{\mathbf{u}} R \bar{\mathbf{u}}$, $\bar{\mathbf{u}}$ is better than $\hat{\mathbf{u}}$; if neither $\bar{\mathbf{u}} R \hat{\mathbf{u}}$ nor $\hat{\mathbf{u}} R \bar{\mathbf{u}}$ is true, then $\bar{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are not ranked.

For each *fixed* population size n, $n = 1, 2, \ldots$, we assume that there is a social ordering – a reflexive, complete, and transitive binary relation – which is represented by a fixed-population social-evaluation function W^n. If, in states x and y, population sizes are the same so that $\mathbf{n}(x) = \mathbf{n}(y) = n$, then, given (2.1) and (2.2), x is socially at least as good as y if and only if

$$\bar{\mathbf{u}} R \hat{\mathbf{u}} \leftrightarrow W^n(\bar{\mathbf{u}}) \geq W^n(\hat{\mathbf{u}}). \tag{2.4}$$

Given an unlimited utility domain, (2.4) applies to all $\bar{\mathbf{u}}$, $\hat{\mathbf{u}}$ in $\mathscr{R}^n$, Euclidean n-space.

Anonymity requires that W^n is symmetric. We assume in addition that it is continuous and increasing. Increasingness is equivalent to the strong Pareto condition. This allows us to define the *representative utility* level υ to be that level of utility which, if enjoyed by each individual, is socially indifferent to the original utility vector $\mathbf{u}$. It is defined formally by

$$W^n(\underbrace{\upsilon, \ldots, \upsilon}_{n \text{ times}}) = W^n(\mathbf{u}). \tag{2.5}$$

Because W^n is continuous and increasing, (2.5) can be solved for υ so that

$$\upsilon = \Upsilon^n(\mathbf{u}). \tag{2.6}$$

Υ^n is ordinally equivalent to W^n;[10] for all $\bar{\mathbf{u}}$, $\hat{\mathbf{u}} \in \mathscr{R}^n$,

$$W^n(\bar{\mathbf{u}}) \geq W^n(\hat{\mathbf{u}}) \leftrightarrow \Upsilon^n(\bar{\mathbf{u}}) \geq \Upsilon^n(\hat{\mathbf{u}}). \tag{2.7}$$

With these tools in hand it can be shown that, if there is a social ordering R, it can be represented by the social value function W; that is, for all $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$ and $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$,

$$\bar{\mathbf{u}} R \hat{\mathbf{u}} \leftrightarrow W(\bar{n}, \bar{\upsilon}) \geq W(\hat{n}, \hat{\upsilon}) \tag{2.8}$$

where $\bar{\upsilon} = \Upsilon^{\bar{n}}(\bar{\mathbf{u}})$ and $\hat{\upsilon} = \Upsilon^{\hat{n}}(\hat{\mathbf{u}})$ and W is increasing in its second argument[11].

[10] The function f is ordinally equivalent to the function g if and only if they are defined on the same domain and there exists an increasing function h such that $f(x) = h(g(x))$ for all x in the common domain

[11] Theorem 2.1 in Blackorby and Donaldson (1984)

To see the role played by the social value function, consider the classical utilitarian ordering where

$$\bar{\mathbf{u}}R^{CU}\hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} \bar{u}^i \geq \sum_{i=1}^{\hat{n}} \hat{u}^i. \tag{2.9}$$

The same-number rule is represented by

$$W^n(\mathbf{u}) \overset{0}{=} \sum_{i=1}^{n} u^i \tag{2.10}$$

where $\overset{0}{=}$ signifies ordinal equivalence. In this case, representative utility is given by

$$\upsilon = \Upsilon^n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n} u^i, \tag{2.11}$$

average utility. The classical utilitarian value function can therefore be written as

$$W^{CU}(n, \upsilon) \overset{0}{=} n\upsilon. \tag{2.12}$$

The average utilitarian rule can be written as

$$\bar{\mathbf{u}}R^{AU}\hat{\mathbf{u}} \leftrightarrow \frac{1}{\bar{n}}\sum_{i=1}^{\bar{n}} \bar{u}^i \geq \frac{1}{\hat{n}}\sum_{i=1}^{\hat{n}} \hat{u}^i. \tag{2.13}$$

Because this principle is utilitarian for same-number comparisons, representative utility is average utility, and the social value function for average utilitarianism is

$$W^{AU}(n, \upsilon) \overset{0}{=} \upsilon. \tag{2.14}$$

The family of critical-level utilitarian principles is defined by

$$\bar{\mathbf{u}}R^{CLU}\hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} [\bar{u}^i - \alpha] \geq \sum_{i=1}^{\hat{n}} [\hat{u}^i - \alpha] \tag{2.15}$$

where α is the critical level of utility. Again, representative utility is average utility, and the CLU social value functions are

$$W^{CLU}(n, \upsilon) \overset{0}{=} n[\upsilon - \alpha]. \tag{2.16}$$

When α is positive, the repugnant conclusion is avoided. Every state with average utility above α is better than every state with average utility below α. Setting α equal to zero produces classical utilitarianism.

It is important to interpret these utility numbers as lifetime utilities. If they are interpreted as utilities in a single period, these principles generate counter-intuitive results on killing. To see this, consider the following simple example[12]. Suppose that the world exists for two periods with two possible states of affairs, x and y, and two people, Yuriko and Bruno. In period one, both are alive in both states; in period two, however, both are alive in state x, but only Yuriko is alive in state y. Utility information is summarized in Table 1.

Using average utilitarianism or critical-level utilitarianism in period one, states x and y are equally good. In period two, however, both average utilitarianism and critical-level utilitarianism with the critical level above five prefer state y to state x.

[12] The example is taken from Blackorby and Donaldson (1991)

Table 1

	State x		State y	
	Period 1	Period 2	Period 1	Period 2
Yuriko	20	20	20	20
Bruno	5	5	5	—

This period-by-period approach to social evaluation suggests that Bruno should be killed at the end of period one. If, however, lifetime utilities are used, no such counter-intuitive results arise. No principle recommends killing Bruno because he is alive in both states of the world and his lifetime utility is higher in state x.

For same-number comparisons, *Generalized Utilitarian* (GU) principles replace individual utilities, u^i, with transformed utilities, $g(u^i)$. The function g can always be chosen so that $g(0) = 0$. If g is concave then inequality aversion (in utilities) is possible. For same-number choices, generalized utilitarianism has the social-evaluation function

$$W^n(\mathbf{u}) \stackrel{o}{=} \sum_{i=1}^{n} g(u^i), \tag{2.17}$$

and representative utility is

$$\upsilon = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(u^i)\right). \tag{2.18}$$

The value function for *Classical Generalized Utilitarianism* (CGU) is

$$W^{CGU}(n, \upsilon) \stackrel{o}{=} ng(\upsilon) = \sum_{i=1}^{n} g(u^i), \tag{2.19}$$

the value function for *Average Generalized Utilitarianism* (AGU) is

$$W^{AGU}(n, \upsilon) \stackrel{o}{=} \upsilon = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(u^i)\right) \stackrel{o}{=} \frac{1}{n}\sum_{i=1}^{n} g(u^i), \tag{2.20}$$

the average of transformed utilities, and the value functions for *Critical-Level Generalized Utilitarianism* (CLGU) are given by

$$W^{CLGU}(n, \upsilon) \stackrel{o}{=} n[g(\upsilon) - g(\alpha)] = \sum_{i=1}^{n} [g(u^i) - g(\alpha)], \tag{2.21}$$

where α is the critical level of utility (Blackorby and Donaldson 1984).

The mathematical result that shows that principles that produce social *orderings* are representable by social value functions does not apply to principles that produce *quasi-orderings* (social preference relations that are reflexive and transitive but not necessarily complete). In the next section, we turn to a discussion of possible justifications for the additive structure of GU and CLGU[13].

[13] The generalized utilitarian principles for same-number choices have additively separable social-evaluation functions

3. Generalized utilitarianism and CLGU

We sketch here two arguments for Generalized Utilitarianism (GU) and for Critical-Level Generalized Utilitarianism (CLGU). We use fixed-population GU as the basis for the quasi-orderings that we derive.

The first argument depends on two axioms, a fixed-numbers axiom and a population expansion axiom (Blackorby and Donaldson 1984). The second argument is employed in an intertemporal variable-population problem and depends on an axiom that rules out the influence of the utilities of those who are dead (such as Socrates and Cleopatra) in all feasible states in the "present" (Blackorby et al. 1995).

We assume that, whatever is true about the completeness of the overall social preference relation, it induces an ordering over all possible same-number alternatives for any fixed population size. This, in turn, implies the existence of a representative-utility function for every population size. Now suppose that, for any particular subgroup of a given population, we replace the utilities of that subgroup with its subgroup representative utility. Such a change would be a matter of indifference if the population subgroup existed on its own. Our axiom requires it to be a matter of social indifference in the larger population.

Population substitution principle. *For all $\bar{n}$, $\hat{n}$, if* $\mathbf{u} = (\bar{\mathbf{u}}, \hat{\mathbf{u}})$, $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$, $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$, *then*

$$\Upsilon^{\bar{n}+\hat{n}}(\bar{\mathbf{u}}, \hat{\mathbf{u}}) = \Upsilon^{\bar{n}+\hat{n}}(\underbrace{\bar{\upsilon}, \ldots, \bar{\upsilon}}_{\bar{n} \text{ times}}, \hat{\mathbf{u}}) \tag{3.1}$$

where $\bar{\upsilon} = \Upsilon^{\bar{n}}(\bar{\mathbf{u}})$.

This principle says that if the representative utility vector of a province or a state replaces its utility subvector in the social-evaluation function of a country, this is a matter of social indifference. It follows that, if a social change affects the well-being of one of these subgroups, its desirability or lack of it is independent of the levels of well-being that individuals outside the subgroup enjoy.

The population substitution principle has strong implications. If the fixed-population orderings satisfy the population substitution principle, then

$$\Upsilon^{n}(\mathbf{u}) = g^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} g(u^{i})\right]. \tag{3.2}$$

The population substitution principle requires that all fixed-population social-evaluation functions be generalized utilitarian[14].

The population-expansion principle that leads to CLU states that *if* the existing population is unaffected by the addition of an extra person, then there is a fixed utility level, α, independent of the utility levels and size of the existing population, that makes the expanded state socially indifferent to its ancestor[15].

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[14] This is Theorem 3.1 in Blackorby and Donaldson (1984). Diewert (1993) points out that it is similar to a result due to Kolmogoroff (1930) and Nagumo (1930)

[15] A similar axiom called the *Pareto plus principle* was introduced by Sikora (1978). It is equivalent to the critical-level population principle with α equal to zero

Critical-level population principle. *For any n and any* $\mathbf{u} \in \mathscr{R}^n$,

$$\mathbf{u}I(\mathbf{u}, \alpha) \tag{3.3}$$

for some fixed $\alpha \in \mathscr{R}$.

These two assumptions, along with our background hypotheses, imply that the ranking of all states is complete and that it is CLGU with an arbitrary critical level – positive, negative, or zero[16]. That is,

$$W^{CLGU}(n, v) \overset{0}{=} n[g(v) - g(\alpha)] = \sum_{i=1}^{n} [g(u^i) - g(\alpha)]. \tag{3.4}$$

Note, as a corollary, that same-number utilitarianism and the critical-level population principle are satisfied if and only if

$$W^{CLU}(n, v) \overset{0}{=} n[v - \alpha] = \sum_{i=1}^{n} [u^i - \alpha], \tag{3.5}$$

the value function for CLU. If, in addition, we require the principle that orders all possible alternatives to avoid the repugnant conclusion, then it must be CLGU (which includes its special case CLU) with a positive critical level.

An alternative characterization of CLGU comes from considering an intertemporal framework (Blackorby et al. 1994, 1995). Again, the principle that ranks states of affairs is welfarist in the sense that the individual lifetime utilities achieved in possible states of the world are sufficient to rank alternatives (when they can be ranked).

When ranking states at time t, however, it seems unreasonable to have to consider the entire history of the world. To avoid this, an axiom called *Independence of the Utilities of the Dead* (IUD) is employed[17]. It allows an evaluator in the present to ignore the utilities of individuals – such as Cleopatra and Socrates – whose lives are over in the states to be compared; lifetime utilities of individuals that continue into the present in at least one of these states *are* permitted to enter the social decision rule. If the social preference relation is an ordering, this axiom, in conjunction with the fixed-population assumptions *strong Pareto, anonymity, continuity*, and the requirement that the repugnant conclusion be avoided, leads to CLGU with a positive critical level[18]. In addition, even when the social ranking is not complete but the same-number alternatives are ranked completely, generalized utilitarianism for these choices is characterized.

Independence of the utilities of the dead permits history to matter to some extent. This is desirable because principles that are based only on present and future utilities lead to the repugnant conclusion if a plausible intertemporal consistency requirement is satisfied[19].

History-dependent principles are able to distinguish between an individual dying just before the period in which the evaluation is made and an individual not being born at all. This is an important distinction, and independence of the utilities of the dead allows it to be made, thereby permitting ethically attractive social decision rules.

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[16] This is Theorem 4.1 in Blackorby and Donaldson (1984)
[17] Hammond (1988) uses a similar axiom in a slightly different framework
[18] This is Corollary 1 in Blackorby et al. (1995)
[19] See Blackorby et al. (1994)

The only rules that satisfy independence of the utilities of the dead and our additional fixed-population assumptions when the overall ranking is complete are the critical-level generalized utilitarian rules. The fact that these principles escape the repugnant conclusion when the critical level is positive provides a strong argument in their favour.

4. CLU and CLGU: strengths and weaknesses

Our approach to population ethics is welfarist: knowledge of individual well-being is sufficient for social evaluations. Values such as freedom matter only in so far as they contribute to welfare. Although critical-level utilitarianism is welfarist, it is only one of many possible welfarist rules.

Critical-level generalized utilitarianism and, for same-number choices, generalized utilitarianism, allow inequality aversion in utilities. The possible choices for the transform g are, however, infinite (g must be increasing, continuous, and concave). There are several simple ways to choose g. One is to make it the identity mapping ($g(u) = u$ for all u). This makes critical-level generalized utilitarianism into critical-level utilitarianism which, in turn, becomes the familiar utilitarian principle for same-number choices. Other options include choosing a greater weight for negative utilities than for positive utilities or employing a family of functions such as

$$g(u) = -e^{-\gamma u} + 1, \tag{4.1}$$

where γ is an inequality-aversion parameter that is greater than zero. The generalized utilitarian principles produced are different for each choice of γ. When γ approaches zero, utilitarian principles result, and inequality aversion increases as γ increases.

As long as the critical lifetime-utility level α is positive, critical-level generalized utilitarianism distinguishes between killing someone and preventing his or her existence. In this case, CLGU suggests that individuals with lifetime-utility levels below α should not be born, other things equal [20]. This means that some lives that are worth living should be prevented if the lifetime utilities experienced are low. Once an individual has been born, however, all positive utility levels have value, and strong Pareto ensures that killing a person or reducing his or her lifetime utility in any way (without changing the utilities of anyone else) is socially undesirable. This does not mean that the existence of all people whose lifetime utilities are below the critical level should have been prevented. If everyone's standard of living in an egalitarian, overpopulated world were below α, *some*, but not all, people should not have been born – it does not matter which ones as long as the utilities of the remaining people rise. If, however, someone is *necessarily* below the critical level (for his or her whole life) because of a serious incurable illness, say, his or her life should have been prevented.

Another effect of a positive α is that it gives weight to individual lives. Suppose, as an example, that states x and y differ for persons 1 and 2 only. In x, person 1

[20] McMahan (1994) investigates the ethics of having children whose lives are likely to be above neutrality in each period but unnaturally short. He argues that length of life is morally irrelevant, a position which is equivalent, in our framework, to advocating a critical level of zero

enjoys a lifetime utility of 100, and person 2 is not alive. In y, both people are alive with lifetime utilities of 50 each. Classical utilitarianism declares x and y to be socially indifferent, but critical-level utilitarianism with $\alpha > 0$ prefers x to y: weight is given to the fact that *one* person experiences the lifetime utility of 100. Therefore, population size cannot be used to substitute for quality of life[21].

CLGU with a positive critical level avoids the repugnant conclusion. Some critics have claimed in response that CLGU suffers from an "α-repugnant conclusion" [22]. As long as representative utility is above the critical level α, CLGU permits numbers to make up for losses of representative utility. We do not find this to be ethically unattractive. If the critical level is chosen in a reasonable way, then if more people enjoy a utility level above α, the outcome must be better. This view is consistent with (but not the same as) Griffin's view of the repugnant conclusion. His argument suggests that the critical level be set at that point where people have the "capacity to appreciate beauty, to form deep loving relationships, to accomplish something with their lives beyond just staying alive" (Griffin 1986, p. 340). This suggests that the ethical judgments needed to choose a critical level require a fairly complex theory of the good. In addition, a critical level expresses not only a minimal level of well-being necessary to make the creation of new people socially desirable, but also the kinds of actions, experiences, and states of mind that we believe to be necessary for a good and valuable life.

Is a single critical level appropriate? As Broome (1992a,b,c) has argued, it should be set high enough so that adding more people above the critical level, other things equal, is a good thing. On the other hand, it should not be set so low that perfectly satisfactory lives would be prevented. This suggests that a single critical level is, perhaps, too strong a demand. Following a suggestion of Parfit (1976, 1982, 1984), we investigate the possibility of a set of critical levels in this paper. If an added person has a utility level in the set, the change is not ranked against the status quo.

Before turning to our characterization of quasi-orderings based on this consideration, it is worth pointing out that such a move, while attractive, does not answer all of the critical-level questions raised by CLGU. Sentient beings other than humans are often given moral standing and, if CLGU is used to solve ethical problems in which decisions change the number of animals, a critical level must be chosen for each species. Blackorby and Donaldson (1992b) investigate the ethics of animal-using research with CLU and separate critical levels for human and non-human animals.

But the problem may be even more complex. In population decisions involving humans only, should men and women have the same critical level? Such a choice would make it possible to argue that, if asexual reproduction were possible and sex could be selected before conception or birth, only women should be born because they have greater life expectancies than men do. This suggests that there are two dimensions to critical levels: the first reflects utility *per diem* and the second reflects life expectancy. It might be desirable to have separate critical levels for human subgroups that differ because of variations in expected lifetimes but are based on the *same* level of well-being *per diem*.

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[21] See Blackorby et al. (1995)

[22] See Broome (1992a,b); Thomas Hurka also made this point when Blackorby and Donaldson (1992b) was presented at the University of Calgary

5. Quasi-orderings in population ethics

In this section, we investigate social quasi-orderings that arise from a set of critical levels. If a single individual is added to an existing population that is unaffected, in utility terms, by the change, the two states are not ranked if the lifetime utility of the added person is in the set. If two states are not ranked, the principle is silent about their relative desirability.

Given our welfarist, Pareto-inclusive, anonymous approach, we look at quasi-orderings of utility vectors – reflexive, transitive, but not necessarily complete social preference relations. Denoting such a relation as $\overset{*}{R}$, $\bar{\mathbf{u}}\overset{*}{R}\hat{\mathbf{u}}$ means that the utility vector $\bar{\mathbf{u}}$ is at least as good as the utility vector $\hat{\mathbf{u}}$. Strict preference and indifference are defined in the usual way, and $\bar{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are not ranked ($\bar{\mathbf{u}}\overset{*}{N}\hat{\mathbf{u}}$) if and only if it is not the case that $\bar{\mathbf{u}}$ is at least as good as $\hat{\mathbf{u}}$ and it is not the case that $\hat{\mathbf{u}}$ is at least as good as $\bar{\mathbf{u}}$;

$$\bar{\mathbf{u}}\overset{*}{N}\hat{\mathbf{u}} \leftrightarrow (\neg\,\bar{\mathbf{u}}\overset{*}{R}\hat{\mathbf{u}}) \wedge (\neg\,\hat{\mathbf{u}}\overset{*}{R}\bar{\mathbf{u}}). \tag{5.1}$$

Suppose that $R_1, \ldots, R_m$ are orderings of the set of all possible utility vectors. If, for every pair of utility vectors, $\overset{*}{R}$ is defined by

$$\bar{\mathbf{u}}\overset{*}{R}\hat{\mathbf{u}} \leftrightarrow (\bar{\mathbf{u}}R_1\hat{\mathbf{u}}) \wedge \cdots \wedge (\bar{\mathbf{u}}R_m\hat{\mathbf{u}}), \tag{5.2}$$

$\overset{*}{R}$ is a quasi-ordering of a particular kind – an *intersection* quasi-ordering.

In this section of the paper, we retain the same-number generalized utilitarian principles discussed above, where the function g is continuous, increasing, and $g(0) = 0$. Their use may be justified by the population substitution principle or by independence of the utilities of the dead in an intertemporal framework.

To investigate quasi-orderings that are consistent with GU for same-number choices, we consider the (hypothetical) possibility of adding a person to a population that is unaffected, in utility terms, by the change. $\mathbf{u}$ is the utility vector of the existing population and u is the utility level of the added person. For every population size n and every utility vector $\mathbf{u} \in \mathscr{R}^n$, let

$$K^n(\mathbf{u}) := \{u \mid (\neg\,(\mathbf{u},u)\overset{*}{R}\mathbf{u}) \wedge (\neg\,\mathbf{u}\overset{*}{R}(\mathbf{u},u))\} = \{u \mid (\mathbf{u},u)\overset{*}{N}\mathbf{u}\} \tag{5.3}$$

be the set of all utility levels for the added person such that the two states are not ranked.

Our basic axiom is the

Critical-set population principle: *For any n and any $\mathbf{u} \in \mathscr{R}^n$, the set $K^n(\mathbf{u})$ is bounded and non-empty.*

It requires that there is always some utility level for the added person such that the two states are not ranked. Together with the strong Pareto principle for fixed populations, this axiom is surprisingly powerful.

Lemma 1. *Suppose that the fixed-population principles are Pareto-inclusive. The critical-set population principle implies, for any n and any $\mathbf{u} \in \mathscr{R}^n$,*

(*i*) *there is no $u \in \mathscr{R}$ such that*

$$(\mathbf{u},u)\overset{*}{I}\mathbf{u}; \tag{5.4}$$

(*ii*) *$K^n(\mathbf{u})$ is an interval;*

(iii) *for all* $u \in \mathscr{R}$,

$$u > \alpha \text{ for all } \alpha \in K^n(\mathbf{u}) \rightarrow (\mathbf{u}, u)\overset{*}{P}\mathbf{u}, \tag{5.5}$$

and

$$u < \alpha \text{ for all } \alpha \in K^n(\mathbf{u}) \rightarrow \mathbf{u}\overset{*}{P}(\mathbf{u}, u). \tag{5.6}$$

Proof: See the appendix.

Part (i) of the lemma is easily seen to be true. Suppose that there is some utility level, u, for the added person that makes the two states socially indifferent so that (5.4) is satisfied. Then, because of strong Pareto and transitivity of $\overset{*}{R}$, the change is good if the added person's utility level exceeds u and bad if the added person's utility is less than u. Such a u therefore precludes noncomparability and implies that $K^n(\mathbf{u})$ is empty. Consequently, no such utility level exists.

In the same situation, if there is any $\bar{u}$ such that the second state is better $((\mathbf{u}, \bar{u})\overset{*}{P}\mathbf{u})$, then it is better in all cases in which the added person has a utility level greater than $\bar{u}$. That is, $(\mathbf{u}, u)\overset{*}{P}\mathbf{u}$ for all $u > \bar{u}$. Similarly, if there exists any $\underline{u}$ such that $\mathbf{u}\overset{*}{P}(\mathbf{u}, \underline{u})$, then $\mathbf{u}\overset{*}{P}(\mathbf{u}, u)$ for all $u < \underline{u}$. This observation is used to prove parts (ii) and (iii) of the lemma.

We define α_L and α_H as

$$\alpha_L := \inf\{K^n(\mathbf{u})\} \text{ and } \alpha_H := \sup\{K^n(\mathbf{u})\}. \tag{5.7}$$

It must be true that $\alpha_L \leq \alpha_H$, and for any n and $\mathbf{u}$, the set $K^n(\mathbf{u})$ is one of the intervals

$$[\alpha_L, \alpha_H], \quad (\alpha_L, \alpha_H], \quad [\alpha_L, \alpha_H), \quad \text{or} \quad (\alpha_L, \alpha_H). \tag{5.8}$$

The nonranked relation of a quasi-ordering is not, in general, transitive. In this context, however, it seems reasonable to assume that, in the critical-level problem, if a succession of people is added, each in the appropriate nonranked set, the resulting vector is not ranked against the original. The axiom that formalizes this intuition is the

Critical-set extension principle: *For any* $\bar{n}$ *and* $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$ *and for any* $\hat{n}$ *and* $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$ *with* $\bar{n} > \hat{n}$ *and* $\bar{u}_i = \hat{u}_i$ *for all* $i = 1, \ldots, \hat{n}$, *if*

$$\bar{u}_{\hat{n}+1} \in K^{\hat{n}}(\hat{\mathbf{u}}) \tag{5.9}$$

and, for $\bar{n} - \hat{n} > 1$, *if*

$$\bar{u}_{\hat{n}+k} \in K^{\hat{n}+k-1}(\hat{\mathbf{u}}, \bar{u}_{\hat{n}+1}, \ldots, \bar{u}_{\hat{n}+k-1}) \text{ for all } k = 2, \ldots, \bar{n} - \hat{n}, \tag{5.10}$$

then

$$\bar{\mathbf{u}}\overset{*}{N}\hat{\mathbf{u}}. \tag{5.11}$$

In Sect. 3, we assumed that critical levels are independent of both n and $\mathbf{u}$. Correspondingly, we use an axiom that makes the non-ranked sets independent of n and $\mathbf{u}$.

Critical-set independence: *For any* n *and any* $\mathbf{u} \in \mathscr{R}^n$,

$$K^n(\mathbf{u}) = \mathscr{K} \tag{5.12}$$

for some fixed set $\mathscr{K}$.

The set $\mathscr{K}$ (which, by Lemma 1, is an interval) can be used to describe ethical principles that are similar to those advocated by Narveson (1967). If adding people at or above neutrality is not a social improvement, it is not possible to rank vectors such as $(\mathbf{u}, u)$ with $u \geq 0$ as indifferent to $\mathbf{u}$. The reason is that this would imply that improvements in the well-being of the added person are a matter of indifference because the indifference relation is transitive, contradicting the strong Pareto condition. It is possible, however, to make additions of this sort not ranked by making the interval $\mathscr{K} = [0, \alpha_H)$ and allowing α_H to approach $+\infty$[23].

The following theorem characterizes *Incomplete Critical-Level Generalized Utilitarianism* (ICLGU).

Theorem 1. *Suppose that* $\overset{*}{R}$ *is a quasi-ordering and the same-number principles are GU. The critical-set population principle, the critical-set extension principle and critical-set independence are satisfied if and only if, for all* $\bar{n}$, $\hat{n}$ *and for all* $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$, $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$, *if* $\bar{n} \neq \hat{n}$,

$$\bar{\mathbf{u}}\overset{*}{P}\hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] > \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}, \tag{5.13}$$

$$\hat{\mathbf{u}}\overset{*}{P}\bar{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] < \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}, \tag{5.14}$$

$$\bar{\mathbf{u}}\overset{*}{N}\hat{\mathbf{u}} \qquad \text{otherwise}, \tag{5.15}$$

and if $\bar{n} = \hat{n}$,

$$\bar{\mathbf{u}}\overset{*}{R}\hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} g(\bar{u}^i) \geq \sum_{i=1}^{\hat{n}} g(\hat{u}^i); \tag{5.16}$$

that is, for all $\bar{n}$, $\hat{n}$ *and for all* $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$, $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$,

$$\bar{\mathbf{u}}\overset{*}{R}\hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] \geq \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}. \tag{5.17}$$

Proof. See the appendix.

Incomplete Critical-Level Utilitarianism (ICLU) is defined by (5.17) with the function g replaced with the identity map. This result indicates that the only quasi-orderings based on GU that satisfy our axioms are intersection quasi-orderings. When population sizes are different, one utility vector is better than another if and only if it is better according to CLGU for all critical levels in the interval $\mathscr{K}$, worse if it is worse according to CLGU for all critical levels in $\mathscr{K}$, and not ranked otherwise. Using Theorem 1, we are able to show how the ICLGU principles produce quasi-orderings (denoted by $\mathbf{R}$) of the set of social states. For states x and y,

$$x\mathbf{R}y \leftrightarrow \sum_{i \in N(x)} [g(U^i(x)) - g(\alpha)] \geq \sum_{i \in N(y)} [g(U^i(y)) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}. \tag{5.18}$$

[23] Morton (1994) discusses this and similar issues using a four-place relation

When the same-number principles are utilitarian, ICLU results, and

$$x\mathbf{R}y \leftrightarrow \sum_{i\in N(x)} [U^i(x)-\alpha] \geq \sum_{i\in N(y)} [U^i(y)-\alpha] \quad \text{for all } \alpha \in \mathscr{K}. \tag{5.19}$$

Social indifference is not possible unless the population sizes are the same in x and y. ICLU and ICLGU avoid the repugnant conclusion as long as at least one value of α in $\mathscr{K}$ is positive.

Theorem 1 allows us to give an answer to the question of adding groups of people to a utility-unaffected population. Consider two people with utilities (v, w) who must be considered together. Then using ICLU which, for same-number choices, is utilitarianism, the vector $(\mathbf{u}, v, w)$ is indifferent to the vector $(\mathbf{u}, \bar{u}, \bar{u})$ where $\bar{u} = (v + w)/2$, the average of the utilities of the two added people. It follows that, if this average is in the interval $\mathscr{K}$, the larger state is not ranked against its parent, if $\bar{u}$ is greater than the utility levels in the interval $\mathscr{K}$, the expanded state is better, and if $\bar{u}$ is less than the utility levels in $\mathscr{K}$, the expanded state is worse. If ICLGU is used, $\bar{u}$ becomes

$$g^{-1}(\tfrac{1}{2}[g(v) + g(w)]), \tag{5.20}$$

the representative utility for (v, w).

6. Undominated population sizes

The incomplete critical-level principles produce quasi-orderings of states of affairs, and this is enough to help guide choices over feasible sets of states with different populations. The way this is done is to find the set of *undominated alternatives* [24]. It is analogous to the set of Pareto optima corresponding to the Pareto quasi-ordering.

Suppose that S is a set of feasible utility vectors corresponding to the feasible states[25]. The set of undominated alternatives in S consists of all the states for which there is no state that is strictly preferred according to ICLGU. It is

$$M(S, \mathscr{K}) = \left\{ \hat{\mathbf{u}} \in S \,|\, \nexists \mathbf{u} \in S \ni \sum_{i=1}^{n} [g(u^i) - g(\alpha)] > \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \forall \alpha \in \mathscr{K} \right\}. \tag{6.1}$$

If $\tilde{P}$ is the set of population sizes in S, then S may be partitioned so that

$$S = \bigcup_{n\in\tilde{P}} S_n, \tag{6.2}$$

where S_n is the subset of S with exactly n people in each state. Over each S_n, ICLGU (or ICLU) is simply GU (or utilitarianism), and the vectors in S_n are ordered completely. It is possible, therefore, to find the set of vectors in S_n which maximize the value function for ICLGU [26]. We can also find the maximized value of the

[24] Sen (1970) calls this the set of maximal elements

[25] We assume that the number of distinct population sizes in S is finite

[26] We assume that the requisite maxima exist

value function for each n. We define the function $\tilde{W}$ as

$$\tilde{W}(n,\alpha) = \max_{u \in S_n} \left\{ \sum_{i=1}^{n} [g(u^i) - g(\alpha)] \right\} \tag{6.3}$$

for each n in $\tilde{P}$ and each critical level α.

The undominated set of population sizes is the set of all n in $\tilde{P}$ for which there is no other n with a higher value of $\tilde{W}(n,\alpha)$ for all α in $\mathscr{K}$. Writing the undominated set as P^M,

$$P^M = \{n^M \in \tilde{P} \mid \nexists n \in \tilde{P} \ni \tilde{W}(n,\alpha) > \tilde{W}(n^M,\alpha) \forall \alpha \in \mathscr{K}\}. \tag{6.4}$$

This is equivalent to

$$P^M = \{n^M \in \tilde{P} \mid \forall n \in \tilde{P} \exists \alpha \in \mathscr{K} \ni \tilde{W}(n^M,\alpha) \geq \tilde{W}(n,\alpha)\}. \tag{6.5}$$

(6.5) allows α to depend on n. It implies that all of the CLGU-best population sizes for the critical levels in $\mathscr{K}$ are in the undominated set P^M.

An example of this procedure is found in the *pure population problem* [27]. There is a fixed amount ω of a single resource, income, to be divided among n people where n is to be chosen. Each person has the same increasing, twice differentiable, strongly concave utility function U with a positive subsistence level s (given by $U(s) = 0$) [28]. Because U is strongly concave, ω must be distributed equally for any population size. Employing ICLU for simplicity,

$$n[v - \alpha] = n[U(\omega/n) - \alpha]. \tag{6.6}$$

Writing $I = \omega/n$ as income per person, we consider the special case where

$$U(I) = 1 - \frac{1}{I} = 1 - \frac{n}{\omega}, \tag{6.7}$$

and the value function becomes

$$n\left[1 - \frac{n}{\omega} - \alpha\right]. \tag{6.8}$$

Treating n as a continuous variable (in this example only), CLU chooses, for $\alpha \leq 1$,

$$\overset{*}{n} = \tfrac{1}{2}\omega(1 - \alpha). \tag{6.9}$$

$\omega/2$ is the optimal population for CU, and, as α increases, $\overset{*}{n}$ decreases until, when $\alpha \geq 1$, $\overset{*}{n} = 0$.

The set of undominated population sizes for incomplete critical-level utilitarianism is the set of values of $\overset{*}{n}$ for all critical levels α in $\mathscr{K}$. Suppose, for example, that $\omega = 24$ and $\mathscr{K} = [1/4, 3/4]$. Then the set of undominated population sizes is $[3, 9]$, all populations from three to nine. 3 is the CLU-optimal population for $\alpha = \alpha_H = 3/4$, and 9 is the CLU-optimal population for $\alpha = \alpha_L = 1/4$.

Quasi-orderings present an important difficulty for sequential decision making that is, perhaps, best explained with an example [29]. Suppose that a decision tree

[27] The pure population problem is discussed in Blackorby and Donaldson (1984) and in Dasgupta (1988). Dasgupta presents a solution that is not rationalizable by a social ordering or quasi-ordering

[28] A function f is strongly concave if and only if it is twice differentiable and $f''(x) < 0$ for all x in its domain

[29] We are indebted to Peter Hammond for this example. See also Hammond (1994)

leads to three alternatives, x, y, and z. The tree has two nodes: the first offers a choice between x and the second node which, in turn, offers a choice between y and z. Suppose that x is better than z but y and z are not ranked. Then the undominated alternatives in the set $\{x, y, z\}$ are x and y. At the first node, there is no reason to prefer either choice – both branches lead to undominated alternatives. But if the branch that leads to y and z is chosen at the first node, information about the ranking of y and z provides no reason for rejecting z although it is dominated by x. Therefore, in general, quasi-orderings do not allow decision makers to dispense with social-preference information about alternatives in paths that have been rejected in the past [30].

When the quasi-ordering is an intersection quasi-ordering, this difficulty can be overcome in certain circumstances. If the set of undominated alternatives is the union of the sets of best alternatives for the generating orderings, then any undominated alternative can be chosen by employing one of the orderings. That is, the recommendations of one of the generating orderings can be followed. In this case, the sequential difficulty described above does not occur.

Although the union of the best sets for all the orderings that generate ICLGU is a subset of the set of undominated alternatives, there are undominated alternatives whose choice is not rationalized by any of the orderings that generate ICLGU. For example, suppose that there are three alternatives with $S = \{\bar{\mathbf{u}}, \hat{\mathbf{u}}, \check{\mathbf{u}}\} = \{(140, 140), (99, 99, 99), (80, 80, 80, 80)\}$. If $\mathscr{K} = [0, 30]$, the undominated set for ICLU is $S = \{\bar{\mathbf{u}}, \hat{\mathbf{u}}, \check{\mathbf{u}}\}$. $\bar{\mathbf{u}}$ is best for $\alpha = 30$, $\check{\mathbf{u}}$ is best for $\alpha = 0$, but there is no value of α such that $\hat{\mathbf{u}}$ is best [31].

The set of *best* alternatives in S according to CLGU for the critical level α is

$$B(S, \alpha) := \left\{ \tilde{u} \in S \,|\, \sum_{i=1}^{\tilde{n}} [g(\tilde{u}^i) - g(\alpha)] \geq \sum_{i=1}^{n} [g(u^i) - g(\alpha)] \,\forall \mathbf{u} \in S \right\}, \tag{6.10}$$

and the set of best alternatives for all the critical levels in $\mathscr{K}$ is

$$B(S, \mathscr{K}) = \bigcup_{\alpha \in \mathscr{K}} B(S, \alpha), \tag{6.11}$$

the union of the CLGU-best sets for all the values of α in $\mathscr{K}$. In addition, the set of best population sizes for all α in $\mathscr{K}$ is

$$P^B := \{n^B \in \tilde{P} \,|\, \exists \alpha \in \mathscr{K} \ni \tilde{W}(n^B, \alpha) \geq \tilde{W}(n, \alpha) \,\forall n \in \tilde{P}\}, \tag{6.12}$$

where $\tilde{W}$ is defined by (6.3).

If the choice of each of the undominated elements of S can be rationalized by CLGU for some critical level in $\mathscr{K}$, then

$$M(S, \mathscr{K}) = B(S, \mathscr{K}). \tag{6.13}$$

Theorem 2 states that the only undominated alternatives that are not rationalized by CLGU for some critical level in $\mathscr{K}$ are in between the best population sizes for $\alpha \in \mathscr{K}$. Equivalently, if the set P^M has no "gaps", then all of the undominated alternatives are best alternatives for some critical level in $\mathscr{K}$.

Theorem 2. *If there exist integers n_L^M and n_H^M, $n_L^M \leq n_H^M$, such that P^M consists of all integers from n_L^M to n_H^M inclusive, then* (6.13) *holds; that is, the undominated*

[30] Orderings always escape the problem

[31] See Blackorby and Donaldson (1977) for several other examples

alternatives in S according to ICLGU are all best choices according to CLGU for some critical level $\alpha \in \mathscr{K}$.

Proof. See the appendix.

Theorem 3 states that if the set of best population sizes for all $\alpha \in \mathscr{K}$ has no gaps the same result is true.

Theorem 3. *If there exist integers* n_L^B *and* n_H^B, $n_L^B \leq n_H^B$, *such that* P^B *consists of all integers from* n_L^B *to* n_H^B *inclusive, then* (6.13) *holds; that is, the undominated alternatives in S according to ICLGU are all best choices according to CLGU for some critical level* $\alpha \in \mathscr{K}$.

Proof. See the appendix.

The pure population problem provides a simple application of Theorem 3. This result requires a concave g and, for simplicity of presentation, we assume that g is twice differentiable. Given this, in the pure population problem, the set of undominated alternatives is equal to the set of best alternatives according to CLGU for all the critical levels in $\mathscr{K}$.

Theorem 4. *In the pure population problem with the function g twice differentiable and concave,* (6.13) *holds; that is, the undominated alternatives in S according to ICLGU are all best choices according to CLGU for some critical level* $\alpha \in \mathscr{K}$.

Proof. See the appendix.

7. Conclusion

Critical-Level Utilitarianism (CLU) and Critical-Level Generalized Utilitarianism (CLGU) are ethical principles for social decisions involving population choices. They are the same as generalized utilitarianism (which allows inequality aversion in utilities) and utilitarianism for same-number choices and, when the critical level is positive, avoid the repugnant conclusion. It can be argued however, that these principles require too much of evaluators – a *single* critical level of utility must be chosen.

It is reasonable, therefore, to employ an interval of critical levels. When a single individual is added to a utility-unaffected population, the resulting state is better if the added person's utility level is above the interval, worse if the added utility is below the interval, and not ranked if the added utility is in the interval.

This procedure produces social quasi-orderings – incomplete rankings of social states with different population sizes – and we call the corresponding principles Incomplete Critical-Level Utilitarianism (ICLU) and Incomplete Critical-Level Generalized Utilitarianism (ICLGU). These principles are easily described. One state is socially better than another with a different population size if and only if it is better than the other according to CLU (CLGU) for all critical levels in the interval. Two such states are not ranked if one is better according to CLU (CLGU) for some critical levels in the interval and worse for others; social indifference is not possible. Same-number comparisons are made with utilitarianism (generalized utilitarianism). The repugnant conclusion is avoided as long as there is at least one positive critical level in the interval.

The set of undominated alternatives is the set of alternatives for which there is no other that is superior according to ICLU (ICLGU). The larger the set $\mathscr{K}$ is, the larger the undominated set is. In general, the set of best alternatives according to CLGU for all the critical levels in $\mathscr{K}$ is contained in, but not equal to, the set of undominated alternatives. When the set of population sizes corresponding to the undominated alternatives or the set of best population sizes for all the critical levels in the interval $\mathscr{K}$ has no gaps, however, the two sets coincide.

In ethical problems as formidable as the ones that involve population changes, it is difficult to make judgments in all cases. The above principles, we suggest, provide a simple and sensible method to make imprecise judgments which, at the same time, satisfy the requirements of rationality.

Appendix

Lemma 1. *Suppose that the fixed-population principles are Pareto-inclusive. The Critical-Set Population Principle implies, for any n and any* $\mathbf{u} \in \mathscr{R}^n$,
(i) there is no $u \in \mathscr{R}$ *such that*

$$(\mathbf{u}, u) \overset{*}{I} \mathbf{u}; \tag{A.1}$$

(ii) $K^n(\mathbf{u})$ *is an interval;*
(iii) for all $u \in \mathscr{R}$,

$$u > \alpha \text{ for all } \alpha \in K^n(\mathbf{u}) \rightarrow (\mathbf{u}, u) \overset{*}{P} \mathbf{u}, \tag{A.2}$$

and

$$u < \alpha \text{ for all } \alpha \in K^n(\mathbf{u}) \rightarrow \mathbf{u} \overset{*}{P} (\mathbf{u}, u). \tag{A.3}$$

Proof. Part (i) follows from the argument of the text.

To establish part (ii), suppose that $K^n(\mathbf{u})$ is not an interval. Then there exist u', $\check{u}$, and u'' in $\mathscr{R}$ with $u' > \check{u} > u''$, u', $u'' \in K^n(\mathbf{u})$, and $\check{u} \notin K^n(\mathbf{u})$. It follows that either $(\mathbf{u}, \check{u}) \overset{*}{P} \mathbf{u}$ or $\mathbf{u} \overset{*}{P} (\mathbf{u}, \check{u})$ (indifference is ruled out by (i)). If the former is true, then, because $u' > \check{u}$, $(\mathbf{u}, u') \overset{*}{P} (\mathbf{u}, \check{u})$, and transitivity implies $(\mathbf{u}, u') \overset{*}{P} \mathbf{u}$ and, therefore, $u' \notin K^n(\mathbf{u})$, a contradiction. If the latter is true, a similar argument establishes that $u'' \notin K^n(\mathbf{u})$. Because $K^n(\mathbf{u})$ is non-empty, it follows that it must be an interval.

To establish (iii), note that, for all $u \notin K^n(\mathbf{u})$, either $(\mathbf{u}, u) \overset{*}{P} \mathbf{u}$ or $\mathbf{u} \overset{*}{P} (\mathbf{u}, u)$. If (A.2) is false, there is a u' such that $u' > \alpha$ for all $\alpha \in K^n(\mathbf{u})$ and $\mathbf{u} \overset{*}{P} (\mathbf{u}, u')$. Pareto-inclusiveness and transitivity of $\overset{*}{R}$ imply that $\mathbf{u} \overset{*}{P} (\mathbf{u}, \alpha)$ for all $\alpha \in K^n(\mathbf{u})$, a contradiction. A similar argument proves (A.3). □

Theorem 1. *Suppose that* $\overset{*}{R}$ *is a quasi-ordering and the same-number principles are GU. The critical-set population principle, the critical-set extension principle, and critical-set independence are satisfied if and only if, for all* $\bar{n}$, $\hat{n}$ *and for all* $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$, $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$, *if* $\bar{n} \neq \hat{n}$,

$$\bar{\mathbf{u}} \overset{*}{P} \hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] > \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}, \tag{A.4}$$

$$\hat{\mathbf{u}} \overset{*}{P} \bar{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] < \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}, \tag{A.5}$$

$$\bar{\mathbf{u}} \overset{*}{N} \hat{\mathbf{u}} \quad \text{otherwise,} \tag{A.6}$$

and if $\bar{n} = \hat{n}$,

$$\bar{\mathbf{u}}\overset{*}{R}\hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} g(\bar{u}^i) \geq \sum_{i=1}^{\hat{n}} g(\hat{u}^i); \tag{A.7}$$

that is, for all $\bar{n}$, $\hat{n}$ *and for all* $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$, $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$,

$$\bar{\mathbf{u}}\overset{*}{R}\hat{\mathbf{u}} \leftrightarrow \sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] \geq \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}. \tag{A.8}$$

Proof. Without loss of generality, assume $\bar{n} > \hat{n}$ and suppose

$$\sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] > \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}. \tag{A.9}$$

Then, because g is continuous and increasing, there is a $\check{u} > \alpha$ for all $\alpha \in \mathscr{K}$ such that

$$\sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\check{u})] \geq \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\check{u})]. \tag{A.10}$$

This is equivalent to

$$\sum_{i=1}^{\bar{n}} g(\bar{u}^i) \geq \sum_{i=1}^{\hat{n}} g(\hat{u}^i) + (\bar{n} - \hat{n})g(\check{u}), \tag{A.11}$$

and because $\bar{\mathbf{u}}$, $(\hat{\mathbf{u}}, \check{u}\mathbf{1}_{\bar{\mathbf{n}}-\hat{\mathbf{n}}}) \in \mathscr{R}^{\bar{\mathbf{n}}}$, GU implies that

$$\bar{\mathbf{u}}\overset{*}{R}(\hat{\mathbf{u}}, \check{u}\mathbf{1}_{\bar{\mathbf{n}}-\hat{\mathbf{n}}}). \tag{A.12}$$

Because $\check{u} > \alpha$ for all $\alpha \in \mathscr{K}$, Lemma 1 and transitivity of $\overset{*}{P}$ imply that

$$(\hat{\mathbf{u}}, \check{u}\mathbf{1}_{\bar{\mathbf{n}}-\hat{\mathbf{n}}})\overset{*}{P}\hat{\mathbf{u}}, \tag{A.13}$$

and transitivity of $\overset{*}{R}$ implies

$$\bar{\mathbf{u}}\overset{*}{P}\hat{\mathbf{u}}. \tag{A.14}$$

Similarly, if

$$\sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\alpha)] < \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\alpha)] \quad \text{for all } \alpha \in \mathscr{K}, \tag{A.15}$$

then

$$\hat{\mathbf{u}}\overset{*}{P}\bar{\mathbf{u}}. \tag{A.16}$$

Now suppose that neither (A.9) nor (A.15) is true. Then, because g is continuous, there exists $\check{\alpha} \in \mathscr{K}$ such that

$$\sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\check{\alpha})] = \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\check{\alpha})] \tag{A.17}$$

or

$$\sum_{i=1}^{\bar{n}} g(\bar{u}^i) = \sum_{i=1}^{\hat{n}} g(\hat{u}^i) + (\bar{n} - \hat{n})g(\check{\alpha}). \tag{A.18}$$

If follows that

$$\bar{\mathbf{u}}\overset{*}{I}(\hat{\mathbf{u}}, \check{\alpha}\mathbf{1}_{\bar{n}-\hat{n}}). \tag{A.19}$$

Because $\check{\alpha} \in \mathscr{K}$, critical-set extension implies

$$(\hat{\mathbf{u}}, \check{\alpha}\mathbf{1}_{\bar{n}-\hat{n}})\overset{*}{N}\hat{\mathbf{u}}. \tag{A.20}$$

If $\bar{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are ranked, transitivity implies that (A.20) is false. It follows, therefore, that

$$\bar{\mathbf{u}}\overset{*}{N}\hat{\mathbf{u}}. \tag{A.21}$$

Sufficiency is easily checked. □

Theorem 2. *If there exist integers n_L^M and n_H^M, $n_L^M \leq n_H^M$, such that P^M consists of all integers from n_L^M to n_H^M inclusive, then* (6.13) *holds; that is, the undominated alternatives in S according to ICLGU are all best choices according to CLGU for some critical level $\alpha \in \mathscr{K}$.*

Proof. First, consider the case of ICLU and CLU and note that, for any $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$, $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$ with $\bar{n} > \hat{n}$,

$$\bar{n}[\bar{\upsilon} - \check{\alpha}] \geq \hat{n}[\hat{\upsilon} - \check{\alpha}] \rightarrow \bar{n}[\bar{\upsilon} - \alpha] > \hat{n}[\hat{\upsilon} - \alpha] \quad \text{for all } \alpha < \check{\alpha}, \tag{A.22}$$

and, for any $\bar{\mathbf{u}} \in \mathscr{R}^{\bar{n}}$, $\hat{\mathbf{u}} \in \mathscr{R}^{\hat{n}}$ with $\bar{n} < \hat{n}$,

$$\bar{n}[\bar{\upsilon} - \check{\alpha}] \geq \hat{n}[\hat{\upsilon} - \check{\alpha}] \rightarrow \bar{n}[\bar{\upsilon} - \alpha] > \hat{n}[\hat{\upsilon} - \alpha] \quad \text{for all } \alpha > \check{\alpha}. \tag{A.23}$$

Consider any $\check{\mathbf{u}} \in S$, with $\check{n} \notin P^M$. By assumption, either $\check{n} < n$ for all $n \in P^M$ or $\check{n} >$ n for all $n \in P^M$. If the former is true, consider $\mathbf{u}_H \in B(S, \alpha_H)$. Then, if $\alpha_H \notin \mathscr{K}$,

$$n_H[\upsilon_H - \alpha_H] \geq \check{n}[\check{\upsilon} - \alpha_H], \tag{A.24}$$

and by (A.22),

$$n_H[\upsilon_H - \alpha] > \check{n}[\check{\upsilon} - \alpha] \quad \text{for all } \alpha < \alpha_H \tag{A.25}$$

and, consequently, for all $\alpha \in \mathscr{K}$. If $\alpha_H \in \mathscr{K}$, then (A.24) holds with a strict inequality and, again, (A.25) holds for all $\alpha \in \mathscr{K}$. It follows that $\check{\mathbf{u}} \notin M(S, \mathscr{K})$.

Now consider any $\check{\mathbf{u}} \in S$, with $\check{n} > n$ for all $n \in P^M$. An analogous argument using (A.23) establishes that, again, $\check{\mathbf{u}} \notin M(S, \mathscr{K})$.

In the case of ICLGU and CLGU, the above argument works with υ replaced by $g(\upsilon)$ and α replaced by $g(\alpha)$. □

Theorem 3. *If there exist integers n_L^B and n_H^B, $n_L^B \leq n_H^B$, such that P^B consists of all integers from n_L^B to n_H^B inclusive, then* (6.13) *holds; that is, the undominated alternatives in S according to ICLGU are all best choices according to CLGU for some critical level $\alpha \in \mathscr{K}$.*

Proof. The proof is similar to the proof of Theorem 2 and is omitted.

Theorem 4. *In the pure population problem with the function g twice differentiable and concave,* (6.13), *holds; that is, the undominated alternatives in S according to ICLGU are all best choices according to CLGU for some critical level $\alpha \in \mathscr{K}$.*

The proof requires two lemmas.

Lemma 2. *In the pure population problem with the function g twice differentiable and concave, there exists, for any feasible population size $\overset{*}{n}$, a value of the critical level $\overset{*}{\alpha}$ (possibly nonpositive) such that population size $\overset{*}{n}$ is best according to CLGU.*

Proof. First, consider the case of ICLU and CLU and define the function $f_\alpha : D \mapsto \mathscr{R}$ where $D := \{t \in \mathscr{R} | t \geq \omega/s\}$ by

$$f_\alpha(t) = t\left[U\left(\frac{\omega}{t}\right) - \alpha\right]. \tag{A.26}$$

Now, for any feasible $\overset{*}{n}$, choose

$$\overset{*}{\alpha} = U\left(\frac{\omega}{\overset{*}{n}}\right) - \frac{\omega}{\overset{*}{n}} U'\left(\frac{\omega}{\overset{*}{n}}\right). \tag{A.27}$$

The first-order condition for maximization of $f_{\overset{*}{\alpha}}(t)$ is

$$f'_{\overset{*}{\alpha}}(t) = U\left(\frac{\omega}{t}\right) - \frac{\omega}{t} U'\left(\frac{\omega}{t}\right) - \overset{*}{\alpha} = 0 \tag{A.28}$$

and it is satisfied at $t = \overset{*}{n}$. The second derivative of f_α is

$$f''_\alpha(t) = \frac{\omega^2}{t^3} U''\left(\frac{\omega}{t}\right) < 0, \tag{A.29}$$

and so $f_\alpha(t)$ is maximized at $t = \overset{*}{n}$. It follows that

$$\overset{*}{n}\left[U\left(\frac{\omega}{\overset{*}{n}}\right) - \overset{*}{\alpha}\right] \geq n\left[U\left(\frac{\omega}{n}\right) - \overset{*}{\alpha}\right] \tag{A.30}$$

for all feasible n. This establishes the theorem in the special case of ICLU and CLU.

In the general case, the above argument works with the function U replaced with $g \circ U$ (which is strongly concave) and α replaced with $g(\alpha)$. □

Lemma 3. *For all* $\bar{\alpha}$, $\hat{\alpha}$ and all $\bar{\mathbf{u}} \in B(S, \bar{\alpha})$, $\hat{\mathbf{u}} \in B(S, \hat{\alpha})$,

$$\bar{\alpha} > \hat{\alpha} \rightarrow \bar{n} \leq \hat{n}. \tag{A.31}$$

Proof. Suppose that $\bar{\mathbf{u}} \in B(S, \bar{\alpha})$ and $\hat{\mathbf{u}} \in B(S, \hat{\alpha})$. Then

$$\sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\bar{\alpha})] \geq \sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\bar{\alpha})] \tag{A.32}$$

and

$$\sum_{i=1}^{\hat{n}} [g(\hat{u}^i) - g(\hat{\alpha})] \geq \sum_{i=1}^{\bar{n}} [g(\bar{u}^i) - g(\hat{\alpha})]. \tag{A.33}$$

Adding (A.32) and (A.33),

$$[\bar{n} - \hat{n}][g(\bar{\alpha}) - g(\hat{\alpha})] \leq 0, \tag{A.34}$$

and if $\bar{\alpha} > \hat{\alpha}$, $g(\bar{\alpha}) > g(\hat{\alpha})$, which implies $\bar{n} \leq \hat{n}$. □

Proof of Theorem 4. Suppose that there is a $\check{\mathbf{u}} \in S$ with $\check{n} \notin P^B$ and there exist n_L^B, $n_H^B \in P^B$ with $n_L^B < \check{n} < n_H^B$. By Lemma 2, there exists $\check{\alpha} \in \mathscr{R}$ such that $\check{\mathbf{u}} \in B(S, \check{\alpha})$. By Lemma 3, it must be true that $\check{\alpha} > \alpha_L$ if $\alpha_L \notin \mathscr{K}$ and $\check{\alpha} \geq \alpha_L$ if $\alpha_L \in \mathscr{K}$. Similarly, $\check{\alpha} < \alpha_H$ if $\alpha_H \notin \mathscr{K}$ and $\check{\alpha} \leq \alpha_H$ if $\alpha_H \in \mathscr{K}$. Consequently, $\check{\alpha} \in \mathscr{K}$, a contradiction, and therefore, P^B has no gaps. Theorem 3 implies that (6.13) holds, and Theorem 4 follows. □

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